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On Complemented Non-Abelian Chief Factors of a Lie Algebra

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Abstract. The number of Frattini chief factors or of chief factors which are complemented by a maximal subalgebra of a finite-dimensional Lie algebra L is the same in every chief series for L, by [4, Theorem 2.3]. However, this is not the case for the number of chief factors which are simply complemented in L. In this paper we determine the possible variation in that number. The same question for groups has been considered by Seral and Lafuente in [2].
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1. Preliminary Results

Throughout L will be a finite-dimensional Lie algebra with product $[,]$ over a field. We say that A is an L*-algebra* if it is a Lie algebra (with product denoted by juxtaposition) and there is a homomorphism $\theta : L \to \text{Der } A$. Then A is also an L-module with action $\cdot$ given by $x.a = \theta(x)(a)$ and we have $x.(a_1a_2) = (x.a_1)a_2 - (x.a_2)a_1$. If A is an ideal of L we will consider it as an L-algebra in the natural way. Given such an L-algebra A, we define the corresponding semi-direct sum $A \rtimes L$ as the set of ordered pairs, where the multiplication is given by

$$(a_1, x_1)(a_2, x_2) = (x_1.a_2 - x_2.a_1 + a_1a_2, [x_1, x_2])$$

for all $a_1,\ a_2 \in A$ and for all $x_1,\ x_2 \in L$.

Let A and B two L-algebras. An (algebra) isomorphism $\theta : A \to B$ is said to be an L*-isomorphism* if it is also an L-module isomorphism. Note that this is stronger than the definition used in [4], where θ is only required to be an L-module isomorphism. However, the results proved there apply equally to this stronger version. When such a θ exists we write $A \cong_L B$. We say that A, B are L*-equivalent,* written $A \sim_L B$ if there is an isomorphism $\Phi : A \rtimes L \to B \rtimes L$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \hookrightarrow & A & \hookrightarrow & A \rtimes L & \twoheadrightarrow & L & \twoheadrightarrow & 0 \\ & & \downarrow\phi & & \downarrow\Phi & & \| & & \\ 0 & \hookrightarrow & B & \hookrightarrow & B \rtimes L & \twoheadrightarrow & L & \twoheadrightarrow & 0 \end{array}$$

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In this case we say that the extensions $A \hookrightarrow A \rtimes L \twoheadrightarrow L$ and $B \hookrightarrow B \rtimes L \twoheadrightarrow L$ are *equivalent.* It is clear that L-equivalence is an equivalence relation.

If $\phi : A \to B$ is an L-isomorphism, then putting $\Phi((a, x)) = (\phi(a), x)$ defines an isomorphism $\Phi : A \rtimes L \to B \rtimes L$ making the above diagram commutative. It follows that L-isomorphic L-algebras are L-equivalent. However, the converse is false. For example, if $L = A \oplus B$, where A and B are isomorphic simple Lie algebras, then A and B are L-equivalent, but they are not L-isomorphic, as $C_L(A) = B$ and $C_L(B) = A$.

If B is an L-algebra we define a 1*-cocycle* of L with values in B to be a map $\beta \in Z^1(L, B)$ such that

$$\beta([x, y]) = x.\beta(y) - y.\beta(x) + \beta(x)\beta(y).$$

Then the map $\theta : L \to \text{Der } B$ given by $\theta(x) = \theta_x$ where $\theta_x(b) = \beta(x)b + x.b$ for all $x \in L$ and $b \in B$ is a homomorphism, and so we can define another L-module structure on B by

$$x \odot b = \beta(x)b + x.b.$$

We denote the L-algebra with this L-module structure by B_β.

The following proposition gives us a useful criterion for two L-algebras to be equivalent.

Proposition 1.1. *Let A and B be two L-algebras. They are L-equivalent if and only if there is a 1-cocycle $\beta \in Z^1(L, B)$ and an L-isomorphism ϕ from A to B_β (that is, $\phi(x.a) = x \odot \phi(a)$ for all $x \in L,\ a \in A$).*

Proof. Suppose first that there is a 1-cocyle $\beta \in Z^1(L, B)$ and an L-isomorphism $\phi : A \to B_\beta$. Then, the map $\Phi : A \rtimes L \to B \rtimes L$ given by

$$\Phi((a, x)) = (\phi(a) + \beta(x), x)$$

shows that A and B are L-equivalent.

Conversely, suppose that they are L-equivalent under the isomorphism $\Phi : A \rtimes L \to B \rtimes L$. Define $\beta : L \to B$ by $\beta(x) = \pi_1(\Phi((0, x)) - (0, x))$ where $\pi_1 : B \rtimes L \to B : (b, x) \mapsto b$ is the projection map onto B. Then it is straightforward to check that $\beta \in Z^1 L, B)$ and that ϕ is an L-isomorphism from A to B_β. ■

If A and B are abelian and L-equivalent, they have the same dimension, and so are L-isomorphic. However, as we have seen, for nonabelian L-algebras, L-equivalence is strictly weaker than L-isomorphism.

Recall that the factor algebra A/B is called a *chief factor* of L if B is an ideal of L and A/B is a minimal ideal of L/B. The *Frattini ideal* of L, denoted by $\phi(L)$, is the largest ideal of L contained in the intersection of all of the maximal subalgebras of L. A chief factor A/B is called *Frattini* if $A/B \subseteq \phi(L/B)$. This concept was introduced in [3].

If there is a subalgebra, M such that $L = A + M$ and $B \subseteq A \cap M$, we say that A/B is a *supplemented* chief factor of L, and that M is a *supplement* of A/B in L. Also, if A/B is a non-Frattini chief factor of L, then A/B is *supplemented* by a maximal subalgebra M of L.

If A/B is a chief factor of L supplemented by a subalgebra M of L, and $A \cap M = B$ then we say that A/B is *complemented* chief factor of L, and M is a *complement* of A/B in L. When L is *solvable*, it is easy to see that a chief factor is Frattini if and only if it is not complemented.

