

Structures of Nichols (Braided) Lie Algebras of Diagonal Type

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Communicated by K. Schmüdgen

Abstract. Let V be a braided vector space of diagonal type. Let $\mathfrak{B}(V)$, $\mathfrak{L}^-(V)$ and $\mathfrak{L}(V)$ be the Nichols algebra, Nichols Lie algebra and Nichols braided Lie algebra over V , respectively. We show that a monomial belongs to $\mathfrak{L}(V)$ if and only if this monomial is connected. We obtain the basis for $\mathfrak{L}(V)$ of arithmetic root systems and the dimension of $\mathfrak{L}(V)$ of finite Cartan type. We give the sufficient and necessary conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$ and $\mathfrak{L}^-(V) = \mathfrak{L}(V)$. We obtain an explicit basis for $\mathfrak{L}^-(V)$ over the quantum linear space V with $\dim V = 2$.

Mathematics Subject Classification 2010: 16W30, 16G10.

Key Words and Phrases: Braided vector space, Nichols algebra, Nichols (braided) Lie algebra, Graph.

1. Introduction

Nichols algebras have found significant applications in various areas of mathematics and mathematical physics including the theories of pointed Hopf algebras and logarithmic quantum fields. In [He05], one-to-one correspondences between Nichols algebras of diagonal type and arithmetic root systems as well as between generalized Dynkin diagrams and twisted equivalence classes of arithmetic root systems were established. The problem of finite-dimensionality of Nichols algebras forms a substantial part of the recent investigations (see e.g. [AHS08, AS10, He05, He06a, He06b, WZZ15a, WZZ15b]). Lie algebra arising from a Nichols algebra was studied in [AAB16].

Braided Lie algebras were studied in [Ka77, BMZP92, Gu86, GRR95, Kh99, Pa98, Sc79, BFM96, ZZ03]. The current paper will focus on Nichols Lie and braided Lie algebras. In [He05] and [He06a], a classification on braided vector spaces of diagonal type with finite-dimensional Nichols algebras was given. In [WZZ15b, WZZ16], we studied the relationship between Nichols algebras and Nichols braided Lie algebras. It was proven that a Nichols algebra is finite-dimensional if and only if the corresponding Nichols (braided) Lie algebra is finite-

* YZZ was partially supported by the Australian Research Council through Discovery-Projects grant DP140101492.

dimensional. This provides a new method for determining when a Nichols algebra is finite dimensional.

In this work we will show that a monomial belongs to Nichols braided Lie algebra $\mathfrak{L}(V)$ of braided vector space V of diagonal type if and only if this monomial is connected. This is one of the main results in this paper, which enables us to obtain the bases for the Nichols braided Lie algebras of arithmetic root systems and the dimensions of the Nichols braided Lie algebras of finite Cartan type. We give the sufficient and necessary conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$ and $\mathfrak{L}^-(V) = \mathfrak{L}(V)$, where $\mathfrak{B}(V)$, $\mathfrak{L}^-(V)$ and $\mathfrak{L}(V)$ denote Nichols algebra, Nichols Lie algebra and Nichols braided Lie algebra over V , respectively. We also obtain an explicit basis for $\mathfrak{L}^-(V)$ over the quantum linear space V with $\dim V = 2$.

This paper is organized as follows. In the remaining part of this section we provide some preliminaries and set our notations. In Section 2 we show that a monomial belongs to $\mathfrak{L}(V)$ if and only if this monomial is connected when V is a braided vector space of diagonal type. Section 3 presents a basis for $\mathfrak{L}(V)$, section 4 gives the basis for $\mathfrak{L}(V)$ of arithmetic root systems and obtains the dimension of $\mathfrak{L}(V)$ of finite Cartan type, and section 5 provides some non-zero monomials for the Nichols algebras $\mathfrak{B}(V)$. In section 6 we present the sufficient and necessary conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$ and $\mathfrak{L}^-(V) = \mathfrak{L}(V)$, respectively. We also give an explicit basis for $\mathfrak{L}^-(V)$ over the quantum linear space V with $\dim V = 2$.

Preliminaries

For any matrix $(q_{ij})_{n \times n}$ over F^* , define a bicharacter χ from $\mathbb{Z}^n \otimes \mathbb{Z}^n$ to F^* such that $\chi(e_i, e_j) = q_{ij}$ for $1 \leq i, j \leq n$, where $\{e_1, e_2, \dots, e_n\}$ is a basis of \mathbb{Z}^n . Let V be a vector space with basis x_1, x_2, \dots, x_n . Define $\alpha(e_i \otimes x_j) = q_{ij}x_j$ and $\delta(x_j) = e_j \otimes x_j$ for $1 \leq i, j \leq n$. It is clear that (V, α, δ) is a Yetter-Drinfeld module over \mathbb{Z}^n and (V, C) is a braided vector space under braiding C , where $C(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ and $C^{-1}(x_i \otimes x_j) = q_{ji}^{-1}x_j \otimes x_i$. In this case, V is called a braided vector space of diagonal type and $(q_{ij})_{n \times n}$ is called a braiding matrix of V . Throughout this paper braided vector space V is of diagonal type with basis x_1, x_2, \dots, x_n and $C(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ without special announcement. Let $\mathfrak{B}(V)$ be the Nichols algebra over the braided vector space V . Define $p_{ij} := q_{ij}$ for $1 \leq i, j \leq n$ and $p_{uv} := \chi(\deg(u), \deg(v))$ for any homogeneous element $u, v \in \mathfrak{B}(V)$. Denote $\text{ord}(p_{uu})$ the order of p_{uu} with respect to multiplication. Let $|u|$ denote length of homogeneous element $u \in \mathfrak{B}(V)$. Let $D =: \{[u] \mid [u] \text{ is a hard super-letter}\}$, $\Delta^+(\mathfrak{B}(V)) := \{\deg(u) \mid [u] \in D\}$, $\Delta(\mathfrak{B}(V)) := \Delta^+(\mathfrak{B}(V)) \cup \Delta^-(\mathfrak{B}(V))$, which is called the root system of V . If $\Delta(\mathfrak{B}(V))$ is finite, then it is called an arithmetic root system. Let $\mathfrak{L}(V)$ denote the braided Lie algebras generated by V in $\mathfrak{B}(V)$ under Lie operations $[x, y] = yx - p_{yx}xy$, for any homogeneous elements $x, y \in \mathfrak{B}(V)$. $(\mathfrak{L}(V), [\])$ is called Nichols braided Lie algebra of V . Let $\mathfrak{L}^-(V)$ denote the Lie algebras generated by V in $\mathfrak{B}(V)$ under Lie operations $[x, y]^- = yx - xy$, for any homogeneous elements $x, y \in \mathfrak{B}(V)$. $(\mathfrak{L}^-(V), [\]^-)$ is called Nichols Lie algebra of V . The other notations are the same as in [WZZ15a].

Recall the dual $\mathfrak{B}(V^*)$ of Nichols algebra $\mathfrak{B}(V)$ of rank n in [He05, Section 1.3]. Let y_i be a dual basis of x_i . $\delta(y_i) = g_i^{-1} \otimes y_i$, $g_i \cdot y_j = p_{ij}^{-1}y_j$ and

$\Delta(y_i) = g_i^{-1} \otimes y_i + y_i \otimes 1$. There exists a bilinear map

$$\langle \cdot, \cdot \rangle: (\mathfrak{B}(V^*) \# FG) \times \mathfrak{B}(V) \longrightarrow \mathfrak{B}(V)$$

such that

$$\langle y_i, uv \rangle = \langle y_i, u \rangle v + g_i^{-1} \cdot u \langle y_i, v \rangle \quad \text{and} \quad \langle y_i, \langle y_j, u \rangle \rangle = \langle y_i y_j, u \rangle$$

for any $u, v \in \mathfrak{B}(V)$. Furthermore, for any $u \in \bigoplus_{i=1}^{\infty} \mathfrak{B}(V)_{(i)}$, one has $u = 0$ if and only if $\langle y_i, u \rangle = 0$ for any $1 \leq i \leq n$. We have

$$[[u, v], w] = [u, [v, w]] + p_{vw}^{-1} [[u, w], v] + (p_{vw} - p_{vw}^{-1}) v \cdot [u, w], \tag{1}$$

$$[u, v \cdot w] = p_{wu} [uv] \cdot w + v \cdot [uw]. \tag{2}$$

We now recall some basic concepts of the graph theory (see [Ha69]). Let Γ_1 be a non-empty set and $\Gamma_2 \subseteq \{\{u, v\} \mid u, v \in \Gamma_1, \text{ with } u \neq v\} \subseteq 2^{\Gamma_1}$. Then $\Gamma = (\Gamma_1, \Gamma_2)$ is called a graph; Γ_1 is called the vertex set of Γ ; Γ_2 is called the edge set of Γ ; Element $\{u, v\} \in \Gamma_2$ is called an edge, written $a_{u,v}$. If $G = (G_1, G_2)$ is a graph and $G_1 \subseteq \Gamma_1$ and $G_2 \subseteq \Gamma_2$, then G is called a subgraph of Γ . If $\emptyset \neq H_1 \subseteq \Gamma_1$ and $H_2 = \{a_{u,v} \in \Gamma_2 \mid u, v \in H_1\}$, then $H = (H_1, H_2)$ is a subgraph, called the subgraph generated by H_1 in Γ .

$a_{u_m u_{m-1}} \cdots a_{u_3 u_2} a_{u_2 u_1}$ is called a walk from u_1 to u_m . We can define an equivalent relation on Γ_1 as follows: for any $u, v \in \Gamma_1$, u and v are equivalent if and only if there exists a walk from u to v or $u = v$. Every subgraph generated by every equivalent class of Γ_1 is called a connected component of Γ .

Let $\Gamma(V)$ be the generalized Dynkin diagram of V with p_{x_i, x_i} and \tilde{p}_{x_i, x_j} for $1 \leq i \neq j \leq n$ omitted. This is called a pure generalized Dynkin graph of V , i.e. $\Gamma(V)_1 = \{1, 2, \dots, n\}$ and $\Gamma(V)_2 = \{a_{ij} \mid p_{ij} p_{ji} \neq 1, i \neq j\}$.

Let $u = h_1 h_2 \cdots h_m$ be a monomial with $h_j = x_{i_j}$ for $1 \leq i_1, i_2, \dots, i_m \leq n, 1 \leq j \leq m$. $\deg(u) = \lambda_1 e_1 + \cdots + \lambda_n e_n$, where $\deg(x_i) = e_i$. Let $\deg_{x_i}(u) := \lambda_i$. Let $\mu(u) := \{x_{i_1}, \dots, x_{i_m}\}$ and $\Gamma(u)$ be a pure generalized Dynkin subgraph generated by $\mu(u)$. If $\Gamma(u)$ is connected, then u is called connected (or $\mu(u)$ is called connected). Otherwise, u is called disconnected (or $\mu(u)$ is called disconnected).

For $u, v \in \mathfrak{B}(V)$, if there exists a non-zero $a \in F$ such that $u = av$, then we write $u \sim v$. This is an equivalent relation. If there exist $x_i \in \mu(u)$ and $x_j \in \mu(v)$ such that $\tilde{p}_{x_i, x_j} \neq 1$, then we say that it is connected between monomial u and monomial v , written $u \diamond v$ in short. Otherwise, we say that it is disconnected between monomial u and monomial v .

Remark: When $u \neq 0$, $\mu(u)$ is independent of the choice of h_1, h_2, \dots, h_m , since $\mathfrak{B}(V)$ is graded and $\deg(u)$ is unique. Therefore, the connectivity of u is independent of the choice of h_1, h_2, \dots, h_m . When $u = 0$, $\mu(u)$ is dependent on the choice of h_1, h_2, \dots, h_m . For example, $\deg(V) > 1, p_{12} p_{21} = 1, p_{11} = -1, u = x_1^2 = x_1^2 x_2 = 0$. Therefore, $\mu(u) = \{x_1\}$ and u is connected. Meantime, $\mu(u) = \{x_1, x_2\}$ and u is disconnected.

Throughout, $\mathbb{Z} =: \{x \mid x \text{ is an integer}\}$. $\mathbb{R} =: \{x \mid x \text{ is a real number}\}$. $\mathbb{N}_0 =: \{x \mid x \in \mathbb{Z}, x \geq 0\}$. $\mathbb{N} =: \{x \mid x \in \mathbb{Z}, x > 0\}$. F denotes the base field, which is an algebraic closed field with characteristic zero. $F^* = F \setminus \{0\}$. S_n denotes symmetric group, $n \in \mathbb{N}$. For any set X , $|X|$ is the cardinal of X . $\text{int}(a)$ means the biggest integer not greater than $a \in \mathbb{R}$.

2. Structure of Nichols braided Lie algebras

In this section we prove that a monomial belongs to $\mathfrak{L}(V)$ if and only if this monomial is connected when V is a braided vector space of diagonal type.

Lemma 2.1. *Assume that u, v, w are homogeneous elements in $\mathfrak{L}(V)$. Set $a = 1 - p_{uv}p_{vw}$, $b = 1 - p_{uw}p_{vu}$, $c = 1 - p_{uv}p_{vu}$, $d = 1 - p_{uv}p_{vu}p_{uw}p_{wu}$, $e = 1 - p_{uv}p_{vu}p_{wv}p_{vw}$ and $f = 1 - p_{wv}p_{vw}p_{uw}p_{wu}$. If*

$$|\{r|r \in \{uv, uw, vw\} \text{ and } r \in \mathfrak{L}(V)\}| \geq 2 \text{ and } |\{t|t \in \{a, b, c\} \text{ and } t \neq 0\}| \geq 1,$$

then $uvw, uvw, vwu, vuw, wuv, wvu \in \mathfrak{L}(V)$.

