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Einstein Solvmanifolds and Two-Step Nilpotent Lie Algebras with a Special Nice Basis

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Abstract. Consider a two-step nilpotent Lie algebra $\mathfrak{n}$ with a special nice basis as introduced in Y. Nikolayevsky, *Einstein solvmanifolds and the pre-Einstein derivation*, Trans. Amer. Math. Soc, **363** (2011), 3935-3958, endowed with an inner product which makes the basis orthonormal. We describe necessary and sufficient conditions for the existence of a rank-one Einstein metric solvable extension of $\mathfrak{n}$. Since every two-step nilpotent Lie algebra attached to a graph (as introduced in S. G. Dani, M. G. Mainkar, *Anosov automorphisms on compact nilmanifolds associated with graphs*, Trans. Amer. Math. Soc. **357** (2005), 2235-2251) has such a nice basis, this Note generalizes the result of H.-R. Fanaï, *Einstein solvmanifolds and graphs*, C. R. Acad. Sci. Paris, Ser. I, **344** (2007), 37-39.
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1. Introduction

Our aim in this Note is the construction of examples of homogeneous Einstein manifolds of negative scalar curvature. The classical examples of Einstein metrics of negative scalar curvature are the symmetric spaces of non-compact type. Some other examples are known, e.g. [[1], [4], [5], [6], [8], [9] and [12]].

It is interesting that all known examples of homogeneous Einstein manifolds of negative scalar curvature are isometric to Einstein Riemannian solvmanifolds, e.g. a simply connected solvable Lie group S together with a left invariant Einstein Riemannian metric g. In these examples, such a solvable Lie group S is a semi-direct product of an abelian Lie group A with a nilpotent normal subgroup N, the nilradical. We consider the class of solvmanifolds for which N is two-step nilpotent with a nice basis as introduced in [11], and A is one-dimensional. The left invariant Riemannian metric g on S defines an inner product $\langle , \rangle$ on the Lie algebra $\mathfrak{s}$ of S. We will refer to a Lie algebra endowed with an inner product as a metric Lie algebra.

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In this Note, we are interested in a metric two-step nilpotent Lie algebra $\mathfrak{n}$ with a special nice basis and study whether there is a metric solvable extension of $(\mathfrak{n}, \langle,\rangle)$ which is Einstein. More precisely, we construct a metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle,\rangle)$ where $\mathfrak{a}$ is one-dimensional orthogonal to $\mathfrak{n}$ and the Lie bracket and the inner product on $\mathfrak{s}$ restricted to $\mathfrak{n}$ are those of $\mathfrak{n}$ and describe necessary and sufficient conditions, in terms of the nice basis, for the manifold $\mathfrak{s}$ to be Einstein. Our result is contained in Theorem 3.2 in the third section. This result generalizes the main result of [3]. We recall some definitions and notations in the next section.

2. Preliminaries

We recall the definition of a nice basis as introduced in [11].

Definition 2.1. Let $\{X_1, ..., X_n\}$ be a basis for a nilpotent Lie algebra $\mathfrak{n}$, with structural constants c_{ij}^k's given by $[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$. Then the basis $\{X_i\}$ is said to be *nice* if the following conditions hold:

- for all $i < j$ there is at most one k such that $c_{ij}^k \neq 0$,
- if c_{ij}^k and $c_{i'j'}^k$ are nonzero then either $\{i, j\} = \{i', j'\}$ or $\{i, j\} \cap \{i', j'\} = \emptyset$.

Let $\mathfrak{n}$ be a two-step n-dimensional nilpotent Lie algebra. As it is explained in [4] p. 189, for our purpose without loss of generality, we can assume that the center of $\mathfrak{n}$ is equal to $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$. Suppose that $\{X_1, ..., X_n\}$ is a nice basis for $\mathfrak{n}$. Now we assume that our nice basis is special, i.e. all nonzero structural constants c_{ij}^k are equal to 1 for $i < j$. In the last section we give many examples of such a special nice basis. We endow $\mathfrak{n}$ with an inner product $\langle,\rangle$ which makes this basis orthonormal.

Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{h}$ where $\mathfrak{h}$ is the vector space such that $\mathfrak{z} \perp \mathfrak{h}$. Since $\mathfrak{n}$ is two-step nilpotent Lie algebra, if $[X_i, X_j] = X_l$ then for all l', we have $[X_l, X_{l'}] = 0$ so $X_l \in \mathfrak{z}$ hence

$$\{X_l : X_l \in \text{nice basis of } \mathfrak{n}, [X_i, X_j] = X_l \ \exists i, j\}$$

is an orthonormal basis for $\mathfrak{z}$ which for simplicity we write it as $\{Z_1, ..., Z_r\}$, with $r = \dim \mathfrak{z}$. So each Z_s $(1 \leq s \leq r)$ is of the form $[X_i, X_j]$ for some $i < j$. Without loss of generality, we can reorder the elements of the basis such that $\{X_1, ..., X_m\}$ is a basis for $\mathfrak{h}$, where $m = \dim \mathfrak{h}$. So we have

$$\{Z_1, ..., Z_r\} = \{X_{m+1}, ..., X_n\}.$$

We can then define a linear map $J : \mathfrak{z} \to \mathfrak{so}(\mathfrak{h})$ by

$$\langle J(z)x, y\rangle = \langle [x, y], z\rangle$$

for $x, y \in \mathfrak{h}$ and $z \in \mathfrak{z}$. The endomorphism J^*J of $\mathfrak{z}$ is represented by the $r \times r$ matrix $\operatorname{tr}({}^tJ(Z_i)J(Z_j))_{i,j}$.

Consider now a metric solvable extension $(\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{n}, \langle\cdot,\cdot\rangle)$ of $\mathfrak{n}$ where the norm of A is one. We let $f := \mathrm{ad}_{\mathrm{A}} : \mathfrak{n} \to \mathfrak{n}$. We can suppose that $\mathrm{tr}f > 0$. According to [6] and [7], if the solvmanifold is Einstein, then the matrix representation of f in the above bases is in the following form:

$$f = \begin{pmatrix} B_r & 0 \\ 0 & D_m \end{pmatrix}.$$

In fact the solvmanifold $\mathfrak{s}$ is Einstein if and only if there exists a (negative) constant μ such that we have the following relations (see [6], [7]):

(E_1) $-(\mathrm{tr}f)B + \frac{1}{4}J^*J = \mu Id_r,$

(E_2) $-(\mathrm{tr}f)D + \frac{1}{2}\sum_{i=1}^{r} J^2 Z_{,i}\,) = \mu Id_m,$

(E_3) $J(B(\cdot)) = J(\cdot)D + DJ(\cdot).$

The scalar curvature of the Einstein solvmanifold is equal to $(n+1)\mu$. In the next section, we consider these relations.

3. Result

Suppose that the relations $(E_1), (E_2)$ and (E_3) are satisfied for f (for a certain nagative constant μ) and thus the solvable extension is Einstein. We are looking for conditions imposed on the two-step nilpotent Lie algebra with a special nice basis by these relations.

The matrix representation of $J(Z_i)$; $1 \le i \le r$, is easy to obtain. According to nice basis' definition, if the element Z_i is only equal to one $[X_j, X_{j'}]$ with $1 \le j < j' \le m$, then the only nonzero entries of $J(Z_i)$ are $J(Z_i)_{jj'} = 1$ and $J(Z_i)_{j'j} = -1$. On the other hand, if the element Z_i is also equal to another $[X_l, X_{l'}]$ such that $\{j, j'\} \neq \{l, l'\}$, then by Definition 2.1, $\{j, j'\} \cap \{l, l'\} = \emptyset$. Hence the nonzero entries of $J(Z_i)$ are increased. So by $\{Z_1, ..., Z_r\} = \{X_{m+1}, ..., X_n\}$ we have

$$J^*J = \mathrm{tr}({}^tJ(Z_i)J(Z_j))_{i,j} = \begin{bmatrix} a_{m+1} & & & 0 \\ & a_{m+2} & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix},$$

where a_s $(m+1 \le s \le n)$ is twice the number of times that X_s appears in all relations $[X_i, X_j] = X_s$ for $1 \le i < j \le m$. Also we obtain

$$\sum_{i=1}^{r} J^2(Z_i) = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_m \end{bmatrix},$$

where a_i $(1 \le i \le m)$ is minus the number of times that X_i appears in all relations $[X_i, X_j] = X_s$ or $[X_j, X_i] = X_s$ for $m+1 \le s \le n, 1 \le j \le m$.