The *centralizer* of an L-algebra A in L is $C_L(A) = \{x \in L \mid x.a = 0$ for all $a \in A\}$. We will denote an algebra direct sum by '$\oplus$', whereas '$\dot{+}$' will denote a direct sum of the underlying vector space only. Then the following proposition gives a criterion for a nonabelian chief factor to be complemented.

Proposition 1.2. *Let A_1/B_1 be a nonabelian chief factor of L. Then, A_1/B_1 is complemented in L if and only if there exists an L-algebra B such that $A_1/B_1 \sim_L B$, and $A_1 \subseteq C_L(B)$.*

Proof. $(\Longrightarrow)$ Suppose that A_1/B_1 is complemented in L and M is a complement of A_1/B_1 in L. Since $L = M + A_1$, for each $x \in L$ we can write $x = m_x + a_x$ for some $m_x \in M$ and $a_x \in A_1$. We consider the L-algebra B whose underlying algebra is A_1/B_1 with the module operation:

$$\begin{array}{rcl} \wedge \; : \; L \times B & \to & B \\ (x, b) & \to & [m_x, a_b] + B_1 \end{array}$$

where $b = a_b + B_1$ $(a_b \in A_1)$. Define the 1-cocycle $\beta \in Z^1(L, B)$ as;

$$\begin{array}{rcl} \beta \; : \; L & \to & B \\ x & \to & \beta(x) = a_x + B_1 \;\; (a_x \in A_1) \end{array}$$

It is immediate that both are well defined mappings and that β is a 1-cocycle. Let $\phi : A_1/B_1 \to B$ be given by, $\phi(a_1 + B_1) = a_1 + B_1$ for all $a_1 + B_1 \in A_1/B_1$. Then we can define another module structure on B using β and, for all $x \in L$ and for all $a_b + B_1 \in A_1/B_1$, we have

$$\begin{array}{rcl} \phi([x, a_b + B_1]) & = & \phi([x, a_b] + B_1) \\ & = & [x, a_b] + B_1 \\ & = & [a_x + B_1, a_b + B_1] + [m_x, a_b + B_1] \\ & = & [\beta(x), a_b + B_1] + x \wedge (a_b + B_1) \\ & = & x \odot (a_b + B_1). \end{array}$$

Hence ϕ is an L-isomorphism and $A_1/B_1 \cong_L B_\beta$. Then, using Proposition 1.1, we have that $A_1/B_1 \sim_L B$. Also

$$\begin{array}{rcl} C_L(B) & = & \{x \in L \mid x \wedge B = 0_B\} \\ & = & \{m_x + a_x \in L \mid [m_x, a_b] + B_1 = B_1 \text{ for all } b \in B\} \\ & = & C_M(A_1/B_1) \oplus A_1 \end{array}$$

whence $A_1 \subseteq C_L(B)$.

$(\Longleftarrow)$ Assume now that B is an L-algebra, $A_1/B_1 \sim_L B$ and $A_1 \subseteq C_L(B)$. We need to show that A_1/B_1 is complemented in L. Since $A_1/B_1 \sim_L B$ we have an L-isomorphism $\phi : B \to (A_1/B_1)_\alpha$ where $\alpha \in Z^1(L, A_1/B_1)$, by Proposition 1.1, and $A_1 \subseteq C_L(B)$. If $b \in B$, then

$$\phi(b) = \phi(b + a_1.b) = \phi(b) + a_1 \odot \phi(b) = \phi(b) + [\alpha(a_1), \phi(b)] + [a_1 + B_1, \phi(b)]$$

so $[\alpha(a_1) + a_1 + B_1, \phi(b)] = B_1$ for all $b \in B$; that is,

$$\alpha(a_1) + a_1 + B_1 \in C_L(A_1/B_1) \cap A_1/B_1 = B_1,$$

since A_1/B_1 is a nonabelian chief factor of L. Hence $\alpha(a_1) = -a_1 + B_1$.

Put $M = Ker(\alpha)$. Let $x \in L$ and $\alpha(x) = a_1 + B_1$. Then

$$\alpha(x + a_1) = \alpha(x) + \alpha(a_1) = (a_1 + B_1) + (-a_1 + B_1) = B_1$$

so $x+a_1 \in M$. Hence $L = M+A_1$. If $m \in M\cap A_1$ we have $B_1 = \alpha(m) = -m+B_1$, so $M \cap A_1 = B_1$ and M is a complement of A_1/B_1 in L. ■

Recall that,

(i) the *socle* of L, $Soc\,(L)$ is the sum of all of the minimal non-zero ideals of L; and

(ii) if U is a subalgebra of L, the *core* of U, U_L, is the largest ideal of L contained in U. We say that U is *core-free* in L if $U_L = 0$.

We shall call L *primitive* if it has a core-free maximal subalgebra. Then we have the following characterisation of primitive Lie algebras.

Theorem 1.3. *([4, Theorem 1.1])*

(i) A Lie algebra L is primitive if and only if there exists a subalgebra M of L such that $L = M + A$ for all minimal ideals A of L.

(ii) Let L be a primitive Lie algebra. Assume that U is a core-free maximal subalgebra of L and that A is a non-trivial ideal of L. Write $C = C_L(A)$. Then $C \cap U = 0$. Moreover, either $C = 0$ or C is a minimal ideal of L.

(iii) If L is a primitive Lie algebra and U is a core-free maximal subalgebra of L, then exactly one of the following statements holds:

(a) $Soc(L) = A$ is a self-centralising abelian minimal ideal of L which is complemented by U; that is, $L = U\dot{+}A$.

(b) $Soc(L) = A$ is a non-abelian minimal ideal of L which is supplemented by U; that is $L = U + A$. In this case $C_L(A) = 0$.

(c) $Soc(L) = A \oplus B$, where A and B are the two unique minimal ideals of L and both are complemented by U; that is, $L = A\dot{+}U = B\dot{+}U$. In this case $A = C_L(B)$, $B = C_L(A)$, and A, B and $(A + B) \cap U$ are nonabelian isomorphic algebras.