Proof. Without loss of generality, we let $uv, uw \in \mathfrak{L}(V)$. If $vw \notin \mathfrak{L}(V)$, then $e = 0, f = 0$ by [WZZ15b, Lemma 3.2(i)] and [WZZ15a, Lemma 4.12]. If $a = 0$, then $b = 0, c = 0$, which is a contradiction. If $a \neq 0$, then $b \neq 0, c \neq 0$, one obtains a contradiction to [WZZ15b, Lemma 3.1]. Consequently, $vw \in \mathfrak{L}(V)$. Similarly, we can obtain others. ■

Lemma 2.2. *Assume that $h_i \in \{x_1, \dots, x_n\}$ for $1 \leq i \leq m$.*

- (i) *If it is disconnected between monomial u and monomial v (i.e. $\tilde{p}_{x_i, x_j} = 1$ for any $x_i \in \mu(u), x_j \in \mu(v)$), then $[u, v] = 0$.*
- (ii) *If $\mu(h_1 h_2 \dots h_m)$ is disconnected, then $\sigma(h_1, h_2, \dots, h_m) = 0$ for any method σ of adding bracket on h_1, h_2, \dots, h_m .*
- (iii) *If $h_1 h_2 \dots h_m \neq 0$ and $\mu(h_1 h_2 \dots h_m)$ is disconnected, then $h_1 h_2 \dots h_m \notin \mathfrak{L}(V)$.*

Proof. (i) u and v are quantum commutative (i.e. $uv = p_{u,v}vu$) since $x_i x_j = p_{x_i, x_j} x_j x_i$ for any $x_i \in \mu(u), x_j \in \mu(v)$.

(ii) We show this by induction on m . $[h_1, h_2] = 0$ for $m = 2$. For $m > 2$, $\sigma(h_1, h_2, \dots, h_m) = [\sigma_1(h_1 h_2 \dots h_t), \sigma_2(h_{t+1} h_{t+2} \dots h_m)]$. If both $h_1 h_2 \dots h_t$ and $h_{t+1} h_{t+2} \dots h_m$ are connected, then it is disconnected between $h_1 h_2 \dots h_t$ and $h_{t+1} h_{t+2} \dots h_m$. By Part (i), $\sigma(h_1, h_2, \dots, h_m) = 0$. If either $h_1 h_2 \dots h_t$ or $h_{t+1} h_{t+2} \dots h_m$ is disconnected, then either $\sigma_1(h_1 h_2 \dots h_t) = 0$ or $\sigma_2(h_{t+1} h_{t+2} \dots h_m) = 0$ by induction hypothesis.

(iii) It follows from Part (ii). ■

Lemma 2.3. (i) *If u is a monomial, then there exist monomials u_1, u_2, \dots, u_r such that $u \sim u_1 u_2 \dots u_r$ and $\{\Gamma(u_1), \Gamma(u_2), \dots, \Gamma(u_r)\}$ is complete set of connected components of $\Gamma(u)$ with $\deg(u) = \sum_{i=1}^r \deg(u_i)$ (which is called a decomposition of connected components of u).*

(ii) *If a monomial u is connected with $|u| > 1$, then there exist two connected monomials v and w such that $v \diamond w$ with $u \sim vw$.*

Proof. (i) If u is connected, it is clear. If u is not connected, we show it by induction on $|u|$. It is clear when $|u| = 1$. Now assume $|u| > 1$. Let

$\Omega := \{v \mid v \text{ is a connected monomial} \}$. Let $u \sim v_1v_2v_3$ and v_2 be in Ω such that $|v_2| = \max\{|v| \mid v \in \Omega, u \sim v_1v_3, v_1 \text{ and } v_3 \text{ are monomials or } 1\}$.

Now we show that v_2 is a connected component of u . In fact, if it does not hold, then $v_1 \diamond v_2$ or $v_3 \diamond v_2$. Without the lose of generality $v_1 \diamond v_2$. Then there exists t such that $h_{i_t} \diamond v_2$ with $v_1 = h_{i_1} \cdots h_{i_s}$ and $v_2h_{i_j} \sim h_{i_j}v_2$ for $t + 1 < j \leq s$. Consequently, $u \sim w_1h_{i_t}v_2w_3$ and $h_{i_t}v_2$ is connected. which is a contradiction.

We have obtained that it is disconnected between v_1 and v_2 in proof above. Thus $u = v_1v_2v_3 \sim v_2v_1v_3$ and $v_1v_3 \sim u_2u_3 \cdots u_r$ is a decomposition of connected components of v_1v_3 since $|v_1v_3| < |u|$. Setting $u_1 := v_2$ we complete proof of Part (i).

(ii) Let $u = h_1h_2 \cdots h_m$ and $h_2 \cdots h_m \sim u_1u_2 \cdots u_r$ be a decomposition of connected components of $h_2 \cdots h_m$. Consequently, $h_1u_1 \cdots u_{r-1}$ is connected since $h_1 \diamond u_i$ for $1 \leq i \leq r$. ■

Theorem 2.4. Assume that $h_i \in \{x_1, \dots, x_n\}$ and $p_{h_1, h_1} \neq 1$ when $\mu(h_1 \cdots h_m) = h_1$.

- (i) If $h_1 \cdots h_b, h_{b+1} \cdots h_m \in \mathfrak{L}(V)$ and $h_1 \cdots h_b \diamond h_{b+1} \cdots h_m$ with $h_1 \cdots h_b \neq 0$ or $h_{b+1} \cdots h_m \neq 0$, then there exist $\tau \in \mathbb{S}_m$ such that $h_1 \cdots h_m \sim h_{\tau(1)} \cdots h_{\tau(m)}$ with $h_{\tau(1)} \cdots h_{\tau(m-1)} \in \mathfrak{L}(V)$ and $h_{\tau(1)} \cdots h_{\tau(m-1)} \diamond h_{\tau(m)}$, or with $h_{\tau(2)} \cdots h_{\tau(m)} \in \mathfrak{L}(V)$ and $h_{\tau(2)} \cdots h_{\tau(m)} \diamond h_{\tau(1)}$.
- (ii) If $h_1h_2 \cdots h_{m-1} \in \mathfrak{L}(V)$ and $h_1h_2 \cdots h_{m-1} \diamond h_m$ or $h_2h_3 \cdots h_m \in \mathfrak{L}(V)$ and $h_2h_3 \cdots h_m \diamond h_1$, then $h_1h_2 \cdots h_m \in \mathfrak{L}(V)$.
- (iii) If $0 \neq h_1h_2 \cdots h_m \in \mathfrak{L}(V)$, then there exists $\tau \in \mathbb{S}_m$ such that $h_1 \cdots h_m \sim h_{\tau(1)} \cdots h_{\tau(m)}$ with $0 \neq h_{\tau(1)}h_{\tau(2)} \cdots h_{\tau(m-1)} \in \mathfrak{L}(V)$ and $h_{\tau(1)} \cdots h_{\tau(m-1)} \diamond h_{\tau(m)}$, or $0 \neq h_{\tau(2)}h_{\tau(3)} \cdots h_{\tau(m)} \in \mathfrak{L}(V)$ and $h_{\tau(2)} \cdots h_{\tau(m)} \diamond h_{\tau(1)}$.
- (iv) If monomial $u = h_1h_2 \cdots h_m$ is connected, then $u \in \mathfrak{L}(V)$.

Proof. We show (i), (ii), (iii) and (iv) by induction on the length of $|h_1h_2 \cdots h_m| = m$. Assume $m = 2$. (i) and (iv) are clear. If $h_1 \neq h_2$, then (ii), (iii) follows from [WZZ15a, Lemma 4.12], [WZZ15a, Lemma 5.2], respectively. If $h_1 = h_2$, then (ii) and (iii) follow from [WZZ15a, Lemma 4.3] and [He05, Lemma 1.3.3(i)].

Now $m > 2$.

(i) If $b = m - 1$ or $b = 1$, let $\tau = id$. Now assume that $1 < b < m - 1$. There exist $1 \leq a \leq b$ and $b + 1 \leq c \leq m$ such that $\tilde{p}_{h_a, h_c} \neq 1$.

(1) Assume $h_1 \cdots h_b \neq 0$. We show this by induction on b . There exist $\tau \in \mathbb{S}_{\{1, \dots, b\}}$ such that $h_1 \cdots h_b \sim h_{\tau(1)} \cdots h_{\tau(b)}$ with case (a): $0 \neq h_{\tau(1)} \cdots h_{\tau(b-1)} \in \mathfrak{L}(V)$, $h_{\tau(1)}h_{\tau(2)} \cdots h_{\tau(b-1)} \diamond h_{\tau(b)}$ or with case (b): $0 \neq h_{\tau(2)} \cdots h_{\tau(b)} \in \mathfrak{L}(V)$, $h_{\tau(2)}h_{\tau(3)} \cdots h_{\tau(b)} \diamond h_{\tau(1)}$ by induction hypothesis of (iii).

(a₁) If $h_{b+1} \cdots h_m \diamond h_{\tau(b)}$, then $h_{\tau(b)}h_{b+1} \cdots h_m \in \mathfrak{L}(V)$ by induction hypothesis of (ii), and $h_{\tau(1)} \cdots h_{\tau(b-1)} \diamond h_{\tau(b)}h_{b+1} \cdots h_m$ since $h_{\tau(1)}h_{\tau(2)} \cdots h_{\tau(b-1)} \diamond h_{\tau(b)}$. It is proved since $|h_{\tau(1)} \cdots h_{\tau(b-1)}| = b - 1 < b$ and induction hypothesis.

(a₂) If $\tilde{p}_{h_{\tau(b)}, h_{j_1}} = 1$ for all $j_1 \in \{b+1, \dots, m\}$, then $h_{\tau(b)}h_{b+1} \cdots h_m \sim h_{b+1} \cdots h_m h_{\tau(b)}$. On the other hand, $h_{\tau(1)} \cdots h_{\tau(b-1)} \diamond h_{b+1} \cdots h_m$ since $h_1 \cdots h_b \diamond h_{b+1} \cdots h_m$. Consequently, $h_{\tau_1\tau(1)} \cdots h_{\tau_1\tau(b-1)}h_{\tau_1(b+1)} \cdots h_{\tau_1(m)} \in \mathfrak{L}(V)$ and $h_{\tau(1)} \cdots h_{\tau(b-1)} \diamond h_{b+1}$

$\cdots h_m \sim h_{\tau_1\tau(1)} \cdots h_{\tau_1\tau(b-1)}h_{\tau_1(b+1)} \cdots h_{\tau_1(m)}$ for some $\tau_1 \in \mathbb{S}_{\{\tau(1), \dots, \tau(b-1), b+1, \dots, m\}}$ by the induction hypotheses of (i) and (ii). Now $h_{\tau_1\tau(1)} \cdots h_{\tau_1\tau(b-1)} h_{\tau_1(b+1)} \cdots h_{\tau_1(m)} \diamond h_{\tau(b)}$ since $h_{\tau(1)}h_{\tau(2)} \cdots h_{\tau(b-1)} \diamond h_{\tau(b)}$, and

$$\begin{aligned} h_1 \cdots h_m &\sim h_{\tau(1)} \cdots h_{\tau(b)}h_{b+1} \cdots h_m \\ &\sim h_{\tau(1)} \cdots h_{\tau(b-1)}h_{b+1} \cdots h_m h_{\tau(b)} \\ &\sim h_{\tau_1\tau(1)} \cdots h_{\tau_1\tau(b-1)}h_{\tau_1(b+1)} \cdots h_{\tau_1(m)}h_{\tau(b)} \\ &= h_{\tau_1\tau(1)} \cdots h_{\tau_1\tau(b-1)}h_{\tau_1\tau(b+1)} \cdots h_{\tau_1\tau(m)}h_{\tau_1\tau(b)} \end{aligned}$$

by $h_{\tau_1(b+1)} \cdots h_{\tau_1(m)} = h_{\tau_1\tau(b+1)} \cdots h_{\tau_1\tau(m)}$ since $\tau \in \mathbb{S}_{\{1, 2, \dots, b\}}$, and $h_{\tau_1\tau(b)} = h_{\tau(b)}$ since $\tau_1 \in \mathbb{S}_{\{\tau(1), \tau(2), \dots, \tau(b-1), b+1, \dots, m\}}$. Consequently,

$$\tau' := \begin{pmatrix} 1 & \cdots & b-1 & b & \cdots & m-1 & m \\ \tau_1\tau(1) & \cdots & \tau_1\tau(b-1) & \tau_1\tau(b+1) & \cdots & \tau_1\tau(m) & \tau_1\tau(b) \end{pmatrix} \in \mathbb{S}_m,$$

$h_{\tau'(1)} \cdots h_{\tau'(m-1)} \in \mathfrak{L}(V)$, $h_{\tau'(1)} \cdots h_{\tau'(m-1)} \diamond h_{\tau'(m)}$ and $h_1 \cdots h_m \sim h_{\tau'(1)} \cdots h_{\tau'(m)}$.
 (b₁) If $h_{\tau(2)} \cdots h_{\tau(b)} \diamond h_{b+1} \cdots h_m$, then $h_{\tau_1\tau(2)} \cdots h_{\tau_1\tau(b)}h_{\tau_1(b+1)} \cdots h_{\tau_1(m)} \in \mathfrak{L}(V)$ and $h_{\tau(2)} \cdots h_{\tau(b)}h_{b+1} \cdots h_m \sim h_{\tau_1\tau(2)} \cdots h_{\tau_1\tau(b)} h_{\tau_1(b+1)} \cdots h_{\tau_1(m)}$ for some $\tau_1 \in \mathbb{S}_{\{\tau(2), \dots, \tau(b), b+1, \dots, m\}}$ by the induction hypotheses of (i) and (ii). We know

$$h_{\tau_1\tau(2)} \cdots h_{\tau_1\tau(b)}h_{\tau_1(b+1)} \cdots h_{\tau_1(m)} \diamond h_{\tau(1)}$$

since $h_{\tau(2)}h_{\tau(3)} \cdots h_{\tau(b)} \diamond h_{\tau(1)}$. We obtain $h_1 \cdots h_m \sim h_{\tau(1)} \cdots h_{\tau(b)}h_{b+1} \cdots h_m \sim h_{\tau(1)}h_{\tau_1\tau(2)} \cdots h_{\tau_1\tau(b)}h_{\tau_1(b+1)} \cdots h_{\tau_1(m)} = h_{\tau_1\tau(1)}h_{\tau_1\tau(2)} \cdots h_{\tau_1\tau(b)} h_{\tau_1\tau(b+1)} \cdots h_{\tau_1\tau(m)}$ by $h_{\tau_1(b+1)} \cdots h_{\tau_1(m)} = h_{\tau_1\tau(b+1)} \cdots h_{\tau_1\tau(m)}$ since $\tau \in \mathbb{S}_{\{1, \dots, b\}}$ and $h_{\tau_1\tau(1)} = h_{\tau(1)}$ since $\tau_1 \in \mathbb{S}_{\{\tau(2), \dots, \tau(b), b+1, \dots, m\}}$. Therefore, $\tau' := \tau_1\tau \in \mathbb{S}_m$, $h_{\tau'(2)} \cdots h_{\tau'(m)} \in \mathfrak{L}(V)$, $h_{\tau'(2)} \cdots h_{\tau'(m)} \diamond h_{\tau'(1)}$ and $h_1 \cdots h_m \sim h_{\tau'(1)} \cdots h_{\tau'(m)}$.

