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The Universal Enveloping Algebra $U(\mathfrak{sl}_2 \ltimes V_2)$, its Prime Spectrum and a Classification of its Simple Weight Modules

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Abstract. For the enveloping algebra A of the Lie algebra $\mathfrak{sl}_2 \ltimes V_2$, explicit descriptions of its prime, primitive, completely prime and maximal spectra are given. A classification of simple weight $\mathfrak{sl}_2 \ltimes V_2$-modules is given. Generators and defining relations are found for the centralizer $C_A(H)$ in A of the Cartan element H of $\mathfrak{sl}_2$. Explicit descriptions of the prime, primitive, completely prime and maximal spectra of $C_A(H)$ are given. Simple $C_A(H)$-modules are classified.
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1. Introduction

In this paper, module means a left module, $\mathbb{K}$ is a field of characteristic zero and $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$. The Lie algebra $\mathfrak{sl}_2 \ltimes V_2$ is the semi-direct product of the simple Lie algebra $\mathfrak{sl}_2$ with the (unique) simple 2-dimensional $\mathfrak{sl}_2$-module V_2 (viewed as an abelian Lie algebra). The Lie algebra $\mathfrak{sl}_2 \ltimes V_2$ admits a 1-dimensional central extension which is called the *Schrödinger algebra* $\mathfrak{s}$. Classifications of simple weight modules with finite dimensional weight spaces over the Schrödinger algebra were given in [10] and [13]. In [11], the authors gave a classification of primitive ideals of the enveloping algebra of the Schrödinger algebra with *nonzero* central charge. A classification of uniserial $\mathfrak{sl}_2 \ltimes V_2$-modules was obtained in [9]. The universal enveloping algebra $A := U(\mathfrak{sl}_2 \ltimes V_2)$ has a close relation with the infinitesimal Hecke algebras of $\mathfrak{sl}_2$, [16]. The first Hochschild cohomology of A was obtained in [16], which is a rank one free module over the center. The description of primitive ideals of the algebra A given in [16, Theorem 6.2] is not correct (for $z = 0$ in that paper), i.e., the claim that all prime ideals are primitive is not correct.

Spectra of the algebra A. In Section 2, an explicit description of the set of prime ideals of the algebra A together with their inclusions is given (Theorem 2.8). Using a classification of prime ideals of A (Theorem 2.8) explicit

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descriptions of the sets of maximal, primitive and completely prime ideals are obtained (Corollary 2.9, Theorem 2.10 and Corollary 2.11, respectively). The group $\mathrm{Aut}_{\mathbb{K}}(A)$ of automorphisms of the algebra A is large as the algebra A contains plenty of ad-locally nilpotent elements. An ideal of an algebra is called a *characteristic ideal* if it is invariant under all the automorphisms of the algebra. Corollary 2.12 is an explicit description of the characteristic prime ideals of A. It says that almost all prime ideals apart from an obvious set are characteristic ones.

The centralizer $C_A(H)$, its generators and defining relations. Let $\mathfrak{h} = \mathbb{K}H$ be the *Cartan* subalgebra of the Lie algebra $\mathfrak{sl}_2$ and $C_A(H)$ be the centralizer of H in A. The aim of Section 3 is to find explicit generators and defining relations for the algebra $C_A(H)$ (Theorem 3.4) (there are three quadratic relations and one cubic), to prove that the centre of the algebra $C_A(H)$ is a polynomial algebra $\mathbb{K}[C, H]$ (Theorem 3.4) and the algebra $C_A(H)$ is a free module over its centre (Proposition 3.6), to realize the algebra $C_A(H)$ as an algebra of differential operators, to prove various properties of the factor algebras $C^{\lambda,\mu}$ of $C_A(H)$. Results of this section are used in many proofs of the paper.

Classification of simple weight A-modules. An A-module M is called a *weight* module if $M = \oplus_{\mu\in\mathbb{K}} M_\mu$ where $M_\mu = \{m \in M \mid Hm = \mu m\}$. Each nonzero component M_μ is a $C_A(H)$-module. If, in addition, the weight A-module M is simple then all nonzero components M_μ are *simple* $C_A(H)$-modules. So, the problem of classification of *simple weight* A-modules is closely related to the problem of classification of *all simple* $C_A(H)$-modules, which can be seen as the first, the more difficult, of two steps. The second one is about how 'to assemble' some of the simple $C_A(H)$-modules into a simple A-module. The difficulty of the first step stems from the fact that the algebra $C_A(H)$ is of comparable size to the algebra A itself ($\mathrm{GK}(C_A(H)) = 4$ and $\mathrm{GK}(A) = 5$ where GK stands for the Gelfand-Kirillov dimension) and the defining relations of the algebra $C_A(H)$ are much more complex than the defining relations of the algebra A (see, (15)–(18)). An advantage is that the algebra $C_A(H)$ has an additional central element H. Moreover, the centre of $C_A(H)$ is a polynomial algebra $\mathbb{K}[C, H]$ (Theorem 3.4) where $C = FX^2 - HXY - EY^2$ is a central element of the algebra A where E, F and H are the canonical basis of the Lie algebra $\mathfrak{sl}_2$. A problem of classification of simple modules is reduced to the one but for the factor algebras $C^{\lambda,\mu} := C_A(H)/(C - \lambda, H - \mu)$ where $\lambda, \mu \in \mathbb{K}$. We assume that the field $\mathbb{K}$ is algebraically closed. There are two distinct cases: $\lambda \neq 0$ and $\lambda = 0$. They require different approaches. The common feature is a discovery of the fact that in order to study simple modules over the algebras $C^{\lambda,\mu}$ we embed them into larger algebras for which classifications of simple modules are known. A surprise is that the sets of simple modules of the algebras $C^{\lambda,\mu}$ and their over-algebras are tightly connected. In the case $\lambda \neq 0$, such an algebra is the first Weyl algebra, but in the second case when $\lambda = 0$, it is a skew polynomial algebra $\mathbb{K}[h][t; \sigma]$ where $\sigma(h) = h - 1$. Classifications of simple $C^{\lambda,\mu}$-modules is given in Section 4 (Theorem 4.8 and Theorem 4.11). Using it a classification of simple weight A-modules is given in Section 6. A typical simple weight A-module depends on an arbitrarily large number of independent parameters. The set of simple A-modules is partitioned into 5 classes each of them is dealt separately with different techniques (Lemma 6.1, Proposition 6.4, Theorem 6.6 and Theorem 6.7).