We say that L is

- *primitive of type* 1 if it has a unique minimal ideal that is abelian;
- *primitive of type* 2 if it has a unique minimal ideal that is non-abelian; and
- *primitive of type* 3 if it has precisely two distinct minimal ideals each of which is non-abelian.

Let A/B and D/E be chief factors of L. We say that they are L*-connected*, if either they are L-isomorphic or there exists an epimorphic image of L which is primitive of type 3 and whose minimal ideals are L-isomorphic to the given factors. The property of being L-connected is an equivalence relation on the set of chief factors. The set of chief factors of L is denoted as:

$$C\mathcal{F}(L) = \{A/B \mid A, B \text{ are ideals of } L, A/B \text{ is a chief factor of } L\}.$$

Let

$$I_L(A) = \{x \in L \mid \operatorname{ad} x \mid_A = \operatorname{ad} a \text{ for some } a \in A\},$$

where A is an L-algebra (and $\operatorname{ad} x \mid_A$ refers to the module action of x on A.)

Lemma 1.4. (i) *Let A, B be ideals of a Lie algebra L with $B \subseteq A$. Then $I_L(A/B) = A + C_L(A/B)$.*

(ii) *Let A be an L-algebra with $C_L(A) \subseteq I_L(A)$. Then $I_L(A)/C_L(A)$ is isomorphic to a subalgebra of $A/Z(A)$.*

(iii) *A is an abelian L-algebra if and only if $I_L(A) = C_L(A)$.*

Proof. (i) We have

$$\begin{aligned} x \in I_L(A/B) &\Leftrightarrow \exists\, a' \in A \text{ such that } [x,a] + B = [a',a] + B \quad \forall a \in A \\ &\Leftrightarrow \exists\, a' \in A \text{ such that } [x - a', a] + B = B \quad \forall a \in A \\ &\Leftrightarrow \exists\, a' \in A \text{ such that } [x - a', a] \in B \quad \forall a \in A \\ &\Leftrightarrow \exists\, a' \in A \text{ such that } x - a' \in C_L(A/B) \\ &\Leftrightarrow x \in A + C_L(A/B) \end{aligned}$$

(ii) For $x \in I_L(A)$ let $a_x \in A$ be such that $x.a = a_x a$ for all $a \in A$. Define $\theta : I_L(A) \to A/Z(A)$ by $\theta(x) = a_x + Z(A)$. Then it is straightforward to check that θ is well-defined and is a homomorphism. Moreover, $\operatorname{Ker}(\theta) = C_L(A)$, whence the result.

(iii) This is straightforward. ∎

Let A, B be two L-algebras. If A and B are L-equivalent, then it is clear from Proposition 1.1 that $I_L(A) = I_L(B)$.

Proposition 1.5. *Let L be a Lie algebra and let F_1, $F_2 \in C\mathcal{F}(L)$. Then the following assertions are equivalent:*

(i) $F_1 \sim_L F_2$;

(ii) *F_1 and F_2 are L-connected;*

(iii) *either* $F_1 \cong_L F_2$ *or there exist* $E_i \in C\mathcal{F}(L)$ *such that* $F_i \cong_L E_i$ *for* $i = 1, 2$ *, and the* E_i *'s have a common complement in* L*, which is a maximal subalgebra of* L*; and*

(iv) *either* $F_1 \cong_L F_2$ *or there exist* $E_i \in C\mathcal{F}(L)$ *such that* $F_i \cong_L E_i$ *for* $i = 1, 2$*, and the* E_i *'s have a common complement in* L*.*

Proof. From [4] we know that two abelian chief factors are L-equivalent if and only if they are L-isomorphic, and if and only if they are L-connected. Moreover, a complement U of an abelian chief factor A/B is a maximal subalgebra and L/U_L is primitive of type 1 with $Soc(L/U_L) = C/U_L$ and $C/U_L \cong_L A/B$, by [4, Remarks following Proposition 2.5]. So we may assume that the chief factors are nonabelian and not L-isomorphic. Let $F_1 = A/B$ and $F_2 = D/E$, where A, B, C, D are ideals of L.

(i) $\Rightarrow$ **(ii)** Put $X = C_L(A/B)$ and $Y = C_L(D/E)$. Since F_1 and F_2 are non-abelian we have that $X \neq Y$, by [4, Theorem 2.1] Also, since $F_1 \sim_L F_2$, we have that $I_L(A/B) = I_L(D/E) := I$. Then $I = A + X = D + Y$, by Lemma 1.4. Also,

$$\frac{X+Y}{X} \subseteq \frac{A+X}{X} \cong_L \frac{A}{B} \text{ and } \frac{X+Y}{Y} \subseteq \frac{D+Y}{Y} \cong_L \frac{D}{E},$$

So $I = X + Y$, since $X \neq Y$ and

$$X/X \cap Y \cong_L I/Y \cong_L D/E \text{ and } Y/Y \cap X \cong_L I/X \cong_L A/B.$$

It thus suffices to show that $L/X \cap Y$ is primitive of type 3. Without loss of generality we can assume that $X \cap Y = 0$. Then $C_L(X) = Y$ and $C_L(Y) = X$. Moreover, since $\sim_L$ is an equivalence relation, we have $Y \sim_L X$. Thus there is a 1-cocyle $\alpha \in Z^1(L, Y)$ and an L-isomorphism, $\phi : Y \to X_\alpha$, by Proposition 1.1. We also have that $U = Ker(\alpha)$ complements X in L, as in the proof of Proposition 1.2. Now let $y \in Y$ and $u \in Y \cap U$. Then

$$[u, \phi(y)] = 0 \text{ since } \phi(y) \in X = C_L(Y) \text{ and } u \in Y, \text{ and}$$

$$[\alpha(u), \phi(y)] = 0 \text{ since } \alpha(u) = 0.$$

But also,

$$\phi([u, y]) = u \odot \phi(y) = [\alpha(u), \phi(y)] + [u, \phi(y)] = 0$$

whence, $[u, y] = 0$, since ϕ is injective. It follows that $u \in C_L(Y) \cap Y = X \cap Y = 0$ and so $Y \cap U = 0$. Thus U is a maximal subalgebra of L with trivial core and F_1 and F_2 are L-connected.