(b₂) If $\tilde{p}_{h_{\tau(j_1)}, h_{j_2}} = 1$ for all $j_1 \in \{2, \dots, b\}, j_2 \in \{b+1, \dots, m\}$, then $h_{\tau(2)} \cdots h_{\tau(b)}h_{b+1} \cdots h_m \sim h_{b+1} \cdots h_m h_{\tau(2)} \cdots h_{\tau(b)}$. On the other hand, $h_{\tau(1)} \diamond h_{b+1} \cdots h_m$ since $h_1 \cdots h_b \diamond h_{b+1} \cdots h_m$. Then $h_{\tau(1)}h_{b+1} \cdots h_m \in \mathfrak{L}(V)$ by induction hypothesis of (ii). See $h_1 \cdots h_m \sim h_{\tau(1)} \cdots h_{\tau(b)} h_{b+1} \cdots h_m \sim h_{\tau(1)}h_{b+1} \cdots h_m h_{\tau(2)} \cdots h_{\tau(b)}$, and $h_{\tau(1)}h_{b+1} \cdots h_m \diamond h_{\tau(2)} \cdots h_{\tau(b)}$ since $h_{\tau(1)} \diamond h_{\tau(2)} \cdots h_{\tau(b)}$. (i) holds since $|h_{\tau(2)} \cdots h_{\tau(b)}| = b-1 < b$ and induction hypothesis on b .

(2). If $h_1 \cdots h_b = 0$, then $h_{b+1} \cdots h_m \neq 0$ and $h_{b+1} \cdots h_m$ is connected by Lemma 2.2(iii). Consequently, $0 = h_1 \cdots h_{m-1} \in \mathfrak{L}(V)$ and $h_1 \cdots h_{m-1} \diamond h_m$.

(ii) Assume that $h_1 h_2 \cdots h_m \notin \mathfrak{L}(V)$. Obviously, $h_1 h_2 \cdots h_{m-1} \neq 0$ and $h_2 h_3 \cdots h_m \neq 0$. Set $i_0 := 1$ and $j_0 := m-1$ when $h_1 h_2 \cdots h_{m-1} \in \mathfrak{L}(V)$; $i_0 := 2$ and $j_0 := m$ when $h_2 h_3 \cdots h_m \in \mathfrak{L}(V)$; $N_0 := \{i_0, i_0+1, \dots, j_0\}$; $A_0 := \{1, \dots, m\} - N_0$,

Now we prove the following **Assertion(k)** by induction on k , $1 \leq k \leq m-2$.

Assertion(k): There exist $1 \leq i_k \leq j_k \leq m$, $\tau_k \in \mathbb{S}_{N_{k-1}}$ such that the following conditions hold:

- (C₁) $0 \neq h_{\tau^k(i_k)} \cdots h_{\tau^k(j_k)} \in \mathfrak{L}(V)$;
- (C₂) $\tilde{p}_{h_r, h_t} = 1$ for $\forall r \neq t \in A_k$;
- (C₃) $h_{\tau^k(i_k)} \cdots h_{\tau^k(j_k)} \diamond h_r$ for $\forall r \in A_k$;
- (C₄) $\tilde{p}_{h_{\tau^k(i_k)} \cdots h_{\tau^k(j_k)}, h_r} = 1$ for $\forall r \in A_k$;

$$(C_5) \quad h_{\tau^k(i_{k-1})}h_{\tau^k(i_{k-1}+1)} \cdots h_{\tau^k(j_{k-1})} \sim h_{\tau^{k-1}(i_{k-1})}h_{\tau^{k-1}(i_{k-1}+1)} \cdots h_{\tau^{k-1}(j_{k-1})} \text{ and} \\ h_{\tau^k(1)} \cdots h_{\tau^k(m)} \sim h_1h_2 \cdots h_m;$$

$$(C_6) \quad i_{k-1} \leq i_k \leq j_k \leq j_{k-1} \text{ with } j_{k-1} - i_{k-1} = j_k - i_k + 1, \text{ where } N_k := \\ \{\tau^k(i_k), \tau^k(i_k + 1), \dots, \tau^k(j_k)\}, A_k := \{1, \dots, m\} - N_k, \tau^k := \tau_k\tau_{k-1} \cdots \tau_1.$$

Step 1. For $k = 1$, now we construct i_1 and j_1 as follows. Considering $0 \neq h_{i_0} \cdots h_{j_0} \in \mathfrak{L}(V)$, we have $\tilde{p}_{h_{i_0} \cdots h_{j_0}, h_r} = 1$ for $r \in A_0$ by [WZZ15a, Lemma 4.12] and there exist $\tau_1 \in \mathbb{S}_{N_0}$ such that $h_{i_0} \cdots h_{j_0} \sim h_{\tau_1(i_0)} \cdots h_{\tau_1(j_0)}$ with case (a) $0 \neq h_{\tau_1(i_0)} \cdots h_{\tau_1(j_0-1)} \in \mathfrak{L}(V)$, $h_{\tau_1(i_0)} \cdots h_{\tau_1(j_0-1)} \diamond h_{\tau_1(j_0)}$, $i_1 := i_0$ and $j_1 := j_0 - 1$ or with case (b) $0 \neq h_{\tau_1(i_0+1)} \cdots h_{\tau_1(j_0)} \in \mathfrak{L}(V)$, $h_{\tau_1(i_0+1)} \cdots h_{\tau_1(j_0)} \diamond h_{\tau_1(i_0)}$, $i_1 := i_0 + 1$ and $j_1 := j_0$ by induction hypothesis of (iii). Obviously, $0 \neq h_{\tau^1(i_1)} \cdots h_{\tau^1(j_1)} \in \mathfrak{L}(V)$ and $h_{\tau^1(r)} = h_r$ for $r \in A_0$.

Obviously, (C₁), (C₅) and (C₆) hold.

$\tilde{p}_{h_t, h_r} = 1$ for any $t \neq r \in A_1$, i.e. (C₂) holds. Indeed, if $\tilde{p}_{h_t, h_r} \neq 1$ for $t \in A_1 - A_0$, $r \in A_0$, then $\tilde{p}_{h_{i_1} \cdots h_{j_1}, h_r} \neq 1$ since $\tilde{p}_{h_{i_0} \cdots h_{j_0}, h_r} = 1$. We obtain $h_{i_1} \cdots h_{j_1} h_t h_r$, $h_t h_{i_1} \cdots h_{j_1} h_r$, $h_r h_{i_1} \cdots h_{j_1} h_t \in \mathfrak{L}(V)$ by Lemma 2.1. However, $h_1 \cdots h_m$ is a quantum equivalent with one among $h_{i_1} \cdots h_{j_1} h_t h_r$, $h_t h_{i_1} \cdots h_{j_1} h_r$, $h_r h_{i_1} \cdots h_{j_1} h_t$, which contradicts $h_1 \cdots h_m \notin \mathfrak{L}(V)$.

(C₃) and (C₄) follow from (C₂).

Step 2. Assuming that **Assertion(k)** holds, we prove that **Assertion(k + 1)** holds, $k \leq m - 3$. Let $i := i_k$ and $j := j_k$ in this proof for convenience.

Considering $0 \neq h_{\tau^k(i)} \cdots h_{\tau^k(j)} \in \mathfrak{L}(V)$, we have that there exists $\tau_{k+1} \in \mathbb{S}_{N_k}$ such that $h_{\tau^k(i)} \cdots h_{\tau^k(j)} \sim h_{\tau^{k+1}(i)} \cdots h_{\tau^{k+1}(j)}$ with case (1°): $0 \neq h_{\tau^{k+1}(i)} \cdots h_{\tau^{k+1}(j-1)} \in \mathfrak{L}(V)$, $h_{\tau^{k+1}(i)} \cdots h_{\tau^{k+1}(j-1)} \diamond h_{\tau^{k+1}(j)}$, $\tau^{k+1}(i_{k+1}) := \tau^{k+1}(i)$ and $\tau^{k+1}(j_{k+1}) := \tau^{k+1}(j - 1)$, or with case (2°): $0 \neq h_{\tau^{k+1}(i+1)} \cdots h_{\tau^{k+1}(j)} \in \mathfrak{L}(V)$, $h_{\tau^{k+1}(i+1)} \cdots h_{\tau^{k+1}(j)} \diamond h_{\tau^{k+1}(i)}$, $\tau^{k+1}(i_{k+1}) := \tau^{k+1}(i + 1)$ and $\tau^{k+1}(j_{k+1}) := \tau^{k+1}(j)$ by induction hypothesis of (iii). We obtain $h_1 \cdots h_m \sim h_{\tau^{k+1}(1)} \cdots h_{\tau^{k+1}(m)}$. For convenience, let $h'_s := h_{\tau^{k+1}(s)}$ for all $s \in \{1, \dots, m\}$. Obviously, $\tau_{k+1}(t) = t$ for all $t \in A_k$.

Clearly, (C₁), (C₅) and (C₆) hold.

(1°) $A_{k+1} = A_k \cup \tau^{k+1}(j)$ and $N_{k+1} = N_k - \tau^{k+1}(j)$. If there exists $r \in A_{k+1}$ such that $\tilde{p}_{h'_\alpha, h_r} = 1$ for all $\alpha \in \{i, \dots, j - 1\}$, then $\tilde{p}_{h'_j, h_r} \neq 1$ by **Assertion(k)** (C₃). Therefore $\tilde{p}_{h'_i \cdots h'_j, h_r} \neq 1$, which contradicts **Assertion(k)** (C₄). For all $r \in A_{k+1}$, there exists $\alpha \in \{i, \dots, j - 1\}$ such that

$$\tilde{p}_{h'_\alpha, h_r} \neq 1 \tag{3}$$

and $h'_1 \cdots h'_{i-1} h'_i \cdots h'_{j-1} \in \mathfrak{L}(V)$ by induction hypothesis of (iv) and Lemma 2.2 (ii).

Set $\{g_1, \dots, g_\beta, l_1, \dots, l_\eta\} = \{\tau^{k+1}(j + 1), \dots, \tau^{k+1}(m - 1)\} \subseteq A_k$ with $g_1 < \dots < g_\beta, l_1 < \dots < l_\eta$ such that $\tilde{p}_{h'_j, h_{g_\lambda}} \neq 1$ and $\tilde{p}_{h'_j, h_{l_\xi}} = 1$ for all $\lambda \in \{1, \dots, \beta\}$, all $\xi \in \{1, \dots, \eta\}$. Then $h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_r} \in \mathfrak{L}(V)$ for all $r \in \{1, \dots, \eta\}$ and $h'_j h'_{g_1} \cdots h'_{g_t} \in \mathfrak{L}(V)$ for all $t \in \{1, \dots, \beta\}$ by induction hypothesis of (iv). If $\tilde{p}_{h'_j, h'_m} \neq 1$, then $\tilde{p}_{h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_\eta}, h'_m} \neq 1$ and $\tilde{p}_{h'_j h'_{g_1} \cdots h'_{g_\beta}, h'_m} \neq 1$ by **Assertion(k)** C₂ and C₄. Then $h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_\eta} h'_j h'_{g_1} \cdots h'_{g_\beta} h'_m \in \mathfrak{L}(V)$

by Lemma 2.1, $h_1 \cdots h_m \sim h'_1 \cdots h'_m \sim h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_\eta} h'_j h'_{g_1} \cdots h'_{g_\beta} h'_m$. It is a contradiction. Thus $\tilde{p}_{h'_j, h'_m} = 1$ and $h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_\eta} h'_m \in \mathfrak{L}(V)$ by induction hypothesis of (ii). If $\beta \geq 1$, then $\tilde{p}_{h'_j, h'_{g_1} \cdots h'_{g_{\beta-1}}, h'_{g_\beta}} \neq 1$ and $\tilde{p}_{h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_\eta} h'_m, h'_{g_\beta}} \neq 1$ by **Assertion(k)** (C₂). $h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_\eta} h'_m h'_j h'_{g_1} \cdots h'_{g_{\beta-1}} h'_{g_\beta} \in \mathfrak{L}(V)$ by Lemma 2.1. $h_1 h_2 \cdots h_m \sim h'_1 \cdots h'_{j-1} h'_{l_1} \cdots h'_{l_\eta} h'_m h'_j h'_{g_1} \cdots h'_{g_{\beta-1}} h'_{g_\beta}$. It is a contradiction. Then $\beta = 0$. $\eta = m - j - 1$, i.e. $\tilde{p}_{h'_j, h'_r} = 1$ for all $r \in \{j+1, \dots, m\}$. So $\tilde{p}_{h'_i \cdots h'_{j-1}, h'_r} = 1$ for all $r \in \{j+1, \dots, m\}$ by **Assertion(k)** (C₄). Assume that there exists $\theta \in \{1, \dots, i-1\}$ such that $\tilde{p}_{h'_\theta, h'_j} \neq 1$. If $\tilde{p}_{h'_j, h'_1 \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_{i-1} h'_i \cdots h'_{j-1} h'_{j+1} \cdots h'_m} = 1$, then $\tilde{p}_{h'_j, h'_1 \cdots h'_{j-1} h'_{j+1} \cdots h'_m} \neq 1$, $h'_1 \cdots h'_{j-1} h'_{j+1} \cdots h'_m \in \mathfrak{L}(V)$ by induction hypothesis of (iv) and (3). $h'_1 \cdots h'_{j-1} h'_{j+1} \cdots h'_m h'_j \in \mathfrak{L}(V)$ by [WZZ15a, Lemma 4.12]. $h_1 h_2 \cdots h_m \sim h'_1 \cdots h'_{j-1} h'_{j+1} \cdots h'_m h'_j$. It is a contradiction. If $\tilde{p}_{h'_j, h'_1 \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_{i-1} h'_i \cdots h'_{j-1} h'_{j+1} \cdots h'_m} \neq 1$, $h'_1 \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_{i-1} h'_i \cdots h'_{j-1} h'_{j+1} \cdots h'_m \in \mathfrak{L}(V)$ by induction hypotheses (iv) and (3). Then $h'_\theta h'_1 \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_{i-1} h'_i \cdots h'_{j-1} h'_{j+1} \cdots h'_m h'_j \in \mathfrak{L}(V)$ by Lemma 2.1. $h_1 h_2 \cdots h_m \sim h'_\theta h'_1 \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_{i-1} h'_i \cdots h'_{j-1} h'_{j+1} \cdots h'_m h'_j$. It is a contradiction. Then $\tilde{p}_{h'_\theta, h'_j} = 1$ for all $\theta \in \{1, \dots, i-1\}$, which implies that (C₂) holds.

