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Hawkins Compatibility Conditions on the Tangent Bundle of a Poisson-Lie Group

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Abstract. Let G be a Poisson-Lie group equipped with a left invariant Riemannian metric compatible with the Poisson structure on G. There are many ways to lift the Poisson structure and the metric to the tangent bundle TG of G. In this paper, we study in different cases the compatibility between the lifted Poisson structure and the lifted metric on TG.
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1. Introduction

The compatibility between Riemannian and Poisson structures introduced geometrically on a differential manifold by Hawkins in [8, 9], and defined algebraically by A.Bahayou and M.Boucetta [1], on a Poisson-Lie group, equipped with a left invariant Riemannian metric, plays an important role in non commutative deformation theory of the graded algebra of differential forms. In fact, Hawkins showed that, if a deformation of the graded algebra $\Omega^*(M)$ of differential forms on a Riemannian manifold $(M, \langle , \rangle)$ comes from a deformation of spectral triplet describing the Riemannian structure, then the Poisson tensor Π on M (associated with the deformation) and the Riemannian metric $\langle , \rangle$ would verify the following compatibility conditions:

(H_1) The metric contravariant connection $\mathcal{D}$ associated to $(\Pi, \langle , \rangle)$ is flat.

(H_2) The metacurvature $\mathcal{M}$ of $\mathcal{D}$ is zero, i.e., the connection $\mathcal{D}$ is metaflat.

(H_3) The Poisson tensor Π is compatible with the Riemannian volume μ, i.e.,

$$d(i_\Pi \mu) = 0.$$

The metric contravariant connection associated naturally to $(\Pi, \langle , \rangle)$ is an analogue of the Levi-Civita connection. It has appeared first in [2]. We also call this

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connection Levi-Civita contravariant connection. The metacurvature, introduced by Hawkins in [9], is a $(2,3)$-tensor field (symmetric in the contravariant indices and antisymmetric in the covariant indices) associated naturally to any torsion-free and flat contravariant connection.
A triple $(M, \Pi, \langle,\rangle)$ satisfying conditions H_1 and H_2 (resp. H_1, H_2 and H_3) is said to be compatible (resp. strongly compatible) in the sense of Hawkins.
A.Bahayou and M.Boucetta [1], showed that, in the case of a Poisson-Lie group, endowed with a left invariant Riemannian metric, satisfying Hawkins compatibility conditions the last geometric problem, is equivalent to an algebraic problem defined as follows:
Let $(G, \Pi_G, \langle,\rangle_G)$ be a Poisson-Lie group endowed with a left invariant Riemannian metric $\langle,\rangle_G$ and let $(\mathfrak{g}, \mathfrak{g}^*)$ its Lie bialgebra. Let $\langle,\rangle_{\mathfrak{g}}$ be the scalar product on $\mathfrak{g}$ associated with the metric $\langle,\rangle_G$. We denote by $\langle,\rangle_{\mathfrak{g}^*}$, the associated scalar product on $\mathfrak{g}^*$ defined by:

$$\langle \alpha, \beta\rangle_{\mathfrak{g}^*} = \langle \sharp(\alpha), \sharp(\beta)\rangle_{\mathfrak{g}},$$

where $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is the isomorphism defined by $\langle,\rangle_{\mathfrak{g}}$. Then, $(G, \Pi_G, \langle,\rangle_G)$ is strongly compatible in the sense of Hawkins if, and only if, the following conditions are satisfied:

(B_1) The Lie algebra $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ is an orthogonal direct sum

$$\mathfrak{g}^* = S_G \overset{\perp}{\oplus} [\mathfrak{g}^*, \mathfrak{g}^*],$$

where $S_G = \{\alpha \in \mathfrak{g}^*, \mathrm{ad}_\alpha + \mathrm{ad}_\alpha^t = 0\}$ is an abelian Lie subalgebra (ad_α^t denotes the adjoint of ad_α with respect to $\langle,\rangle_{\mathfrak{g}^*}$) and $[\mathfrak{g}^*, \mathfrak{g}^*]$ the derived ideal is also abelian. In this case A.Bahayou and M.Boucetta called this algebra $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ a Milnor Lie algebra.

(B_2) For any $\alpha, \beta, \gamma \in S_G$;

$$\mathcal{M}^G(\alpha, \beta, \gamma) = -\mathcal{D}^G_\alpha \mathcal{D}^G_\beta d\gamma = \mathrm{ad}_\alpha \mathrm{ad}_\beta \rho(\gamma) = 0 :$$

$\mathcal{D}^G$ is the Levi-Civita contravariant connection associated to $(\Pi_G, \langle,\rangle_G)$, $\mathcal{M}^G$ is the metacurvature of $\mathcal{D}^G$ and $\rho = -d : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ is the 1-cocycle, dual of the Lie bracket $[,]_{\mathfrak{g}}$ on $\mathfrak{g}$, extended as a differential to $\wedge^{\dim \mathfrak{g} - 2}\mathfrak{g}^*$.

(B_3) If (G, Π_G) is connected and unimodular Poisson-Lie group, then the Lie algebra $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*})$ is unimodular (i.e., for all $\alpha \in \mathfrak{g}^*, \mathrm{tr}(\mathrm{ad}_\alpha) = 0$), and for all $X \in \mathfrak{g}$,

$$\rho(i_{\xi(X)}\mu_e) = 0,$$

where ξ is the 1-cocycle associated with Π_G and μ_e is a left invariant volume form on G.

The condition H_1 is equivalent to the condition B_1, H_2 is equivalent to B_2 and H_3 is equivalent to B_3.
The algebraic name of a Milnor Lie algebra comes from the fact that, in [10], Milnor showed that a left invariant Riemannian metric on a Lie group G is flat if, and only if, the Lie algebra $\mathfrak{g}$ of G split as an orthogonal direct sum $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$

of an abelian Lie subalgebra $\mathfrak{b}$ with an abelian ideal $\mathfrak{u}$ and for any $b \in \mathfrak{b}$, ad_b is skew-symmetric.

Now, let $\langle,\rangle_{\mathfrak{g}^*}$ be a scalar product on $\mathfrak{g}^*$ and $\langle,\rangle^*_G$ be the left invariant contravariant Riemannian metric on G induced by $\langle,\rangle_{\mathfrak{g}^*}$. We say that $(G, \Pi_G, \langle,\rangle^*_G)$ is a Riemannian Poisson-Lie group if, and only if, the Poisson tenseur Π_G and the metric $\langle,\rangle^*_G$ are compatible in the sense given by M.Boucetta in [3], as follows:

$$[Ad^*_g(A_\alpha\gamma + \mathrm{ad}^*_{\Pi^G_l(g)(\alpha)}\gamma), Ad^*_g(\beta)]_{\mathfrak{g}^*} + [Ad^*_g(\alpha), Ad^*_g(A_\beta\gamma + \mathrm{ad}^*_{\Pi^G_l(g)(\beta)}\gamma)]_{\mathfrak{g}^*} = 0,$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$ and for any $g \in G$, where $[,]_{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$, A is the infinitesimal Levi-Civita connection associated with $([,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ and $\Pi^G_l(g)$ is defined by $\Pi^G_l(g) = (L_{g^{-1}})_*\Pi_G(g)$, where $(L_g)_*$ denotes the tangent map of the left translation of G by g.

The notion of Riemannian Lie algebra is introduced by M.Boucetta [3], as Lie algebra $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ equipped with a scalar product $\langle,\rangle_{\mathfrak{g}}$ satisfying the following condition:

$$[B_X Y, Z]_{\mathfrak{g}} + [X, B_Z Y]_{\mathfrak{g}} = 0,$$

for all $X, Y, Z \in \mathfrak{g}$, where B is the infinitesimal Levi-Civita connection associated with $([,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$.

Let (G, Π_G) be a Poisson-Lie group with multiplicative Poisson tensor Π_G. By M.Boumaiza and N.Zaalani [5], the tangent bundle TG of G with its tangent Poisson structure Π_{TG} defined in the sense of Sanchez de Alvarez [11] is a Poisson-Lie group,with Lie-bialgebra $(\mathfrak{g} \rtimes \mathfrak{g}, \mathfrak{g}^* \ltimes \mathfrak{g}^*)$, where $\mathfrak{g} \rtimes \mathfrak{g}$ is the semi-direct product Lie algebra with bracket:

$$[(X,Y),(X',Y')]_{\mathfrak{g}\rtimes\mathfrak{g}} = ([X,X']_{\mathfrak{g}}, [X,Y']_{\mathfrak{g}} + [Y,X']_{\mathfrak{g}}), \quad (X,Y),\ (X',Y') \in \mathfrak{g}\times\mathfrak{g}, \quad (1)$$

and $\mathfrak{g}^* \ltimes \mathfrak{g}^*$ is the semi-direct product Lie algebra with bracket:

$$[(\alpha,\beta),(\alpha',\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*} = ([\alpha,\beta']_{\mathfrak{g}^*} + [\beta,\alpha']_{\mathfrak{g}^*}, [\beta,\beta']_{\mathfrak{g}^*}), \quad (\alpha,\beta),\ (\alpha',\beta') \in \mathfrak{g}^*\times\mathfrak{g}^*. \quad (2)$$

We call (TG, Π_{TG}) the Sanchez de Alvarez tangent Poisson-Lie group of G.

The tangent Lie group TG is isomorphic to $G \times \mathfrak{g}$ ($\mathfrak{g}$ viewed as an abelian Lie group with respect to the addition).

On the tangent bundle TG, we can define the following three metrics:

- The natural left invariant Riemannian metric $\langle,\rangle_{TG}$ associated to $\langle,\rangle_G$ defined on $\mathfrak{g} \times \mathfrak{g}$ by:

$$\langle(X,Y),(X',Y')\rangle_{TG}(e,0) = \langle X, X'\rangle_G(e) + \langle Y, Y'\rangle_G(e), \quad (X,Y),(X',Y') \in \mathfrak{g}\times\mathfrak{g}. \quad (3)$$

Since TG is isomorphic to $G \times \mathfrak{g}$, this metric coincides with the product metric $\langle,\rangle_{G\times\mathfrak{g}}$ on $G \times \mathfrak{g}$.

- The natural lifted Riemannian metric $\langle,\rangle^S_{TG}$, introduced by Sasaki [12].
- The natural lifted Riemannian metric $\langle,\rangle^C_{TG}$ introduced by J. Cheeger and D. Gromoll [6].

We show that these three metrics $\langle,\rangle_{TG}$, $\langle,\rangle^S_{TG}$ and $\langle,\rangle^C_{TG}$ on TG, coincides at the neutral element $(e,0)$ of TG.

We show that, if $(G, \Pi_G, \langle,\rangle_G)$ is compatible in the sense of Hawkins, then

$(TG, \Pi_{TG}, \langle,\rangle_{TG})$ is compatible in the sense of Hawkins if, and only if, (G, Π_G) is a trivial Poisson-Lie group. We also show that, if $(G, \Pi_G, \langle,\rangle_G)$ is a Riemannian Poisson-Lie group, then $(TG, \Pi_{TG}, \langle,\rangle_{TG})$ is a Riemannian Poisson-Lie group if, and only if, (G, Π_G) is a trivial Poison-Lie group.
If (G, Π_G) is a Poisson-Lie group, there exists a linear Poisson structure $\Pi_{\mathfrak{g}}$ on $\mathfrak{g}$, induced by Π_G and making $\mathfrak{g}$ an abelian Poisson-Lie group. Let $TG \equiv G \times \mathfrak{g}$ identified with the direct product Poisson-Lie group of (G, Π_G) and $(\mathfrak{g}, \Pi_{\mathfrak{g}})$, with Lie-bialgebra $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}^* \times \mathfrak{g}^*)$, where $\mathfrak{g} \times \mathfrak{g}$ is the direct product Lie algebra with bracket:

$$[(X,Y),(X',Y')]_{\mathfrak{g}\times\mathfrak{g}} = ([X,X']_{\mathfrak{g}}, 0), \quad (X,Y),\ (X',Y') \in \mathfrak{g} \times \mathfrak{g}, \tag{4}$$

and $\mathfrak{g}^* \times \mathfrak{g}^*$ is the direct product Lie algebra with bracket:

$$[(\alpha,\beta),(\alpha',\beta')]_{\mathfrak{g}^*\times\mathfrak{g}^*} = ([\alpha,\alpha']_{\mathfrak{g}^*}, [\beta,\beta']_{\mathfrak{g}^*}), \ \ (\alpha,\beta),\ (\alpha',\beta') \in \mathfrak{g}^* \times \mathfrak{g}^*. \tag{5}$$

We show that, $(G, \Pi_G, \langle,\rangle_G)$ is strongly compatible in the sense of Hawkins if, and only if, $(TG \equiv G\times\mathfrak{g}, \Pi_{G\times\mathfrak{g}}, \langle,\rangle_{G\times\mathfrak{g}})$ is strongly compatible in the sense of Hawkins. We also show that, $(G, \Pi_G, \langle,\rangle_G)$ is a Riemannian Poison-Lie group if, and only if, $(TG \equiv G \times \mathfrak{g}, \Pi_{G\times\mathfrak{g}}, \langle,\rangle_{G\times\mathfrak{g}})$ is a Riemannian Poisson-Lie group.
Note that, in [9], Hawkins showed that a deformation of the differential graded algebra of differential forms $\Omega^*(M)$, defines a generalized Poisson bracket $\{,\}$ on this space. Hawkins also showed that a generalized Poisson bracket making $\Omega^*(M)$ a differential graded Poisson algebra exists if, and only if, $(M, \Pi, \langle,\rangle)$ is compatible in the sense of Hawkins. This lifting of Hawkins compatibility conditions to TG defines a generalized Poisson bracket on $\Omega^*(TG)$, the graded algebra of differential forms on TG.
Finally, note that, by geometric approach it is more difficult to construct examples of compatible Riemannian metrics and Poisson structures. Nevertheless, the algebraic approach allows us to construct some interesting examples of Poisson-Lie groups, strongly compatible in the sense of Hawkins.