Using the relation (E_1) we have

$$B = \frac{-1}{\mathrm{trf}} \begin{bmatrix} \mu - \frac{1}{4}a_{m+1} & & 0 \\ & \ddots & \\ 0 & & \mu - \frac{1}{4}a_n \end{bmatrix}.$$

This implies from (E_2) that

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} \mu - \frac{1}{2}a_1 & & 0 \\ & \ddots & \\ 0 & & \mu - \frac{1}{2}a_m \end{bmatrix}.$$

Now suppose that $[X_i, X_j] = X_{m+1} = Z_1$. So the entry ij in $J(Z_1)$ is nonzero, it follows from (E_3) that

$$\frac{-1}{\mathrm{trf}}(\mu - \frac{1}{4}a_{m+1}) = \frac{-1}{\mathrm{trf}}(\mu - \frac{1}{2}a_j) + \frac{-1}{\mathrm{trf}}(\mu - \frac{1}{2}a_i).$$

So

$$\mu = -\frac{1}{4}a_{m+1} + \frac{1}{2}a_j + \frac{1}{2}a_i. \tag{1}$$

In the same way, we can write similar relations for other nonzero entries of $J(Z_1)$ and other $J(Z_i)$s $(1 \leq i \leq r)$ as well. So for being Einstein, all the μs obtained by relation 1 for every $J(Z_i), (1 \leq i \leq r)$ should be equal. Now it is easy to deduce from 1 that we have

$$\begin{aligned} -2\mu = &(\mathrm{the\ number\ of\ X_{m+1}\ in\ all\ relations\ [X_i, X_j] = X_{m+1},\ for\ all\ i, j}) \\ &+ (\mathrm{the\ number\ of\ X_j\ in\ all\ relations\ [X_i, X_j] = X_s,\ for\ all\ i, s}) \\ &+ (\mathrm{the\ number\ of\ X_i\ in\ all\ relations\ [X_i, X_j] = X_s,\ for\ all\ j, s}). \end{aligned}$$

For all i, j, k this summation is easy to compute. If all are equal, the Einstein conditions are satisfied and the metric solvable extension of $\mathfrak{n}$ is Einstein. In this case from $[X_i, X_j] = X_k$ we use the simple notation

$$\mu = -\frac{(\sharp X_i) + (\sharp X_j) + (\sharp X_k)}{2}.$$

So the following definition makes sense.

Definition 3.1. Let $\mathfrak{n}$ be a metric two-step nilpotent Lie algebra with a special nice basis as explained before. For all i, j, k with $[X_i, X_j] = X_k$, we associate the number a_{ij}^k such that

$$\begin{aligned} a_{ij}^k = &(\mathrm{the\ number\ of\ X_i\ in\ all\ relations\ of\ basis}) \\ &+ (\mathrm{the\ number\ of\ X_j\ in\ all\ relations\ of\ basis}) \\ &+ (\mathrm{the\ number\ of\ X_k\ in\ all\ relations\ of\ basis}) \\ = &(\sharp X_i) + (\sharp X_j) + (\sharp X_k). \end{aligned}$$

So we have shown the following theorem:

Theorem 3.2. *Let $(\mathfrak{n}, \langle , \rangle)$ be a metric two-step nilpotent Lie algebra with a special nice basis as explained before. Then there is a metric solvable extension of $\mathfrak{n}$ which is Einstein if and only if all the numbers a_{ij}^k are equal. In this case we have $\mu = -\frac{a_{ij}^k}{2}$ for any i, j, k with $[X_i, X_j] = X_k$.*

Now we give some examples and apply our result.

Example 3.3. According to the notation of [10], let $\mathfrak{n} = \mathcal{G}_{7,3,24}$ be a metric two-step nilpotent Lie algebra with a nice basis with these relations

1. $[X_1, X_2] = X_5$,
2. $[X_2, X_3] = X_6$,
3. $[X_2, X_4] = X_7$,
4. $[X_3, X_4] = X_5$.

We compute the number of $X_i, (1 \leq i \leq 7)$ in above relations. From 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_5) = 1 + 3 + 2 = 6$, from 2 we have $(\sharp X_2) + (\sharp X_3) + (\sharp X_6) = 3 + 2 + 1 = 6$, from 3 we have $(\sharp X_2) + (\sharp X_4) + (\sharp X_7) = 3 + 2 + 1 = 6$ and from 4 we have $(\sharp X_3) + (\sharp X_4) + (\sharp X_5) = 2 + 2 + 2 = 6$, all the numbers are equal so the metric solvable extension of $\mathfrak{n}$ is Einstein and $\mu = -\frac{6}{2} = -3$.