(C₃) and (C₄) follows from (C₂).

(2°) $A_{k+1} = A_k \cup \tau^{k+1}(i)$ and $N_{k+1} = N_k - \tau^{k+1}(i)$. If there exists $r \in A_{k+1}$ such that $\tilde{p}_{h'_\alpha, h'_r} = 1$ for all $\alpha \in \{i+1, \dots, j\}$, then $\tilde{p}_{h'_i, h'_r} \neq 1$ by **Assertion(k)** and $\tilde{p}_{h'_i \cdots h'_j, h'_r} \neq 1$, which contradicts **Assertion(k)**. Thus for all $r \in A_{k+1}$, there exists $\alpha \in \{i+1, \dots, j\}$ such that

$$\tilde{p}_{h'_\alpha, h'_r} \neq 1 \tag{4}$$

and $h'_{i+1} \cdots h'_j h'_{j+1} \cdots h'_m \in \mathfrak{L}(V)$ by induction hypothesis of (iv).

Set $\{g_1, \dots, g_\beta, l_1, \dots, l_\eta\} = \{\tau^{k+1}(2), \dots, \tau^{k+1}(i-1)\}$ with

$$g_1 < \cdots < g_\beta, l_1 < \cdots < l_\eta \text{ such that } \tilde{p}_{h'_i, h'_{g_\lambda}} \neq 1 \text{ and } \tilde{p}_{h'_i, h'_{l_\xi}} = 1$$

for all $\lambda \in \{1, \dots, \beta\}$, all $\xi \in \{1, \dots, \eta\}$. Then $h'_{l_1} \cdots h'_{l_r} h'_{i+1} \cdots h'_m \in \mathfrak{L}(V)$ for all $r \in \{1, \dots, \eta\}$ and $h'_{g_1} \cdots h'_{g_t} h'_i \in \mathfrak{L}(V)$ for all $t \in \{1, \dots, \beta\}$ by induction hypothesis of (iv). If $\tilde{p}_{h'_i, h'_1} \neq 1$, then $\tilde{p}_{h'_{l_1} \cdots h'_{l_\eta} h'_{i+1} \cdots h'_m, h'_1} \neq 1$ and $\tilde{p}_{h'_{g_1} \cdots h'_{g_\beta} h'_i, h'_1} \neq 1$ by **Assertion(k)** (C₄). Then $h'_1 h'_{g_1} \cdots h'_{g_\beta} h'_i h'_{l_1} \cdots h'_{l_\eta} h'_{i+1} \cdots h'_m \in \mathfrak{L}(V)$ by Lemma 2.1, $h_1 \cdots h_m \sim h'_1 \cdots h'_m \sim h'_1 h'_{g_1} \cdots h'_{g_\beta} h'_i h'_{l_1} \cdots h'_{l_\eta} h'_{i+1} \cdots h'_m$. It is a contradiction. Thus $\tilde{p}_{h'_i, h'_1} = 1$. $h'_1 h'_{l_1} \cdots h'_{l_\eta} h'_{i+1} \cdots h'_m \in \mathfrak{L}(V)$ by induction hypothesis of (iv). If $\beta \geq 1$, then $\tilde{p}_{h'_{g_2} \cdots h'_{g_\beta} h'_i, h'_{g_1}} \neq 1$ and $\tilde{p}_{h'_1 h'_{l_1} \cdots h'_{l_\eta} h'_{i+1} \cdots h'_m, h'_{g_1}} \neq 1$ by **Assertion(k)** (C₃). then $h'_{g_1} h'_{g_2} \cdots h'_{g_\beta} h'_i h'_{l_1} h'_{l_1} \cdots h'_{l_\eta} h'_{i+1} \cdots h'_m \in \mathfrak{L}(V)$ by Lemma 2.1. $h_1 h_2 \cdots h_m \sim h'_{g_1} h'_{g_2} \cdots h'_{g_\beta} h'_i h'_{l_1} h'_{l_1} \cdots h'_{l_\eta} h'_{i+1} \cdots h'_m$, which is a contradiction. Then $\beta = 0$. $\eta = i - 2$, i.e. $\tilde{p}_{h'_i, h'_r} = 1$ for all $r \in \{1, \dots, i-1\}$. So $\tilde{p}_{h'_{i+1} \cdots h'_j, h'_r} = 1$ for all $r \in \{1, \dots, i-1\}$ by **Assertion(k)** (C₄). Assume that there exists $\theta \in \{j+1, \dots, m\}$ such that $\tilde{p}_{h'_\theta, h'_i} \neq 1$. If $\tilde{p}_{h'_i, h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_j h'_{j+1} \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_m} = 1$, then $\tilde{p}_{h'_i, h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_m} \neq 1$ and $h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_m \in \mathfrak{L}(V)$ by induction hypothesis (iv) and (4). $h'_i h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_m \in \mathfrak{L}(V)$ by [WZZ15a, Lemma 4.12]. $h_1 h_2 \cdots h_m \sim h'_i h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_m$, which is a contradiction.

If $\tilde{p}_{h'_\theta, h'_i} \neq 1$, then $h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_j h'_{j+1} \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_m \in \mathfrak{L}(V)$ by induction hypothesis of (iv) and (4). $h'_i h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_j h'_{j+1} \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_m h'_\theta \in \mathfrak{L}(V)$ by Lemma 2.1. $h_1 h_2 \cdots h_m \sim h'_i h'_1 \cdots h'_{i-1} h'_{i+1} \cdots h'_j h'_{j+1} \cdots h'_{\theta-1} h'_{\theta+1} \cdots h'_m h'_\theta$, which is a contradiction. Then $\tilde{p}_{h'_\theta, h'_i} = 1$ for all $\theta \in \{j + 1, \dots, m\}$, which implies (C₂).

(C₃) and (C₄) follows from (C₂).

Step 3. In **Assertion(m - 2)**, It is a contradiction by (C₃) and (C₄).

(iii) By Lemma 2.2, u is connected. which implies that there exist two connected monomials v and w such that $u \sim vw$. By inductive assumption, v and w belong to $\mathfrak{L}(V)$. Consequently, (iii) holds by (i).

(iv) By Lemma 2.3, $u \sim vw$ such that v and w are connected, as well as, $v \diamond w$. By inductive assumption, $v, w \in \mathfrak{L}(V)$. It follows from (i) and (ii). ■

Corollary 2.5. *If $h_1 h_2 \cdots h_m \neq 0$ and $p_{h_1, h_1} \neq 1$ when $\mu(h_1 h_2 \cdots h_m) = h_1$, then $h_1 h_2 \cdots h_m \in \mathfrak{L}(V)$ if and only if $\mu(h_1 h_2 \cdots h_m)$ is connected.*

Proof. It follows from Lemma 2.2 and Theorem 2.4 (iv). ■

3. A basis of $\mathfrak{L}(V)$

Lemma 3.1. *Assume that V is a braided vector space of diagonal type. If $u \neq 0$ are homogeneous elements in $\mathfrak{B}(V)$ and $p_{i, i} \neq 1$ when $\mu(u) = x_i, i \in \{1, \dots, n\}$, then $\mu(u)$ is connected (i.e. every monomial of u is connected) if and only if $u \in \mathfrak{L}(V)$.*

Proof. The necessity follows from Corollary 2.5. We now prove the sufficiency. If u is not connected and $u = \sum_{i=1}^r k_i \sigma_i(u_i)$ with $k_i \in F^*$, where u_i is a non-zero disconnected monomial and σ_i is a method of bracket on letters of u_i for $1 \leq i \leq r$. By Lemma 2.2, $\sigma_i(u_i) = 0$ for $1 \leq i \leq r$ and $u = 0$, which is a contradiction. ■

Lemma 3.2. *If $[u] \in D$, then u is connected and $u \neq 0$.*

Proof. By [Kh99, Cor. 1], $u \neq 0$. Obviously $[u] \neq 0$ and $[u] \in \mathfrak{L}(V)$. Considering Lemma 3.1 we complete the proof. ■

Theorem 3.3. *If $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type with $\dim V \geq 2$ and $p_{i, i} \neq 1$ for $1 \leq i \leq n$, then the set*

$$\{[u_1]^{k_1} [u_2]^{k_2} \cdots [u_s]^{k_s} \mid [u_i] \in D, |D| = s; 0 \leq k_i < h_{u_i}; 1 \leq i \leq s; u_s < u_{s-1} < \cdots < u_1, \mu([u_1]^{k_1} [u_2]^{k_2} \cdots [u_s]^{k_s}) \text{ is connected, } \sum_{i=1}^s k_i > 0\}$$

is a basis of $\mathfrak{L}(V)$.

Proof. It follows from [He05, Th. 1.4.6], and Corollary 2.5 and Lemma 3.1. ■

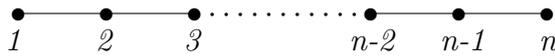
4. Dimension of $\mathfrak{L}(V)$

In this section we give the basis for $\mathfrak{L}(V)$ of arithmetic root systems and obtain the dimensions of $\mathfrak{L}(V)$ of finite Cartan type.

Let V_{i_1, \dots, i_r} denote the braided vector subspace generated by $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$ of V and let D_{i_1, \dots, i_r} denote $\{[u] \mid [u] \text{ is a hard super-letter of } \mathfrak{B}(V_{i_1, i_2, \dots, i_r})\}$; $L_{i_1, \dots, i_r} := \{[u_1]^{k_1} [u_2]^{k_2} \dots [u_s]^{k_s} \mid [u_j] \in D_{i_1, \dots, i_r}; 0 \leq k_j < \text{ord}(p_{u_j, u_j}); 1 \leq j \leq s; |D_{i_1, \dots, i_r}| = s; u_s < u_{s-1} < \dots < u_1, \mu([u_1]^{k_1} [u_2]^{k_2} \dots [u_s]^{k_s}) \text{ is connected, } \sum_{j=1}^s k_j > 0\}$; further set $B_{i_1, \dots, i_r} := \{[u_1]^{k_1} [u_2]^{k_2} \dots [u_s]^{k_s} \mid [u_j] \in D_{i_1, \dots, i_r}; 0 \leq k_j < \text{ord}(p_{u_j, u_j}); 1 \leq j \leq s, \text{ and let } |D_{i_1, \dots, i_r}| = s, \sum_{j=1}^s k_j > 0\}$. Let $V_{s;t}$ be a short notation for $V_{s, s+1, \dots, t}$. Similarly we use $B_{s;t}$ and $L_{s;t}$. Let $B_{i;j} := \emptyset$ and $L_{i;j} := \emptyset$ when $i > j$.

Lemma 4.1. *Assume that $\mathfrak{B}(V)$ is connected Nichols algebra of diagonal type with $\dim V > 2$ and $\Delta(\mathfrak{B}(V))$ is an arithmetic root system. Then the following statements hold.*

(i) ($n \geq 1$) *If the pure generalized Dynkin graph is*



then

$$L_{1;n} = B_{1;n} - \cup_{i=1}^{n-2} L_{1;i} B_{i+2;n} \tag{5}$$

$$= B_{1;n} - \cup_{i=2}^{n-2} (L_{1;i} - L_{1;i-1}) B_{i+2;n} - L_{1;1} B_{3;n} \tag{6}$$

$$= B_{1;n} - \cup_{i=1}^{n-3} L_{1;i} (B_{i+2;n} - B_{i+3;n}) - L_{1;n-2} B_{n;n}, \tag{7}$$

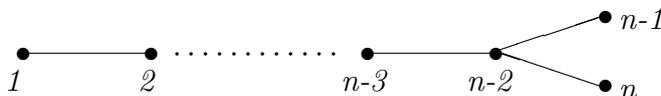
$$|L_{1;n}| = |B_{1;n}| - \sum_{i=1}^{n-2} |L_{1;i}| (|B_{i+2;n}| - |B_{i+3;n}|). \tag{8}$$

$$= \sum_{j=0}^{\text{int}(\frac{n-1}{2})} (-1)^j u_j, \tag{9}$$

where $u_0 = |B_{1;n}|$ and where for $j > 0$,

$$u_j = \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \dots \sum_{n_j=1}^{n_{j-1}-2} |B_{1;n_j}| (|B_{n_j+2;n_{j-1}}| - |B_{n_j+3;n_{j-1}}|) \dots \\ \dots (|B_{n_2+2;n_1}| - |B_{n_2+3;n_1}|) (|B_{n_1+2;n}| - |B_{n_1+3;n}|).$$

(ii) ($n \geq 4$) *If the pure generalized Dynkin graph is*



then

$$L_{1;n} = B_{1;n} - B_{n-1;n-1}B_{n;n} - \cup_{i=1}^{n-3} L_{1;i}B_{i+2;n} \tag{10}$$

$$= B_{1;n} - \cup_{i=2}^{n-3} (L_{1;i} - L_{1;i-1})B_{i+2;n} - L_{1;1}B_{3;n} - B_{n-1;n-1}B_{n;n} \tag{11}$$

$$= B_{1;n} - L_{1;n-3}B_{n-1;n} - B_{n-1;n-1}B_{n;n} - \cup_{i=1}^{n-3} L_{1;i}(B_{i+2;n} - B_{i+3;n}), \tag{12}$$

$$\begin{aligned} |L_{1;n}| = & |B_{1;n}| - |B_{n-1;n-1}| |B_{n;n}| - |L_{1;n-3}| |B_{n-1;n}| \\ & - \sum_{i=1}^{n-3} |L_{1;i}| (|B_{i+2;n}| - |B_{i+3;n}|). \end{aligned} \tag{13}$$

where $|L_{1;i}|$ is obtained by the formula (9) when $1 \leq i \leq n - 3$.