2. Preliminaries

Contravariant connection. Contravariant connections on Poisson manifolds were defined by Vaismann [13] and studied in detail by Fernandes [7]. This notion appears extensively in the context of noncommutative deformations (see[8, 9]). Let (M, Π) be a Poisson manifold. We associate with the Poisson tensor Π the anchor map $\Pi_\sharp : T^*M \to TM$ defined by $\beta(\Pi_\sharp(\alpha)) = \Pi(\alpha, \beta)$ and the Koszul bracket $[,]_\Pi$ on differential 1-forms given by:

$$[\alpha,\beta]_\Pi \ = \ L_{\Pi_\sharp(\alpha)}\beta - L_{\Pi_\sharp(\beta)}\alpha - d(\Pi(\alpha,\beta)).$$

A contravariant connection on M, with respect to Π, is a $\mathbb{R}$-bilinear map

$$\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M), \quad (\alpha,\beta) \mapsto \mathcal{D}_\alpha\beta,$$

such that for all $f \in \mathcal{C}^\infty(M)$

$$\mathcal{D}_{f\alpha}\beta = f\mathcal{D}_\alpha\beta, \quad \mathcal{D}_\alpha(f\beta) = f\mathcal{D}_\alpha\beta + \Pi_\sharp(\alpha)(f)\beta.$$

The torsion and the curvature of a contravariant connection $\mathcal{D}$ are formally identical to the usual ones:

$$T(\alpha,\beta) = \mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha - [\alpha,\beta]_\Pi$$

$$R(\alpha,\beta)\gamma = \mathcal{D}_\alpha\mathcal{D}_\beta\gamma - \mathcal{D}_\beta\mathcal{D}_\alpha\gamma - \mathcal{D}_{[\alpha,\beta]_\Pi}\gamma. \tag{6}$$

These are respectively (2,1) and (3,1)-type tensor fields. When $T\equiv 0$ (resp. $R\equiv 0$), $\mathcal{D}$ is called torsion-free (resp. flat).
Let (M,Π) be a Poisson manifold. Let $\langle,\rangle$ be a covariant Riemannian metric on M and let $\langle,\rangle^*$ be the contravariant Riemannian metric associated to $\langle,\rangle$. The metric contravariant connection associated to $(\Pi,\langle,\rangle^*)$ is the unique contravariant connection $\mathcal{D}$ such that $\mathcal{D}$ is torsion-free and the metric $\langle,\rangle^*$ is parallel with respect to $\mathcal{D}$, i.e.,

$$\Pi_\sharp(\alpha)\langle\beta,\gamma\rangle^* = \langle\mathcal{D}_\alpha\beta,\gamma\rangle^* + \langle\beta,\mathcal{D}_\alpha\gamma\rangle^*.$$

The connection $\mathcal{D}$ is the contravariant analogue of the Levi-Civita connection and can be defined by the Koszul formula:

$$\begin{aligned}2\langle\mathcal{D}_\alpha\beta,\gamma\rangle^* \;=\;& \Pi_\sharp(\alpha)\langle\beta,\gamma\rangle^* + \Pi_\sharp(\beta)\langle\alpha,\gamma\rangle^* - \Pi_\sharp(\gamma)\langle\alpha,\beta\rangle^* \\ &+ \langle[\alpha,\beta]_\Pi,\gamma\rangle^* + \langle[\gamma,\alpha]_\Pi,\beta\rangle^* + \langle[\gamma,\beta]_\Pi,\alpha\rangle^*.\end{aligned} \tag{7}$$

We call $\mathcal{D}$ the Levi-Civita contravariant connection associated to $(\Pi,\langle,\rangle^*)$.

The metacurvature. In this subsection we recall briefly the definition of the metacurvature introduced by Hawkins in [9].
Let (M,Π) be a Poisson manifold and $\mathcal{D}$ a torsion-free and flat contravariant connection on M with respect to Π. In [9], Hawkins showed that such a connection defines a bracket $\{,\}$ on the space of differential forms $\Omega^*(M)$ such that:

1. The bracket $\{,\}$ is $\mathbb{R}$-bilinear, antisymmetric of degree 0, i.e.,

$$\{\sigma,\upsilon\} = -(-1)^{\deg(\sigma)\deg(\upsilon)}\{\upsilon,\sigma\}.$$

2. The exterior differential d is a derivation with respect to $\{,\}$, i.e.,

$$d\{\sigma,\upsilon\} = \{d\sigma,\upsilon\} + (-1)^{\deg(\sigma)}\{\sigma,d\upsilon\}.$$

3. $\{,\}$, satisfies the product rule,

$$\{\sigma,\upsilon\wedge\nu\} = \{\sigma,\upsilon\}\wedge\nu + (-1)^{\deg(\sigma)\deg(\upsilon)}\upsilon\wedge\{\sigma,\nu\}.$$

4. For any $f,g\in\mathcal{C}^\infty(M)$ and for any $\sigma\in\Omega^*(M)$, the bracket $\{f,g\}$ coincides with the initial Poisson bracket on M and

$$\{f,\sigma\} = \mathcal{D}_{df}\sigma.$$

Hawkins called this bracket $\{,\}$ a generalized Poisson bracket and showed that there exists a (2,3) tensor $\mathcal{M}$ symmetric in the contravariant indices and antisymmetric in the covariant indices such that the generalized Poisson bracket satisfies the graded Jacobi identity,

$$\{\sigma,\{\upsilon,\nu\}\}-\{\{\sigma,\upsilon\},\nu\}-(-1)^{\deg(\sigma)\deg(\upsilon)}\{\upsilon,\{\sigma,\nu\}\}=0,$$

if, and only if, $\mathcal{M}$ is identically zero.
$\mathcal{M}$ is called metacurvature of $\mathcal{D}$ and given by

$$\mathcal{M}(df,\alpha,\beta)=\{f,\{\alpha,\beta\}\}-\{\{f,\alpha\},\beta\}-\{\{f,\beta\},\alpha\}. \tag{8}$$

Hawkins showed in [9] that, for any parallel 1-form α with respect to $\mathcal{D}$ (i.e., $\mathcal{D}\alpha=0$) and every 1-form β, the value of the generalized Poisson bracket at α and β is given by,

$$\{\alpha,\beta\}=-\mathcal{D}_\beta d\alpha.$$

Thus, it is easily deduced from the equation (8) that for any parallel 1-forms α,γ and any 1-form β,

$$\mathcal{M}(\alpha,\beta,\gamma)=-\mathcal{D}_\beta\mathcal{D}_\gamma d\alpha. \tag{9}$$

The connection $\mathcal{D}$ is said metaflat if, $\mathcal{M}$ is identically zero.

Now, let (G,Π_G) be a Poisson-Lie group with Lie-bialgebra $(\mathfrak{g},\mathfrak{g}^*)$. A covariant Riemannian metric $\langle,\rangle_G$ on G (that acts on the tangent bundle TG), is said to be left invariant if, the action of G by left translations $L_g:G\to G, g\in G$ is an isometry action. This covariant metric $\langle,\rangle_G$ defines a left invariant contravariant metric $\langle,\rangle_G^*$ on G (acting on the cotangent bundle T^*G), given by:

$$\langle\alpha,\beta\rangle_G^*=\langle\sharp(\alpha),\sharp(\beta)\rangle_G.$$

Where $\sharp:T^*G\to TG$ is the isomorphism defined by the metric $\langle,\rangle_G$.
Note that, the Koszul bracket of two left invariant 1-forms is a left invariant 1-form (see [14]) and, if one identifies $\mathfrak{g}^*$ with the space of left invariant 1-forms, the Koszul bracket $[,]_\Pi$ coincides with the Lie bracket $[,]_{\mathfrak{g}^*}$ of $\mathfrak{g}^*$.
Denote by $\mathcal{D}^G$ the Levi-Civita contravariant connection associated to $(\Pi_G,\langle,\rangle_G^*)$. Since the Riemannian metric $\langle,\rangle_G^*$ is left invariant, for any $\alpha,\beta,\gamma\in\mathfrak{g}^*$, (7) becomes:

$$2\langle\mathcal{D}^G_\alpha\beta,\gamma\rangle_G^*(e)=\langle[\alpha,\beta]_{\mathfrak{g}^*},\gamma\rangle_G^*(e)+\langle[\gamma,\alpha]_{\mathfrak{g}^*},\beta\rangle_G^*(e)+\langle[\gamma,\beta]_{\mathfrak{g}^*},\alpha\rangle_G^*(e), \tag{10}$$

where $[,]_{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$.
Note that, the restriction of $\mathcal{D}^G$ to $\mathfrak{g}^*\times\mathfrak{g}^*$ defines a product on $\mathfrak{g}^*$ (we also call this product, infinitesimal Levi-Civita connection). The vanishing of the curvature of $\mathcal{D}^G$ is equivalent to the vanishing of the restriction of the curvature of $\mathcal{D}^G$ to $\mathfrak{g}^*$.
A.Bahayou and M.Boucetta [1], showed that $\mathcal{D}^G$ is flat if, and only if, the Lie algebra, $\mathfrak{g}^*=S_G\overset{\perp}{\oplus}[\mathfrak{g}^*,\mathfrak{g}^*]$ is a Milnor Lie algebra and the Levi-Civita contravariant connection is given, for any $\beta\in\mathfrak{g}^*$ by :

$$\mathcal{D}^G_\alpha\beta=\begin{cases}0 & if \quad \alpha\in[\mathfrak{g}^*,\mathfrak{g}^*]\\ \mathrm{ad}_\alpha\beta=[\alpha,\beta]_{\mathfrak{g}^*} & if \quad \alpha\in S_G,\end{cases} \tag{11}$$

and for any $\alpha \in S_G$, $\mathcal{D}^G\alpha = 0$.
If $\mathfrak{g}^* = S_G \overset{\perp}{\oplus} [\mathfrak{g}^*, \mathfrak{g}^*]$ is a Milnor Lie algebra the metacurvature $\mathcal{M}^G$ is given by:

$$\mathcal{M}^G(\alpha,\beta,\gamma) = \begin{cases} 0 & if \quad \alpha,\beta \; or \; \gamma \in [\mathfrak{g}^*,\mathfrak{g}^*] \\ -\mathcal{D}_\alpha\mathcal{D}_\beta d\gamma = \mathrm{ad}_\alpha \mathrm{ad}_\beta \rho(\gamma) & for\; all \quad \alpha,\beta,\gamma \in S_G, \end{cases} \tag{12}$$

where $\rho : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ is the dual of the Lie bracket $[,]_{\mathfrak{g}}$ on $\mathfrak{g}$ (see[1]).

The Sasaki metric and the Cheeger-Gromoll metric on TG. Let $(G, \langle,\rangle_G)$ be a Lie group equipped with a Riemannian metric and let ∇^G the Levi-Civita connection associated to $\langle,\rangle_G$.
The differential of the projection $\pi : TG \to G$ is a smooth map $d\pi : TTG \to TG$. For $(g,u) \in TG$, we denote by $\mathbf{V}_{(g,u)}$, the Kernel of $d\pi$ at (g,u),

$$\mathbf{V}_{(g,u)} = \ker(d\pi(g,u)),$$

and call it the vertical subspace of $T_{(g,u)}TG$.
Now we define a smooth linear map depending of the Levi-Civita connection ∇^G. Let V be an open neighborhood of g in G and W an open neighborhood of 0 in T_gG such that the exponential map $exp_g : T_gG \to G$ is a diffeomorphism of W onto V. Furthermore, let $\tau : \pi^{-1}(V) \to T_gG$ be the translation map which translates every $Y \in \pi^{-1}(V)$ in a parallel manner form $y = \pi(Y)$ to g along the unique geodesic arc in V connecting g and y. Then the connection map $K_{(g,u)} : T_{(g,u)}TG \to T_gG$ of the Levi-Civita connection ∇^G defined by

$$K_{(g,u)}(Z) = d(exp_g \circ R_{-u} \circ \tau)(Z),$$

where $Z \in T_{(g,u)}TG$ and R_{-u} is the translation map defined by $R_{-u}(X) = X - u$, for $X \in T_gG$. The horizontal subspace $\mathbf{H}_{(g,u)}$ of $T_{(g,u)}TG$ is defined by

$$\mathbf{H}_{(g,u)} = \ker(K_{(g,u)}).$$

The tangent space $T_{(g,u)}TG$ of the tangent bundle TG at the point (g,u) is a direct sum of its vertical and horizontal subspaces, i.e.,

$$T_{(g,u)}TG = \mathbf{H}_{(g,u)} \oplus \mathbf{V}_{(g,u)},$$

and each tangent vector $Z \in T_{(g,u)}TG$ can be decomposed as

$$Z = X^h + Y^v,$$

where X^h is the horizontal lift of $X \in T_gG$ and Y^v is the vertical lift of $Y \in T_gG$, such that X and Y are uniquely determined by $X = d\pi(Z)$ and $Y = K(Z)$.
The Sasaki metric $\langle,\rangle^S_{TG}$ on the tangent bundle TG is the natural Riemannian metric on TG (see [12]) given by

$$\begin{aligned} \langle X^h, Y^h\rangle^S_{TG}(g,u) &= \langle X, Y\rangle_G(g) \\ \langle X^v, Y^v\rangle^S_{TG}(g,u) &= \langle X, Y\rangle_G(g) \\ \langle X^h, Y^v\rangle^S_{TG}(g,u) &= 0, \end{aligned}$$

where $X, Y \in T_gG$, $X^h_{(g,u)} \in \mathbf{H}_{(g,u)}$ is the horizontal lift of X and $Y^v_{(g,u)} \in \mathbf{V}_{(g,u)}$ is the vertical lift of Y.
The Cheeger-Gromoll metric $\langle , \rangle^C_{TG}$ is the natural Riemannian metric on the tangent bundle TG (see [6]) such that:

$$\begin{array}{rcl}\langle X^h, Y^h\rangle^C_{TG}(g,u) & = & \langle X, Y\rangle_G(g)\\ \langle X^h, Y^v\rangle^C_{TG}(g,u) & = & 0\\ \langle X^v, Y^v\rangle^C_{TG}(g,u) & = & \frac{1}{1+r^2}\big(\langle X, Y\rangle_G(g) + \langle X, u\rangle_G(g)\langle Y, u\rangle_G(g)\big),\end{array}$$

where r denotes $\mid u \mid = \sqrt{\langle u, u\rangle_G}$.

3. Compatibility conditions in the sense of Hawkins.

Proposition 3.1. *Let $(G, \langle , \rangle_G)$ be a Lie group equipped with a left invariant Riemannian metric $\langle , \rangle_G$, the left invariant Sasaki metric $\langle , \rangle^S_{TG}$ and the left invariant Cheeger-Gromoll metric $\langle , \rangle^C_{TG}$ on the tangent bundle TG coincide with the natural Riemannian metric $\langle , \rangle_{TG}$ at the neutral element $(e, 0)$ of TG, given by:*

$$\begin{array}{rcl}\langle (X,0), (X',0)\rangle_{TG}(e,0) & = & \langle X, X'\rangle_G(e)\\ \langle (0,Y), (0,Y')\rangle_{TG}(e,0) & = & \langle Y, Y'\rangle_G(e)\\ \langle (X,0), (0,Y')\rangle_{TG}(e,0) & = & 0,\end{array}$$

for all $(X, Y), (X', Y') \in \mathfrak{g} \times \mathfrak{g}$.