Example 3.4. Let $\mathfrak{n} = \mathcal{G}_{7,3,12}$ be a metric two-step nilpotent Lie algebra with a nice basis like:

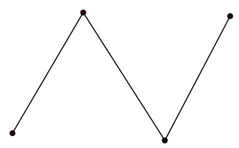
1. $[X_1, X_2] = X_5$,
2. $[X_1, X_3] = X_6$,
3. $[X_2, X_4] = X_6$,
4. $[X_3, X_4] = X_7$.

We compute the number of $X_i, (1 \leq i \leq 7)$ in above relations. From 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_5) = 2 + 2 + 1 = 5$ and from 2 we have $(\sharp X_1) + (\sharp X_3) + (\sharp X_6) = 2 + 2 + 2 = 6$, since the numbers are different then the metric solvable extension of $\mathfrak{n}$ is not Einstein.

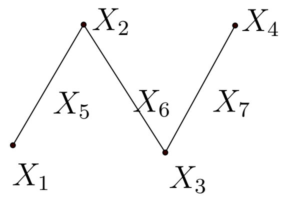
Corollary 3.5. *Let $(\mathfrak{n}, \langle , \rangle)$ be a metric two-step nilpotent Lie algebra with a special nice basis as explained before with $r = \dim \mathfrak{z} = 1$. Then there is always a metric solvable extension of $\mathfrak{n}$ which is Einstein.*

Actually we just have $Z_1 = X_{m+1}$ and by Definition 2.1 from $[X_i, X_j] = Z_1 = [X_l, X_s]$ we have $\{i, j\} = \{l, s\}$ or $\{i, j\} \cap \{l, s\} = \emptyset$. So all the numbers a_{ij}^1 are equal.

Remark 3.6. Let G be the following graph



and let $\mathfrak{n}$ be the metric two-step nilpotent Lie algebra associated with G (ref. [2], [9]). We can write the basis relations:



1. $[X_1, X_2] = X_5$,
2. $[X_2, X_3] = X_6$,
3. $[X_3, X_4] = X_7$,

Now we investigate the condition of Theorem 3.2: from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_5) = 1{+}2{+}1 = 4$ and from 2 we have $(\sharp X_2){+}(\sharp X_3){+}(\sharp X_6) = 2{+}2{+}1 = 5$, since the numbers are different then the metric solvable extension of $\mathfrak{n}$ is not Einstein. As this graph is not regular or not a bipartite graph such that all vertices in each partite set have the same degree, the result of [3] gives the same nonexistence answer.

4. Investigation on two-step nilpotent Lie algebras with a nice basis of dimension$\leq$ 7

By using notations and classifications given in [10], we present all metric two-step nilpotent Lie algebras with a nice basis of dimension ≤ 7. Fortunately all these spaces have a nice basis which is special as we considered before. Hence we can apply our result for all of them.

Example 4.1. Let $\mathfrak{n}$ be $\mathcal{G}_3$ the 3-dimensional Heisenberg Lie algebra. We have only one relation for the nice basis:

1. $[X_1, X_2] = X_3$,

so we have $Z_1 = X_3$ and by Corollary 3.5, as there is a unique $Z_1 = X_3$ the metric solvable extension of this Lie algebra is Einstein and from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_3) = 1 + 1 + 1 = 3 = a$ so $\mu = -\frac{3}{2}$. In fact we have

$$J^*J = (2), \quad \sum_{i=1}^{1} J^2(Z_i) = -Id_2.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2}) = \lambda$ and from (E_2) we have $D = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})Id_2$ and let $d = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})$. This implies from (E_3) that

$$\lambda J(Z_1) = J(Z_1)D + DJ(Z_1)$$

and so $\lambda = 2d$. It follows again that $\mu = -\frac{3}{2}$.

Example 4.2. Let $\mathfrak{n}$ be $\mathcal{G}_{5,1}$ then we have these relations for the nice basis:

1. $[X_1, X_3] = X_5$,
2. $[X_2, X_4] = X_5$,

again there is a unique $Z_1 = X_5$, so the metric solvable extension of this Lie algebra is Einstein and from 1 we have $(\sharp X_1) + (\sharp X_3) + (\sharp X_5) = 1 + 1 + 2 = 4 = a$ so $\mu = -\frac{4}{2} = -2$. In fact we have $Z_1 = X_5$ and

$$J^*J = (4), \quad \sum_{i=1}^{1} J^2(Z_i) = -Id_4.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - 1) = \lambda$ and from (E_2) we have $D = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})Id_4$ and let $d = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})$. This implies from (E_3) that

$$\lambda J(Z_1) = J(Z_1)D + DJ(Z_1)$$

and so $\lambda = 2d$. It follows that $\mu = -2$ as we saw above.