(ii) $\Rightarrow$ **(iii)** This follows immediately from the definition.

(iii) $\Rightarrow$ **(iv)** This is trivial.

(iv) $\Rightarrow$ (i) If, $F_1 \cong_L F_2$ then it is clear that $F_1 \sim_L F_2$. So suppose that there exist $E_i \in C\mathcal{F}(L)$ such that $F_i \cong_L E_i$ $(i = 1, 2)$, and E_i's have a common complement in L. Assume that the subalgebra U of L complements both A/B and D/E where $A/B \cong_L F_1$ and $D/E \cong_L F_2$. So U also complements $(U_L + A)/U_L$ and $(U_L + D)/U_L$. Let

$$\phi : \frac{U_L + A}{U_L} \to \frac{U_L + D}{U_L} \quad \text{and} \quad \beta : L \to \frac{U_L + A}{U_L}$$

be given by $\phi(a + U_L) = d + U_L$ if $a \in A$, $d \in D$ and $a + d \in U$, and $\beta(x) = a + U_L$ if $x \in L$, $a \in A$ and $x + a \in U$. Then it is straightforward to check that $\beta \in Z^1(L, (U_L + A)/U_L)$, and that ϕ is an L-isomorphism. Thus $(U_L{+}A)/U_L \sim_L (U_L{+}D)/U_L$ and $A/B{\sim_L}D/E$. This completes the proof. ■

Now we will give a definition for the A-crown of L, which is a generalization of a concept introduced by Towers in [4].

Let A be an irreducible L-algebra (that is, A is a Lie algebra and an irreducible L-module). Put $I = I_L(A)$. We set

$$D_L(A) = \cap\{R \mid R \subseteq I, R \text{ is an ideal of } L, A \sim_L I/R \text{ and } I/R \text{ is non-Frattini}\}$$

and

$$E_L(A) = \{x \in L \mid \alpha(x) = 0 \quad \text{for all } \alpha \in Z^1(L, A)\}.$$

Obviously, if $A \sim_L B$, then $D_L(A) = D_L(B)$ and $E_L(A) = E_L(B)$. The quotient, $I_L(A)/D_L(A)$ is then called the A*-crown* of L.

In [4] the *crown* of a supplemented chief factor A/B of L was defined to be C/R, where

$$C = A + C_L(A/B)$$

and

$$R = \cap\{M_L \mid M \in \mathcal{J}\},$$

where $\mathcal{J}$ is the set of all maximal subalgebras which supplement a chief factor L-connected to A/B. Clearly $C = I_L(A/B)$ and $R = D_L(A/B)$ where A/B is considered to be an L-algebra in the natural way.

Let A be an L-algebra. Then the set of 1-coboundaries, $Z^1(L, A)$ and the 1-dimensional cohomology space, $H^1(L, A)$, are defined in the usual way (see, for example [1]). We put $A^L := \{a \in A \mid L.a = 0\}$. Then $A^L = H^0(L, A)$. Let N be an ideal of L. Then A is, by restriction, an N-algebra, and $Z^1(N, A)$, $H^1(N, A)$ become L-modules. Moreover, we have the following inflation-restriction exact sequences.

Lemma 1.6. *Let*

$$N \rightarrowtail L \twoheadrightarrow L/N$$

be a short exact sequence of Lie algebras, where N is an ideal of L and the arrows are the canonical inclusion and projection. If A is an L-algebra, we have the following exact sequences:

$$0 \longrightarrow Z^1(L/N, A^N) \xrightarrow{inf} Z^1(L, A) \xrightarrow{res} Z^1(N, A)$$

$$0 \longrightarrow H^1(L/N, A^N) \xrightarrow{inf} H^1(L,A) \xrightarrow{res} H^1(N,A)^{L/N}$$

where inf and res denote the corresponding inflation and restriction maps.

Theorem 1.7. *Let A be an irreducible L-algebra and let N be an ideal of L with $N \subseteq C_L(A)$. Then the following are equivalent:*

$$(1)\ N \subseteq E_L(A) \quad (2)\ Z^1(L,A) = Z^1(L/N,A) \quad (3)\ H^1(L,A) = H^1(L/N,A)$$

Proof. This follows from the above lemma. Note that the inflation is bijective if and only if the restriction is null and that is equivalent to $N \subseteq Ker(\alpha)$ for all $\alpha \in Z^1(L,A)$. ■

The analogue of the following result for groups was proved using cohomology theory. Here we give a more direct proof for the Lie algebra case.

Theorem 1.8. *If A is an abelian irreducible L-algebra, then $E_L(A) = D_L(A)$.*

Proof. Put $I = I_L(A) = C_L(A)$, by Lemma 1.4. Let $\alpha \in Z^1(L,A)$. First note that $\alpha|_I$ is an L-homomorphism from I into A, since

$$\alpha([x,y]) = x.\alpha(y) - y.\alpha(x) + \alpha(x)\alpha(y) = 0 \text{ for all } x, y \in I, \text{and}$$

$$\alpha([x,i]) = x.\alpha(i) - i.\alpha(x) + \alpha(x)\alpha(i) = x.\alpha(i) \text{ for all } x \in L, i \in I.$$

It follows that $\alpha(I)$ is an L-submodule of A, and so $\alpha(I) = 0$ or A, by the irreducibility of A. The former implies that $D_L(A) \subseteq I \subseteq \ker(\alpha)$. So suppose that $\alpha(I) = A$. Then $I/I \cap \ker(\alpha) \cong_L A$. Moreover,

$$\begin{aligned}\dim(I + \ker(\alpha)) &= \dim I + \dim\ker(\alpha) - \dim I \cap \ker(\alpha)\\ &= \dim A + \dim\ker(\alpha)\\ &= \dim \operatorname{im}(\alpha) + \dim\ker(\alpha)\\ &= \dim L.\end{aligned}$$

It follows that $L = I + \ker(\alpha)$, and $I/I \cap \ker(\alpha)$ is complemented by $\ker(\alpha)$ (which is a subalgebra of L) and so is non-Frattini. Hence $D_L(A) \subseteq I \cap \ker(\alpha)$.