Proof. Let $\pi : TG \to G : (g, u) \mapsto g$ be the natural projection, the differential mapping $d\pi(e, 0)$ at the point $(e, 0)$ is given by

$$d\pi(e,0) : \mathfrak{g} \rtimes \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto X,$$

then the vertical subspace $\mathbf{V}_{(e,0)}$ of $\mathfrak{g} \rtimes \mathfrak{g}$ is given by

$$\mathbf{V}_{(e,0)} = \ker(d\pi(e,0)) = \{0\} \rtimes \mathfrak{g}.$$

The connection map $K_{(e,0)} : \mathfrak{g} \rtimes \mathfrak{g} \to \mathfrak{g}$ at the point $(e, 0)$ is given, for all $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ by

$$K_{(e,0)}(X, Y) = d(exp_e \circ R_{-0} \circ \tau)(X, Y) = Y,$$

then the horizontal subspace $\mathbf{H}_{(e,0)}$ of $\mathfrak{g} \rtimes \mathfrak{g}$ is given by

$$\mathbf{H}_{(e,0)} = \ker(K_{(e,0)}) = \mathfrak{g} \rtimes \{0\}.$$ ■

The natural left invariant contravariant metric $\langle , \rangle^*_{TG}$ on TG associated to $\langle , \rangle_{TG}$ at the point $(e, 0)$ is given, for all $(\alpha, \beta), (\alpha', \beta') \in \mathfrak{g}^* \times \mathfrak{g}^*$ by,

$$\begin{array}{rcl}\langle (\alpha,0), (\alpha',0)\rangle^*_{TG}(e,0) & = & \langle \alpha, \alpha'\rangle^*_G(e)\\ \langle (0,\beta), (0,\beta')\rangle^*_{TG}(e,0) & = & \langle \beta, \beta'\rangle^*_G(e)\\ \langle (\alpha,0), (0,\beta')\rangle^*_{TG}(e,0) & = & 0.\end{array} \tag{13}$$

Compatibility between the natural metric and the Sanchez de Alvarez Poisson structure. This subsection is devoted to the compatibility between the natural metric and the Sanchez de Alvarez Poisson structure on TG.

Proposition 3.2. *Let $(G, \Pi_G, \langle,\rangle_G^*)$ be a Poisson-Lie group equipped with the left invariant contravariant Riemannian metric $\langle,\rangle_G^*$ and let $(TG, \Pi_{TG}, \langle,\rangle_{TG}^*)$ be the Sanchez de Alvarez tangent Poisson-Lie group of G equipped with the natural left invariant contravariant metric $\langle,\rangle_{TG}^*$ (13). Let $\mathcal{D}^G$ and $\mathcal{D}^{TG}$ be the Levi-Civita contravariant connections associated with the pairs $(\Pi_G, \langle,\rangle_G^*)$ and $(\Pi_{TG}, \langle,\rangle_{TG}^*)$ respectively. Then for all $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'') \in \mathfrak{g}^* \times \mathfrak{g}^*$, we have:*

1. $\langle \mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(\alpha'',0)\rangle_{TG}^* = 0.$
2. $\langle \mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(0,\beta'')\rangle_{TG}^* = \langle (0, \frac{1}{2}(\mathcal{D}^G_\alpha \alpha' + \mathcal{D}^G_{\alpha'}\alpha)),(0,\beta'')\rangle_{TG}^*.$
3. $\langle \mathcal{D}^{TG}_{(\alpha,0)}(0,\beta'),(\alpha'',0)\rangle_{TG}^* = \langle (\mathcal{D}^G_\alpha \beta' + \frac{1}{2}\mathrm{ad}^t_\alpha \beta', 0),(\alpha'',0)\rangle_{TG}^*.$
4. $\langle \mathcal{D}^{TG}_{(\alpha,0)}(0,\beta'),(0,\beta'')\rangle_{TG}^* = 0.$
5. $\langle \mathcal{D}^{TG}_{(0,\beta)}(\alpha',0),(\alpha'',0)\rangle_{TG}^* = \langle (\mathcal{D}^G_\beta \alpha' + \frac{1}{2}\mathrm{ad}^t_{\alpha'} \beta, 0),(\alpha'',0)\rangle_{TG}^*.$
6. $\langle \mathcal{D}^{TG}_{(0,\beta)}(\alpha',0),(0,\beta'')\rangle_{TG}^* = 0.$
7. $\langle \mathcal{D}^{TG}_{(0,\beta)}(0,\beta'),(\alpha'',0)\rangle_{TG}^* = 0.$
8. $\langle \mathcal{D}^{TG}_{(0,\beta)}(0,\beta'),(0,\beta'')\rangle_{TG}^* = \langle (0, \mathcal{D}^G_\beta \beta'),(0,\beta'')\rangle_{TG}^*.$

Proof. According to equations (2), (10) and (13) we obtain:

1.

$$\begin{aligned} 2\langle \mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(\alpha'',0)\rangle_{TG}^* &= \langle [(\alpha,0),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha'',0)\rangle_{TG}^* \\ &+ \langle [(\alpha'',0),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha,0)\rangle_{TG}^* \\ &+ \langle [(\alpha'',0),(\alpha,0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha',0)\rangle_{TG}^* \\ &= 0. \end{aligned}$$

2.

$$\begin{aligned} 2\langle \mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(0,\beta'')\rangle_{TG}^* &= \langle [(\alpha,0),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta'')\rangle_{TG}^* \\ &+ \langle [(0,\beta''),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha,0)\rangle_{TG}^* \\ &+ \langle [(0,\beta''),(\alpha,0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha',0)\rangle_{TG}^* \\ &= \langle ([\beta'',\alpha']_{\mathfrak{g}^*},0),(\alpha,0)\rangle_{TG}^* + \langle ([\beta'',\alpha]_{\mathfrak{g}^*},0),(\alpha',0)\rangle_{TG}^* \\ &= \langle [\beta'',\alpha']_{\mathfrak{g}^*},\alpha\rangle_G^* + \langle [\beta'',\alpha]_{\mathfrak{g}^*},\alpha'\rangle_G^*, \end{aligned}$$

since $\mathcal{D}^G$ is a torsion-free and the metric $\langle,\rangle_G^*$ is parallel with respect to $\mathcal{D}^G$, then

$$\begin{aligned} 2\langle \mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(0,\beta'')\rangle_{TG}^* &= \langle \mathcal{D}^G_\alpha \alpha' + \mathcal{D}^G_{\alpha'}\alpha, \beta''\rangle_G^* \\ &= \langle (0, \mathcal{D}^G_\alpha \alpha' + \mathcal{D}^G_{\alpha'}\alpha),(0,\beta'')\rangle_G^*. \end{aligned}$$

3.

$$\begin{aligned}
2\langle \mathcal{D}^{TG}_{(\alpha,0)}(0,\beta'),(\alpha'',0)\rangle^*_{TG} &= \langle [(\alpha,0),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha'',0)\rangle^*_{TG}\\
&\quad+\ \langle [(\alpha'',0),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha,0)\rangle^*_{TG}\\
&\quad+\ \langle [(\alpha'',0),(\alpha,0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta')\rangle^*_{TG}\\
&= \langle [\alpha,\beta']_{\mathfrak{g}^*},\alpha''\rangle^*_G + \langle [\alpha'',\beta']_{\mathfrak{g}^*},\alpha\rangle^*_G\\
&= \langle \mathcal{D}^G_\alpha\beta',\alpha''\rangle^*_G + \langle \mathcal{D}^G_{\alpha''}\beta',\alpha\rangle^*_G\\
&= \langle \mathcal{D}^G_\alpha\beta',\alpha''\rangle^*_G - \langle \beta',\mathcal{D}^G_{\alpha''}\alpha\rangle^*_G\\
&= \langle \mathcal{D}^G_\alpha\beta',\alpha''\rangle^*_G - \langle \beta',[\alpha'',\alpha]_{\mathfrak{g}^*}+\mathcal{D}^G_{\alpha}\alpha''\rangle^*_G\\
&= \langle \mathcal{D}^G_\alpha\beta',\alpha''\rangle^*_G + \langle \mathcal{D}^G_\alpha\beta',\alpha''\rangle^*_G + \langle \beta',\mathrm{ad}_\alpha\alpha''\rangle^*_G\\
&= 2\langle (\mathcal{D}^G_\alpha\beta' + \frac{1}{2}\mathrm{ad}^t_\alpha\beta',0),(\alpha'',0)\rangle^*_{TG}.
\end{aligned}$$

4.

$$\begin{aligned}
2\langle \mathcal{D}^{TG}_{(\alpha,0)}(0,\beta'),(0,\beta'')\rangle^*_{TG} &= \langle ([\alpha,\beta']_{\mathfrak{g}^*},0),(0,\beta'')\rangle^*_{TG} + \langle ([\beta'',\alpha]_{\mathfrak{g}^*},0),(0,\beta')\rangle^*_{TG}\\
&\quad+\ \langle (0,[\beta'',\beta']_{\mathfrak{g}^*}),(\alpha,0)\rangle^*_{TG}\\
&= 0.
\end{aligned}$$

5.

$$\begin{aligned}
2\langle \mathcal{D}^{TG}_{(0,\beta)}(\alpha',0),(\alpha'',0)\rangle^*_{TG} &= \langle [(0,\beta),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha'',0)\rangle^*_{TG}\\
&\quad+\ \langle [(\alpha'',0),(0,\beta)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha',0)\rangle^*_{TG}\\
&\quad+\ \langle [(\alpha'',0),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta)\rangle^*_{TG}\\
&= \langle [\beta,\alpha']_{\mathfrak{g}^*},\alpha''\rangle^*_G + \langle [\alpha'',\beta]_{\mathfrak{g}^*},\alpha'\rangle^*_G\\
&= \langle \mathcal{D}^G_\beta\alpha',\alpha''\rangle^*_G - \langle \mathcal{D}^G_\beta\alpha'',\alpha'\rangle^*_G + \langle \beta,\mathcal{D}^G_{\alpha'}\alpha''\rangle^*_G - \langle \beta,\mathcal{D}^G_{\alpha''}\alpha'\rangle^*_G\\
&= \langle \mathcal{D}^G_\beta\alpha',\alpha''\rangle^*_G + \langle \alpha'',\mathcal{D}^G_\beta\alpha'\rangle^*_G + \langle \beta,[\alpha',\alpha'']_{\mathfrak{g}^*}\rangle^*_G\\
&= 2\langle (\mathcal{D}^G_\beta\alpha' + \frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta,0),(\alpha'',0)\rangle^*_{TG}.
\end{aligned}$$

6.

$$\begin{aligned}
2\langle \mathcal{D}^{TG}_{(0,\beta)}(\alpha',0),(0,\beta'')\rangle^*_{TG} &= \langle [(0,\beta),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta'')\rangle^*_{TG}\\
&\quad+\ \langle [(0,\beta''),(0,\beta)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha',0)\rangle^*_{TG}\\
&\quad+\ \langle [(0,\beta''),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta)\rangle^*_{TG}\\
&= 0
\end{aligned}$$

7.

$$\begin{aligned}
2\langle \mathcal{D}^{TG}_{(0,\beta)}(0,\beta'),(\alpha'',0)\rangle^*_{TG} &= \langle [(0,\beta),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha'',0)\rangle^*_{TG}\\
&\quad+\ \langle [(\alpha'',0),(0,\beta)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta')\rangle^*_{TG}\\
&\quad+\ \langle [(\alpha'',0),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta)\rangle^*_{TG}\\
&= \langle (0,[\beta,\beta']_{\mathfrak{g}^*}),(\alpha'',0)\rangle^*_{TG} + \langle ([\alpha'',\beta]_{\mathfrak{g}^*},0),(0,\beta')\rangle^*_{TG}\\
&\quad+ \langle ([\alpha'',\beta']_{\mathfrak{g}^*},0),(0,\beta)\rangle^*_{TG}\\
&= 0.
\end{aligned}$$

8.

$$\begin{aligned}2\langle\mathcal{D}^{TG}_{(0,\beta)}(0,\beta'),(0,\beta'')\rangle^*_{TG} &= \langle[(0,\beta),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta'')\rangle^*_{TG}\\ &+\ \langle[(0,\beta''),(0,\beta)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta')\rangle^*_{TG}\\ &+\ \langle[(0,\beta''),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(0,\beta)\rangle^*_{TG}\\ &= \langle[\beta,\beta']_{\mathfrak{g}^*},\beta''\rangle^*_G+\langle[\beta'',\beta)]_{\mathfrak{g}^*},\beta'\rangle^*_G+\langle[\beta'',\beta']_{\mathfrak{g}^*},\beta\rangle^*_G\\ &= 2\langle\mathcal{D}^G_\beta\beta',\beta''\rangle^*_G\\ &= 2\langle(0,\mathcal{D}^G_\beta\beta'),(0,\beta'')\rangle^*_{TG}.\end{aligned}$$

■

Proposition 3.3. *Let $\mathcal{D}^G$ and $\mathcal{D}^{TG}$ be the Levi-Civita contravariant connections associated with the pairs $(\Pi_G,\langle,\rangle^*_G)$ and $(\Pi_{TG},\langle,\rangle^*_{TG})$ respectively. For any $(\alpha,\beta),(\alpha',\beta')\in\mathfrak{g}^*\times\mathfrak{g}^*$ we can get:*

1) $\mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0)=(0,\frac{1}{2}(\mathcal{D}^G_\alpha\alpha'+\mathcal{D}^G_{\alpha'}\alpha))$,

2) $\mathcal{D}^{TG}_{(\alpha,0)}(0,\beta')=(\mathcal{D}^G_\alpha\beta'+\frac{1}{2}\mathrm{ad}^t_\alpha\beta',0)$,

3) $\mathcal{D}^{TG}_{(0,\beta)}(\alpha',0)=(\mathcal{D}^G_\beta\alpha'+\frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta,0)$,

4) $\mathcal{D}^{TG}_{(0,\beta)}(0,\beta')=(0,\mathcal{D}^G_\beta\beta')$.