The algebras $C_A(H), C^{\lambda,\mu}$ and their spectra. It is proved that the algebra $C_A(H)$ is a free module over its centre and an explicit basis is given (Proposition 3.6). One of the important moments is a realization of the algebras $C_A(H)$ and $C^{\lambda,\mu}$ as algebras of differential operators (Proposition 3.7). The algebras $C^{\lambda,\mu}$ are simple iff $\lambda \neq 0$ (Theorem 4.5 and Proposition 4.9). For every $\lambda \neq 0$, the algebra $C^{\lambda,\mu}$ is a subalgebra of the first Weyl algebra A_1'. Theorem 4.8 classifies all simple $C^{\lambda,\mu}$-modules, it shows that the algebra $C^{\lambda,\mu}$ has *exactly one more* simple module than the Weyl algebra A_1'. A similar result holds for the algebras $C^{0,\mu}$ (Theorem 4.11) but the Weyl algebra A_1' is replaced by the skew polynomial algebra $R = \mathbb{K}[h][t;\sigma]$ where $\sigma(h) = h - 1$. In this case, *all* simple t-torsionfree R-modules are *also* simple t-torsionfree $C^{0,\mu}$-modules, and vice versa (Theorem 4.11.(2)).

In Section 5, the prime, completely prime, maximal and primitive spectra of the algebra $C_A(H)$ are classified together with inclusions of primes (Theorem 5.3, Corollary 5.4, Corollary 5.5 and Theorem 5.6, respectively).

2. The prime ideals of A

The aim of this section is to describe the prime ideals of the algebra A (Theorem 2.8). As a result, the sets of maximal, primitive, completely prime and prime characteristic ideals are described (Corollary 2.9, Theorem 2.10, Corollary 2.11 and Corollary 2.12, respectively). An explicit classification of prime ideals that are invariant under all automorphisms of the algebra A is given (Corollary 2.12).

The n-th *Weyl algebra* $A_n = A_n(\mathbb{K})$ is an associative algebra which is generated by elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the defining relations: $[x_i, x_j] = 0$, $[y_i, y_j] = 0$ and $[y_i, x_j] = \delta_{ij}$ where $[a, b] := ab - ba$ and δ_{ij} is the Kronecker delta function. The Weyl algebra A_n is a simple Noetherian domain of Gefand-Kirillov dimension $2n$. For an algebra R, we denote by $Z(R)$ its centre. For an element $r \in R$, we denote by (r) the ideal of R generated by the element r.

Lemma 2.1. *[15, Lemma 14.6.5] Let B be a $\mathbb{K}$-algebra, $S = B \otimes A_n$ be the tensor product of the algebra B and the Weyl algebra A_n, δ be a $\mathbb{K}$-derivation of S and $T = S[t;\delta]$. Then there exists an element $s \in S$ such that the algebra $T = B[t';\delta'] \otimes A_n$ is a tensor product of algebras where $t' = t + s$ and $\delta' = \delta + \mathrm{ad}_s$.*

Recall that the Lie algebra $\mathfrak{sl}_2 = \mathbb{K}F \oplus \mathbb{K}H \oplus \mathbb{K}E$ is a simple Lie algebra over $\mathbb{K}$ where the Lie bracket is given by the rule: $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. Let $V_2 = \mathbb{K}X \oplus \mathbb{K}Y$ be the 2-dimensional simple $\mathfrak{sl}_2$-module with basis X and Y: $H \cdot X = X$, $H \cdot Y = -Y$, $E \cdot X = 0$, $E \cdot Y = X$, $F \cdot X = Y$ and $F \cdot Y = 0$. Let $\mathfrak{a} := \mathfrak{sl}_2 \ltimes V_2$ be the semi-direct product of Lie algebras where V_2 is viewed as an abelian Lie algebra. In more detail, the Lie algebra $\mathfrak{a}$ admits the basis $\{H, E, F, X, Y\}$ and the Lie bracket is defined as follows

$$\begin{array}{lllll} [H,E]=2E, & [H,F]=-2F, & [E,F]=H, & [E,X]=0, & [E,Y]=X, \\ [F,X]=Y, & [F,Y]=0, & [H,X]=X, & [H,Y]=-Y, & [X,Y]=0. \end{array}$$

Recall that $A = U(\mathfrak{a})$ is the enveloping algebra of the Lie algebra $\mathfrak{a}$.

An involution $*$ of A. Let Λ be an algebra. An anti-isomorphism τ of the algebra Λ (i.e., $\tau(ab) = \tau(b)\tau(a)$ for all $a, b \in \Lambda$) is called an *involution* if $\tau^2 = \mathrm{id}_\Lambda$. The algebra A admits the following involution $*$:

$$F^* = -E, \quad H^* = H, \quad E^* = -F, \quad Y^* = X, \quad X^* = Y. \tag{1}$$

There is an automorphism S of the algebra A such that

$$S(F) = E, \quad S(H) = -H, \quad S(E) = F, \quad S(Y) = -X, \quad S(X) = -Y, \tag{2}$$

and $S^2 = \mathrm{id}_A$.