Thus, in either case, $D_L \subseteq \ker(\alpha)$, and $D_L(A) \subseteq E_L(A)$.

Finally suppose that there exists $x \in E_L(A)$ such that $x \notin D_L(A)$. Then there exists $R \in D_L(A)$ such that $x \notin R$ but $x \in \ker(\alpha)$ for all $\alpha \in Z^1(L,A)$. Since I/R is non-Frattini, there is a maximal subalgebra M of L such that $L = I + M$ and $I \cap M = R$. Now there is a cocycle $\beta \in Z^1(L,B)$ and an L-isomorphism ϕ from I/R onto A_β, by Proposition 1.1. Moreover $A_\beta = A$, since A is abelian. So define $\alpha : L \to A$ by $\alpha(m) = 0, \alpha(i) = \phi(i + R)$. Then it is straightforward to check that $\alpha \in Z^1(L,A)$ and that $M = \ker(\alpha)$. Furthermore, $x \in I \cap M = R$, contradiction. Hence $E_L(A) \subseteq D_L(A)$ and equality results. ■

In the rest of this section we investigate the case where A is nonabelian.

Recall that, if A is an L-algebra, then $\alpha : L \to A$ is a 1-cocyle if and only if $\alpha^* : L \to A \rtimes L$ given by;

$$\alpha^*(x) = (\alpha(x), x)$$

is a homomorphism and $\alpha \longmapsto \alpha^*(L)$ defines a bijection between $Z^1(L,A)$ and the set of all complements of A in $A \rtimes L$. Then

$$\ker(\alpha) = \alpha^*(L) \cap L$$

We can give the following characterization:

Theorem 1.9. *Let A be a nonabelian irreducible L-algebra. Then;*

$$E_L(A)_{A \rtimes L} = \cap\{C_L(B) \mid B \sim_L A\}.$$

Proof. By Proposition 1.1 we have that

$$\cap\{C_L(B) \mid B \sim_L A\} = \cap\{C_L(A_\alpha) \mid \alpha \in Z^1(L,A)\}.$$

Consider the semi-direct sum $A \rtimes L$. From the remark above this theorem, we have immediately that

$$E_L(A) = \cap\{H \mid H \text{ is a complement of } A \text{ in } A \rtimes L\}.$$

In particular $E_L(A)_{A \rtimes L}$ is an ideal of $A \rtimes L$ and $E_L(A) \cap A = 0$. As $E_L(A) \subseteq L$, we have that $E_L(A)_{A \rtimes L} \subseteq C_L(A)$. On the other hand, if $\alpha \in Z^1(L,A)$ and $x \in \ker(\alpha)$, then $x \in C_L(A)$ if and only if $x \in C_L(A_\alpha)$. So we have that

$$E_L(A)_{A \rtimes L} \subseteq \cap\{C_L(A_\alpha) \mid \alpha \in Z^1(L,A)\}.$$

Suppose now that $x \in \cap\{C_L(A_\alpha) \mid \alpha \in Z^1(L,A)\}$. Then, for all $\alpha \in Z^1(L,A)$ and for all $a \in A$ we have

$$0 = x \odot a = \alpha(x)a + x.a.$$

Putting $\alpha = 0$ we obtain that $x.a = 0$ for all $a \in A$. Thus, $\alpha(x)a = 0$ for all $a \in A$, and so $\alpha(x) \in Z(A) = 0$ as A is irreducible and nonabelian. Hence, $x \in E_L(A)$. The reverse inclusion follows. ■

Lemma 1.10. *Let A be an irreducible L-algebra such that $C_L(A) \subset I_L(A)$. Then,*

$$D_L(A) \subseteq C_L(A) \Longleftrightarrow I_L(A)/C_L(A) \cong_L A$$

Proof. Put $I := I_L(A)$, etc. If $I/C \cong_L A$, then it is not Frattini, since it is nonabelian. It follows from the definition of $D_L(A)$ that $D_L(A) \subseteq C_L(A)$. Suppose now that $D_L(A) \subseteq C_L(A)$. Then A is nonabelian, so $I \neq C_L(A)$. Moreover, I/D is completely reducible (as in [4, Theorem 3.2]), so $I_L(A)/C_L(A) \cong_L A$. ■

Corollary 1.11. *Let A be a nonabelian irreducible L-algebra such that*

$$\{B \in C\mathcal{F}(L) \mid B \sim_L A\} \neq \emptyset.$$

Then

$$D_L(A) = \cap\{C_L(B) \mid B \sim_L A,\ B \in C\mathcal{F}(L)\}$$

Let A be a nonabelian irreducible L-algebra. We set

$$J_L(A) = \cap\{C_L(B) \mid B \sim_L A,\ B \ncong_L F,\ F \in C\mathcal{F}(L)\}$$

if $\{B \mid B \sim_L A,\ B \ncong_L F,\ F \in C\mathcal{F}(L)\} \neq \varnothing$ and we put $J_L(A) = I_L(A)$, otherwise.

Proposition 1.12. *Let A be a nonabelian irreducible L-algebra. Then*

$$I_L(A) = J_L(A) + D_L(A)$$

and

$$J_L(A) \cap D_L(A) = E_L(A)_{A \rtimes L}$$

Proof. It is clear that $J_L(A) \cap D_L(A) = E_L(A)_{A \rtimes L}$. Let $B \sim_L A$ be such that $B \ncong_L F$ if $F \in C\mathcal{F}(L)$, and put $S := C_L(B)$, $I := I_L(A)$, etc. Then there is a 1-cocyle $\alpha \in Z^1(L, B)$ and an L-isomorphism $\phi : A \to B_\alpha$, by Proposition 1.1. Let $x \in S$ and $a \in A$. Then

$$\phi(x.a) = x \odot \phi(a) = \alpha(x)\phi(a) + x.\phi(a) = \alpha(x)\phi(a),$$

so $x.a = \phi^{-1}\alpha(x)a$. It follows that $x \in I$ and $S \subseteq I$. Now, $Z(B) = 0$, since B is nonabelian and an irreducible L-algebra, so there is an monomorphism $\theta : I/S = I_L(B)/S \to B$, by Lemma 1.4. Moreover, θ cannot be surjective, since $I/S \in C\mathcal{F}(L)$. Hence I/S is isomorphic to a proper subalgebra of B.