Example 4.3. Let $\mathfrak{n}$ be $\mathcal{G}_{5,2}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_4$,
2. $[X_1, X_3] = X_5$,

from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_4) = 2 + 1 + 1 = 4$ and from 2 we have $(\sharp X_1) + (\sharp X_3) + (\sharp X_6) = 2 + 1 + 1 = 4$. So the numbers are equal and the extension is Einstein and $\mu = -\frac{4}{2} = -2$. In fact we have $Z_1 = X_4$, $Z_2 = X_5$ and and also

$$J^*J = 2Id_2, \quad \sum_{i=1}^{2} J^2(Z_i) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})Id_2$, let $\lambda = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})$ and from (E_2) we have

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} 1+\mu & 0 & 0 \\ 0 & \frac{1}{2}+\mu & 0 \\ 0 & 0 & \frac{1}{2}+\mu \end{bmatrix},$$

let $d = \frac{-1}{\mathrm{trf}}(1 + \mu)$ and $d' = \frac{-1}{\mathrm{trf}}(\frac{1}{2} + \mu)$. This implies from (E_3) that

$$\lambda J(Z_i) = J(Z_i)D + DJ(Z_i), (1 \leq i \leq 2)$$

so $d + d' = \lambda$. It follows that $\mu = -2$ as above.

Example 4.4. Let $\mathfrak{n}$ be $\mathcal{G}_{6,1}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_5$,
2. $[X_1, X_4] = X_6$,
3. $[X_2, X_3] = X_6$,

so we have $Z_1 = X_5$, $Z_2 = X_6$ and

$$J^*J = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad \sum_{i=1}^{2} J^2(z_i) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

From (E_1) we have

$$B = -\frac{1}{\mathrm{trf}} \begin{bmatrix} \mu - \frac{1}{2} & 0 \\ 0 & \mu - 1 \end{bmatrix},$$

let $\lambda_1 = \frac{-1}{\mathrm{trf}}(\mu - \frac{1}{2})$ and $\lambda_2 = \frac{-1}{\mathrm{trf}}(\mu - 1)$. Furthermore, from (E_2) we have

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} \mu + 1 & 0 & 0 & 0 \\ 0 & \mu + 1 & 0 & 0 \\ 0 & 0 & \mu + \frac{1}{2} & 0 \\ 0 & 0 & 0 & \mu + \frac{1}{2} \end{bmatrix},$$

let $d_1 = \frac{-1}{\mathrm{trf}}(\mu + 1)$ and $d_2 = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})$. This implies from (E_3) that

$$\lambda_i J(Z_i) = J(Z_i)D + DJ(Z_i), (1 \le i \le 2)$$

and then we have $\lambda_1 = 2d_1$ and $\lambda_2 = d_1 + d_2$. It follows that $\mu = -\frac{5}{2}$, hence the metric solvable extension of $\mathfrak{n}$ is Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_5) = 2 + 2 + 1 = 5$, from 2 we have $(\sharp X_1) + (\sharp X_4) + (\sharp X_6) = 2 + 1 + 2 = 5$ and from 3 we have $(\sharp X_2) + (\sharp X_3) + (\sharp X_6) = 2 + 1 + 2 = 5$ so the numbers are equal and the extension is Einstein and $\mu = -\frac{5}{2}$.

Example 4.5. Let $\mathfrak{n}$ be $\mathcal{G}_{6,3}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_4$,
2. $[X_1, X_3] = X_5$,
3. $[X_2, X_3] = X_6$,

so we have $Z_1 = X_4$, $Z_2 = X_5$, $Z_3 = X_6$ and

$$J^*J = 2Id_3, \quad \sum_{i=1}^{3} J^2(z_i) = -2Id_3.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})Id_3$, let $\lambda = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})$ and from (E_2) we have $D = \frac{-1}{\mathrm{trf}}(\mu + 1)Id_3$, let $d = \frac{-1}{\mathrm{trf}}(\mu + 1)$. This implies from (E_3) that