The 1-spatial ageing algebra $\mathcal{A}$. Let $\mathcal{A}$ be the subalgebra of A generated by the elements H, E, X and Y. The algebra $\mathcal{A}$ is called the *1-spatial ageing algebra* and is studied in details in [7]. A classification of simple weight modules over the 1-spatial ageing algebra was given in [14]. Let $\mathcal{A}_X$ be the localization of $\mathcal{A}$ at the powers of X. Let $\partial := HX^{-1} + EYX^{-2} \in \mathcal{A}_X$. Then $[\partial, X] = 1$ and so the subalgebra $A_1 := \mathbb{K}\langle \partial, X\rangle$ of $\mathcal{A}_X$ is the first Weyl algebra. The algebra $\mathcal{A}_X$ is a central simple algebra of Gelfand-Kirillov dimension 4 (see [7, Lemma 2.2.(1)]). In fact, $\mathcal{A}_X$ is a tensor product of two central simple algebras

$$\mathcal{A}_X = A_{1,X} \otimes A_1^+ \tag{3}$$

where $A_{1,X}$ is the localization of A_1 at the powers of X and $A_1^+ := \mathbb{K}\langle EX^{-1}, Y\rangle$ is the first Weyl algebra since $[EX^{-1}, Y] = 1$ (see [7, (4)]).

The centre of the algebra A. Using the defining relations of the algebra A, the algebra A is a skew polynomial algebra

$$A = \mathcal{A}[F; \sigma, \delta] \tag{4}$$

where σ is an automorphism of $\mathcal{A}$ such that $\sigma(H) = H+2, \sigma(E) = E, \sigma(Y) = Y$, $\sigma(X) = X$; and δ is a σ-derivation of the algebra $\mathcal{A}$ such that $\delta(H) = 0, \delta(E) = -H, \delta(Y) = 0$ and $\delta(X) = Y$. Then the localization A_X of A at the powers of X is a skew polynomial algebra

$$A_X = \mathcal{A}_X[F; \sigma, \delta] \tag{5}$$

where σ and δ are defined as in (4). The key idea of finding the centre of A is by 'deleting the automorphism' σ first and then using Lemma 2.1 'deleting the derivation'. In more detail, let $\Phi := FX^2$, then by (5) and (3),

$$A_X = \mathcal{A}_X[\Phi; \delta'] = (A_{1,X} \otimes A_1^+)[\Phi; \delta'] \tag{6}$$

is an Ore extension where δ' is a derivation of the algebra $\mathcal{A}_X$ given by the rule: $\delta'(\partial) = -2\partial YX$, $\delta'(X) = YX^2$, $\delta'(EX^{-1}) = -\partial X^2$ and $\delta'(Y) = 0$. The element $s = -\partial X^2 Y$ satisfies the conditions of Lemma 2.1. Specifically, the element $C := \Phi + s = FX^2 - HXY - EY^2$ commutes with the elements of A_1^+, moreover, the element C commutes with the elements of $A_{1,X}$ and hence,

$$A_X = \mathbb{K}[C] \otimes A_{1,X} \otimes A_1^+ = \mathbb{K}[C] \otimes \mathcal{A}_X \tag{7}$$

is a tensor product of algebras. By (7), the skew field $\mathrm{Frac}(A)$ is isomorphic to the skew field of fractions of the second Weyl algebra $A_2\big(\mathbb{K}(C)\big)$ over the field $\mathbb{K}(C)$ of rational functions. Moreover, $Z\big(\mathrm{Frac}(A)\big) = \mathbb{K}(C)$.

Lemma 2.2. *1. $Z(A) = Z(A_X) = \mathbb{K}[C]$ where $C = FX^2 - HXY - EY^2$.*

2. $S(C) = -C$ where S is the automorphism (2) of A.

Proof. 1. By (7), $Z(A_X) = Z(\mathbb{K}[C]) \otimes Z(A_{1,X}) \otimes Z(A_1^+) = \mathbb{K}[C]$. Since $\mathbb{K}[C] \subseteq Z(A) \subseteq A \cap Z(A_X) = \mathbb{K}[C]$, we have $Z(A) = \mathbb{K}[C]$.

2. Statement 2 is obvious. ∎

The result of Lemma 2.2.(1) was proved in Corollary 23 of [11], they use a different approach.

Lemma 2.3. *1. In the algebra A, $(X) = (Y) = AX + AY = XA + YA$.*

2. Let $U := U(\mathfrak{sl}_2)$. Then $A/(X) \simeq U$.

Proof. 1. The equality $(X) = (Y)$ follows from the equalities $FX - XF = Y$ and $EY - YE = X$. So, $(X) = (Y) = (X, Y)$. Let us show that $XA \subseteq AX + AY$ and $YA \subseteq AX + AY$. Recall that $A = \mathcal{A}[F; \sigma, \delta]$ (see (4)) and X is a normal element of $\mathcal{A}$, $XA = X\sum_{i\geqslant 0} \mathcal{A}F^i = \sum_{i\geqslant 0} \mathcal{A}XF^i = \mathcal{A}X + \sum_{i\geqslant 1} \mathcal{A}XF^i = \mathcal{A}X + \sum_{i\geqslant 1} \mathcal{A}(F^iX - iF^{i-1}Y) \subseteq AX + AY$. The second inclusion follows from the first one by applying the automorphism S (see (2)). So, $(X, Y) = AX + AY$. By applying the involution $*$ to this equality we obtain that $(X, Y) = XA + YA$.