Now $D + S$ is an ideal of I and I/D is completely reducible L-algebra, with its irreducible components L-equivalent to A (as in [4, Theorem 3.2]), and thus to B. If A_i/D is an irreducible component of I/D, then $A_i \subseteq S$, as in Proposition 1.2. It follows that $D + S = I$.

Suppose that $D+J \subset I$. Let I/R be a chief factor of L such that $D+J \subseteq R$. Then, $I/R \cong_L A$ because $D \subseteq R$. As $J \subseteq R$, there exists $B \sim_L A$ with $B \ncong_L F$ if $F \in C\mathcal{F}(L)$, such that $I/C_L(B)$ has a factor isomorphic to I/R, contradicting the fact that $\dim(I/C_L(B) < \dim A$. ■

2. On Complemented Chief Factors

Let L be a Lie algebra. We say that a chief factor of L is a *c-factor* if it is complemented in L by a subalgebra, and that it is an *m-factor* if it is complemented by a maximal subalgebra of L; otherwise we say that it is a *c'-factor*, respectively an *m'-factor*

Observe that, an abelian chief factor is an m-factor (respectively an m'-factor) if and only if it is a c-factor (respectively, a *Frattini* chief factor).

Let A/B and C/D be chief factors of L. We write $A/B \searrow C/D$ if $A = B + C$ and $B \cap C = D$. If $A/B \searrow C/D$, A/B is a Frattini chief factor and C/D is supplemented by a maximal subalgebra of L, then we call this situation an *m-crossing*, and denote it by $[A/B \searrow C/D]$.

We say that two chief factors A/B and C/D of L are *m-related* if one of the following holds.

1. There is a supplemented chief factor R/S such that $A/B \swarrow R/S \searrow C/D$.
2. There is an m-crossing $[U/V \searrow W/X]$ such that $A/B \swarrow V/X$ and $W/X \searrow C/D$.
3. There is a Frattini chief factor Y/Z such that $A/B \searrow Y/Z \swarrow C/D$.
4. There is an m-crossing $[U/V \searrow W/X]$ such that $A/B \searrow U/V$ and $U/W \swarrow C/D$.

Then we have the following result.

Proposition 2.1. *Let L be a Lie algebra over any field, let H and K be ideals of L with $H \subseteq K$, and let*

$$H = X_0 < X_1 < X_2 < \ldots < X_n = K$$

and

$$H = Y_0 < Y_1 < Y_2 < \ldots < Y_m = K$$

be two sections of chief series of L between H and K. Then $n = m$ and there exists a unique permutation π in S_n such that X_i/X_{i-1} and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are m-related. In particular,

(i) $X_i/X_{i-1} \sim_L Y_{\pi(i)}/Y_{\pi(i)-1}$

(ii) X_i/X_{i-1} and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are simultaneously m-factors or m'-factors.

(iii) If X_i/X_{i-1} and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are m-factors, then they have a maximal subalgebra of L as a common complement.

Proof. This follows from [5, Theorems 2.9 and 2.7]. ■

In particular, the number of *m-factors* in any chief series of L are the same. But this is no longer true for *c-factors,* in spite of the equivalence between (3) and (4) in Proposition 1.5, as we shall see in a later example.

If S is a subalgebra of L, the *normalizer of S in L* is defined as

$$N_L(S) = \{x \in L \,|\, [x, S] \subseteq S\}.$$

Lemma 2.2. *Assume that B^*/B is a c'-factor and that A^*/A is a c-factor of L, both of which are nonabelian and such that $B^*/B \searrow A^*/A$. Let $I = I_L(A^*/A)$ and $C = C_L(A^*/A)$. Then*

(i) $I/C \searrow B^*/B$ and I/C is a c'-factor;

(ii) there exists an ideal X of L with $X \subseteq I$ such that $X/N \searrow A^*/A$, where $N = X \cap C$, $I/C \searrow X/N$ and X/N is a c-factor;

(iii) there exists a supplement F of I/C in L such that L/N is isomorphic to the natural semi-direct sum of I/C by F/C; and

(iv) L/C is a primitive Lie algebra of type 2 and $Soc(L/C) = I/C$.

Proof. We have that $B^*/B \searrow A^*/A$, so $A^* + B = B^*$ and $A^* \cap B = A$. Also, $[B, A^*] + A = A$ or A^*. But the latter implies that $A^* \subseteq A + B \cap A^* = A$, a contradiction, so $[B, A^*] \subseteq A$; that is, $B \subseteq C$. Hence $B^* + C = A^* + B + C = A^* + C = I$, by Lemma 1.4, and $B^* \cap C = (A^* + B) \cap C = B + A^* \cap C = B + A = B + A^* \cap B = B$. Thus $I/C \searrow B^*/B$. Suppose that I/C is a c-factor of L. Then there is a subalgebra U of L such that $L = I + U$ and $I \cap U = C$. But now $L = B^* + C + U = B^* + U$ and $B^* \cap U = B^* \cap I \cap U = B^* \cap C = B$, so B^*/B is a c-factor, a contradiction. Thus, I/C is a c'-factor of L and we have (i).