$$\lambda J(Z_i) = J(Z_i)D + DJ(Z_i), (1 \le i \le 3)$$

and so $2d = \lambda$. It follows that $\mu = -\frac{5}{2}$, hence the metric solvable extension of $\mathfrak{n}$ is Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_4) = 2 + 2 + 1 = 5$, from 2 we have $(\sharp X_1) + (\sharp X_3) + (\sharp X_5) = 2 + 2 + 1 = 5$ and also from 3 we have $(\sharp X_2) + (\sharp X_3) + (\sharp X_6) = 2 + 2 + 1 = 5$ so the numbers are equal and the extension is Einstein and $\mu = -\frac{5}{2}$.

Example 4.6. Let $\mathfrak{n}$ be $\mathcal{G}_{7,3,12}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_5$,
2. $[X_1, X_3] = X_6$,
3. $[X_2, X_4] = X_6$,
4. $[X_3, X_4] = X_7$,

so we have $Z_1 = X_5,\ Z_2 = X_6,\ Z_3 = X_7$ and

$$J^*J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \sum_{i=1}^{3} J^2(z_i) = -2Id_4.$$

From (E_1) we have

$$B = -\frac{1}{\mathrm{trf}} \begin{bmatrix} \mu - \frac{1}{2} & 0 & 0 \\ 0 & \mu - 1 & 0 \\ 0 & 0 & \mu - \frac{1}{2} \end{bmatrix},$$

let $\lambda_1 = \frac{-1}{\mathrm{trf}}(\mu - \frac{1}{2})$ and $\lambda_2 = \frac{-1}{\mathrm{trf}}(\mu - 1)$. Moreover, from (E_2) we have $D = \frac{-1}{\mathrm{trf}}(\mu + 1)Id_4$, let $d = \frac{-1}{\mathrm{trf}}(\mu + 1)$. This implies from (E_3) that

$$\lambda_1 J(Z_1) = J(Z_1)D + DJ(Z_1)$$

so $\lambda_1 = 2d$ and then $\mu = -\frac{5}{2}$. From

$$\lambda_2 J(Z_2) = J(Z_2)D + DJ(Z_2)$$

we have $\lambda_2 = 2d$ and so $\mu = -3$. It follows that the μs are different, hence the metric solvable extension of $\mathfrak{n}$ is not Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_5) = 2 + 2 + 1 = 5$ and from 2 we have $(\sharp X_1) + (\sharp X_3) + (\sharp X_6) = 2 + 2 + 2 = 6$, so the numbers are different and the extension is not Einstein.

Example 4.7. Let $\mathfrak{n}$ be $\mathcal{G}_{7,3,24}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_5$,
2. $[X_3, X_4] = X_5$,
3. $[X_2, X_3] = X_6$,
4. $[X_2, X_4] = X_7$,

so we have $Z_1 = X_5,\ Z_2 = X_6,\ Z_3 = X_7$ and

$$J^*J = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \sum_{i=1}^{3} J^2(z_i) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

From (E_1) we have

$$B = -\frac{1}{\mathrm{trf}} \begin{bmatrix} \mu - 1 & 0 & 0 \\ 0 & \mu - \frac{1}{2} & 0 \\ 0 & 0 & \mu - \frac{1}{2} \end{bmatrix},$$

let $\lambda_1 = \frac{-1}{\mathrm{trf}}(\mu - 1)$ and $\lambda_2 = \frac{-1}{\mathrm{trf}}(\mu - \frac{1}{2})$. Now, from (E_2) we have

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} \mu + \frac{1}{2} & 0 & 0 & 0 \\ 0 & \mu + \frac{3}{2} & 0 & 0 \\ 0 & 0 & \mu + 1 & 0 \\ 0 & 0 & 0 & \mu + 1 \end{bmatrix},$$

let $d_1 = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})$, $d_2 = \frac{-1}{\mathrm{trf}}(\mu + \frac{3}{2})$ and $d_3 = \frac{-1}{\mathrm{trf}}(\mu + 1)$. This implies from (E_3) that

$$\lambda_i J(Z_i) = J(Z_i)D + DJ(Z_i), (1 \leq i \leq 3)$$

and then we have $\lambda_1 = d_2 + d_1$, $\lambda_1 = 2d_3$ and $\lambda_2 = d_2 + d_3$. It follows that $\mu = -3$, hence the metric solvable extension of $\mathfrak{n}$ is Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_5) = 1 + 3 + 2 = 6$, from 2 we have $(\sharp X_3) + (\sharp X_4) + (\sharp X_5) = 2 + 2 + 2 = 6$, from 3 we have $(\sharp X_2) + (\sharp X_3) + (\sharp X_6) = 3 + 2 + 1 = 6$ and also from 4 we have $(\sharp X_2) + (\sharp X_4) + (\sharp X_7) = 3 + 2 + 1 = 6$ so the numbers are equal and the extension is Einstein and $\mu = -\frac{6}{2} = -3$.