(iii) ($n \geq 6$) If the pure generalized Dynkin graph is



then

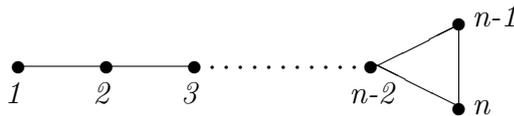
$$L_{1;n} = B_{1;n} - \cup_{i=1}^{n-4} L_{1;i}B_{i+2;n} - L_{1;n-2}B_{n;n} - B_{n-2;n-2}(B_{n-1;n} - B_{n;n}) \tag{14}$$

$$\begin{aligned} = & B_{1;n} - \cup_{i=2}^{n-4} (L_{1;i} - L_{1;i-1})B_{i+2;n} - L_{1;1}B_{3;n} \\ & - (L_{1;n-2} - L_{1;n-4})B_{n;n} - B_{n-2;n-2}(B_{n-1;n} - B_{n;n}), \end{aligned} \tag{15}$$

$$\begin{aligned} |L_{1;n}| = & |B_{1;n}| - \sum_{i=2}^{n-4} (|L_{1;i}| - |L_{1;i-1}|) |B_{i+2;n}| - |L_{1;1}| |B_{3;n}| \\ & - (|L_{1;n-2}| - |L_{1;n-4}|) |B_{n;n}| - |B_{n-2;n-2}| (|B_{n-1;n}| - |B_{n;n}|) \end{aligned} \tag{16}$$

where $|L_{1;i}|$ is obtained by the formula (9) when $1 \leq i \leq n - 2$.

(iv) ($n \geq 4$) If the pure generalized Dynkin graph is



then

$$L_{1;n} = B_{1;n} - L_{1;1}B_{3;n} - (L_{1;2} - L_{1;1})B_{4;n} \cdots - (L_{1;n-3} - L_{1;n-4})B_{n-1;n} \tag{17}$$

$$= B_{1;n} - \cup_{i=2}^{n-3} (L_{1;i} - L_{1;i-1})B_{i+2;n} - L_{1;1}B_{3;n}, \tag{18}$$

$$|L_{1;n}| = |B_{1;n}| - \sum_{i=2}^{n-3} (|L_{1;i}| - |L_{1;i-1}|) |B_{i+2;n}| - |L_{1;1}| |B_{3;n}|. \tag{19}$$

where $|L_{1;i}|$ is obtained by the formula (9) when $1 \leq i \leq n - 2$.

(v) ($n \geq 5$) If the pure generalized Dynkin graph is



then

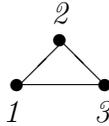
$$L_{1;n} = B_{1;n} - L_{1;1}B_{3;n} - (L_{1;2} - L_{1;1})B_{4;n} \cdots - (L_{1;n-5} - L_{1;n-6})B_{n-3;n} - (L_{1;n-4} - L_{1;n-5})B_{n-2;n} - (L_{1;n-2} - L_{1;n-4})B_{n;n} \tag{20}$$

$$= B_{1;n} - \cup_{i=2}^{n-4} (L_{1;i} - L_{1;i-1})B_{i+2;n} - L_{1;1}B_{3;n} - (L_{1;n-2} - L_{1;n-4})B_{n;n}, \tag{21}$$

$$|L_{1;n}| = |B_{1;n}| - \sum_{i=2}^{n-4} (|L_{1;i}| - |L_{1;i-1}|) |B_{i+2;n}| - |L_{1;1}| |B_{3;n}| - (|L_{1;n-2}| - |L_{1;n-4}|) |B_{n;n}|, \tag{22}$$

where $|L_{1;i}|$ is obtained by the formula (9) when $1 \leq i \leq n - 2$.

(vi) If the pure generalized Dynkin graph is



then $L_{1;3} = B_{1;3}$.

Proof. (i) We only determine which element in $B_{1;n}$ is connected. It is clear that the left hand of (5) \subseteq the right hand of (5). If $u \in B_{1;n} - L_{1;n}$, let $i_u := \min\{j \mid x_j \notin \mu(u) \text{ and there exists } x_i \in \mu(u) \text{ such that } 1 \leq i < j \leq n\}$. By Lemma 3.2, there exist $v \in L_{1;i_u-1}$ and $w \in B_{i_u+1;n}$ such that $u = vw$. Consequently, the right hand of (5) \subseteq the left hand of (5). therefore (5) holds.

$$\begin{aligned} |L_{1;n}| &= |B_{1;n}| - \sum_{n_1=1}^{n-2} |L_{1;n_1}| (|B_{n_1+2;n}| - |B_{n_1+3;n}|) \\ &= |B_{1;n}| - \sum_{n_1=1}^{n-2} \left(|B_{1;n_1}| - \sum_{n_2=1}^{n_1-2} |L_{1;n_2}| (|B_{n_2+2;n_1}| - |B_{n_2+3;n_1}|) \right) \\ &\quad \times (|B_{n_1+2;n}| - |B_{n_1+3;n}|) \\ \dots &= \sum_{j=0}^{\text{int}(\frac{n-1}{2})} (-1)^j u_j. \end{aligned}$$

Similarly, we can show (ii)–(vi). ■

Theorem 4.2. *The bases for the Nichols braided Lie algebras of arithmetic root systems are given by Lemma 4.1.*

Proof. We can check that all pure generalized Dynkin diagrams in [He05, Table A.1, A.2], [He06a, Table B, C] are in Lemma 4.1. ■

By [Hu78], $|D(A_n)| = C_{n+1}^2$, $|D(B_n)| = n^2 = |D(C_n)|$, $|D(D_n)| = n^2 - n$, $|D(E_6)| = 36$, $|D(E_7)| = 63$, $|D(E_8)| = 120$, $|D(F_4)| = 24$, $|D(G_2)| = 6$. By Lemma 4.1 and [WZZ15a, Lemma 6.4], we have the following results.

Theorem 4.3. *Let $\text{ord}(q) := N$.*

(i) For A_n , $n \geq 1$,

$$\begin{array}{cccccccc} & q & q^{-1} & q & q^{-1} & q & \dots & q & q^{-1} & q \\ \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & & & n-1 & n & & & \end{array}, \quad q \in F^*/\{1\}.$$

then $\dim \mathfrak{L}(V) = \sum_{j=0}^{\text{int}(\frac{n-1}{2})} (-1)^j u_j$, where $u_0 = |B_{1;n}|$ and, for $j > 0$,

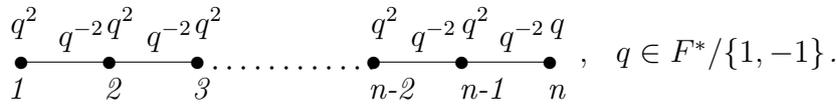
$$u_j = \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \cdots \sum_{n_j=1}^{n_{j-1}-2} |B_{1;n_j}| (|B_{n_j+2;n_{j-1}}| - |B_{n_j+3;n_{j-1}}|) \cdots \\ \cdots (|B_{n_2+2;n_1}| - |B_{n_2+3;n_1}|) (|B_{n_1+2;n}| - |B_{n_1+3;n}|)$$

and $|B_{i;k}| = N^{C_{k-i+2}^2} - 1$ for $1 \leq i \leq k \leq n$.

Furthermore, $\dim \mathfrak{L}(V) =$

$$|L_{1;n}| = N^{C_{n+1}^2} - 1 + \sum_{j=1}^{\text{int}(\frac{n-1}{2})} (-1)^j \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \cdots \sum_{n_j=1}^{n_{j-1}-2} (N^{C_{n_j+1}^2} - 1) \\ (N^{C_{n_{j-1}-n_j}^2} - N^{C_{n_{j-1}-n_j-1}^2}) \cdots (N^{C_{n_1-n_2}^2} - N^{C_{n_1-n_2-1}^2}) (N^{C_{n-n_1}^2} - N^{C_{n-n_1-1}^2}). \quad (23)$$

(ii) For B_n , $n \geq 2$,



then $\dim \mathfrak{L}(V) = \sum_{j=0}^{\text{int}(\frac{n-1}{2})} (-1)^j u_j$, where $u_0 = |B_{1;n}|$ and, for $1 \leq i < n$,

$$u_j = \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \cdots \sum_{n_j=1}^{n_{j-1}-2} |B_{1;n_j}| (|B_{n_j+2;n_{j-1}}| - |B_{n_j+3;n_{j-1}}|) \cdots \\ \cdots (|B_{n_2+2;n_1}| - |B_{n_2+3;n_1}|) (|B_{n_1+2;n}| - |B_{n_1+3;n}|)$$

for $j > 0$; $|B_{i;n}| = N^{(n-i+1)^2} - 1$ and $|B_{i;k}| = N^{C_{k-i+2}^2} - 1$ for $1 \leq i \leq k < n$, when N is odd; $|B_{i;n}| = (\frac{N}{2})^{(n-i+1)^2-n+i-1} N^{n-i+1} - 1$ for $1 \leq i < n$ and $|B_{i;k}| = (\frac{N}{2})^{C_{k-i+2}^2} - 1$ for $1 \leq i \leq k < n$, when N is even.

Furthermore, $\dim \mathfrak{L}(V) =$

$$|L_{1;n}| = N^{n^2} - 1 + \sum_{j=1}^{\text{int}(\frac{n-1}{2})} (-1)^j \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \cdots \sum_{n_j=1}^{n_{j-1}-2} (N^{C_{n_j+1}^2} - 1) \\ (N^{C_{n_{j-1}-n_j}^2} - N^{C_{n_{j-1}-n_j-1}^2}) \cdots (N^{C_{n_1-n_2}^2} - N^{C_{n_1-n_2-1}^2}) \\ (N^{(n-n_1-1)^2-n+n_1+1} - N^{(n-n_1-2)^2-n+n_1+2}), \quad (24)$$

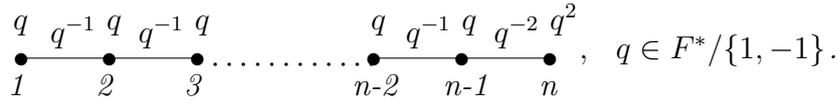
when N is odd; $\dim \mathfrak{L}(V) =$

$$|L_{1;n}| = (\frac{N}{2})^{n^2-n} N^n - 1 + \sum_{j=1}^{\text{int}(\frac{n-1}{2})} (-1)^j \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \cdots \sum_{n_j=1}^{n_{j-1}-2} ((\frac{N}{2})^{C_{n_j+1}^2} - 1) \quad (25)$$

$$((\frac{N}{2})^{C_{n_{j-1}-n_j}^2} - (\frac{N}{2})^{C_{n_{j-1}-n_j-1}^2}) \cdots ((\frac{N}{2})^{C_{n_1-n_2}^2} - (\frac{N}{2})^{C_{n_1-n_2-1}^2}) \\ ((\frac{N}{2})^{(n-n_1-1)^2-n+n_1+1} N^{n-n_1-1} - (\frac{N}{2})^{(n-n_1-2)^2-n+n_1+2} N^{n-n_1-2}), \quad (26)$$

when N is even.

(iii) For C_n , $n > 2$,



then $\dim \mathfrak{L}(V) = \sum_{j=0}^{\text{int}(\frac{n-1}{2})} (-1)^j u_j$, where $u_0 = |B_{1;n}|$ and, for $j > 0$,

$$u_j = \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \dots \sum_{n_j=1}^{n_{j-1}-2} |B_{1;n_j}| (|B_{n_j+2;n_{j-1}}| - |B_{n_j+3;n_{j-1}}|) \dots (|B_{n_2+2;n_1}| - |B_{n_2+3;n_1}|) (|B_{n_1+2;n}| - |B_{n_1+3;n}|);$$

$|B_{i;n}| = N^{(n-i+1)^2} - 1$ for $1 \leq i < n$ and $|B_{i;k}| = N^{C_{k-i+2}^2} - 1$ for $1 \leq i \leq k < n$, when N is odd; $|B_{i;n}| = N^{(n-i+1)^2 - n + i - 1} (\frac{N}{2})^{n-i+1} - 1$ for $1 \leq i < n$ and $|B_{i;k}| = N^{C_{k-i+2}^2} - 1$ for $1 \leq i \leq k < n$, when N is even.