Let

$$A = A_0 < A_1 < \ldots < A_n = C$$

be part of a chief series of L between A and C. Then

$$A^* = A^* + A_0 < A^* + A_1 < \ldots < A^* + A_n = I$$

is part of a chief series of L between A^* and I. Suppose that $(A^* + A_i)/A_i$ is a c-factor for some $1 \leq i \leq n-1$. Then there is a subalgebra U such that $L = A^* + A_i + U$ and $(A^* + A_i) \cap U = A_i$. Then $A_i \subseteq U$ so $L = A^* + U = A^* + A_{i-1} + U$ and $(A^* + A_{i-1}) \cap U = A^* \cap U + A_{i-1} = A_{i-1}$, since $A^* \cap U \subseteq A^* \cap A_i = A$. Thus $(A^* + A_{i-1})/A_{i-1}$ is a c-factor. It follows that $(A^* + A_k)/A_k$ is a c-factor and $(A^* + A_{k+1})A_{k+1}$ is a c'-factor for some $0 \leq k \leq n-1$, since A^*/A is a c-factor and I/C is a c'-factor. Put $N = A_k$, $X = A^* + A_k$, $Y = A_{k+1}$ and $M = A^* + A_{k+1}$. Then it is straightforward to check that

$$I/C \searrow M/Y \searrow X/N \searrow A^*/A$$

where M/Y is a c'-factor and X/N is a c-factor and we have (ii).

Without loss of generality we may assume that $N = 0$. Let U be a complement of X in L, so $L = X + U$ and $X \cap U = 0$, and consider, $K = U \cap C$. Then $[X, C] \subseteq X \cap C = N = 0$, since $I/C \searrow X/N$, so K is an ideal of L. We have $U + (K + X) = L$ and $U \cap (K + X) = K + (U \cap X) = K$. It follows that $(K + X)/K$ is a c-factor of L. Also $K + X + C = X + C = A^* + A_k + C = A^* + C = I$ and $(K + X) \cap C = K + X \cap C = K$, so $I/C \searrow (K + X)/K$ and we may assume that $K = 0$.

Observe that the map

$$u + x \longrightarrow (x + C, u + C)$$

where $u \in U$ and $x \in X$ defines an epimorphism between $L = X + U$ and the natural semidirect sum $I/C \rtimes ((U + C)/C)$. Furthermore, it is easy to check that the kernel of this map is N, so putting $F = U + C$ proves (iii).

We have

$$\frac{I}{C} = \frac{A^* + C}{C} \cong \frac{A^*}{A^* \cap C} = \frac{A^*}{A},$$

and $[U_L, I] \subseteq C$, whence $[U_L, A^*] \subseteq A$ and $U_L \subseteq C$. This establishes (iv). ■

We can construct an example of the situation in Lemma 2.2 as follows.

Example 2.3. Let L_0 be a primitive Lie algebra of type 2 with $Soc(L_0) = X_0$, where X_0 is not complemented in L_0, and let U_0 be a supplement to X_0 in L_0. So, for example, we could take $L_0 = sl(2) \otimes \mathcal{O}_m + 1 \otimes \mathcal{D}$, $X_0 = sl(2) \otimes \mathcal{O}_m$, $U_0 = (Fe_0 + Fe_1) \otimes \mathcal{O}_m + 1 \otimes \mathcal{D}$ where $\mathcal{O}_m$ is the truncated polynomial algebra in m indeterminates, $\mathcal{D}$ is a solvable subalgebra of $\mathrm{Der}(\mathcal{O}_m)$, $\mathcal{O}_m$ has no $\mathcal{D}$-invariant ideals, and the ground field is algebraically closed of characteristic $p > 5$ (see [6, Theorem 6.4]).

Put $Y_0 = U_0 \cap X_0$. Then X_0 is a U_0-algebra and so we can form the semi-direct sum

$$L = X_0 \rtimes U_0 = \{(x, u) \mid x \in X_0, u \in U_0\}.$$

Put $X = \{(x, 0) \mid x \in X_0\}$, $U = \{(0, u) \mid u \in U_0\}$. Then $L = X + U$, $X \cap U = 0$ and X is an ideal of L.

Now let $B = \{(0, y) \mid y \in Y_0\}$, $W = \{(y, 0) \mid y \in Y_0\}$. Putting $C = \{(y, -y) \mid y \in Y_0\}$ and $I = X + C$, we have that

(1) C is an ideal of L, $X \cap C = 0$, $I = X + B$ and $B = U \cap I$;

(2) $X \cap (B + C) = W$, $B + W = B + C$, W is an ideal of $C + U$ and $[W, C] = 0$;

(3) $W \cong_U B \cong_U C$.

Consider the following chief series of L.

$$0 < C < I < \ldots < L \text{ and } 0 < X < I < \ldots < L.$$

We have the situation of Lemma 2.2 with $N = 0$. Then I/C is a c'-factor and $X/0$ is a c-factor as in the lemma. Suppose that C is complemented in L, so there is a maximal subalgebra M of L such that $L = C \dot{+} M$. Then $[C, X] = 0$ so $X \cap M$ is an ideal of L. But X is a minimal ideal of L, so $X \cap M = 0$ or $X \subseteq M$. The former implies that $\dim(X + M) = \dim X + \dim M > \dim Y_0 + \dim M = \dim C + \dim M = \dim L$, which is impossible. The latter implies that $L = I + M$ and $I \cap M = (C + X) \cap M = C \cap M + X = X$, so M is a complement for I/X. If the chief factors between I and L are the same in each series then the second series has one more complemented chief factor than the first.

Then we have the following proposition.

Proposition 2.4. *Suppose that, in the situation of Proposition 2.1, X_i/X_{i-1} and $Y_{\pi(i)}/Y_{\pi(i)-1}$ are m'-factors. Then we have:*

(a) both factors are c'-factors; or

(b) both factors are nonabelian c-factors; or

(c) both factors are nonabelian, one of them is a c-factor, the other one is a c'-factor, and there exist ideals I, C, X, and N of L of satisfying (i)-(iv) of Lemma 2.2.

Proof. Assume that X_i/X_{i-1} is a c'-factor and that $Y_{\pi(i)}/Y_{\pi(i)-1}$ is a c-factor. As these two chief factors are m-related, one of the following situations arises.

1. There is a supplemented chief factor R/S such that
$$X_i/X_{i-1} \swarrow R/S \searrow Y_{\pi(i)}/Y_{\pi(i)-1}.$$
Since X_i/X_{i-1} is a c'-factor, so is R/S. We are thus in the situation of Lemma 2.2 with $B^* = R$, $B = S$, $A^* = Y_{\pi(i)}$, $A^* = Y_{\pi(i)-1}$.