2. By statement 1, $A/(X) = A/(X, Y) \simeq U$. ∎

Lemma 2.4. *For all $i \geqslant 1$, $(X^i) = (X)^i$.*

Proof. To prove the statement we use induction on i. The case $i = 1$ is obvious. Suppose that $i > 1$ and the equality $(X^j) = (X)^j$ holds for all $1 \leqslant j \leqslant i - 1$. By Lemma 2.3.(1), $AX \subseteq XA + YA$. It follows from the equality $FX^i = X^iF + iX^{i-1}Y$ that $X^{i-1}Y \in (X^i)$. Now, $(X)^i = (X)^{i-1}(X) = (X^{i-1})(X) = AX^{i-1}AXA \subseteq AX^{i-1}(XA + YA) \subseteq (X^i) + AX^{i-1}YA \subseteq (X^i)$. Therefore, $(X)^i = (X^i)$. ∎

Proposition 2.5. *Let $\mathfrak{q} \in \mathrm{Max}(\mathbb{K}[C]) \setminus \{(C)\}$. Then*

1. *The ideal $(\mathfrak{q}) := A\mathfrak{q}$ of A is a maximal, completely prime ideal.*
2. *The factor algebra $A/(\mathfrak{q})$ is a simple algebra.*

Proof. Notice that $\mathfrak{q} = \mathbb{K}[C]q$ where $q = q(C) \in \mathbb{K}[C]$ is an irreducible polynomial such that $q(0) \in \mathbb{K}^*$.

(i) *The factor algebra $A/(\mathfrak{q})$ is a simple algebra, i.e., $(\mathfrak{q})$ is a maximal ideal of A*: By (7), $A_X/(\mathfrak{q})_X \simeq L_\mathfrak{q} \otimes A_{1,X} \otimes A_1^+$ is a central simple algebra where $L_\mathfrak{q} := \mathbb{K}[C]/\mathfrak{q}$ is a finite field extension of $\mathbb{K}$. Hence, the algebra $A/(\mathfrak{q})$ is a simple algebra iff $(X^i, \mathfrak{q}) = A$ for all $i \geqslant 1$. By Lemma 2.4, $(X^i) = (X)^i$ for all $i \geqslant 1$. Therefore, $(X^i, \mathfrak{q}) = (X^i) + (\mathfrak{q}) = (X)^i + (\mathfrak{q})$ for all $i \geqslant 1$. It remains to show that $(X)^i + (\mathfrak{q}) = A$ for all $i \geqslant 1$. By Lemma 2.3.(1), $(X) = (X, Y)$. If $i = 1$ then $(X) + (\mathfrak{q}) = (X, Y, \mathfrak{q}) = \big(X, Y, q(0)\big) = A$, since $q(0) \in \mathbb{K}^*$. Now, $A = A^i = \big((X) + (\mathfrak{q})\big)^i \subseteq (X)^i + (\mathfrak{q}) \subseteq A$, i.e., $(X)^i + (\mathfrak{q}) = A$, as required.

(ii) *$(\mathfrak{q})$ is a completely prime ideal of A*: Since $A_X/(\mathfrak{q})_X \simeq L_\mathfrak{q} \otimes A_{1,X} \otimes A_1^+$ is a domain, the ideal $A \cap (\mathfrak{q})_X$ is a completely prime ideal of A. Now, it suffices to show that $(\mathfrak{q}) = A \cap (\mathfrak{q})_X$. But this is obvious since by statement (i), the ideal $(\mathfrak{q})$ is a maximal ideal of A.

(iii) $Z\big(A/(\mathfrak{q})\big) = L_\mathfrak{q}$: Since $L_\mathfrak{q} \subseteq Z\big(A/(\mathfrak{q})\big) \subseteq Z\big(A_X/(\mathfrak{q})_X\big) = L_\mathfrak{q}$, we have $Z\big(A/(\mathfrak{q})\big) = L_\mathfrak{q}$. ■

Proposition 2.6. *$A \cap (C)_X = (C)$ and the ideal (C) of A is a completely prime ideal.*

Proof. Recall that $A = \mathcal{A}[F; \sigma, \delta]$ (see (4)), X is a normal element in $\mathcal{A}$ and $C = X^2F + s'$ where $s' = -X(H-1)Y - EY^2$.
(i) *If $Xf \in (C)$ for some $f \in A$ then $f \in (C)$*: Notice that $Xf = Cg$ for some $g \in A$. To prove the statement (i) we use induction on the degree $m = \deg_F(f)$ of the element $f \in A$. Since A is a domain, $\deg_F(fg) = \deg_F(f) + \deg_F(g)$ for all $f, g \in A$. The case when $m \leqslant 0$ (i.e., $f \in \mathcal{A}$) is obvious since the equality $Xf = Cg$ holds iff $f = g = 0$ (since $\deg_F(Xf) \leqslant 0$ and $\deg_F(Cg) \geqslant 1$ providing $g \neq 0$). So, we may assume that $m \geqslant 1$. We can write the element f as a sum $f = f_0 + f_1F + \cdots + f_mF^m$ where $f_i \in \mathcal{A}$ and $f_m \neq 0$. The equality $Xf = Cg$ implies that $\deg_F(g) = \deg_F(Xf) - \deg_F(C) = m - 1$. Therefore, $g = g_0 + g_1F + \cdots + g_{m-1}F^{m-1}$ for some $g_i \in \mathcal{A}$ and $g_{m-1} \neq 0$. Then

$$\begin{aligned} &Xf_0 + Xf_1F + \cdots + Xf_mF^m = (X^2F + s')\big(g_0 + g_1F + \cdots + g_{m-1}F^{m-1}\big) \\ &= X^2\big(\sigma(g_0)F + \delta(g_0)\big) + X^2\big(\sigma(g_1)F + \delta(g_1)\big)F \cdots + X^2\big(\sigma(g_{m-1})F + \delta(g_{m-1})\big)F^{m-1} \\ &\qquad + s'g_1F + \cdots + s'g_{m-1}F^{m-1} \\ &= X^2\delta(g_0) + s'g_0 + \big(X^2\sigma(g_0) + X^2\delta(g_1) + s'g_1\big)F + \cdots + X^2\sigma(g_{m-1})F^m. \end{aligned}$$