Furthermore, $\dim \mathfrak{L}(V) =$

$$\begin{aligned} |L_{1;n}| = & N^{n^2} - 1 + \sum_{j=1}^{\text{int}(\frac{n-1}{2})} (-1)^j \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \dots \sum_{n_j=1}^{n_{j-1}-2} (N^{C_{n_j+1}^2} - 1) \\ & (N^{C_{n_{j-1}-n_j}^2} - N^{C_{n_{j-1}-n_j-1}^2}) \dots (N^{C_{n_1-n_2}^2} - N^{C_{n_1-n_2-1}^2}) \\ & (N^{(n-n_1-1)^2 - n + n_1 + 1} - N^{(n-n_1-2)^2 - n + n_1 + 2}), \end{aligned} \tag{27}$$

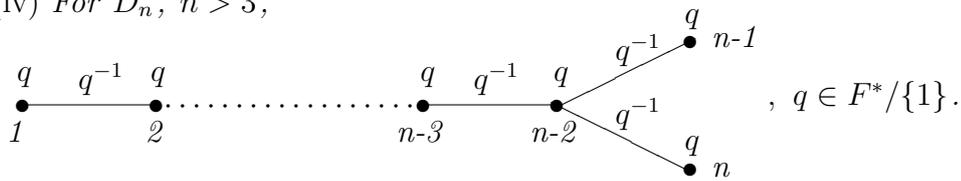
when N is odd; $\dim \mathfrak{L}(V) =$

$$\begin{aligned} |L_{1;n}| = & (\frac{N}{2})^n N^{n^2-n} - 1 + \sum_{j=1}^{\text{int}(\frac{n-1}{2})} (-1)^j \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \dots \sum_{n_j=1}^{n_{j-1}-2} (N^{C_{n_j+1}^2} - 1) \\ & (N^{C_{n_{j-1}-n_j}^2} - N^{C_{n_{j-1}-n_j-1}^2}) \dots (N^{C_{n_1-n_2}^2} - N^{C_{n_1-n_2-1}^2}) \\ & (N^{(n-n_1-1)^2 - n + n_1 + 1} (\frac{N}{2})^{n-n_1-1} - N^{(n-n_1-2)^2 - n + n_1 + 2} (\frac{N}{2})^{n-n_1-2}), \end{aligned} \tag{28}$$

$$\tag{29}$$

when N is even.

(iv) For D_n , $n > 3$,



then

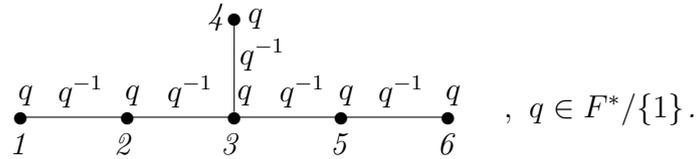
$$\begin{aligned} \dim \mathfrak{L}(V) = & |B_{1;n}| - |B_{n-1;n-1}| |B_{n;n}| - \\ & - \sum_{i=1}^{n-3} |L_{1;i}| (|B_{i+2;n}| - |B_{i+3;n}|) - |L_{1;n-3}| |B_{n-1;n}|, \end{aligned}$$

where $|B_{k;n}| = N^{(n-k+1)^2 - n + k - 1} - 1$ for $n - k + 1 > 2$, $|B_{n-1;n}| = N^3 - 1$, $|B_{n;n}| = |B_{n-1;n-1}| = N - 1$; $|L_{1;t}|$ is obtained by the formula (23) when $1 \leq t \leq n - 3$.

Furthermore,

$$\begin{aligned}
 \dim \mathfrak{L}(V) = & |L_{1;n}| = N^{n^2-n} - 1 - (N-1)^2 - \sum_{i=1}^{n-3} \left\{ N^{C_{i+1}^2} - 1 \right. \\
 & + \sum_{j=1}^{\text{int}(\frac{i-1}{2})} (-1)^j \sum_{n_1=1}^{n-2} \sum_{n_2=1}^{n_1-2} \cdots \sum_{n_j=1}^{n_{j-1}-2} (N^{C_{n_j+1}^2} - 1)(N^{C_{n_{j-1}-n_j}^2} - N^{C_{n_{j-1}-n_j-1}^2}) \\
 & \left. \cdots (N^{C_{n_1-n_2}^2} - N^{C_{n_1-n_2-1}^2})(N^{C_{i-n_1}^2} - N^{C_{i-n_1-1}^2}) \right\} \\
 & (N^{(n-i-1)^2-n+i+1} - N^{(n-i-2)^2-n+i+2}) \\
 & - \left\{ N^{C_{n-2}^2} - 1 + \sum_{j=1}^{\text{int}(\frac{n-4}{2})} (-1)^j \sum_{n_1=1}^{n-5} \sum_{n_2=1}^{n_1-2} \cdots \sum_{n_j=1}^{n_{j-1}-2} (N^{C_{n_j+1}^2} - 1) \right. \\
 & (N^{C_{n_{j-1}-n_j}^2} - N^{C_{n_{j-1}-n_j-1}^2}) \cdots (N^{C_{n_1-n_2}^2} - N^{C_{n_1-n_2-1}^2}) \\
 & \left. (N^{C_{n-3-n_1}^2} - N^{C_{n-4-n_1}^2}) \right\} (N^2 - 1). \tag{30}
 \end{aligned}$$

(v) For E_6 ,

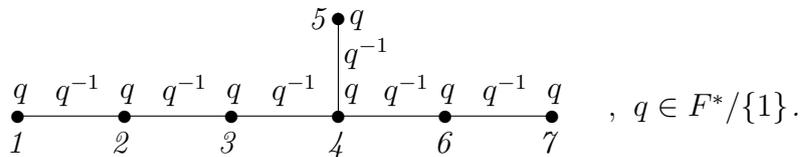


then

$$\begin{aligned}
 \dim \mathfrak{L}(V) = & |B_{1;6}| - \sum_{i=2}^{6-4} (|L_{1;i}| - |L_{1;i-1}|) |B_{i+2;6}| - |L_{1;1}| |B_{3;6}| \\
 & - (|L_{1;6-2}| - |L_{1;6-4}|) |B_{6;6}| - |B_{6-2;6-2}| (|B_{6-1;6}| - |B_{6;6}|) \\
 = & N^{36} - 1 - (|L_{1;2}| - |L_{1;1}|)(NN^3 - 1) - (N-1)(N^{C_5^2} - 1) \\
 & - (|L_{1;4}| - |L_{1;2}|)(N-1) - (N-1)(N^3 - N),
 \end{aligned}$$

where $|L_{1;t}|$ is obtained by the formula (23) when $1 \leq t \leq 4$.

(vi) For E_7 ,



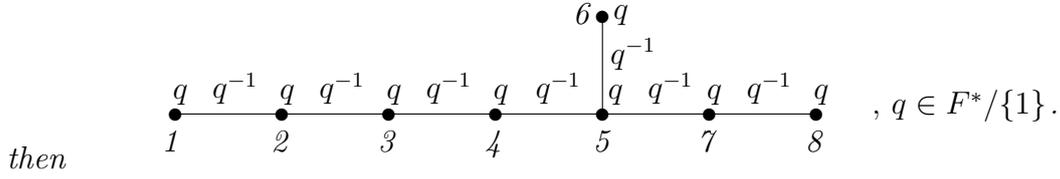
then

$$\begin{aligned}
 \dim \mathfrak{L}(V) = & |B_{1;7}| - \sum_{i=2}^{7-4} (|L_{1;i}| - |L_{1;i-1}|) |B_{i+2;7}| - |L_{1;1}| |B_{3;7}| \\
 & - (|L_{1;7-2}| - |L_{1;7-4}|) |B_{7;7}| - |B_{7-2;7-2}| (|B_{7-1;7}| - |B_{7;7}|) \\
 = & |B_{1;7}| - (|L_{1;2}| - |L_{1;1}|) |B_{3;7}| - (|L_{1;3}| - |L_{1;2}|) |B_{5;7}| \\
 & - |L_{1;1}| |B_{3;7}| - (|L_{1;7-2}| - |L_{1;7-4}|) |B_{7;7}| \\
 & - |B_{7-2;7-2}| (|B_{7-1;7}| - |B_{7;7}|)
 \end{aligned}$$

$$= N^{63} - 1 - (|L_{1;2}| - |L_{1;1}|)(N^{10} - 1) - (|L_{1;3}| - |L_{1;2}|)(NN^3 - 1) - (N - 1)(N^{20} - 1) - (|L_{1;5}| - |L_{1;3}|)(N - 1) - (N - 1)(N^3 - N),$$

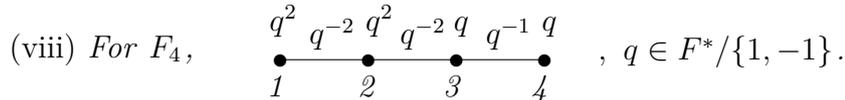
where $|L_{1;t}|$ is obtained by the formula (23) when $1 \leq t \leq 5$.

(vii) For E_8 ,



$$\begin{aligned} \dim \mathfrak{L}(V) &= |B_{1;8}| - \sum_{i=2}^{8-4} (|L_{1;i}| - |L_{1;i-1}|) |B_{i+2;8}| - |L_{1;1}| |B_{3;8}| \\ &\quad - (|L_{1;8-2}| - |L_{1;8-4}|) |B_{8;8}| - |B_{8-2;8-2}| (|B_{8-1;8}| - |B_{8;8}|) \\ &= |B_{1;8}| - (|L_{1;2}| - |L_{1;1}|) |B_{4;8}| - (|L_{1;3}| - |L_{1;2}|) |B_{5;8}| \\ &\quad - (|L_{1;4}| - |L_{1;3}|) |B_{6;8}| - |L_{1;1}| |B_{3;8}| \\ &\quad - (|L_{1;6}| - |L_{1;4}|) |B_{8;8}| - |B_{6;6}| (|B_{7;8}| - |B_{8;8}|) \\ &= N^{120} - 1 - (|L_{1;2}| - |L_{1;1}|)(N^{20} - 1) \\ &\quad - (|L_{1;3}| - |L_{1;2}|)(N^{C_5^2} - 1) \\ &\quad - (|L_{1;4}| - |L_{1;3}|)(NN^3 - 1) - (N - 1)(N^{36} - 1) \\ &\quad - (|L_{1;6}| - |L_{1;4}|)(N - 1) - (N - 1)(N^3 - N), \end{aligned}$$

where $|L_{1;t}|$ is obtained by the formula (23) when $1 \leq t \leq 6$.



then $\dim \mathfrak{L}(V) = \sum_{j=0}^1 (-1)^j u_j$, where $u_0 = |B_{1;4}|$,

$$u_1 = \sum_{n_1=1}^2 (|B_{1;n_1}|)(|B_{n_1+2;4}| - |B_{n_1+3;4}|),$$

$|B_{1;4}| = N^{24} - 1$, $|B_{1;1}| = N - 1$, $|B_{1;2}| = N^{C_3^2} - 1 = N^3 - 1$, $|B_{3;4}| = N^{C_3^2} - 1 = N^3 - 1$ and $|B_{4;4}| = N - 1$, when N is odd;

$|B_{1;4}| = N^{12}(\frac{N}{2})^{12} - 1$, $|B_{1;1}| = \frac{N}{2} - 1$, $|B_{1;2}| = (\frac{N}{2})^{C_3^2} - 1 = (\frac{N}{2})^3 - 1$, $|B_{3;4}| = N^{C_3^2} - 1 = N^3 - 1$ and $|B_{4;4}| = N - 1$, when N is even.

Furthermore,

$$\dim \mathfrak{L}(V) = N^{24} - 1 - (N - 1)(N^3 - N) - (N^3 - 1)(N - 1), \tag{31}$$

when N is odd;

$$\dim \mathfrak{L}(V) = \left(\frac{N}{2}\right)^{12} N^{12} - 1 - \left(\frac{N}{2} - 1\right)(N^3 - N) - \left(\left(\frac{N}{2}\right)^3 - 1\right)(N - 1), \quad (32)$$

when N is even.

(ix) For G_2 ,
$$\begin{array}{ccc} q & q^{-3} & q^3 \\ \bullet & \xrightarrow{\quad} & \bullet \\ 1 & & 2 \end{array}, \quad q \in F^*/\{1, -1\}, \quad q^3 \neq 1.$$

then $\dim \mathfrak{L}(V) = N^6 - 1$ when $3 \nmid N$; $\dim \mathfrak{L}(V) = \left(\frac{N}{3}\right)^3 N^3 - 1$ when $3 \mid N$.

5. Non-zero monomials in Nichols algebras

Let $(s)_q := 1 + q + q^2 + \dots + q^{s-1}$ and $(s)_q! := (1)_q(2)_q \dots (s)_q$.

Lemma 5.1. *Assume that $h_i \in \{x_1, \dots, x_n\}$ for $1 \leq i \leq m$ and u_j is 1 or a monomial with $x_k \notin \mu(u_j)$ for $1 \leq j \leq l + 1$. Let $q := p_{x_k, x_k}^{-1}$ for convenience.*

(i)
$$\begin{aligned} \langle y_k, u_1 x_k u_2 x_k \cdots u_l x_k u_{l+1} \rangle &= \\ &= \sum_{j=1}^l q^{j-1} p_{k, u_1 u_2 \cdots u_j}^{-1} u_1 x_k u_2 x_k \cdots x_k (u_j u_{j+1}) x_k \cdots x_k u_{l+1}. \end{aligned} \quad (33)$$

(ii)
$$\langle y_k^l, u_1 x_k u_2 x_k \cdots u_l x_k u_{l+1} \rangle = (l)_q! p_{k, u_1}^{-1} p_{k, u_1 u_2}^{-1} \cdots p_{k, u_1 u_2 \cdots u_l}^{-1} u_1 u_2 \cdots u_{l+1}. \quad (34)$$

(iii) *If $\text{ord}(p_{h_i, h_i}) > |\text{deg}_{h_i}(h_1 \cdots h_m)|$ or $p_{h_i, h_i} = 1$ for all $i \in \{1, \dots, m\}$, then $h_1 \cdots h_m \neq 0$.*

Proof. (i) can be obtained by induction on l .