Proof. Using the proposition (3.2) we obtain:

1)

$$\begin{aligned}\langle\mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(\alpha'',\beta'')\rangle^*_{TG} &= \langle\mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(\alpha'',0)\rangle^*_{TG}+\langle\mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0),(0,\beta'')\rangle^*_{TG}\\ &= \langle(0,\frac{1}{2}(\mathcal{D}^G_\alpha\alpha'+\mathcal{D}^G_{\alpha'}\alpha)),(\alpha'',\beta'')\rangle^*_{TG},\end{aligned}$$

then, $\mathcal{D}^{TG}_{(\alpha,0)}(\alpha',0)=(0,\frac{1}{2}(\mathcal{D}^G_\alpha\alpha'+\mathcal{D}^G_{\alpha'}\alpha))$.
In the same way, we can obtain 2), 3) and 4). ■

Proposition 3.4. *Let R^G and R^{TG} be the curvatures of $\mathcal{D}^G$ and $\mathcal{D}^{TG}$ respectively. Then for all $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have:*

1.

$$\begin{aligned}&R^{TG}((\alpha,0),(\alpha',0))(\alpha'',0)\\ &=\frac{1}{2}\Big(R^G(\alpha,\alpha')\alpha''+\mathcal{D}^G_{[\alpha,\alpha']_{\mathfrak{g}^*}}\alpha''+\mathcal{D}^G_\alpha\mathcal{D}^G_{\alpha''}\alpha'-\mathcal{D}^G_{\alpha'}\mathcal{D}^G_{\alpha''}\alpha\\ &\quad+\ \mathrm{ad}^t_\alpha(\frac{1}{2}(\mathcal{D}^G_{\alpha'}\alpha''+\mathcal{D}^G_{\alpha''}\alpha'))-\mathrm{ad}^t_{\alpha'}(\frac{1}{2}(\mathcal{D}^G_\alpha\alpha''+\mathcal{D}^G_{\alpha''}\alpha)),0\Big).\end{aligned}$$

2.

$$\begin{aligned}&R^{TG}((\alpha,0),(\alpha',0))(0,\beta'')\\ &=\frac{1}{2}\Big(0,(R^G(\alpha,\alpha')\beta''+\mathcal{D}^G_{[\alpha,\alpha']_{\mathfrak{g}^*}}\beta'')+\frac{1}{2}(\mathcal{D}^G_\alpha\mathrm{ad}^t_{\alpha'}\beta''-\mathcal{D}^G_{\alpha'}\mathrm{ad}^t_\alpha\beta'')\\ &\quad+\ (\mathcal{D}^G_{(\mathcal{D}^G_{\alpha'}\beta''+\frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta'')}\alpha-\mathcal{D}^G_{(\mathcal{D}^G_\alpha\beta''+\frac{1}{2}\mathrm{ad}^t_\alpha\beta'')}\alpha')\Big).\end{aligned}$$

3.

$$
\begin{aligned}
&R^{TG}((0,\beta),(\alpha',0))(\alpha'',0)\\
&=\frac{1}{2}\Big(0, R^G(\beta,\alpha')\alpha''+R^G(\beta,\alpha'')\alpha'+\mathcal{D}^G_{[\beta,\alpha'']_{\mathfrak{g}^*}}\alpha'+\mathcal{D}^G_{\alpha''}\mathcal{D}^G_{\alpha'}\beta\\
&-\ \frac{1}{2}\mathcal{D}^G_{\alpha'}ad^t_{\alpha''}\beta-\mathcal{D}^G_{(\mathcal{D}^G_\beta\alpha''+\frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta)}\alpha'\Big).
\end{aligned}
$$

4.

$$
\begin{aligned}
&R^{TG}((0,\beta),(\alpha',0))(0,\beta'')\\
&=\Big(R^G(\beta,\alpha')\beta''+\frac{1}{2}(\mathcal{D}^G_\beta\mathrm{ad}^t_{\alpha'}\beta''+\mathrm{ad}^t_{\alpha'}\mathcal{D}^G_\beta\beta''\\
&+\ \mathrm{ad}^t_{[\beta,\alpha']_{\mathfrak{g}^*}}\beta''+\mathrm{ad}^t_{(\mathcal{D}^G_{\alpha'}\beta''+\frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta'')}\beta),0\Big).
\end{aligned}
$$

5.

$$
\begin{aligned}
&R^{TG}((0,\beta),(0,\beta'))(\alpha'',0)\\
&=\Big(R^G(\beta,\beta')\alpha''+\frac{1}{2}(\mathcal{D}^G_\beta\mathrm{ad}^t_{\alpha''}\beta'+\mathcal{D}^G_{\beta'}\mathrm{ad}^t_{\alpha''}\beta+\mathrm{ad}^t_{\alpha''}[\beta,\beta']_{\mathfrak{g}^*}\\
&+\ \mathrm{ad}^t_{(\mathcal{D}^G_{\beta'}\alpha''+\frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta')}\beta+\mathrm{ad}^t_{(\mathcal{D}^G_\beta\alpha''+\frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta)}\beta'),0\Big).
\end{aligned}
$$

6. $R^{TG}((0,\beta),(0,\beta'))(0,\beta'')=(0,R^G(\beta,\beta')\beta'')$.

Proof. Using the equation (6) and the proposition (3.3), we find:

1.

$$
\begin{aligned}
&R^{TG}((\alpha,0),(\alpha',0))(\alpha'',0)\\
&=\mathcal{D}^{TG}_{(\alpha,0)}\mathcal{D}^{TG}_{(\alpha',0)}(\alpha'',0)-\mathcal{D}^{TG}_{(\alpha',0)}\mathcal{D}^{TG}_{(\alpha,0)}(\alpha'',0)\\
&-\ \mathcal{D}^{TG}_{[(\alpha,0),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}}(\alpha'',0)\\
&=\mathcal{D}^{TG}_{(\alpha,0)}(0,\frac{1}{2}(\mathcal{D}^G_{\alpha'}\alpha''+\mathcal{D}^G_{\alpha''}\alpha'))-\mathcal{D}^{TG}_{(\alpha',0)}(0,\frac{1}{2}(\mathcal{D}^G_{\alpha}\alpha''+\mathcal{D}^G_{\alpha''}\alpha))\\
&=\Big(\frac{1}{2}\mathcal{D}^G_\alpha\mathcal{D}^G_{\alpha'}\alpha''+\frac{1}{2}\mathcal{D}^G_\alpha\mathcal{D}^G_{\alpha''}\alpha'+\frac{1}{2}\mathrm{ad}^t_\alpha(\frac{1}{2}(\mathcal{D}^G_{\alpha'}\alpha''+\mathcal{D}^G_{\alpha''}\alpha')),0\Big)\\
&-\ \Big(\frac{1}{2}\mathcal{D}^G_{\alpha'}\mathcal{D}^G_{\alpha}\alpha''+\frac{1}{2}\mathcal{D}^G_{\alpha'}\mathcal{D}^G_{\alpha''}\alpha+\frac{1}{2}\mathrm{ad}^t_{\alpha'}(\frac{1}{2}(\mathcal{D}^G_{\alpha}\alpha''+\mathcal{D}^G_{\alpha''}\alpha)),0\Big)\\
&=\frac{1}{2}\Big(R^G(\alpha,\alpha')\alpha''+\mathcal{D}^G_{[\alpha,\alpha']_{\mathfrak{g}^*}}\alpha''+\mathcal{D}^G_\alpha\mathcal{D}^G_{\alpha''}\alpha'-\mathcal{D}^G_{\alpha'}\mathcal{D}^G_{\alpha''}\alpha\\
&+\ \mathrm{ad}^t_\alpha(\frac{1}{2}(\mathcal{D}^G_{\alpha'}\alpha''+\mathcal{D}^G_{\alpha''}\alpha'))-\mathrm{ad}^t_{\alpha'}(\frac{1}{2}(\mathcal{D}^G_{\alpha}\alpha''+\mathcal{D}^G_{\alpha''}\alpha)),0\Big).
\end{aligned}
$$

2.

$$\begin{aligned}
&R^{TG}((\alpha,0),(\alpha',0))(0,\beta'')\\
&= \mathcal{D}^{TG}_{(\alpha,0)}\mathcal{D}^{TG}_{(\alpha',0)}(0,\beta'') - \mathcal{D}^{TG}_{(\alpha',0)}\mathcal{D}^{TG}_{(\alpha,0)}(0,\beta'')\\
&\quad - \mathcal{D}^{TG}_{[(\alpha,0),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}}(0,\beta'')\\
&= \mathcal{D}^{TG}_{(\alpha,0)}(\mathcal{D}^{G}_{\alpha'}\beta'' + \frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta'',0) - \mathcal{D}^{TG}_{(\alpha',0)}(\mathcal{D}^{G}_{\alpha}\beta'' + \frac{1}{2}\mathrm{ad}^t_{\alpha}\beta'',0)\\
&= \frac{1}{2}\Big(0,(R^G(\alpha,\alpha')\beta'' + \mathcal{D}^G_{[\alpha,\alpha']_{\mathfrak{g}^*}}\beta'') + \frac{1}{2}(\mathcal{D}^G_\alpha\mathrm{ad}^t_{\alpha'}\beta'' - \mathcal{D}^G_{\alpha'}\mathrm{ad}^t_{\alpha}\beta'')\\
&\quad + (\mathcal{D}^G_{(\mathcal{D}^G_{\alpha'}\beta''+\frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta'')}\alpha - \mathcal{D}^G_{(\mathcal{D}^G_{\alpha}\beta''+\frac{1}{2}\mathrm{ad}^t_{\alpha}\beta'')}\alpha')\Big)
\end{aligned}$$

3.

$$\begin{aligned}
&R^{TG}((0,\beta),(\alpha',0))(\alpha'',0)\\
&= \mathcal{D}^{TG}_{(0,\beta)}\mathcal{D}^{TG}_{(\alpha',0)}(\alpha'',0) - \mathcal{D}^{TG}_{(\alpha',0)}\mathcal{D}^{TG}_{(0,\beta)}(\alpha'',0)\\
&\quad - \mathcal{D}^{TG}_{[(0,\beta),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}}(\alpha'',0)\\
&= \mathcal{D}^{TG}_{(0,\beta)}(0,\frac{1}{2}(\mathcal{D}^G_{\alpha'}\alpha'' + \mathcal{D}^G_{\alpha''}\alpha')) - \mathcal{D}^{TG}_{(\alpha',0)}(\mathcal{D}^G_\beta\alpha'' + \frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta,0)\\
&\quad - \mathcal{D}^{TG}_{([\beta,\alpha']_{\mathfrak{g}^*},0)}(\alpha'',0)\\
&= \frac{1}{2}\Big(0,R^G(\beta,\alpha')\alpha'' + R^G(\beta,\alpha'')\alpha' + \mathcal{D}^G_{[\beta,\alpha'']_{\mathfrak{g}^*}}\alpha' + \mathcal{D}^G_{\alpha''}\mathcal{D}^G_{\alpha'}\beta\\
&\quad - \frac{1}{2}\mathcal{D}^G_{\alpha'}\mathrm{ad}^t_{\alpha''}\beta - \mathcal{D}^G_{(\mathcal{D}^G_\beta\alpha''+\frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta)}\alpha'\Big).
\end{aligned}$$

4.

$$\begin{aligned}
&R^{TG}((0,\beta),(\alpha',0))(0,\beta'')\\
&= \mathcal{D}^{TG}_{(0,\beta)}\mathcal{D}^{TG}_{(\alpha',0)}(0,\beta'') - \mathcal{D}^{TG}_{(\alpha',0)}\mathcal{D}^{TG}_{(0,\beta)}(0,\beta'')\\
&\quad - \mathcal{D}^{TG}_{[(0,\beta),(\alpha',0)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}}(0,\beta'')\\
&= \mathcal{D}^{TG}_{(0,\beta)}(\mathcal{D}^G_{\alpha'}\beta'' + \frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta'',0) - \mathcal{D}^{TG}_{(\alpha',0)}(0,\mathcal{D}^G_\beta\beta'')\\
&\quad - \mathcal{D}^{TG}_{([\beta,\alpha']_{\mathfrak{g}^*},0)}(0,\beta'')\\
&= \Big(R^G(\beta,\alpha')\beta'' + \frac{1}{2}(\mathcal{D}^G_\beta\mathrm{ad}^t_{\alpha'}\beta'' + \mathrm{ad}^t_{\alpha'}\mathcal{D}^G_\beta\beta''\\
&\quad + \mathrm{ad}^t_{[\beta,\alpha']_{\mathfrak{g}^*}}\beta'' + \mathrm{ad}^t_{(\mathcal{D}^G_{\alpha'}\beta''+\frac{1}{2}\mathrm{ad}^t_{\alpha'}\beta'')}\beta),0\Big).
\end{aligned}$$

5.

$$\begin{aligned}
&R^{TG}((0,\beta),(0,\beta'))(\alpha'',0)\\
&= \mathcal{D}^{TG}_{(0,\beta)}\mathcal{D}^{TG}_{(0,\beta')}(\alpha'',0) - \mathcal{D}^{TG}_{(0,\beta')}\mathcal{D}^{TG}_{(0,\beta)}(\alpha'',0)\\
&\quad - \mathcal{D}^{TG}_{[(0,\beta),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}}(\alpha'',0)\\
&= \mathcal{D}^{TG}_{(0,\beta)}(\mathcal{D}^{G}_{\beta'}\alpha'' + \frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta',0) - \mathcal{D}^{TG}_{(0,\beta')}(\mathcal{D}^{G}_{\beta}\alpha'' + \frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta,0)\\
&\quad - (\mathcal{D}^{G}_{[\beta,\beta']_{\mathfrak{g}^*}}\alpha'' + \frac{1}{2}\mathrm{ad}^t_{\alpha''}[\beta,\beta']_{\mathfrak{g}^*},0)\\
&= \Big(R^G(\beta,\beta')\alpha'' + \frac{1}{2}(\mathcal{D}^{G}_{\beta}\mathrm{ad}^t_{\alpha''}\beta' + \mathcal{D}^{G}_{\beta'}\mathrm{ad}^t_{\alpha''}\beta + \mathrm{ad}^t_{\alpha''}[\beta,\beta']_{\mathfrak{g}^*}\\
&\quad + \mathrm{ad}^t_{(\mathcal{D}^{G}_{\beta'}\alpha''+\frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta')}\beta + \mathrm{ad}^t_{(\mathcal{D}^{G}_{\beta}\alpha''+\frac{1}{2}\mathrm{ad}^t_{\alpha''}\beta)}\beta'),0\Big).
\end{aligned}$$

6.

$$\begin{aligned}
&R^{TG}((0,\beta),(0,\beta'))(0,\beta'')\\
&= \mathcal{D}^{TG}_{(0,\beta)}\mathcal{D}^{TG}_{(0,\beta')}(0,\beta'') - \mathcal{D}^{TG}_{(0,\beta')}\mathcal{D}^{TG}_{(0,\beta)}(0,\beta'')\\
&\quad - \mathcal{D}^{TG}_{[(0,\beta),(0,\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}}(0,\beta'')\\
&= (0,\mathcal{D}^{G}_{\beta}\mathcal{D}^{G}_{\beta'}\beta'') - (0,\mathcal{D}^{G}_{\beta'}\mathcal{D}^{G}_{\beta}\beta'') - (0,\mathcal{D}^{G}_{[\beta,\beta']_{\mathfrak{g}^*}}\beta'')\\
&= (0,R^G(\beta,\beta')\beta'').
\end{aligned}$$