Comparing the terms of degree zero we have the equality $Xf_0 = X^2\delta(g_0) + s'g_0 = X^2\delta(g_0) - \big(X(H-1)Y + EY^2\big)g_0$, i.e., $X\big(f_0 - X\delta(g_0) + (H-1)Yg_0\big) = -EY^2g_0$. All terms in the equality belong to the algebra $\mathcal{A}$. Since X is a normal element of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{A}X$ is a domain and the element EY^2 does not belong to the ideal $\mathcal{A}X$ (see [7]), we have $g_0 \in \mathcal{A}X$, i.e., $g_0 = Xh_0$ for some $h_0 \in \mathcal{A}$. Now, the element

g can be written as $g = Xh_0 + g'F$ where $g' = 0$ if $m = 1$, and $\deg_F(g') = m - 2$ if $m \geqslant 2$. Then $Xf = C(Xh_0 + g'F)$ and so $X(f - Ch_0) = Cg'F$. Notice that $Cg'F$ has zero constant term as a noncommutative polynomial in F (where the coefficients are written on the left). Therefore, the element $f - Ch_0$ has zero constant term, and hence can be written as $f - Ch_0 = f'F$ for some $f' \in A$ with $\deg_F(f') < \deg_F(f)$. Now, $Xf'F = Cg'F$, hence $Xf' = Cg' \in (C)$ (by deleting F). By induction, $f' \in (C)$, and then $f = Ch_0 + f'F \in (C)$, as required.

(ii) $A \cap (C)_X = (C)$: Let $u \in A \cap (C)_X$. Then $X^i u \in (C)$ for some $i \in \mathbb{N}$. By statement (i), $u \in (C)$.

(iii) *The ideal (C) of A is a completely prime ideal*: By (7), $A_X/(C)_X \simeq A_{1,X} \otimes A_1^+$ is a domain. By statement (ii), the algebra $A/(C)$ is a subalgebra of $A_X/(C)_X$, so $A/(C)$ is a domain. This means that the ideal (C) is a completely prime ideal of A. ∎

For an algebra R, let $\mathrm{Spec}\,(R)$ be the set of its prime ideals. The set $(\mathrm{Spec}\,(R), \subseteq)$ is a partially ordered set (poset) with respect to inclusion of prime ideals. Each element $r \in R$ determines two maps from R to R, $r\cdot : x \mapsto rx$ and $\cdot r : x \mapsto xr$ where $x \in R$. Recall that, for an element $r \in R$, we denote by (r) the ideal of R generated by the element r.

Proposition 2.7. ([6]) *Let R be a Noetherian ring and s be an element of R such that $\mathcal{S}_s := \{s^i \mid i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., s is a normal element such that $\ker(\cdot s_R) \subseteq \ker(s_R \cdot)$). Then $\mathrm{Spec}\,(R) = \mathrm{Spec}(R, s) \sqcup \mathrm{Spec}_s(R)$ where $\mathrm{Spec}(R, s) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) = \{\mathfrak{q} \in \mathrm{Spec}\,(R) \mid s \notin \mathfrak{q}\}$ and*

(a) the map $\mathrm{Spec}(R, s) \mapsto \mathrm{Spec}\,(R/(s)), \mathfrak{p} \mapsto \mathfrak{p}/(s)$, is a bijection with the inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ where $\pi : R \to R/(s), r \mapsto r + (s)$,

(b) the map $\mathrm{Spec}_s(R) \to \mathrm{Spec}\,(R_s)$, $\mathfrak{p} \mapsto \mathcal{S}_s^{-1}\mathfrak{p}$, is a bijection with the inverse $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$ where $\sigma : R \to R_s := \mathcal{S}_s^{-1}R$, $r \mapsto \frac{r}{1}$.

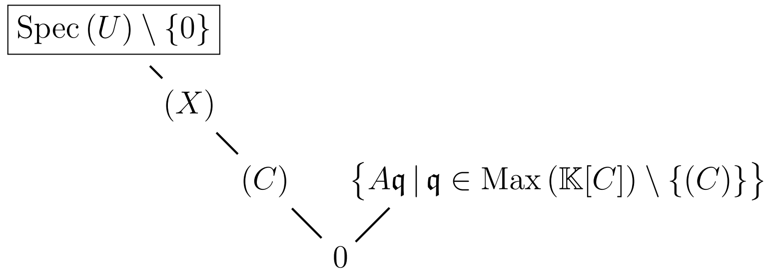
(c) For all $\mathfrak{p} \in \mathrm{Spec}\,(R, s)$ and $\mathfrak{q} \in \mathrm{Spec}_s(R), \mathfrak{p} \not\subseteq \mathfrak{q}$.

In this paper, we will identify the sets in the statements (a) and (b) above via the bijections given there. The next theorem gives an explicit description of the poset $(\mathrm{Spec}\,(A), \subseteq)$.