(ii) We show this by induction on l . If $l = 1$, $\langle y_k, u_1 x_k u_2 \rangle = p_{k, u_1}^{-1} u_1 u_2$. Assume $l > 1$. Then

$$\begin{aligned} \langle y_k^l, u_1 x_k u_2 x_k \cdots u_l x_k u_{l+1} \rangle &= \langle y_k^{l-1}, \langle y_k, u_1 x_k u_2 x_k \cdots u_l x_k u_{l+1} \rangle \rangle \\ &= \langle y_k^{l-1}, \sum_{j=1}^l q^{j-1} p_{k, u_1 u_2 \cdots u_j}^{-1} u_1 x_k u_2 x_k \cdots x_k (u_j u_{j+1}) x_k \cdots x_k u_{l+1} \rangle \\ &= \sum_{j=1}^l q^{j-1} p_{k, u_1 u_2 \cdots u_j}^{-1} \langle y_k^{l-1}, u_1 x_k u_2 x_k \cdots x_k (u_j u_{j+1}) x_k \cdots x_k u_{l+1} \rangle \\ &= (l)_q! p_{k, u_1}^{-1} p_{k, u_1 u_2}^{-1} \cdots p_{k, u_1 u_2 \cdots u_l}^{-1} u_1 u_2 \cdots u_{l+1}. \end{aligned}$$

(iii) We show this by induction on $t := |\mu(h_1 h_2 \cdots h_m)|$. If $t = 1$, we obtain $h_1 \cdots h_m \neq 0$ by [He05, Lemma 1.3.3 (i)]. Assume $t > 1$ and $h_1 h_2 \cdots h_m = u_1 x_k u_2 x_k \cdots u_l x_k u_{l+1}$ with $|\mu(u_1 u_2 \cdots u_{l+1})| = t - 1$. Thus $u_1 u_2 \cdots u_{l+1} \neq 0$ by induction hypothesis, $(1 + p_{kk}^{-1})(1 + p_{kk}^{-1} + p_{kk}^{-2}) \cdots (1 + p_{kk}^{-1} + \cdots + p_{kk}^{-l+1}) \neq 0$ since $\text{ord}(p_{x_k, x_k}) > |\text{deg}_{x_k}(h_1 \cdots h_m)| = l$ or $p_{h_i, h_i} = 1$. Hence $u_1 x_k u_2 x_k \cdots u_l x_k u_{l+1} \neq 0$, completing the proof. ■

Corollary 5.2. *If u and v are monomials in $\mathfrak{B}(V)$, $\mu(u) \cap \mu(v) = \emptyset$, $p_{ij}p_{ji} = 1$ for any $x_i \in \mu(u)$, $x_j \in \mu(v)$, then the following conditions are equivalent:*

- (i) $u \neq 0$, $v \neq 0$, (ii) $uv \neq 0$, (iii) $uv \notin \mathfrak{L}(V)$.

Proof. We know (ii) \iff (iii) by Lemma 2.2. (ii) \implies (i) is clear. (ii) \impliedby (i): Assume $|u| = k$, then $\exists y_{i_1}, \dots, y_{i_k} \in d(u)$ such that $\langle y_{i_1} \cdots y_{i_k}, u \rangle \in F^*$ and $\langle y_{i_1} \cdots y_{i_k}, uv \rangle = \langle y_{i_1} \cdots y_{i_k}, u \rangle v \neq 0$. ■

Lemma 5.3. *If u and v are two homogeneous elements with $\mu(u) \cap \mu(v) = \emptyset$, then $uv = 0$ implies $u = 0$ or $v = 0$.*

Proof. Without loss of generality, there exists i_0 such that $\mu(v) \subseteq \{1, 2, \dots, i_0\}$ and $\mu(u) \subseteq \{i_0 + 1, i_0 + 2, \dots, n\}$. Assume $u \neq 0$ and $v \neq 0$ with $u = \sum_{i=1}^s k_i u_i$ and $v = \sum_{j=1}^t k'_j v_j$, where $k_i \neq 0$, $k'_j \neq 0$, $u_i \in B_{1, i_0}$, $v_j \in B_{i_0+1, n}$ for $1 \leq i \leq i_0$, $1 + i_0 \leq j \leq n$. Consequently, $uv = \sum_{i=1}^s \sum_{j=1}^t k_i k'_j u_i v_j \neq 0$ with $u_i v_j \in B_{1, n}$ and $k_i k'_j \neq 0$ for $1 \leq i \leq s$, $1 \leq j \leq t$. ■

6. Bases of Nichols Lie algebras $\mathfrak{L}^-(V)$

In this section we give the sufficient and necessary conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$ and $\mathfrak{L}^-(V) = \mathfrak{L}(V)$. We also obtain an explicit basis for $\mathfrak{L}^-(V)$ over the quantum linear space V with $\dim V = 2$.

6.1. Conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$.

Lemma 6.1. *Assume that u_i is a homogeneous element and $u_i u_j = r_{u_i, u_j} u_j u_i$ with $r_{u_i, u_j} \in F^*$ for $1 \leq i, j \leq k$ and $r_{u_i, u_j} = 1$ when $u_i = u_j$. Then*

$$[u_1, \dots, u_m]^- = (r_{u_1, u_2 \cdots u_m} - 1)(r_{u_2, u_3 \cdots u_m} - 1) \cdots (r_{u_{m-1}, u_m} - 1) u_m \cdots u_2 u_1,$$

where $[u_1, \dots, u_m]^- := [u_1, [u_2, \dots, [u_{m-1}, u_m]^- \cdots]^-]^-$.

Proof. Since $[u_1, [u_2, \dots, u_m]^-] = 0$, we get $u_1 [u_2, \dots, u_m]^- = r_{u_1, u_2 \cdots u_m} [u_2, \dots, u_m]^- u_1$. Therefore,

$$\begin{aligned} [u_1, \dots, u_m]^- &= u_1 [u_2, \dots, u_m]^- - [u_2, \dots, u_m]^- u_1 \\ &= (r_{u_1, u_2 \cdots u_m} - 1) [u_2, \dots, u_m]^- u_1 \\ &= (r_{u_1, u_2 \cdots u_m} - 1)(r_{u_2, u_3 \cdots u_m} - 1) \cdots (r_{u_{m-1}, u_m} - 1) u_m u_{m-1} \cdots u_1. \end{aligned} \quad \blacksquare$$

Lemma 6.2. *Let $(L, [\]^-)$ be a Lie algebra and $u_1, u_2, \dots, u_m \in L$. If σ is a method of adding bracket $[\]^-$ on u_1, u_2, \dots, u_m , then there exist some $\tau_j \in \mathbb{S}_m$, $\xi_j = 1$ or -1 such that*

$$\sigma(u_1, u_2, \dots, u_m) = \sum_{j=1}^r \xi_j [u_{\tau_j(1)}, \dots, u_{\tau_j(m)}]^- \tag{35}$$

Proof. We show (35) by induction on m . Obviously, (35) holds for $m = 2$. Assume $m > 2$. Let $\sigma(u_1, u_2, \dots, u_m) = [\sigma_1(u_1, u_2, \dots, u_s), \sigma_2(u_{s+1}, u_{s+2}, \dots, u_m)]^-$. Now we show (35) by induction on s . In case $s = 1$,

$$\begin{aligned} \sigma(u_1, u_2, \dots, u_m) &= [u_1, \sigma_2(u_2, u_3, \dots, u_m)]^- = \\ &= \sum_{j=1}^r \xi_j [u_1, [u_{\tau_j(2)} \cdots u_{\tau_j(m)}]^-]^- = \sum_{j=1}^r \xi_j [u_{\tau_j(1)}, u_{\tau_j(2)} \cdots u_{\tau_j(m)}]^- , \end{aligned}$$

by the induction assumption, where $\tau_j \in \mathbb{S}_{\{2,3,\dots,m\}}$ for $1 \leq j \leq r$. Therefore, (35) holds. Assume $s > 1$ and $\sigma_1(u_1, u_2, \dots, u_s) = [\sigma_3(u_1, u_2, \dots, u_k), \sigma_4(u_{k+1}, u_{k+2}, \dots, u_s)]^-$. Then

$$\begin{aligned} \sigma(u_1, u_2, \dots, u_m) &= \\ &= [[\sigma_3(u_1, u_2, \dots, u_k), \sigma_4(u_{k+1}, u_{k+2}, \dots, u_s)]^-, \sigma_2(u_{s+1}, u_{s+2}, \dots, u_m)]^- \\ &= [[\sigma_3(u_1, u_2, \dots, u_k), \sigma_2(u_{s+1}, u_{s+2}, \dots, u_m)]^-, \sigma_4(u_{k+1}, u_{k+2}, \dots, u_s)]^- + \\ &\quad + [\sigma_3(u_1, u_2, \dots, u_k), [\sigma_4(u_{k+1}, u_{k+2}, \dots, u_s), \sigma_2(u_{s+1}, u_{s+2}, \dots, u_m)]^-]^- \\ &= \sum_{j=1}^{r_1} \xi_j [[u_{\tau_j(1)}, u_{\tau_j(2)} \cdots u_{\tau_j(k)}, u_{\tau_j(s+1)}, u_{\tau_j(s+2)} \cdots u_{\tau_j(m)}]^- , \sigma_4(u_{k+1}, u_{k+2}, \dots, u_s)]^- \\ &\quad + \sum_{j=r_1+1}^{r_2} [\sigma_3(u_1, u_2, \dots, u_k), \xi_j [u_{\tau_j(k+1)}, u_{\tau_j(k+2)} \cdots u_{\tau_j(s)}, u_{\tau_j(s+1)}, u_{\tau_j(s+2)} \cdots u_{\tau_j(m)}]^-]^- \\ &\quad \text{(by the first inductive assumption, where } \tau_j \in \mathbb{S}_{\{2,3,\dots,k,s+1,s+2,\dots,m\}} \\ &\quad \text{for } 1 \leq j \leq r_1; \tau_j \in \mathbb{S}_{\{k+1,k+2,\dots,m\}} \text{ for } r_1 + 1 \leq j \leq r_2) \\ &= \sum_{j=r_2+1}^{r_3} \xi_j [u_{\tau_j(1)}, \dots, u_{\tau_j(m)}]^- \quad \text{(by the second inductive assumption, where} \\ &\quad \tau_j \in \mathbb{S}_{\{1,2,\dots,m\}} \text{ for } r_2 + 1 \leq j \leq r_3). \end{aligned}$$

Therefore, (35) holds. ■

Proposition 6.3. *If $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type, then $\mathfrak{L}(V) = \mathfrak{L}^-(V)$ if and only if $p_{ii}^2 = 1$, $p_{ij} = p_{ji} = 1$ for $1 \leq i \neq j \leq n$. In this case, $\mathfrak{L}(V) = \mathfrak{L}^-(V) = V$.*

Proof. Sufficiency. By Corollary 2.5, $\mathfrak{L}(V) = V$. By [WZZ15b], $[x_i, x_j]^- = 0$ for $i \neq j$ and $\mathfrak{L}^-(V) = V$.

Necessity. We show this by following three steps.

(i) $p_{ii}^2 = 1$ since $x_i^k \in \mathfrak{L}(V)$ for $k \leq \text{ord}(p_{ii})$ and $x_i^m \notin \mathfrak{L}^-(V)$ when $0 \neq x_i^m$ and $m > 1$.

(ii) If $p_{ij}p_{ji} = 1$ and $p_{ij} \neq 1$ with $i \neq j$, then $[x_i, x_j]^- \neq 0$ and $[x_i, x_j] = 0$. Consequently, $[x_i, x_j]^- \in \mathfrak{L}^-(V) - \mathfrak{L}(V)$, which is a contradiction.

(iii) If $p_{ij}p_{ji} \neq 1$ with $i < j$, then $0 \neq [x_i, x_j] \in \mathfrak{L}(V) = \mathfrak{L}^-(V)$ and $[x_i, x_j] = k[x_i, x_j]^-$ with $0 \neq k \in F$. By Corollary 2.5, $x_j x_i \in \mathfrak{L}(V)$, which implies $x_j x_i = k'[x_i, x_j]^-$. This is a contradiction since $x_j x_i$ and $[x_i, x_j]$ are linearly independent. ■

Proposition 6.4. *Assume that $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type. Then $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$ if and only if $p_{ii} = -1, p_{ij}p_{ji} = 1$ for all $1 \leq i \neq j \leq n$, and there exists a $\tau \in \mathbb{S}_m$ such that*

$$(p_{h_{\tau(1)}, h_{\tau(2)} \cdots h_{\tau(m)}} - 1)(p_{h_{\tau(2)}, h_{\tau(3)} \cdots h_{\tau(m)}} - 1) \cdots (p_{h_{\tau(m-1)}, h_{\tau(m)}} - 1) \neq 0$$

for all $h_1 > h_2 > \cdots > h_m$ with $h_i \in \{x_1, \dots, x_n\}, 1 \leq i \leq m$.

Proof. Necessity. If there exists $1 \leq i \leq n$ such that $p_{ii} \neq -1$, then $0 \neq x_i^2 \in \mathfrak{B}(V)$ and $x_i^2 \notin \mathfrak{L}^-(V)$, which is a contradiction. If there exist i, j such that $p_{ij}p_{ji} \neq 1$ with $1 \leq i < j \leq n$, then $[x_i, x_j] \neq 0$ and $[x_i, x_j]^- \neq 0$. Since $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$, we have that there exist $k, k' \in F^*$ such that $[x_i, x_j] = k[x_i, x_j]^-$, and $x_j x_i = k'[x_i, x_j]^-$, which contradicts to that $[x_i, x_j]$ and $x_j x_i$ are linearly independent. Therefore, V is a quantum linear space.