■

Theorem 3.5. *Let $(G,\Pi_G,\langle,\rangle_G)$ be a Poisson-Lie group equipped with a left invariant Riemannian metric $\langle,\rangle_G$ and let $(TG,\Pi_{TG},\langle,\rangle_{TG})$ be the Sanchez de Alvarez tangent Poisson-Lie group of G equipped with the natural left invariant Riemannian metric $\langle,\rangle_{TG}$. If $(G,\Pi_G,\langle,\rangle_G)$ is compatible in the sense of Hawkins, then $(TG,\Pi_{TG},\langle,\rangle_{TG})$ is compatible in the sense of Hawkins if, and only if, (G,Π_G) is a trivial Poisson-Lie group.*

Proof. The Levi-Civita contravariant connection $\mathcal{D}^G$ is flat if, and only if, $\mathfrak{g}^* = S_G \overset{\perp}{\oplus} [\mathfrak{g}^*,\mathfrak{g}^*]$ is a Milnor Lie algebra. If $\mathcal{D}^G$ is flat then, for any $\beta\in\mathfrak{g}^*$,

$$\mathcal{D}^G_\alpha\beta = \begin{cases} 0 & if \quad \alpha\in[\mathfrak{g}^*,\mathfrak{g}^*]\\ \mathrm{ad}_\alpha\beta = [\alpha,\beta]_{\mathfrak{g}^*} & if \quad \alpha\in S_G\end{cases} \tag{14}$$

and for any $\alpha\in S_G$, $\mathcal{D}^G\alpha = 0$ [1].
According to proposition (3.3), if $\mathcal{D}^G$ is flat then, $\mathcal{D}^{TG}$ is flat if, and only if, $\Pi_G = 0$. In fact, if $\Pi_G\neq 0$, for any $(\alpha,\beta)\in[\mathfrak{g}^*\ltimes\mathfrak{g}^*,\mathfrak{g}^*\ltimes\mathfrak{g}^*] = [\mathfrak{g}^*,\mathfrak{g}^*]\ltimes[\mathfrak{g}^*,\mathfrak{g}^*]$ and for any $(\alpha',\beta')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we can deduce from proposition (3.3) that, if $\mathcal{D}^G$ is flat,

$$\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha',\beta') = \frac{1}{2}(\mathrm{ad}^t_\alpha\beta' + \mathrm{ad}^t_{\alpha'}\beta,\mathcal{D}^G_{\alpha'}\alpha).$$

Since $\Pi_G\neq 0$ there exists $\alpha\in[\mathfrak{g}^*,\mathfrak{g}^*]$ and $\alpha'\in S_G$ such that $[\alpha',\alpha]_{\mathfrak{g}^*}\neq 0$. Then there exists α and α' such that, $\mathcal{D}^G_{\alpha'}\alpha = \mathrm{ad}_{\alpha'}\alpha = [\alpha',\alpha]_{\mathfrak{g}^*}\neq 0$. Hence

$$\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha',\beta') = \frac{1}{2}(\mathrm{ad}^t_\alpha\beta' + \mathrm{ad}^t_{\alpha'}\beta,\mathcal{D}^G_{\alpha'}\alpha)\neq(0,0).$$

Then D^{TG} does not verify the first condition of (14) and D^{TG} is not flat.
If $\Pi_G = 0$, then $D^G = 0$ and $D^{TG} = 0$.
We can deduce:
If $\Pi_G \neq 0$, then $(TG, \Pi_{TG}, \langle,\rangle_{TG})$ is not compatible in the sense of Hawkins.
If $\Pi_G = 0$, then $\Pi_{TG} = 0$ and $(TG, \Pi_{TG}, \langle,\rangle_{TG})$ is compatible in the sense of Hawkins. ■

Corollary 3.6. *Let $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ be a Lie algebra equipped with a scalar product and let $(\mathfrak{g}^* \ltimes \mathfrak{g}^*, [,]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*\ltimes\mathfrak{g}^*})$ be the semi-direct product Lie algebra equipped with the natural scalar product. If $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ is a Milnor Lie algebra, then $(\mathfrak{g}^* \ltimes \mathfrak{g}^*, [,]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*\ltimes\mathfrak{g}^*})$ is a Milnor Lie algebra if, and only if, $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*})$ is an abelian Lie algebra.*

Proof. It is a direct consequence of the previous theorem. ■

Compatibility between the natural metric and the product Poisson structure on TG. Let (G, Π_G) be a Poisson-Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. The linearized Poisson structure of Π_G at e is the linear Poisson structure $\Pi_{\mathfrak{g}}$ on $\mathfrak{g}$, whose value at $X \in \mathfrak{g}$ is given by:

$$\Pi_{\mathfrak{g}}(X) = d_e\Pi_G(X).$$

The linear Poisson structure $\Pi_{\mathfrak{g}}$ on $\mathfrak{g} = T_eG$, making $(\mathfrak{g}, \Pi_{\mathfrak{g}})$ an abelian Poisson-Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ such that the Lie bracket of $\mathfrak{g}$ is zero and the Lie bracket of $\mathfrak{g}^*$ is $[,]_{\mathfrak{g}^*}$.
Let $(TG \equiv G \times \mathfrak{g}, \Pi_{G\times\mathfrak{g}})$ identified with the direct product Poisson-Lie group of (G, Π_G) and $(\mathfrak{g}, \Pi_{\mathfrak{g}})$, with Lie-bialgebra $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}^* \times \mathfrak{g}^*)$, where $\mathfrak{g} \times \mathfrak{g}$ is the direct product Lie algebra with bracket defined in (4) and $\mathfrak{g}^* \times \mathfrak{g}^*$ is the direct product Lie algebra with bracket defined in (5).
Let $\langle,\rangle^*_{G\times\mathfrak{g}}$ be the natural left invariant contravariant product metric on $G \times \mathfrak{g}$ defined for all $(\alpha, \beta), (\alpha', \beta') \in \mathfrak{g}^* \times \mathfrak{g}^*$ by:

$$\langle(\alpha, \beta), (\alpha', \beta')\rangle^*_{G\times\mathfrak{g}} = \langle\alpha, \alpha'\rangle^*_G + \langle\beta, \beta'\rangle^*_{\mathfrak{g}}, \tag{15}$$

where $\langle,\rangle^*_G$ is the left invariant Riemannian metric on G and $\langle,\rangle^*_{\mathfrak{g}} = \langle,\rangle_{\mathfrak{g}^*}$ is the scalar product on $\mathfrak{g}^*$ considered as a left invariant contravariant Riemannian metric on $\mathfrak{g}$.

Proposition 3.7. *Let $\mathcal{D}^{G\times\mathfrak{g}}$, $\mathcal{D}^G$ and $\mathcal{D}^{\mathfrak{g}}$ be the Levi-Civita connections associated respectively to $(\Pi_{G\times\mathfrak{g}}, \langle,\rangle^*_{G\times\mathfrak{g}})$, $(\Pi_G, \langle,\rangle^*_G)$ and $(\Pi_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}^*})$. Then for all $(\alpha, \beta), (\alpha', \beta') \in \mathfrak{g}^* \times \mathfrak{g}^*$, we have:*

$$\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\beta)}(\alpha', \beta') = (\mathcal{D}^G_\alpha\alpha', \mathcal{D}^{\mathfrak{g}}_\beta\beta').$$

Proof. According to equations (5), (10) and (15) we obtain:

$$\begin{aligned}
2\langle \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\beta)}(\alpha',\beta'),(\alpha'',\beta'')\rangle^*_{G\times\mathfrak{g}} &= \langle[(\alpha,\beta),(\alpha',\beta')]_{\mathfrak{g}^*\times\mathfrak{g}^*},(\alpha'',\beta'')\rangle^*_{G\times\mathfrak{g}}\\
&\quad+\ \langle[(\alpha'',\beta''),(\alpha',\beta')]_{\mathfrak{g}^*\times\mathfrak{g}^*},(\alpha,\beta)\rangle^*_{G\times\mathfrak{g}}\\
&\quad+\ \langle[(\alpha'',\beta''),(\alpha,\beta)]_{\mathfrak{g}^*\times\mathfrak{g}^*},(\alpha',\beta')\rangle^*_{G\times\mathfrak{g}}\\
&= \langle[\alpha,\alpha']_{\mathfrak{g}^*},\alpha''\rangle^*_G + \langle([\alpha'',\alpha',]_{\mathfrak{g}^*},\alpha\rangle^*_G + \langle[\alpha'',\alpha]_{\mathfrak{g}^*},\alpha'\rangle^*_G\\
&\quad+\ \langle[\beta,\beta']_{\mathfrak{g}^*},\beta''\rangle^*_{\mathfrak{g}} + \langle([\beta'',\beta',]_{\mathfrak{g}^*},\beta\rangle^*_{\mathfrak{g}} + \langle[\beta'',\beta]_{\mathfrak{g}^*},\beta'\rangle^*_{\mathfrak{g}}\\
&= 2\langle\mathcal{D}^G_\alpha\alpha',\alpha''\rangle^*_G + 2\langle\mathcal{D}^{\mathfrak{g}}_\beta\beta',\beta''\rangle^*_{\mathfrak{g}}\\
&= 2\langle(\mathcal{D}^G_\alpha\alpha',\mathcal{D}^{\mathfrak{g}}_\beta\beta'),(\alpha'',\beta'')\rangle^*_{G\times\mathfrak{g}}. \qquad\blacksquare
\end{aligned}$$

Proposition 3.8. *Let $R^{G\times\mathfrak{g}}$, R^G and $R^{\mathfrak{g}}$ be the curvatures of $\mathcal{D}^{G\times\mathfrak{g}}$,$\mathcal{D}^G$ and $\mathcal{D}^{\mathfrak{g}}$ respectively. Then for all $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have:*

$$R^{G\times\mathfrak{g}}((\alpha,\beta),(\alpha',\beta'))(\alpha'',\beta'') = (R^G(\alpha,\alpha')\alpha'',R^{\mathfrak{g}}(\beta,\beta')\beta'').$$

Proof. Using the equation (6) and the proposition (3.7), we obtain:

$$\begin{aligned}
R^{G\times\mathfrak{g}}((\alpha,\beta),(\alpha',\beta'))(\alpha'',\beta'') &= \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\beta)}\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha',\beta')}(\alpha'',\beta'') - \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha',\beta')}\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\beta)}(\alpha'',\beta'')\\
&\quad-\ \mathcal{D}^{G\times\mathfrak{g}}_{[(\alpha,\beta),(\alpha',\beta')]_{\mathfrak{g}^*\times\mathfrak{g}^*}}(\alpha'',\beta'')\\
&= \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\beta)}(\mathcal{D}^G_{\alpha'}\alpha'',\mathcal{D}^{\mathfrak{g}}_{\beta'}\beta'') - \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha',\beta')}(\mathcal{D}^G_\alpha\alpha'',\mathcal{D}^{\mathfrak{g}}_\beta\beta'')\\
&\quad-\ \mathcal{D}^{G\times\mathfrak{g}}_{([\alpha,\alpha']_{\mathfrak{g}^*},[\beta,\beta']_{\mathfrak{g}^*})}(\alpha'',\beta'')\\
&= (\mathcal{D}^G_\alpha\mathcal{D}^G_{\alpha'}\alpha'' - \mathcal{D}^G_{\alpha'}\mathcal{D}^G_\alpha\alpha'' - \mathcal{D}^G_{[\alpha,\alpha']_{\mathfrak{g}^*}}\alpha'',\mathcal{D}^{\mathfrak{g}}_\beta\mathcal{D}^{\mathfrak{g}}_{\beta'}\beta'' - \mathcal{D}^{\mathfrak{g}}_{\beta'}\mathcal{D}^{\mathfrak{g}}_\beta\beta''\\
&\quad-\ \mathcal{D}^{\mathfrak{g}}_{[\beta,\beta']_{\mathfrak{g}^*}}\beta'')\\
&= (R^G(\alpha,\alpha')\alpha'',R^{\mathfrak{g}}(\beta,\beta')\beta''). \qquad\blacksquare
\end{aligned}$$

Remark 3.9. The restriction of $\mathcal{D}^G$ to $\mathfrak{g}^*\times\mathfrak{g}^*$ coincides with $\mathcal{D}^{\mathfrak{g}}$ and the restriction of the curvature R^G of $\mathcal{D}^G$ to $\mathfrak{g}^*$ coincides with $R^{\mathfrak{g}}$, i.e.,

$$\mathcal{D}^G_\alpha\beta = \mathcal{D}^{\mathfrak{g}}_\alpha\beta,\ \ R^G(\alpha,\beta)\gamma = R^{\mathfrak{g}}(\alpha,\beta)\gamma,$$

for all $\alpha,\beta,\gamma\in\mathfrak{g}^*$.

Lemma 3.10. *The direct product Lie algebra $(\mathfrak{g}^*\times\mathfrak{g}^*,[,]_{\mathfrak{g}^*\times\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*})$ is a Milnor Lie algebra if, and only if, $(\mathfrak{g}^*,[,]_{\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*})$ is a Milnor Lie algebra. Furthermore*

$$\mathfrak{g}^*\times\mathfrak{g}^* = \Big(S_G\times S_G\Big)\overset{\perp}{\oplus}\Big([\mathfrak{g}^*,\mathfrak{g}^*]\times[\mathfrak{g}^*,\mathfrak{g}^*]\Big).$$

Proof. According to proposition (3.8), $\mathcal{D}^{G\times\mathfrak{g}}$ is flat if, and only if, $\mathcal{D}^G$ is flat, which is equivalent to $(\mathfrak{g}^*\times\mathfrak{g}^*,[,]_{\mathfrak{g}^*\times\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*})$ is a Milnor Lie algebra if, and only if, $(\mathfrak{g}^*,[,]_{\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*})$ is a Milnor Lie algebra. Furthermore if $(\mathfrak{g}^*\times\mathfrak{g}^*,[,]_{\mathfrak{g}^*\times\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*})$ is a Milnor Lie algebra, then,

$$\mathfrak{g}^*\times\mathfrak{g}^* = S_{G\times\mathfrak{g}}\overset{\perp}{\oplus}[\mathfrak{g}^*\times\mathfrak{g}^*,\mathfrak{g}^*\times\mathfrak{g}^*],$$

where

$$\begin{array}{rcl} S_{G\times\mathfrak{g}} & = & \{(\alpha,\beta)\in\mathfrak{g}^*\times\mathfrak{g}^*, \mathrm{ad}_{(\alpha,\beta)}+\mathrm{ad}^t_{(\alpha,\beta)}=(0,0)\} \\ & = & \{(\alpha,\beta)\in\mathfrak{g}^*\times\mathfrak{g}^*, (\mathrm{ad}_\alpha,\mathrm{ad}_\beta)+(\mathrm{ad}^t_\alpha,\mathrm{ad}^t_\beta)=(0,0)\} \\ & = & S_G\times S_G, \end{array}$$

and $[\mathfrak{g}^*\times\mathfrak{g}^*,\mathfrak{g}^*\times\mathfrak{g}^*]=[\mathfrak{g}^*,\mathfrak{g}^*]\times[\mathfrak{g}^*,\mathfrak{g}^*]=S^\perp_{G\times\mathfrak{g}}$, is the orthogonal of $S_{G\times\mathfrak{g}}$ with respect to the scalar product $\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*}$ on $\mathfrak{g}^*\times\mathfrak{g}^*$. It's clear that $[\mathfrak{g}^*,\mathfrak{g}^*]\times[\mathfrak{g}^*,\mathfrak{g}^*]$ is an abelian ideal. ∎

Lemma 3.11. *Let $\mathcal{M}^{G\times\mathfrak{g}}$, $\mathcal{M}^G$ and $\mathcal{M}^\mathfrak{g}$ be the metacurvatures of $\mathcal{D}^{G\times\mathfrak{g}}$, $\mathcal{D}^G$ and $\mathcal{D}^\mathfrak{g}$ respectively. Then, if $(\mathfrak{g}^*\times\mathfrak{g}^*,[,]_{\mathfrak{g}^*\times\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*})$ is a Milnor Lie algebra we have:*

1. *For all $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in S_G\times S_G$,*
$$\mathcal{M}^{G\times\mathfrak{g}}((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta''))=(\mathcal{M}^G(\alpha,\alpha',\alpha''),0).$$
2. *For all $(\alpha,\beta),(\alpha',\beta')$ or $(\alpha'',\beta'')\in[\mathfrak{g}^*,\mathfrak{g}^*]\times[\mathfrak{g}^*,\mathfrak{g}^*]$,*
$$\mathcal{M}^{G\times\mathfrak{g}}((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta''))=0.$$

Proof.