Theorem 2.8. *Let $U := U(\mathfrak{sl}_2)$. The prime spectrum of the algebra A is a disjoint union*

$$\mathrm{Spec}(A)=\mathrm{Spec}(U)\sqcup\mathrm{Spec}(A_X) = \{(X,\mathfrak{p})|\mathfrak{p}\in\mathrm{Spec}(U)\}\sqcup\{A\mathfrak{q}|\mathfrak{q}\in\mathrm{Spec}(\mathbb{K}[C])\}. \quad (8)$$

Furthermore,



(9)

Proof. By Lemma 2.3.(2), $A/(X) \simeq U$. By Lemma 2.4 and Proposition 2.7,

$$\operatorname{Spec}(A) = \operatorname{Spec}(A, X) \ \sqcup \ \operatorname{Spec}(A_X). \tag{10}$$

Therefore, $\operatorname{Spec}(A) = \{(X, \mathfrak{p}) \,|\, \mathfrak{p} \in \operatorname{Spec}(U)\} \sqcup \{A \cap A_X\mathfrak{q} \,|\, \mathfrak{q} \in \operatorname{Spec}(\mathbb{K}[C])\}$. By Proposition 2.5.(1), $A \cap A_X\mathfrak{q} = (\mathfrak{q})$ for all $\mathfrak{q} \in \operatorname{Max}(\mathbb{K}[C]) \setminus \{(C)\}$. By Proposition 2.6, $A \cap A_X C = (C)$. Therefore, (8) holds. For all $\mathfrak{q} \in \operatorname{Max}(\mathbb{K}[C]) \setminus \{(C)\}$, the ideals $A\mathfrak{q}$ of A are maximal. Notice that $(C) \subseteq (X)$. Therefore, (9) holds. ■

The next result is an explicit description of the set of maximal ideals of the algebra A.

Corollary 2.9. $\operatorname{Max}(A) = \operatorname{Max}(U) \ \sqcup \ \big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \operatorname{Max}(\mathbb{K}[C]) \setminus \{(C)\}\big\}$.

Proof. It is clear by (9). ■

An annihilator of a simple module is called a *primitive ideal*. Every primitive ideal is a prime ideal but the reverse does not hold, in general. The next theorem is a description of the set of primitive ideals of the algebra A.

Theorem 2.10. $\operatorname{Prim}(A) = \operatorname{Prim}(U) \ \sqcup \ \big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \operatorname{Spec}\mathbb{K}[C] \setminus \{0\}\big\}$.

Proof. Clearly, $\operatorname{Prim}(U) \subseteq \operatorname{Prim}(A)$ and $\{A\mathfrak{q} \,|\, \mathfrak{q} \in \operatorname{Max}(\mathbb{K}[C]) \setminus \{(C)\}\} \subseteq \operatorname{Prim}(A)$ since $A\mathfrak{q}$ is a maximal ideal (Corollary 2.9). The ideal (X) is not a primitive ideal, since the factor algebra $A/(X) \simeq U$ contains central elements. 0 is not a primitive ideal since the centre of A is non-trivial. In view of (9) it suffices to show that $(C) \in \operatorname{Prim}(A)$. The algebra A is a Jacobson algebra since it is a universal enveloping algebra of a finite dimensional Lie algebra [15, Corollary 9.1.8]. Therefore, any prime ideal of A is an intersection of primitive ideals lying over it. Clearly, $(X) = \bigcap_{(X) \subseteq P, P \in \operatorname{Spec}(U) \setminus \{0\}} P$. Since (C) is a prime ideal it must be primitive, by (9). ■

Recall that a prime ideal $\mathfrak{p}$ of an algebra A is called a *completely prime ideal* if the factor algebra $A/\mathfrak{p}$ is a domain. The next corollary is a description of the set $\operatorname{Spec}_c(A)$ of completely prime ideals of the algebra A.

Corollary 2.11. *The set* $\mathrm{Spec}_c(A)$ *of completely prime ideals of* A *is equal to*

$$\begin{aligned}\mathrm{Spec}_c(A) &= \mathrm{Spec}_c(U) \ \sqcup\ \{A\mathfrak{q}\,|\,\mathfrak{q}\in \mathrm{Spec}\,(\mathbb{K}[C])\}\\ &= \{(X,\mathfrak{p})\,|\,\mathfrak{p}\in\mathrm{Spec}\,(U),\ \mathfrak{p}\neq \mathrm{ann}_U(M) \text{ for some simple finite dimensional}\\ &\qquad U\text{-module } M \text{ of } \dim_{\mathbb{K}}(M)\geqslant 2\}\ \sqcup\ \{A\mathfrak{q}\,|\,\mathfrak{q}\in\mathrm{Spec}\,(\mathbb{K}[C])\}.\end{aligned}$$

Proof. The result follows from Proposition 2.5.(1) and Proposition 2.6. ∎

Let A be a $\mathbb{K}$-algebra and $\mathrm{Aut}_{\mathbb{K}}(A)$ be its group of automorphisms. An ideal $\mathfrak{a}$ of the algebra A is called a *characteristic ideal* if $\sigma(\mathfrak{a})=\mathfrak{a}$ for all $\sigma\in\mathrm{Aut}_{\mathbb{K}}(A)$. Let $\mathrm{Spec}_{ch}(A)$ be the set of prime characteristic ideals of A, the, so-called, *characteristic prime spectrum* of A.

For each element $(\lambda,\mu)\in(\mathbb{K}^*)^2$, there is an automorphism $t_{\lambda,\mu}$ of the algebra A given by the rule

$$t_{\lambda,\mu}: A\to A,\quad E\mapsto \lambda E,\ \ F\mapsto\lambda^{-1}F,\ \ H\mapsto H,\ \ X\mapsto \mu X,\ \ Y\mapsto \lambda^{-1}\mu Y. \quad (11)$$

Clearly, $t_{\lambda,\mu}t_{\lambda',\mu'}=t_{\lambda\lambda',\mu\mu'}$ and $t^{-1}_{\lambda,\mu}=t_{\lambda^{-1},\mu^{-1}}$. So, the 2-dimensional algebraic torus $\mathbb{T}^2:=\{t_{\lambda,\mu}\,|\,(\lambda,\mu)\in(\mathbb{K}^*)^2\}\simeq(\mathbb{K}^*)^2$ is a subgroup of $\mathrm{Aut}_{\mathbb{K}}(A)$. We note that $t_{\lambda,\mu}(C)=\lambda^{-1}\mu^2C$.