If there exist $h_1 > h_2 > \cdots > h_m$ with $h_i \in \{x_1, \dots, x_n\}, 1 \leq i \leq m$, such that $(p_{h_{\tau(1)}, h_{\tau(2)} \cdots h_{\tau(m)}} - 1)(p_{h_{\tau(2)}, h_{\tau(3)} \cdots h_{\tau(m)}} - 1) \cdots (p_{h_{\tau(m-1)}, h_{\tau(m)}} - 1) = 0$ for any $\tau \in \mathbb{S}_m$. By Lemma 6.1 and Lemma 3.2, $\sigma(h_{\tau(1)}, h_{\tau(2)}, \dots, h_{\tau(m)}) = 0$ for any $\tau \in \mathbb{S}_m$ and any method σ of adding bracket $[\]^-$ on $h_{\tau(1)}, h_{\tau(2)}, \dots, h_{\tau(m)}$. Consequently, $0 \neq h_1 h_2 \cdots h_m \notin \mathfrak{L}^-(V)$, which is a contradiction.

Sufficiency. Obviously, V is a quantum linear space. For $\forall h_1 > h_2 > \cdots > h_m$ with $h_i \in \{x_1, \dots, x_n\}, 1 \leq i \leq m$, there exist $\tau \in \mathbb{S}_m$ such that $(p_{h_{\tau(1)}, h_{\tau(2)} \cdots h_{\tau(m)}} - 1)(p_{h_{\tau(2)}, h_{\tau(3)} \cdots h_{\tau(m)}} - 1) \cdots (p_{h_{\tau(m-1)}, h_{\tau(m)}} - 1) \neq 0$. By Lemma 6.1, $h_1 h_2 \cdots h_m \sim h_{\tau(m)} h_{\tau(m-1)} \cdots h_{\tau(1)} \in \mathfrak{L}^-(V)$. ■

6.2. Relationships between graphs and $\mathfrak{L}^-(V)$.

Let $\Gamma_a(V)_1 = \{1, 2, \dots, n\}$ and $\Gamma_a(V)_2 = \{a_{ij} \mid p_{ij} \neq 1, i \neq j\}$. $\Gamma_a(V) = (\Gamma_a(V)_1, \Gamma_a(V)_2)$ is called the augmented Dynkin graph of V .

Proposition 6.5. *Assume that $h_i \in \{x_1, \dots, x_n\}$ for $1 \leq i \leq m$ and $u = h_1 h_2 \cdots h_m$.*

- (i) *If it is disconnected between monomial u and monomial v in $\Gamma_a(V)$ (i.e. $p_{x_i, x_j} = 1$ for any $x_i \in \mu(u), x_j \in \mu(v)$ and $i \neq j$), then $[u, v]^- = 0$.*
- (ii) *If $\Gamma_a(u)$ is weakly disconnected in $\Gamma_a(V)$ (i.e. $\Gamma_a(u)$ is disconnected in $\Gamma_a(V)$ or $|\mu(u)| = 1$), then $\sigma(h_1, h_2, \dots, h_m) = 0$ for any method σ of adding bracket $[\]^-$ on h_1, h_2, \dots, h_m .*

Proof. (i) u and v are commutative (i.e. $uv = vu$) since $x_i x_j = x_j x_i$ for any $x_i \in \mu(u), x_j \in \mu(v)$.

(ii) We show this by induction on m . $[h_1, h_2]^- = 0$ for $m = 2$. For $m > 2$, $\sigma(h_1, h_2, \dots, h_m) = [\sigma_1(h_1 h_2 \cdots h_t), \sigma_2(h_{t+1} h_{t+2} \cdots h_m)]^-$. If both $h_1 h_2 \cdots h_t$ and $h_{t+1} h_{t+2} \cdots h_m$ are connected in $\Gamma_a(V)$, then it is disconnected between $h_1 h_2 \cdots h_t$ and $h_{t+1} h_{t+2} \cdots h_m$ in $\Gamma_a(V)$. By Part (i), $\sigma(h_1, h_2, \dots, h_m) = 0$. If either $h_1 h_2 \cdots h_t$ or $h_{t+1} h_{t+2} \cdots h_m$ is disconnected in $\Gamma_a(V)$, then either $\sigma_1(h_1 h_2 \cdots h_t) = 0$ or $\sigma_2(h_{t+1} h_{t+2} \cdots h_m) = 0$ by induction hypothesis. ■

6.3. $\mathfrak{L}^-(V)$ over quantum linear space V with $\dim V = 2$.

Let $R_m := \{\alpha \mid \alpha \text{ is a primitive } m\text{-th root of } 1\}$. For $a, b \in \mathbb{Z}$ with $b \neq 0$, if there exists $c \in \mathbb{Z}$ such that $a = bc$, then we say that b is a factor of a and a is divisible by b , written $b \mid a$. Otherwise, we say that a is not divisible by b , written $b \nmid a$. For convenience, assume $b \nmid \infty$ and $\infty \nmid b$.

Lemma 6.6. *Assume that u_i is a homogeneous element and $u_i u_j = r_{u_i, u_j} u_j u_i$ with $r_{u_i, u_j} \in F^*$ for $1 \leq i, j \leq k$ and $r_{u_i, u_j} = 1$ when $u_i = u_j$. Set $l_i := \bar{l}_{u_i}$ and $r_{i,j} := r_{u_i, u_j}$ for $\forall 1 \leq j, i \leq k$.*

$$(i) \quad l_k^{m_k} l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_1]^- = (r_{2,1} - 1)^{m_2} (r_{3,1} r_{3,2}^{m_2} - 1)^{m_2} \cdots \\ \cdots (r_{k,1} r_{k,2}^{m_2} \cdots r_{k,k-1}^{m_{k-1}} - 1)^{m_k} u_1 u_2^{m_2} \cdots u_k^{m_k} \tag{36}$$

for $\forall m_2, \dots, m_k \in \mathbb{N}$, i.e.

$$l_k^{m_k} l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_1]^- = \lambda_2^{m_2} \lambda_3^{m_3} \cdots \lambda_k^{m_k} u_1 u_2^{m_2} \cdots u_k^{m_k} \tag{37}$$

where $\lambda_i := r_{i,1} r_{i,2}^{m_2} \cdots r_{i,i-1}^{m_{i-1}} - 1$, $2 \leq i \leq k$

- (ii) If $u_1, u_i \in \mathfrak{L}^-(V)$ and $\lambda_i \neq 0$ for $2 \leq i \leq k$, then $u_1 u_2 \cdots u_k \in \mathfrak{L}^-(V)$.
- (iii) If $u_{2i} = u_2$ and $u_{2i-1} = u_1$ for $1 \leq i \leq \text{int}(\frac{k+1}{2})$, then $u_1 u_2^{m_1} \cdots u_k^{m_k} \sim u_2^{\alpha_2} u_1^{\alpha_1}$; $\lambda_k = r_{u_2, u_1}^{\alpha_1} - 1$ with $\alpha_1 = 1 + m_1 + m_2 + \cdots + m_{k-1}$ when k is even; $\lambda_k = r_{u_1, u_2}^{\alpha_2} - 1$ with $\alpha_2 = m_2 + m_4 + \cdots + m_{k-1}$ when k is odd.
- (iv) Assume that $u_1, u_2 \in \mathfrak{L}^-(V)$ and $u_1^{\alpha_1} u_2^{\alpha_2} \neq 0$. $\text{ord}(r_{u_2, u_1}) \nmid \alpha_1$, or $\text{ord}(r_{u_1, u_2}) \nmid \alpha_2$, then $u_1^{\alpha_1} u_2^{\alpha_2} \in \mathfrak{L}^-(V)$.
- (v) Assume that $u_1, u_2 \in \{x_1, x_2, \dots, x_n\}$. If $u_1^{\alpha_1} u_2^{\alpha_2} \neq 0$, then $u_1^{\alpha_1} u_2^{\alpha_2} \in \mathfrak{L}^-(V)$ if and only if $\text{ord}(r_{u_2, u_1}) \nmid \alpha_1$, or $\text{ord}(r_{u_1, u_2}) \nmid \alpha_2$.

Proof. (i) We show this by induction on k . For $k = 1$, this is similar to [WZZ15b, Lemma 4.1 (iii)]. Assume that $k = 2$. We show this by induction on m_2 . It is clear when $m_2 = 1$. If $m_2 > 1$, then

$$l_2^{m_2} [u_1]^- = [u_2, l_2^{m_2-1} [u_1]^-]^- = (r_{21} - 1)^{m_2-1} [u_2, u_1 u_2^{m_2-1}]^- \\ = (r_{21} - 1)^{m_2-1} (r_{21} u_1 u_2^{m_2} - u_1 u_2^{m_2}) = (r_{21} - 1)^{m_2} u_1 u_2^{m_2}.$$

Now $k > 2$. We show this by induction on m_k . If $m_k = 1$, then

$$l_k^{m_k} l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_1]^- = [u_k, l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_1]^-]^- \\ = (r_{2,1} - 1)^{m_2} (r_{3,1} r_{3,2}^{m_2} - 1)^{m_3} \cdots (r_{k-1,1} r_{k-1,2}^{m_2} \cdots r_{k-1,k-2}^{m_{k-2}} - 1)^{m_{k-1}} \\ [u_k, u_1 u_2^{m_2} \cdots u_{k-1}^{m_{k-1}}]^- \\ = (r_{2,1} - 1)^{m_2} (r_{3,1} r_{3,2}^{m_2} - 1)^{m_3} \cdots (r_{k-1,1} r_{k-1,2}^{m_2} \cdots r_{k-1,k-2}^{m_{k-2}} - 1)^{m_{k-1}} \\ (r_{k,1} r_{k,2}^{m_2} \cdots r_{k,k-1}^{m_{k-1}} - 1) u_1 u_2^{m_2} \cdots u_{k-1}^{m_{k-1}} u_k.$$

If $m_k > 1$, then, by the induction hypothesis,

$$\begin{aligned} l_k^{m_k} l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_1]^- &= [u_k, l_k^{m_k-1} l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_1]^-]^- \\ &= (r_{2,1} - 1)^{m_2} (r_{3,1} r_{3,2}^{m_2} - 1)^{m_3} \cdots (r_{k-1,1} r_{k-1,2}^{m_2} \cdots r_{k-1,k-2}^{m_{k-2}} - 1)^{m_{k-1}} \\ &\quad (r_{k,1} r_{k,2}^{m_2} \cdots r_{k,k-1}^{m_{k-1}} - 1)^{m_k-1} [u_k, u_1 u_2^{m_2} \cdots u_{k-1}^{m_{k-1}} u_k^{m_k-1}]^- \\ &= (r_{2,1} - 1)^{m_2} (r_{3,1} r_{3,2}^{m_2} - 1)^{m_3} \cdots (r_{k-1,1} r_{k-1,2}^{m_2} \cdots r_{k-1,k-2}^{m_{k-2}} - 1)^{m_{k-1}} \\ &\quad (r_{k,1} r_{k,2}^{m_2} \cdots r_{k,k-1}^{m_{k-1}} - 1)^{m_k} u_1 u_2^{m_2} \cdots u_{k-1}^{m_{k-1}} u_k^{m_k}. \end{aligned}$$

Consequently, (36) holds. (ii) follows from (i). (iii) follows from (i).

(iv) If $\text{ord}(r_{u_2, u_1}) \nmid \alpha_1$, then $\lambda_2 = (r_{u_2, u_1}^{\alpha_1} - 1) \neq 0$ by Part (iii) for $k = 2$. Consequently, $u_1^{\alpha_1} u_2^{\alpha_2} \in \mathfrak{L}^-(V)$ by formula (37). Similarly, $u_1^{\alpha_1} u_2^{\alpha_2} \in \mathfrak{L}^-(V)$ when $\text{ord}(r_{u_1, u_2}) \nmid \alpha_2$.

(v) We only need show the necessity. Assume that it does not holds. By (iii), $l_k^{m_k} l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_1]^- = 0$ and $l_k^{m_k} l_{k-1}^{m_{k-1}} \cdots l_2^{m_2} [u_2]^- = 0$ for any $m_2, m_3, \dots, m_k \in \mathbb{N}$. Considering Lemma 3.2, $u_1^{\alpha_1} u_2^{\alpha_2} = 0$, which is a contradiction. ■

Theorem 6.7. *If $\mathfrak{L}^-(V)$ is Nichols Lie algebra over quantum linear space V with $\dim V = 2$, then*

$$\mathfrak{L}^-(V) = \text{span} \left\{ x_1, x_2, x_2^{\alpha_2} x_1^{\alpha_1} \mid \begin{array}{l} 1 \leq \alpha_2 < N_2, \ 1 \leq \alpha_1 < N_1, \\ \text{ord}(p_{12}) \nmid \alpha_2 \ \text{or} \ \text{ord}(p_{12}) \nmid \alpha_1 \end{array} \right\}, \quad (38)$$

where $N_i := \begin{cases} \text{ord}(p_{ii}), & \text{when } \text{ord}(p_{ii}) > 1 \\ \infty, & \text{when } \text{ord}(p_{ii}) = 1 \end{cases}$, for $1 \leq i \leq 2$.

Proof. Let $u_i \in \{x_1, x_2\}$ and $r_{u_i, u_j} := \begin{cases} 1, & \text{when } u_j = u_i \\ p_{u_i, u_j}, & \text{when } u_j \neq u_i \end{cases}$.

Then $r_{u_i, u_j} r_{u_j, u_i} = 1$ and $u_i u_j = r_{u_i, u_j} u_j u_i$ for $1 \leq i, j \leq k$. Considering Lemma 6.5 (v) we complete the proof. ■

Example 6.8. Assume $p_{12} = -1$ and $\text{ord}(p_{11}) = 3$, $\text{ord}(p_{22}) = 5$. Then $\mathfrak{L}^-(V) = F\{x_1, x_2, x_2 x_1, x_2 x_1^2, x_2^2 x_1, x_2^3 x_1, x_2^3 x_1^2, x_2^4 x_1\}$. In this case,

$$\mathfrak{B}(V) = \mathfrak{L}^-(V) \oplus \text{span}\{x_i^{m_i} \mid 1 < m_i \in \mathbb{N}\} \oplus F \oplus \text{span}\{x_2^2 x_1^2, x_2^4 x_1^2\}.$$

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Received June 5, 2017
and in final form October 9, 2017