1. If $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in S_G\times S_G$, we deduce from the equation (12) and the proposition (3.7) that:
$$\begin{array}{rcl} \mathcal{M}^{G\times\mathfrak{g}}((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')) & = & -\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\beta)}\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha',\beta')}d(\alpha'',\beta'') \\ & = & -\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\beta)}(\mathcal{D}^G_{\alpha'}d\alpha'',\mathcal{D}^\mathfrak{g}_{\beta'}d\beta'') \\ & = & -(\mathcal{D}^G_\alpha\mathcal{D}^G_{\alpha'}d\alpha'',\mathcal{D}^\mathfrak{g}_\beta\mathcal{D}^\mathfrak{g}_{\beta'}d\beta'') \\ & = & (\mathrm{ad}_\alpha\mathrm{ad}_{\alpha'}\rho(\alpha''),\mathrm{ad}_\beta\mathrm{ad}_{\beta'}\rho_1(\beta'')) \\ & = & (\mathcal{M}^G(\alpha,\alpha',\alpha''),0), \end{array}$$
where $\rho:\mathfrak{g}^*\to\mathfrak{g}^*\wedge\mathfrak{g}^*$ is the dual of the Lie bracket on the Lie algebra $\mathfrak{g}$ of G and $\rho_1:\mathfrak{g}^*\to\mathfrak{g}^*\wedge\mathfrak{g}^*$ is the dual on the Lie bracket of the abelian Lie algebra $\mathfrak{g}$ of the abelian Lie group $\mathfrak{g}$.
2. If $(\alpha,\beta),(\alpha',\beta')$ or $(\alpha'',\beta'')\in[\mathfrak{g}^*\times\mathfrak{g}^*,\mathfrak{g}^*\times\mathfrak{g}^*]=[\mathfrak{g}^*,\mathfrak{g}^*]\times[\mathfrak{g}^*,\mathfrak{g}^*]$, we deduce from (12) that:
$$\mathcal{M}^{G\times\mathfrak{g}}((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta''))=0.$$ ∎

We call a Lie algebra $(\mathfrak{g},[,]_\mathfrak{g})$ unimodular if for any $X\in\mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_X)=0$. We call a Lie group G unimodular if its Lie algebra $(\mathfrak{g},[,]_\mathfrak{g})$ is unimodular.

Proposition 3.12. *Let (G,Π_G) be a Poisson-Lie group and $(\mathfrak{g},\Pi_\mathfrak{g})$ the abelian Poisson-Lie group equipped with the linear Poisson structure $\Pi_\mathfrak{g}$. Let $(G\times\mathfrak{g},\Pi_{G\times\mathfrak{g}})$ be the product Poisson-Lie group of (G,Π_G) and $(\mathfrak{g},\Pi_\mathfrak{g})$. Then:*

1. *The product Lie group $G\times\mathfrak{g}$ is unimodular if, and only if, G is unimodular.*
2. *The direct product Lie algebra $\mathfrak{g}^*\times\mathfrak{g}^*$ is unimodular if, and only if, $\mathfrak{g}^*$ is unimodular.*

Proof. Let $(\mathfrak{g}\times\mathfrak{g},\mathfrak{g}^*\times\mathfrak{g}^*)$ be the Lie bialgebra of the product Poisson-Lie group $(G\times\mathfrak{g},\Pi_{G\times\mathfrak{g}})$.

1. The abelian Lie group $\mathfrak{g}$ is unimodular, since its Lie algebra is abelian, i.e., for all $X,Y\in\mathfrak{g}$, $\mathrm{ad}_X Y=[X,Y]_{\mathfrak{g}}=0$, thus for all $X\in\mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_X)=0$. Then for all $(X,Y)\in\mathfrak{g}\times\mathfrak{g}$,
$$\mathrm{tr}(\mathrm{ad}_{(X,Y)})=\mathrm{tr}(\mathrm{ad}_X).$$
2. For any $(\alpha,\beta)\in\mathfrak{g}^*\times\mathfrak{g}^*$,
$$\mathrm{tr}(\mathrm{ad}_{(\alpha,\beta)})=\mathrm{tr}(\mathrm{ad}_\alpha\times\mathrm{ad}_\beta)=\mathrm{tr}(\mathrm{ad}_\alpha)+\mathrm{tr}(\mathrm{ad}_\beta).$$

If $\mathfrak{g}^*\times\mathfrak{g}^*$ is unimodular, then for any $(\alpha,\beta)\in\mathfrak{g}^*\times\mathfrak{g}^*$,
$$\mathrm{tr}(\mathrm{ad}_{(\alpha,\beta)})=\mathrm{tr}(\mathrm{ad}_\alpha)+\mathrm{tr}(\mathrm{ad}_\beta)=0,$$
and for $\alpha=\beta, 2\mathrm{tr}(\mathrm{ad}_\alpha)=0$, then $\mathfrak{g}^*$ is unimodular.
If $\mathfrak{g}^*$ is unimodular then for any $(\alpha,\beta)\in\mathfrak{g}^*\times\mathfrak{g}^*$,
$$\mathrm{tr}(\mathrm{ad}_\alpha)=\mathrm{tr}(\mathrm{ad}_\beta)=0\Rightarrow\mathrm{tr}(\mathrm{ad}_{(\alpha,\beta)})=0,$$
then $\mathfrak{g}^*\times\mathfrak{g}^*$ is unimodular. ∎

Theorem 3.13. *Let $(G,\Pi_G,\langle,\rangle_G)$ be a Poisson-Lie group equipped with a left invariant Riemannian metric $\langle,\rangle_G$ and let $(TG\equiv G\times\mathfrak{g},\Pi_{G\times\mathfrak{g}},\langle,\rangle_{G\times\mathfrak{g}})$ be the tangent bundle identified with the product Poisson Lie group of (G,Π_G) and $(\mathfrak{g},\Pi_{\mathfrak{g}})$, with $\langle,\rangle_{G\times\mathfrak{g}}$ is the natural Riemannian metric on $G\times\mathfrak{g}$. Then, the triple $(G,\Pi_G,\langle,\rangle_G)$ is strongly compatible in the sense of Hawkins if, and only if, $(TG\equiv G\times\mathfrak{g},\Pi_{G\times\mathfrak{g}},\langle,\rangle_{G\times\mathfrak{g}})$ is strongly compatible in the sense of Hawkins.*

Proof. According to lemma (3.10), $\mathfrak{g}^*$ is a Milnor Lie algebra if, and only if, the direct product Lie algebra $\mathfrak{g}^*\times\mathfrak{g}^*$ is a Milnor Lie algebra. By lemma (3.11), $\mathcal{D}^G$ is metaflat if and only if, $\mathcal{D}^{G\times\mathfrak{g}}$ is metaflat.
According to proposition (3.12), the product Lie group $G\times\mathfrak{g}$ is unimodular if, and only if, G is unimodular, moreover the direct product Lie algebra $\mathfrak{g}^*\times\mathfrak{g}^*$ is unimodular if, and only if, $\mathfrak{g}^*$ is unimodular.
The projection maps $\pi_1: G\times\mathfrak{g}\to G;(g,Y)\mapsto g$ and $\pi_2: G\times\mathfrak{g}\to\mathfrak{g};(g,Y)\mapsto Y$, are Poisson-Lie group homomorphisms.
Denote by ξ, ξ_1 and ξ_2, the 1-cocycle associated to Π_G, $\Pi_{\mathfrak{g}}$ and $\Pi_{G\times\mathfrak{g}}$ respectively. We denote also by ρ, ρ_1 and ρ_2, the dual of the Lie brackets on the Lie algebras of G, $\mathfrak{g}$ and $G\times\mathfrak{g}$ respectively.
If μ (resp. ν) is a left invariant volume form on G (resp. on $\mathfrak{g}$), its pull back

$\pi_1^*(\mu) = (\mu, 0)$, (resp. $\pi_2^*(\nu) = (0, \nu)$) is also a left invariant volume form on $G \times \{0\}$ (resp. on $\{e\} \times \mathfrak{g}$.) Then for all $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ we find:

$$\begin{array}{rcl} \rho_2(i_{\xi_2(X,Y)}(\mu, \nu)) & = & -d(i_{\xi_2(X,Y)}(\pi_1^*(\mu) + \pi_2^*(\nu))) \\ & = & -d(i_{\xi_2(X,Y)}\pi_1^*(\mu)) - d(i_{\xi_2(X,Y)}\pi_2^*(\nu)) \\ & = & -\pi_1^*(d(i_{\pi_{*1}\xi_2(X,Y)}\mu) - \pi_2^*(d(i_{\pi_{*2}\xi_2(X,Y)}\nu) \\ & = & (\rho(i_{\xi(X)}\mu), 0) + (0, \rho_1(i_{\xi_1(Y)}\nu)) \\ & = & (\rho(i_{\xi(X)}\mu), 0), \end{array}$$

since $\rho_1 = 0$, ($\mathfrak{g}$ is an abelian Lie group). Hence,

$$\rho(i_{\xi(X)}\mu) = 0 \Leftrightarrow \rho_2(i_{\xi_2(X,Y)}(\mu, \nu)) = 0. \qquad \blacksquare$$

Remark 3.14. 1. According to proposition (3.1), if $(G, \Pi_G, \langle, \rangle_G)$ is compatible in the sense of Hawkins, then the Sanchez de Alvarez tangent Poisson Lie group (TG, Π_{TG}) equipped with the left invariant Sasaki metric $\langle, \rangle_{TG}^S$ or the left invariant Cheeger-Gromoll metric $\langle, \rangle_{TG}^C$ is compatible in the sense of Hawkins if, and only if, (G, Π_G) is a trivial Poisson-Lie group.

2. $(G, \Pi_G, \langle, \rangle_G)$ is strongly compatible in the sense of Hawkins if, and only if, $(TG \equiv G \times \mathfrak{g}, \Pi_{G\times\mathfrak{g}})$ equipped with the left invariant Sasaki metric $\langle, \rangle_{TG}^S$ or the left invariant Cheeger-Gromoll metric $\langle, \rangle_{TG}^C$ is also strongly compatible in the sense of Hawkins.

4. Riemannian Lie algebra

Let $(G, \langle, \rangle_G)$ be a Lie group endowed with a left invariant Riemannian metric $\langle, \rangle_G$ and $(\mathfrak{g}, [,]_\mathfrak{g}, \langle, \rangle_\mathfrak{g})$ its Lie algebra endowed with the scalar product $\langle, \rangle_\mathfrak{g}$ associated to $\langle, \rangle_G$. The infinitesimal Levi-Civita connection associated with $([,]_\mathfrak{g}, \langle, \rangle_\mathfrak{g})$ is the bilinear map $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by

$$2\langle B_X Y, Z\rangle_\mathfrak{g} = \langle [X, Y]_\mathfrak{g}, Z\rangle_\mathfrak{g} + \langle [Z, X]_\mathfrak{g}, Y\rangle_\mathfrak{g} + \langle [Z, Y]_\mathfrak{g}, X\rangle_\mathfrak{g}, \qquad (16)$$

for all $X, Y, Z \in \mathfrak{g}$.
Note that B is the unique bilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ which verifies:

1. $B_X Y - B_Y X = [X, Y]_\mathfrak{g}$;
2. for any $X \in \mathfrak{g}$, $B_X : \mathfrak{g} \to \mathfrak{g}$ is skew-adjoint i.e.,

$$\langle B_X Y, Z\rangle_\mathfrak{g} + \langle Y, B_X Z\rangle_\mathfrak{g} = 0, \;\; Y, Z \in \mathfrak{g}.$$

The triple $(\mathfrak{g}, \langle, \rangle_\mathfrak{g}, [,]_\mathfrak{g})$ is said Riemannian Lie algebra [3, 4] if, for all $X, Y, Z \in \mathfrak{g}$:

$$[B_X Y, Z]_\mathfrak{g} + [X, B_Z Y]_\mathfrak{g} = 0. \qquad (17)$$

Since $\langle, \rangle_G$ is left invariant, the Levi-Civita covariant connection ∇^G of $\langle, \rangle_G$ is entirely determined by B. We denote $\nabla^\mathfrak{g} = \nabla^G(e)$. Then, for all $X, Y \in \mathfrak{g}$, we have,

$$\nabla^\mathfrak{g}_X Y = B_X Y.$$

Let $(\mathfrak{g} \rtimes \mathfrak{g}, [,]_{\mathfrak{g}\rtimes\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{g}})$ be the semi-direct product Lie algebra with bracket (1), equipped with the natural scalar product, given by:

$$\langle (X,Y),(X',Y')\rangle_{\mathfrak{g}\times\mathfrak{g}} = \langle X,X'\rangle_{\mathfrak{g}} + \langle Y,Y'\rangle_{\mathfrak{g}}, \;\; (X,Y),(X',Y') \in \mathfrak{g}\times\mathfrak{g}. \tag{18}$$

Proposition 4.1. *Let $\nabla^{\mathfrak{g}}$ and $\nabla^{\rtimes}$ be the Levi-Civita connections associated respectively to $([,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ and $([,]_{\mathfrak{g}\rtimes\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{g}})$. Then for all $(X,Y),(X',Y') \in \mathfrak{g}\times\mathfrak{g}$ we have:*

1. $\nabla^{\rtimes}_{(X,0)}(X',0) = (\nabla^{\mathfrak{g}}_X X', 0)$.
2. $\nabla^{\rtimes}_{(X,0)}(0,Y') = (0, \nabla^{\mathfrak{g}}_X Y' + \frac{1}{2}\mathrm{ad}^t_{Y'}X)$.
3. $\nabla^{\rtimes}_{(0,Y)}(X',0) = (0, \nabla^{\mathfrak{g}}_Y X' + \frac{1}{2}\mathrm{ad}^t_{Y}X')$.
4. $\nabla^{\rtimes}_{(0,Y)}(0,Y') = (\frac{1}{2}(\nabla^{\mathfrak{g}}_Y Y' + \nabla^{\mathfrak{g}}_{Y'}Y), 0)$.