Corollary 2.12. *Let* $G:=\mathrm{Aut}_{\mathbb{K}}(A)$ *and* $\mathcal{Q}:=\{A\mathfrak{q}\,|\,\mathfrak{q}\in\mathrm{Max}\,(\mathbb{K}[C])\setminus\{(C)\}\}$. *Then the set* $\mathcal{Q}$ *is* G*-invariant and* $\mathrm{Spec}_{ch}(A)=\mathrm{Spec}(A)\setminus\mathcal{Q}$.

Proof. By (9), the ideals $0,(C)$ and (X) are characteristic. Then, for each $\sigma\in G$, $\sigma(C)=\lambda_\sigma C$ for some $\lambda_\sigma\in\mathbb{K}^*$. Hence, the subset $\mathcal{Q}$ of $\mathrm{Spec}(A)$ is G-invariant. Since (X) is a characteristic ideal of A, there is a group homomorphism

$$\mathrm{Aut}_{\mathbb{K}}(A)\to\mathrm{Aut}_{\mathbb{K}}(U),\quad \sigma\mapsto\bar\sigma: a+(X)\mapsto\sigma(a)+(X).$$

All ideals of U are characteristic ideals, [3]. To finish the proof notice that none of the ideals in $\mathcal{Q}$ is $\mathbb{T}^2$-invariant (since $t_{\lambda,\mu}(C)=\lambda^{-1}\mu^2C$).

∎

3. The centralizer $C_A(H)$ and its defining relations

The aim of this section is to find explicit generators and defining relations for the centralizer $C_A(H)$ of the element H in A (Theorem 3.4), to prove that the centre of the algebra $C_A(H)$ is a polynomial algebra $\mathbb{K}[C,H]$ (Theorem 3.4) and the algebra $C_A(H)$ is a free module over its centre (Proposition 3.6), to realize the algebra $C_A(H)$ as an algebra of differential operators and to prove various properties of the factor algebra $C^{\lambda,\mu}$ of $C_A(H)$. Results of this section is used in many proofs of the paper.

The centralizer of the element H**.** The subalgebra $\mathbb{E}$ of A is generated by the elements E,X and Y that satisfy the defining relations:

$$EY-YE=X,\quad EX=XE\quad \text{and } YX=XY.$$

So, the algebra $\mathbb{E}$ is isomorphic to the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. Recall that the 1-spatial ageing algebra $\mathcal{A}$ is the subalgebra of A generated by the elements H, E, X and Y. Then

$$\mathcal{A} = \mathbb{E}[H; \delta] \tag{12}$$

is an Ore extension where δ is the derivation of the algebra $\mathbb{E}$ such that $\delta(E) = 2E, \delta(X) = X$ and $\delta(Y) = -Y$. The localization $\mathcal{A}_X$ of the algebra $\mathcal{A}$ at the powers of the element X is an Ore extension

$$\mathcal{A}_X = \mathbb{E}_X[H; \delta] \tag{13}$$

where $\mathbb{E}_X$ is the localization of the algebra $\mathbb{E}$ at the powers of the element X and δ is the derivation of the algebra $\mathbb{E}_X$ given by the rule: $\delta(E) = 2E, \delta(X) = X$ and $\delta(Y) = -Y$.

For an algebra R and a non-empty subset $S \subseteq R$, $C_R(S) := \{r \in R \,|\, rs = sr \text{ for all } s \in S\}$ is the *centralizer* of S in R. The next lemma describes the structure of the algebras A_X and $C_{A_X}(H)$.

Lemma 3.1. *1. $C_{\mathcal{A}_X}(H) = \mathbb{K}[H] \otimes A_1'$ is a tensor product of algebras where $A_1' := \mathbb{K}\langle e, t\rangle$ is the (first) Weyl algebra with canonical generators $e := EX^{-2}$ and $t := XY$ (where $[e, t] = 1$).*

2. *$C_{A_X}(H) = \mathbb{K}[C, H] \otimes A_1'$ and $Z\big(C_{A_X}(H)\big) = \mathbb{K}[C, H]$.*

3. *$A_X = C_{A_X}(H)[X^{\pm 1}; \sigma]$ is a skew polynomial algebra where $\sigma(C) = C, \sigma(H) = H-1, \sigma(e) = e$ and $\sigma(t) = t$. In particular, the algebra $A_X = \mathbb{K}[C] \otimes A_1' \otimes B_1$ is a tensor product of algebras where $B_1 = \mathbb{K}[H][X^{\pm 1}; \sigma]$ is a central simple algebra and $\sigma(H) = H - 1$.*

Proof. 1. By (13), $\mathcal{A}_X = \mathbb{E}_X[H; \delta]$. So, $C_{\mathcal{A}_X}(H) = \mathbb{E}_X^\delta[H]$ where $\mathbb{E}_X^\delta = \{a \in \mathbb{E}_X \,|\, \delta(a) = 0\}$. Let us show that $\mathbb{E}_X^\delta = A_1'$. By the explicit nature of the derivation δ,

$$\mathbb{E}_X^\delta = \bigoplus_{i,k \in \mathbb{N}; j \in \mathbb{Z}} \{\mathbb{K}E^i X^j Y^k \,|\, \delta(E^i X^j Y^k) = 0\}.$$

Now, $\delta(E^i X^j Y^k) = (2i + j - k) E^i X^j Y^k = 0$, i.e., $j = k - 2i$. So, $E^i X^j Y^k = E^i X^{k-2i} Y^k = (EX^{-2})^i \cdot (XY)^k$. Therefore, $\mathbb{E}_X^\delta = A_1'$.

2. By (7), $A_X = \mathbb{K}[C] \otimes \mathcal{A}_X$. So, $C_{A_X}(H) = \mathbb{K}[C] \otimes C_{\mathcal{A}_X}(H) = \mathbb{K}[C, H] \otimes A_1'$, by statement 1. The Weyl algebra A_1' is a central algebra, hence $Z\big(C_{A_X}(H)\big) = \mathbb{K}[C, H]$.