Example 4.8. Let $\mathfrak{n}$ be $\mathcal{G}_{7,4,1}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_5$,
2. $[X_1, X_3] = X_6$,
3. $[X_3, X_4] = X_7$,

so we have $Z_1 = X_5,\ Z_2 = X_6,\ Z_3 = X_7$ and

$$J^*J = 2Id_3, \quad \sum_{i=1}^{3} J^2(z_i) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})Id_3$, let $\lambda = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})$ and from (E_2) we have

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} 1 + \mu & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \mu & 0 & 0 \\ 0 & 0 & 1 + \mu & 0 \\ 0 & 0 & 0 & \frac{1}{2} + \mu \end{bmatrix},$$

let $d = -\frac{1}{\mathrm{trf}}(1+\mu)$ and $d' = -\frac{1}{\mathrm{trf}}(\frac{1}{2}+\mu)$. This implies from (E_3) that

$$\lambda J(Z_1) = J(Z_1)D + DJ(Z_1)$$

and so $d + d' = \lambda$ it follows that $\mu = -2$. Moreover, from

$$\lambda J(Z_2) = J(Z_2)D + DJ(Z_2)$$

we have $d + d = \lambda$ and hence $\mu = -\frac{5}{2}$, so the metric solvable extension of $\mathfrak{n}$ is not Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_5) = 2 + 1 + 1 = 4$ and from 2 we have $(\sharp X_1) + (\sharp X_3) + (\sharp X_6) = 2 + 2 + 1 = 5$ so the numbers are different and the extension is not Einstein.

Example 4.9. Let $\mathfrak{n}$ be $\mathcal{G}_{7,4,2}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_5$,
2. $[X_1, X_3] = X_6$,
3. $[X_1, X_4] = X_7$,

so we have $Z_1 = X_5$, $Z_2 = X_6$, $Z_3 = X_7$ and

$$J^*J = 2Id_3, \quad \sum_{i=1}^{3} J^2(z_i) = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})Id_3$ and let $\lambda = -\frac{1}{\mathrm{trf}}(\mu - \frac{1}{2})$ and from (E_2) we have

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} \frac{3}{2}+\mu & 0 & 0 & 0 \\ 0 & \frac{1}{2}+\mu & 0 & 0 \\ 0 & 0 & \frac{1}{2}+\mu & 0 \\ 0 & 0 & 0 & \frac{1}{2}+\mu \end{bmatrix},$$

so let $d = \frac{-1}{\mathrm{trf}}(\frac{3}{2}+\mu)$ and $d' = \frac{-1}{\mathrm{trf}}(\frac{1}{2}+\mu)$. This implies from (E_3) that

$$\lambda J(Z_i) = J(Z_i)D + DJ(Z_i), (1 \le i \le 3)$$

and so $d + d' = \lambda$. It follows that $\mu = -\frac{5}{2}$, hence the metric solvable extension of $\mathfrak{n}$ is Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1)+(\sharp X_2)+(\sharp X_5) = 3+1+1 = 5$, from 2 we have $(\sharp X_1)+(\sharp X_3)+(\sharp X_6) = 3+1+1 = 5$ and also from 3 we have $(\sharp X_1) + (\sharp X_4) + (\sharp X_7) = 3 + 1 + 1 = 5$ so the numbers are equal and the extension is Einstein and $\mu = -\frac{5}{2}$.