Proof. According to equations (1), (16) and (18), the techniques used in the proof of this proposition are the same techniques used in the proof of proposition (3.3). ∎

Proposition 4.2. *Let $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ be a Lie algebra equipped with a scalar product $\langle,\rangle_{\mathfrak{g}}$ and let $(\mathfrak{g} \rtimes \mathfrak{g}, [,]_{\mathfrak{g}\rtimes\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{g}})$ be the semi-direct product Lie algebra equipped with the natural scalar product $\langle,\rangle_{\mathfrak{g}\times\mathfrak{g}}$. If $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ is Riemannian Lie algebra then, $(\mathfrak{g} \rtimes \mathfrak{g}, [,]_{\mathfrak{g}\rtimes\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{g}})$ is a Riemannian Lie algebra if, and only if, $(\mathfrak{g}, [,]_{\mathfrak{g}})$ is an abelian Lie algebra.*

Proof. By M.Boucetta [4], $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ is a Riemannian Lie algebra if and only if $\nabla^{\mathfrak{g}}$ is flat.
The Levi-Civita connection $\nabla^{\mathfrak{g}}$ is flat if, and only if, $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \overset{\perp}{\oplus} S$ is a Milnor Lie algebra. If $\nabla^{\mathfrak{g}}$ is flat, then for all $Y \in \mathfrak{g}$,

$$\nabla^{\mathfrak{g}}_X Y = \begin{cases} 0 & if \quad X \in [\mathfrak{g},\mathfrak{g}] \\ \mathrm{ad}_X Y = [X,Y]_{\mathfrak{g}} & if \quad X \in S, \end{cases} \tag{19}$$

and for all $X \in S = \{X \in \mathfrak{g}; \mathrm{ad}^t_X + \mathrm{ad}_X = 0\}$, $\nabla^{\mathfrak{g}} X = 0$ [1].
According to proposition (4.1), if $\nabla^{\mathfrak{g}}$ is flat then, $\nabla^{\rtimes}$ is flat if, and only if, $\mathfrak{g}$ is abelian. In fact, if $\mathfrak{g}$ is not abelian, for any $(X,Y) \in [\mathfrak{g}\rtimes\mathfrak{g}, \mathfrak{g}\rtimes\mathfrak{g}] = [\mathfrak{g},\mathfrak{g}] \rtimes [\mathfrak{g},\mathfrak{g}]$ and for any $(X',Y') \in \mathfrak{g}\rtimes\mathfrak{g}$, we can deduce from proposition (4.1) that, if $\nabla^{\mathfrak{g}}$ is flat then,

$$\nabla^{\rtimes}_{(X,Y)}(X',Y') = \frac{1}{2}(\nabla^{\mathfrak{g}}_{Y'}Y, \mathrm{ad}^t_{Y'}X + \mathrm{ad}^t_Y X).$$

Since $\mathfrak{g}$ is not abelian there exists $Y \in [\mathfrak{g},\mathfrak{g}]$ and $Y' \in S$ such that $[Y',Y]_{\mathfrak{g}} \neq 0$. Then there exists Y and Y' such that, $\nabla^{\mathfrak{g}}_{Y'}Y = \mathrm{ad}_{Y'}Y = [Y',Y]_{\mathfrak{g}} \neq 0$. Hence

$$\nabla^{\rtimes}_{(X,Y)}(X',Y') = \frac{1}{2}(\nabla^{\mathfrak{g}}_{Y'}Y, \mathrm{ad}^t_{Y'}X + \mathrm{ad}^t_Y X) \neq (0,0).$$

Then $\nabla^{\rtimes}$ does not verify the first condition of (19) and $\nabla^{\rtimes}$ is not flat.
If $\mathfrak{g}$ is abelian, then $\mathfrak{g}\rtimes\mathfrak{g}$ is also abelian and $(\mathfrak{g} \rtimes \mathfrak{g}, [,]_{\mathfrak{g}\rtimes\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{g}})$ is a Riemannian Lie algebra. ∎

Proposition 4.3. *Let $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ and $(\mathfrak{k}, [,]_{\mathfrak{k}}, \langle,\rangle_{\mathfrak{k}})$ be two Lie algebras equipped with a scalar product $\langle,\rangle_{\mathfrak{g}}$ and $\langle,\rangle_{\mathfrak{k}}$ respectively and let $(\mathfrak{g}\times\mathfrak{k}, [,]_{\mathfrak{g}\times\mathfrak{k}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{k}})$ be the direct product Lie algebra equipped with the natural scalar product $\langle,\rangle_{\mathfrak{g}\times\mathfrak{k}}$. Then:*

1. *$(\mathfrak{g}\times\mathfrak{k}, [,]_{\mathfrak{g}\times\mathfrak{k}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{k}})$ is a Riemannian Lie algebra if, and only if, $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ and $(\mathfrak{k}, [,]_{\mathfrak{k}}, \langle,\rangle_{\mathfrak{k}})$ are Riemannian Lie algebras.*

2. *$(\mathfrak{g}\times\mathfrak{k}, [,]_{\mathfrak{g}\times\mathfrak{k}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{k}})$ is a Milnor Lie algebra if, and only if, $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ and $(\mathfrak{k}, [,]_{\mathfrak{k}}, \langle,\rangle_{\mathfrak{k}})$ are Milnor Lie algebras.*

Proof.

1. For all $(X,Y),(X',Y')\in\mathfrak{g}\times\mathfrak{k}$, the infinitesimal Levi-Civita connection $B^{\mathfrak{g}\times\mathfrak{k}}$ associated with $([,]_{\mathfrak{g}\times\mathfrak{k}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{k}})$ is given by

$$B^{\mathfrak{g}\times\mathfrak{k}}_{(X,Y)}(X',Y') = (B^{\mathfrak{g}}_X X', B^{\mathfrak{k}}_Y Y'),$$

where $B^{\mathfrak{g}}$ and $B^{\mathfrak{k}}$ are the infinitesimal Levi-Civita connections associated respectively to $([,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ and $([,]_{\mathfrak{k}}, \langle,\rangle_{\mathfrak{k}})$.
Then for all $(X,Y),(X',Y'),(X'',Y'')\in\mathfrak{g}\times\mathfrak{k}$ we find:

$$\begin{aligned}
&[B^{\mathfrak{g}\times\mathfrak{k}}_{(X,Y)}(X',Y'),(X'',Y'')]_{\mathfrak{g}\times\mathfrak{k}} + [(X,Y), B^{\mathfrak{g}\times\mathfrak{k}}_{(X'',Y'')}(X',Y')]_{\mathfrak{g}\times\mathfrak{k}}\\
&= [(B^{\mathfrak{g}}_X X', B^{\mathfrak{k}}_Y Y'),(X'',Y'')]_{\mathfrak{g}\times\mathfrak{k}} + [(X,Y),(B^{\mathfrak{g}}_{X''}X', B^{\mathfrak{k}}_{Y''}Y')]_{\mathfrak{g}\times\mathfrak{k}}\\
&= ([B^{\mathfrak{g}}_X X', X'']_{\mathfrak{g}}, [B^{\mathfrak{k}}_Y Y', Y'']_{\mathfrak{k}}) + ([X, B^{\mathfrak{g}}_{X''}X']_{\mathfrak{g}}, [Y, B^{\mathfrak{k}}_{Y''}Y']_{\mathfrak{k}})\\
&= ([B^{\mathfrak{g}}_X X', X'']_{\mathfrak{g}} + [X, B^{\mathfrak{g}}_{X''}X']_{\mathfrak{g}}, [B^{\mathfrak{k}}_Y Y', Y'']_{\mathfrak{k}} + [Y, B^{\mathfrak{k}}_{Y''}Y']_{\mathfrak{k}}).
\end{aligned}$$

From equation (17), $\mathfrak{g}\times\mathfrak{k}$ is a Riemannian Lie algebra if, and only if, $\mathfrak{g}$ and $\mathfrak{k}$ are Riemannian Lie algebras.

2. Denote by $\nabla^{\mathfrak{g}\times\mathfrak{k}}$, $\nabla^{\mathfrak{g}}$ and $\nabla^{\mathfrak{k}}$ the Levi-Civita connections associated respectively to $([,]_{\mathfrak{g}\times\mathfrak{k}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{k}})$, $([,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ and $([,]_{\mathfrak{k}}, \langle,\rangle_{\mathfrak{k}})$. Let $R^{\mathfrak{g}\times\mathfrak{k}}$, $R^{\mathfrak{g}}$ and $R^{\mathfrak{k}}$ be the curvatures of $\nabla^{\mathfrak{g}\times\mathfrak{k}}$, $\nabla^{\mathfrak{g}}$ and $\nabla^{\mathfrak{k}}$ respectively. Then for all $(X,Y),(X',Y'),(X'',Y'')\in\mathfrak{g}\times\mathfrak{k}$, we find:

$$R^{\mathfrak{g}\times\mathfrak{k}}((X,Y),(X',Y'))(X'',Y'') = (R^{\mathfrak{g}}(X,X')X'', R^{\mathfrak{k}}(Y,Y')Y'').$$

Then $\nabla^{\mathfrak{g}\times\mathfrak{k}}$ is flat if, and only if, $\nabla^{\mathfrak{g}}$ and $\nabla^{\mathfrak{k}}$ are flat, which is equivalent to: $(\mathfrak{g}\times\mathfrak{k}, [,]_{\mathfrak{g}\times\mathfrak{k}}, \langle,\rangle_{\mathfrak{g}\times\mathfrak{k}})$ is a Milnor Lie algebra if and only if $(\mathfrak{g}, [,]_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ and $(\mathfrak{k}, [,]_{\mathfrak{k}}, \langle,\rangle_{\mathfrak{k}})$ are Milnor Lie algebras. ■

5. Riemannian Poisson-Lie group

Let (G,Π_G) be a Poisson Lie-group and $(\mathfrak{g},\mathfrak{g}^*)$ its Lie bialgebra. Let $\langle,\rangle_{\mathfrak{g}^*}$ be a scalar product on $\mathfrak{g}^*$. A left invariant contravariant metric $\langle,\rangle^*_G$ on G is given by:

$$\langle\alpha,\beta\rangle^*_G(g) = \langle T^*_e L_g(\alpha), T^*_e L_g(\beta)\rangle_{\mathfrak{g}^*}(e),$$

where $\alpha, \beta \in \Omega^1(G), g \in G$ and e the neutral element of G.
Let $A : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ be the infinitesimal Levi-Civita connection associated with $(\langle,\rangle_{\mathfrak{g}^*}, [,]_{\mathfrak{g}^*})$ given for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$ by:

$$2\langle A_\alpha \beta, \gamma\rangle_{\mathfrak{g}^*} = \langle [\alpha, \beta]_{\mathfrak{g}^*}, \gamma\rangle_{\mathfrak{g}^*} + \langle [\gamma, \alpha]_{\mathfrak{g}^*}, \beta\rangle_{\mathfrak{g}^*} + \langle [\gamma, \beta]_{\mathfrak{g}^*}, \alpha\rangle_{\mathfrak{g}^*},$$

where $[,]_{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$.
Since the Riemannian metric $\langle,\rangle_G^*$ on G is left invariant, then the Levi-Civita contravariant connection $\mathcal{D}^G$ associated to $(\Pi_G, \langle,\rangle_G^*)$ is given by

$$\mathcal{D}^G_\alpha \beta = A_\alpha \beta, \ \ \alpha, \beta \in \mathfrak{g}^*.$$

Let $(G, \Pi_G, \langle,\rangle_G^*)$ be a Poisson Lie-group equipped with a left invariant contravariant Riemannian metric $\langle,\rangle_G^*$ and let $\Pi_l^G : G \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$, the mapping defined by $\Pi_l^G(g) = (L_{g^{-1}})_* \Pi_G(g)$ where $(L_g)_*$ denotes the tangent map of the left translation of G by g.
By M.Boucetta [3], $(G, \Pi_G, \langle,\rangle_G^*)$ is a Riemannian Poisson-Lie group if, and only if:

$$[Ad_g^*(A_\alpha\gamma + ad^*_{\Pi_l^G(g)(\alpha)}\gamma), Ad_g^*(\beta)]_{\mathfrak{g}^*} + [Ad_g^*(\alpha), Ad_g^*(A_\beta\gamma + ad^*_{\Pi_l^G(g)(\beta)}\gamma)]_{\mathfrak{g}^*} = 0, \quad (20)$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}^*, g \in G$, where A is the infinitesimal Levi-Civita connection associated with $(\langle,\rangle_{\mathfrak{g}^*}, [,]_{\mathfrak{g}^*})$.

Lemma 5.1. *Let $(G, \Pi_G, \langle,\rangle_G^*)$ be a Poisson-Lie group equipped with a left invariant contravariant Riemannian metric. Then, the dual Lie algebra $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ is a Riemannian Lie algebra if and only if $\mathcal{D}^G$ is flat.*

Proof. Let G^* be the dual of G: the connected and simply connected Poisson Lie group associated with $\mathfrak{g}^*$. The connection on $\mathfrak{g}^*$ identifies with the Levi Civita covariant connection associated with the Riemannian metric on G^* (where $\mathfrak{g}^*$ is regarded as Lie algebra of left invariant vector fields on G^*), hence by [4], one can deduce that the dual Lie algebra $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ is a Riemannian Lie algebra if, and only if, $\mathcal{D}^G$ is flat. ■

Theorem 5.2. *Let $(G, \Pi_G, \langle,\rangle_G^*)$ be a Poisson-Lie group equipped with a left invariant contravariant Riemannian metric $\langle,\rangle_G^*$ and let $(TG, \Pi_{TG}, \langle,\rangle_{TG}^*)$ the Sanchez de Alvarez tangent Poisson-Lie group of G with the natural left invariant contravariant Riemannian metric $\langle,\rangle_{TG}^*$. If $(G, \Pi_G, \langle,\rangle_G^*)$ is a Riemannian Poisson Lie group then $(TG, \Pi_{TG}, \langle,\rangle_{TG}^*)$ is a Riemannian Poisson-Lie group if, and only if, (G, Π_G) is a trivial Poisson-Lie group.*

Proof. If $(G, \Lambda_G, \langle,\rangle_G^*)$ is a Riemannian Poisson-Lie group then, $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ is a Riemannian Lie algebra [3].
According to proposition (3.3), if $\mathcal{D}^G$ is flat, then $\mathcal{D}^{TG}$ is flat if, and only if, Π_G is trivial. Then by lemma (5.1) we deduce that, if $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ is a Riemannian Lie algebra, then $\mathfrak{g}^* \ltimes \mathfrak{g}^*$ is a Riemannian Lie algebra if, and only if, $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*})$ is an abelian Lie algebra.