2. There is an m-crossing $[U/V \searrow W/X]$ such that
$$X_i/X_{i-1} \swarrow V/X \text{ and } W/X \searrow Y_{\pi(i)}/Y_{\pi(i)-1}.$$
Then [5, Theorem 2.4] implies that $[U/W \searrow V/X]$ is also an m-crossing.
Suppose that W/X is supplemented by the maximal subalgebra M of L, so $L = W + M$ and $X \subseteq W \cap M$. Then $L = W + M = U + M$. If $V \subseteq M$ then $V \subseteq U \cap M$ and U/V is supplemented by M, contradicting the fact that U/V is Frattini. Hence $V \not\subseteq M$. It follows that $L = V + M$. Moreover, $X \subseteq V \cap M \subseteq V$. As V/X is a chief factor of L we have $V \cap M = X$, and so V/X is a c-factor. But then X_i/X_{i-1} is a c-factor, which is a contradiction. Thus this case cannot occur.

3. There is a Frattini chief factor Y/Z such that
$$X_i/X_{i-1} \searrow Y/Z \swarrow Y_{\pi(i)}/Y_{\pi(i)-1}.$$
Since $Y_{\pi(i)}/Y_{\pi(i)-1}$ is a c-factor, so is Y/Z. But Y/Z is Frattini, so this is impossible and this case cannot occur.

4. There is an m-crossing $[U/V \searrow W/X]$ such that
$$X_i/X_{i-1} \searrow U/V \text{ and } U/W \swarrow Y_{\pi(i)}/Y_{\pi(i)-1}.$$
Then [5, Theorem 2.4] implies that $[U/W \searrow V/X]$ is also an m-crossing, so U/W is a c'-factor, contradicting the fact that $Y_{\pi(i)}/Y_{\pi(i)-1}$ is a c-factor. Hence this case cannot occur either. ∎

Let A be an irreducible L-algebra. We say that A is of *cc'-type* in L if there exist two chief series of L in which case (c) of Proposition 2.4 holds with $A \sim_L X_i/X_{i-1}$ (Clearly this forces A to be nonabelian.)

Proposition 2.5. *Let v be the number of equivalence classes of irreducible L-algebras of cc'-type. Then the number of complemented chief factors on two chief series of L differs by at most v.*

Proof. A consequence of Proposition 2.4 is that, on a chief series of L, for each non-abelian crown there is at most one m'-factor. If the crown corresponds to a factor of cc'-type, this shows that on each chief series there is at most one c'-factor corresponding to the crown. ∎

Theorem 2.6. *Let A be a nonabelian irreducible L-algebra. Then A is of cc'-type in L if and only if*

$$E_L(A) \subset D_L(A) \subset I_L(A)$$

and $Soc(P)$ is a c'-factor of P, where P is the corresponding primitive epimorphic image of L.

Proof. Put $E = E_L(A)$, $D = D_L(A)$ and $I = I_L(A)$ and suppose that $E \subset D \subset I$. Then, since $D_L(A) \neq \emptyset$, there is an ideal R of L such that $I/R \sim_L A$. Also, $J_L(A) \neq I$, since otherwise $D = E$, by Proposition 1.12, so there is an ideal B of L with $B \sim_L A$ and $B \ncong_L F$ if $F \in \mathcal{CF}(L)$. Put $H = C_L(B)$. Then $H \subseteq I_L(B) = I$, by Proposition 1.1, and $H \neq I$, by Lemma 1.4 (iii), so $H \subset I$. Put $K = H \cap R$. Then $H/K \cong_L (H+R)/R$. Moreover, if $H + R = R$ then $H = R$, $H \subset I = I_L(B)$ and $D_L(B) = D \subseteq R = H$, so $I_L(B)/C_L(B) \cong_L B$, by Lemma 1.10, contradicting the fact that $B \ncong_L F$ if $F \in \mathcal{CF}(L)$. It follows that $H + R = I$ and $H/K \cong_L I/R$, whence $A \sim_L H/K$. By Proposition 1.2, H/K is a c-factor of L and I/R is a c'-factor and so we have that A is of cc'-type.
Conversely, if A is of cc'-type, from the definition we obtain ideals I, C, X, and N of L and a subalgebra U of L such that $I/C \sim_L A$, I/C and X/N are m'-factors, I/C is a c'-factor, $I/C \searrow X/N$, and U complements X/N in L (using the same notation as Proposition 2.4).Note that $I = C + X$ and $C \cap X = N$, so $I/C \cong_L X/N$, whence $C_L(X/N) = C_L(I/C) = C$, by Lemma 2.2 (iii). Now $[L, U \cap C] = [U + X, U \cap C] \subseteq U \cap C$ since $[X, C] \subseteq N \subseteq U \cap C$, so $U \cap C$ is an ideal of L. Also

$$\frac{X + U \cap C}{U \cap C} \searrow \frac{X}{N}.$$

As in the proof of Proposition 1.2, we obtain an L-algebra B, $B \sim_L A$ such that

$$C_L(B) = X + C_U(X/N) = X + U \cap C.$$

Suppose now that there exists $F \in \mathcal{CF}(L)$, such that $F \cong_L B$. Then $I_L(F) = I_L(B) = I$ and $C_L(F) = C_L(B) = X + U \cap C$, and so $F \cong_L I/(X + U \cap C)$. It follows that $I/(X + U \cap C) \sim_L I/C$, which is a primitive Lie algebra of type 2, by Lemma 2.2 (iv). But

$$\frac{L}{X + U \cap C} \cong_L \frac{L}{U \cap C} \Big/ \frac{X + U \cap C}{U \cap C},$$

so $L/U \cap C$ is primitive of type 3. It follows that $(X + U \cap C)/U \cap C$ is an m-factor, and hence so is X/N, by [5, Lemma 2.1], a contradiction.

Moreover we have that

$$D_L(A) \subseteq C \subset I, \; J_L(A) \subseteq X \subset I,$$

which completes the proof. ∎

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