Example 4.10. Let $\mathfrak{n}$ be $\mathcal{G}_{7,4,3}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_6$,
2. $[X_3, X_5] = X_6$,
3. $[X_4, X_5] = X_7$,

so we have $Z_1 = X_6$, $Z_2 = X_7$ and

$$J^*J = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \quad \sum_{i=1}^{2} J^2(z_i) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

From (E_1) we have

$$B = -\frac{1}{\mathrm{trf}} \begin{bmatrix} \mu - 1 & 0 \\ 0 & \mu - \frac{1}{2} \end{bmatrix},$$

let $\lambda_1 = \frac{-1}{\mathrm{trf}}(\mu - 1)$ and $\lambda_2 = \frac{-1}{\mathrm{trf}}(\mu - \frac{1}{2})$. Moreover, from (E_2) we have

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} \mu + \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \mu + \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \mu + \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \mu + \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \mu + 1 \end{bmatrix},$$

let $d_1 = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})$ and $d_2 = \frac{-1}{\mathrm{trf}}(\mu + 1)$. This implies from (E_3) that

$$\lambda_1 J(Z_1) = J(Z_1)D + DJ(Z_1)$$

so $\lambda_1 = 2d_1$ hence $\mu = -2$ and also $\lambda_1 = d_2 + d_1$ then $\mu = -\frac{5}{2}$. It follows that the μs are different hence the metric solvable extension of $\mathfrak{n}$ is not Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1) + (\sharp X_2) + (\sharp X_6) = 1 + 1 + 2 = 4$ and from 2 we have $(\sharp X_3) + (\sharp X_5) + (\sharp X_6) = 1 + 2 + 2 = 5$, so the numbers are different and the extension is not Einstein.

Example 4.11. Let $\mathfrak{n}$ be $\mathcal{G}_{7,4,4}$ then we have these relations for the nice basis:

1. $[X_1, X_4] = X_7$,
2. $[X_2, X_5] = X_7$,
3. $[X_3, X_6] = X_7$,

so we have $Z_1 = X_7$ and

$$J^*J = (6), \quad \sum_{i=1}^{1} J^2(z_i) = -Id_6.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - \frac{3}{2}) = \lambda$ and from (E_2) we have $D = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})Id_6$ so let $d = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})$. This implies from (E_3) that

$$\lambda J(Z_1) = J(Z_1)D + DJ(Z_1)$$

and then $\lambda = 2d$. It follows that $\mu = -\frac{5}{2}$, hence the metric solvable extension of $\mathfrak{n}$ is Einstein. On the other hand, by Corollary 3.5, there is a unique $Z_1 = X_7$ so the metric solvable extension of this Lie algebra is Einstein and from 1 we have $(\sharp X_1) + (\sharp X_4) + (\sharp X_7) = 1 + 1 + 3 = 5 = a$ so $\mu = -\frac{5}{2}$.

Example 4.12. Let $\mathfrak{n}$ be $\mathcal{G}_{7,8,19}$ then we have these relations for the nice basis:

1. $[X_1, X_2] = X_6$,
2. $[X_3, X_4] = X_6$,
3. $[X_1, X_3] = X_7$,
4. $[X_4, X_5] = X_7$,

so we have $Z_1 = X_6$, $Z_2 = X_7$ and

$$J^*J = 4Id_2, \quad \sum_{i=1}^{2} J^2(z_i) = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

From (E_1) we have $B = -\frac{1}{\mathrm{trf}}(\mu - 1)Id_2$, let $\lambda = \frac{-1}{\mathrm{trf}}(\mu - 1)$. Furthermore, from (E_2) we have

$$D = \frac{-1}{\mathrm{trf}} \begin{bmatrix} \mu+1 & 0 & 0 & 0 & 0 \\ 0 & \mu+\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \mu+1 & 0 & 0 \\ 0 & 0 & 0 & \mu+1 & 0 \\ 0 & 0 & 0 & 0 & \mu+\frac{1}{2} \end{bmatrix},$$

let $d_1 = \frac{-1}{\mathrm{trf}}(\mu + 1)$ and $d_2 = \frac{-1}{\mathrm{trf}}(\mu + \frac{1}{2})$. This implies from (E_3) that

$$\lambda_1 J(Z_1) = J(Z_1)D + DJ(Z_1)$$

so $\lambda = d_2 + d_1$ hence $\mu = -\frac{5}{2}$ and $\lambda = 2d_1$ then $\mu = -3$. It follows that the μs are different hence the metric solvable extension of $\mathfrak{n}$ is not Einstein. On the other hand, by Theorem 3.2, from 1 we have $(\sharp X_1)+(\sharp X_2)+(\sharp X_6) = 2+1+2 = 5$ and from 2 we have $(\sharp X_3) + (\sharp X_4) + (\sharp X_6) = 2 + 2 + 2 = 6$, so the numbers are different and the extension is not Einstein.

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