If $\Pi_G \neq 0$, then $\mathfrak{g}^* \ltimes \mathfrak{g}^*$ is not a Riemannian Lie algebra and $(TG, \Pi_{TG}, \langle,\rangle^*_{TG})$ is not a Riemannian Poisson-Lie group.
If $\Pi_G = 0$, then $\Pi_{TG} = 0$ and $(TG, \Pi_{TG}, \langle,\rangle^*_{TG})$ is a Riemannian Poisson-Lie group. ■

Theorem 5.3. *Let $(G, \Pi_G, \langle,\rangle^*_G)$ be a Poisson-Lie group equipped with the left invariant contravariant Riemannian metric $\langle,\rangle^*_G$ and let $(TG \equiv G\times\mathfrak{g}, \Pi_{G\times\mathfrak{g}}, \langle,\rangle^*_{G\times\mathfrak{g}})$ be the tangent bundle identified with the product Poisson Lie group of (G, Π_G) and $(\mathfrak{g}, \Pi_\mathfrak{g})$, with $\langle,\rangle^*_{G\times\mathfrak{g}}$ is the natural contravariant Riemannian metric on $G \times \mathfrak{g}$. Then, $(TG \equiv G \times \mathfrak{g}, \Pi_{G\times\mathfrak{g}}, \langle,\rangle^*_{G\times\mathfrak{g}})$ is a Riemannian Poisson-Lie group if, and only if, $(G, \Pi_G, \langle,\rangle^*_G)$ is a Riemannian Poisson Lie group.*

Proof. The infinitesimal Levi-Cevita connection $A^{G\times\mathfrak{g}}$ associated to $([,]_{\mathfrak{g}^*\times\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*})$ is given for any $(\alpha,\alpha'),(\beta,\beta') \in \mathfrak{g}^*\times\mathfrak{g}^*$ by:

$$A^{G\times\mathfrak{g}}_{(\alpha,\alpha')}(\beta,\beta') = \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,\alpha')}(\beta,\beta') = (\mathcal{D}^G_\alpha\beta, \mathcal{D}^\mathfrak{g}_{\alpha'}\beta') = (A_\alpha\beta, A_{\alpha'}\beta').$$

Since the Lie group $\mathfrak{g}$ is abelian then for all $(g,X) \in G\times\mathfrak{g}, (X,Y) \in \mathfrak{g}\times\mathfrak{g}$ and for any $(\alpha,\alpha') \in \mathfrak{g}^*\times\mathfrak{g}^*$ we get:

$$Ad^*_{(g,X)}(\alpha,\alpha') = (Ad^*_g\alpha,\alpha'), \quad ad^*_{(X,Y)}(\alpha,\alpha') = (ad^*_X\alpha, 0).$$

Moreover for all $(g,X) \in G\times\mathfrak{g}$ we find:

$$\Pi^{G\times\mathfrak{g}}_l(g,X)(\alpha,\alpha') = (\Pi^G_l(g)(\alpha), \Pi_\mathfrak{g}(X)(\alpha')).$$

Then for any $(\alpha,\alpha'),(\beta,\beta'),(\beta,\beta') \in \mathfrak{g}^*\times\mathfrak{g}^*$, we obtain:

$$\begin{aligned}
&[Ad^*_{(g,X)}(A^{G\times\mathfrak{g}}_{(\alpha,\alpha')}(\gamma,\gamma') + ad^*_{\Pi^{G\times\mathfrak{g}}_l(g,X)(\alpha,\alpha')}(\gamma,\gamma')), Ad^*_{(g,X)}(\beta,\beta')]_{\mathfrak{g}^*\times\mathfrak{g}^*} \\
&+[Ad^*_{(g,X)}(\alpha,\alpha'), Ad^*_{(g,X)}(A^{G\times\mathfrak{g}}_{(\beta,\beta')}(\gamma,\gamma') + ad^*_{\Pi^{G\times\mathfrak{g}}_l(g,X)(\beta,\beta')}(\gamma,\gamma'))]_{\mathfrak{g}^*\times\mathfrak{g}^*} \\
= \; &[Ad^*_{(g,X)}((A_\alpha\gamma, A_{\alpha'}\gamma') + ad^*_{(\Pi^G_l(g)(\alpha),\Pi_\mathfrak{g}(X)(\alpha'))}(\gamma,\gamma')), (Ad^*_g\beta,\beta')]_{\mathfrak{g}^*\times\mathfrak{g}^*} \\
&+[(Ad^*_g\alpha,\alpha'), Ad^*_{(g,X)}((A_\beta\gamma, A_{\beta'}\gamma') + ad^*_{(\Pi^G_l(g)(\beta),\Pi_\mathfrak{g}(X)(\beta'))}(\gamma,\gamma'))]_{\mathfrak{g}^*\times\mathfrak{g}^*} \\
= \; &([Ad^*_g(A_\alpha\gamma + ad^*_{\Pi^G_l(g)(\alpha)}\gamma), Ad^*_g\beta]_{\mathfrak{g}^*}, [A_{\alpha'}\gamma',\beta']_{\mathfrak{g}^*}) \\
&+([Ad^*_g\alpha, Ad^*_g(A_\beta\gamma + ad^*_{\Pi^G_l(g)(\beta)}\gamma)]_{\mathfrak{g}^*}, [\alpha', A_{\beta'}\gamma']_{\mathfrak{g}^*}) \\
= \; &\big([Ad^*_g(A_\alpha\gamma + ad^*_{\Pi^G_l(g)(\alpha)}\gamma), Ad^*_g\beta]_{\mathfrak{g}^*} + [Ad^*_g\alpha, Ad^*_g(A_\beta\gamma + ad^*_{\Pi^G_l(g)(\beta)}\gamma)]_{\mathfrak{g}^*}, \\
&[A_{\alpha'}\gamma',\beta']_{\mathfrak{g}^*} + [\alpha', A_{\beta'}\gamma']_{\mathfrak{g}^*}\big).
\end{aligned}$$

Then from equations (17) and (20), $(TG \equiv G\times\mathfrak{g}, \Pi_{G\times\mathfrak{g}}, \langle,\rangle^*_{G\times\mathfrak{g}})$ is a Riemannian Poisson Lie group if, and only if, $(G, \Pi_G, \langle,\rangle^*_G)$ is a Riemannian Poisson-Lie group. ■

Remark 5.4. 1. If $(G, \Pi_G, \langle,\rangle^*_G)$ is a Riemannian Poisson-Lie group then by the proposition (3.1), the Sanchez de Alvarez tangent Poisson Lie group (TG, Π_{TG}) equipped with the left invariant contravariant Sasaki metric $\langle,\rangle^{*S}_{TG}$ or the left invariant contravariant Cheeger-Gromoll metric $\langle,\rangle^{*C}_{TG}$ is a Riemannian Poisson-Lie group if, and only if, (G, Π_G) is a trivial Poisson-Lie group.

2. $(G, \Pi_G, \langle,\rangle_G^*)$ is a Riemannian Poisson-Lie group if, and only if, the pair $(TG \equiv G \times \mathfrak{g}, \Pi_{G\times\mathfrak{g}})$ equipped with the left invariant contravariant Sasaki metric $\langle,\rangle_{TG}^{*S}$ or the left invariant contravariant Cheeger-Gromoll metric $\langle,\rangle_{TG}^{*C}$ is also a Riemannian Poisson Lie group.

6. Example

A.Bahayou and M.Boucetta [1] proved that a connected and simply connected 3-dimensional Poisson-Lie group $(G, \Pi_G, \langle,\rangle_G)$ equipped with a left invariant Riemannian metric $\langle,\rangle_G$ is strongly compatible in the sense of Hawkins if and only if, it is isomorphic to one of the following triples:

1. $(\mathbb{R}^3, \Pi_{\mathbb{R}^3}, \langle,\rangle_{\mathbb{R}^3})$, where $\mathbb{R}^3$ is equipped with its abelian Lie group structure, $\langle,\rangle_{\mathbb{R}^3}$ is the canonical Euclidian metric and

$$\Pi_{\mathbb{R}^3} = \lambda\frac{\partial}{\partial_x} \wedge (z\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_z}),$$

where $\lambda \in \mathbb{R}$ or,

2. $(H_3, \Pi_{H_3}, \langle,\rangle_{H_3})$, where $H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} (x, y, z) \in \mathbb{R}^3 \right\}$, and

$$\Pi_{H_3} = \lambda(x\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_x}) \wedge \frac{\partial}{\partial_z}, \quad \langle,\rangle_{H_3} = dx^2 + dy^2 + a(dz - xdy)^2,$$

where $\lambda \in \mathbb{R}$ and $a > 0$.
Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. The Lie algebra $\mathbb{R}^3$ with bracket,

$$[e_1, e_2]_{\mathbb{R}^3} = \lambda e_3, \ \ [e_1, e_3]_{\mathbb{R}^3} = -\lambda e_2, \ \ [e_2, e_3]_{\mathbb{R}^3} = 0,$$

is a Milnor Lie algebra. Furthermore

$$[\mathbb{R}^3, \mathbb{R}^3] =< \{e_2, e_3\} >, \text{the subspace spanned by} \{e_2, e_3\}, \ S_{\mathbb{R}^3} =< \{e_1\} >, \ \text{and}$$

$$\mathbb{R}^3 =< \{e_1\} > \overset{\perp}{\oplus} < \{e_2, e_3\} >= \mathbb{R} \times \{0_{\mathbb{R}^2}\} \overset{\perp}{\oplus} \{0\} \times \mathbb{R}.$$

Let $(\mathfrak{h}_3, \mathfrak{h}_3^*)$ be the Lie bialgebra of (H_3, Π_{H_3}) and (e_1, e_2, e_3) be an orthonormal basis of $\mathfrak{h}_3^*$. The Lie algebra $\mathfrak{h}_3^*$ with bracket

$$[e_1, e_3]_{\mathfrak{h}_3^*} = -\lambda e_2, \ \ [e_2, e_3]_{\mathfrak{h}_3^*} = \lambda e_1, \ \ [e_1, e_2]_{\mathfrak{h}_3^*} = 0,$$

is a Milnor Lie algebra. Moreover $[\mathfrak{h}_3^*, \mathfrak{h}_3^*] =< \{e_1, e_2\} >$, $S_{H_3} =< \{e_3\} >$ and

$$\mathfrak{h}_3^* =< \{e_3\} > \overset{\perp}{\oplus} < \{e_1, e_2\} > .$$

The direct product Lie algebra $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$ of the Milnor Lie algebra $(\mathbb{R}^3, [,]_{\mathbb{R}^3})$ is a Milnor Lie algebra. Moreover $\mathbb{R}^3 \times \mathbb{R}^3 =$

$$\mathbb{R}^6 =< \{e_1\} > \times < \{e_4\} > \overset{\perp}{\oplus} < \{e_2, e_3\} > \times < \{e_5, e_6\} >= \mathbb{R}^2 \times \{0_{\mathbb{R}^4}\} \overset{\perp}{\oplus} \{0_{\mathbb{R}^2}\} \times \mathbb{R}^4,$$

where (e_4, e_5, e_6) is an orthonormal basis of $\mathbb{R}^3$.
The direct product Lie algebra $\mathfrak{h}_3^* \times \mathfrak{h}_3^*$ of the Milnor Lie algebra $(\mathfrak{h}_3^*, [,]_{\mathfrak{h}_3^*})$ is a Milnor Lie algebra. Furthermore

$$\mathfrak{h}_3^* \times \mathfrak{h}_3^* =< \{e_3\} > \times < \{e_6\} > \overset{\perp}{\oplus} < \{e_1, e_2\} > \times < \{e_4, e_5\} >,$$

where (e_4, e_5, e_6) is an orthonormal basis of $\mathfrak{h}_3^*$.
If we identify $(TG \equiv G \times \mathfrak{g}, \Pi_{G\times\mathfrak{g}})$ with the direct product Poisson-Lie group of (G, Π_G) and $(\mathfrak{g}, \Pi_\mathfrak{g})$, then $(TG \equiv G \times \mathfrak{g}, \Pi_{G\times\mathfrak{g}}, \langle , \rangle_{G\times\mathfrak{g}})$ is strongly compatible in the sense of Hawkins if and only if, it is isomorphic to one of the following triples:

1. $(\mathbb{R}^6, \Pi_{\mathbb{R}^6}, \langle , \rangle_{\mathbb{R}^6})$, where $\mathbb{R}^6$ is equipped with its abelian Lie group structure, $\langle , \rangle_{\mathbb{R}^6}$ is the canonical Euclidian metric, with coordinate (x, y, z, u, v, w) and Poisson tensor

$$\Pi_{\mathbb{R}^6} = \lambda \frac{\partial}{\partial_x} \wedge (z\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_z}) + \lambda \frac{\partial}{\partial_u} \wedge (w\frac{\partial}{\partial_v} - v\frac{\partial}{\partial_w}),$$

   where $\lambda \in \mathbb{R}$ or,

2. $(H_3 \times \mathfrak{h}_3, \Pi_{H_3\times\mathfrak{h}_3}, \langle , \rangle_{H_3\times\mathfrak{h}_3})$, where

$$H_3 \times \mathfrak{h}_3 = \left\{ \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \right), (x, y, z, u, v, w) \in \mathbb{R}^6 \right\},$$

   and

$$\Pi_{H_3\times\mathfrak{h}_3} = \lambda(x\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_x}) \wedge \frac{\partial}{\partial_z} + \lambda(u\frac{\partial}{\partial_v} - v\frac{\partial}{\partial_u}) \wedge \frac{\partial}{\partial_w},$$

$$\langle , \rangle_{TH_3} = dx^2 + dy^2 + a(dz - xdy)^2 + du^2 + dv^2 + a(dw - udv)^2, \ \ a > 0,$$

where $\lambda \in \mathbb{R}$ and $(x, y, z, u, v, w) \in \mathbb{R}^6$.

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