3. Statement 3 follows from statement 2. ∎

Lemma 3.2. *Let $t := XY$. For $i \geqslant 1$, the following identities hold in the algebra A.*

1. *$F^i X^{2i} = FX^2\big(FX^2 + 2t\big)\big(FX^2 + 4t\big) \cdots \big(FX^2 + 2(i-1)t\big)$.*
2. *$E^i Y^{2i} = EY^2\big(EY^2 + 2t\big)\big(EY^2 + 4t\big) \cdots \big(EY^2 + 2(i-1)t\big)$.*

Proof. 1. We use induction on $i \geqslant 1$. The initial case when $i=1$ is obvious. So, let $i>1$ and suppose that the identity holds for all integers $<i$. Then

$$\begin{aligned} F^iX^{2i} &= F\cdot FX^2\big(FX^2+2t\big)\big(FX^2+4t\big)\cdots\big(FX^2+2(i-2)t\big)\cdot X^2\\ &= FX^2\big(FX^2+2t\big)\big(FX^2+4t\big)\cdots\big(FX^2+2(i-1)t\big)\end{aligned}$$

since $FX^2\cdot X^2 = X^2\cdot\big(FX^2+2t\big)$.

2. Statement 2 follows from statement 1 by applying the automorphism S, see (2). ■

The reader is referred to [1, 2, 4] for information about generalized Weyl algebras. The algebra U is a generalized Weyl algebra,

$$U \simeq \mathbb{K}[H,\Delta]\big(\sigma, a=\frac{1}{4}\big(\Delta - H(H+2)\big)\big) \tag{14}$$

where $\Delta := 4FE + H(H+2)$ is the Casimir element of the enveloping algebra U and σ is the automorphism of the algebra $\mathbb{K}[H,\Delta]$ defined by $\sigma(H)=H-2$ and $\sigma(\Delta)=\Delta$, [1]. In particular, U is a $\mathbb{Z}$-graded algebra $U=\bigoplus_{i\in\mathbb{Z}} Dv_i$ where $D:=\mathbb{K}[H,\Delta]=\mathbb{K}[H,FE]$, $v_i=E^i$ if $i\geqslant 1$, $v_0=1$ and $v_i=F^{|i|}$ if $i\leqslant -1$. The polynomial algebra $\mathbb{K}[X,Y]\subset A$ is also a $\mathbb{Z}$-graded algebra $\mathbb{K}[X,Y]=\bigoplus_{j\in\mathbb{Z}}\mathbb{K}[t]w_j$ where $t=XY$, $W_j=X^j$ if $j\geqslant 1$, $w_0=1$ and $w_j=Y^j$ if $j\leqslant -1$. Note that the algebra A is a $\mathbb{Z}$-graded algebra $A=\bigoplus_{i\in\mathbb{Z}} A_i$ where $A_i := \{a\in A \mid [H,a]=ia\}$. Clearly, $C_A(H)=A_0$. The following lemma gives the generators of the algebra $C_A(H)$.

Lemma 3.3. *The algebra*
$$C_A(H)=\mathbb{K}\langle H, FE, XY, FX^2, EY^2\rangle = \mathbb{K}\langle C, H, FE, XY, FX^2\rangle$$
is a Noetherian algebra.

Proof. Since $A=\oplus_{i\in\mathbb{Z}}A_i$ is a $\mathbb{Z}$-graded Noetherian algebra, the algebra $A_0 = C_A(H)$ is a Noetherian algebra. The algebra $A=U\otimes\mathbb{K}[X,Y]$ is a tensor product of vector spaces. Hence $A=\bigoplus_{i\in\mathbb{Z}} Dv_i\otimes\bigoplus_{j\in\mathbb{Z}}\mathbb{K}[t]w_j$ where v_i, t and w_j are as above. Using the relations $[E,t]=X^2$ and $[F,t]=Y^2$, we see that $A=\sum_{i,j\in\mathbb{Z}} D[t]v_iw_j$ where $D[t]=\bigoplus_{i\geqslant 0} Dt^i$ is a vector space. Notice that $A_k=\sum\{D[t]v_iw_j \mid i,j\in\mathbb{Z}; 2i+j=k\}$. In particular, $C_A(H)=$

$$A_0=\sum_{\substack{i,j\in\mathbb{Z};\\ 2i+j=0}} D[t]v_iw_j=\sum_{i\in\mathbb{Z}} D[t]v_iw_{-2i}=\sum_{i\geqslant 1} D[t]F^iX^{2i}+D[t]+\sum_{i\geqslant 1}D[t]E^iY^{2i}.$$

Now, using Lemma 3.2 and the equalities $[FX^2,t]=t^2$ and $[EY^2,t]=t^2$, we see that

$$C_A(H)=\sum_{i\geqslant 1} D[t]\big(FX^2\big)^i + D[t] + \sum_{i\geqslant 1} D[t]\big(EY^2\big)^i.$$

Hence, $C_A(H)=\mathbb{K}\langle H, FE, XY, FX^2, EY^2\rangle$. Since $C=FX^2-HXY-EY^2$, the second equality in the lemma follows. ■

The next theorem describes defining relations of the algebra $C_A(H)$ and shows that its centre is a polynomial algebra $\mathbb{K}[H,C]$.

Theorem 3.4. *Let $\Phi := FX^2$ and $\Theta := FE$. Then the algebra $C_A(H)$ is of Gelfand-Kirillov dimension 4 and generated by the elements C, H, t, Φ and Θ subject to the following defining relations (where C and H are central in the algebra $C_A(H)$):*

$$[\Phi, t] = t^2, \tag{15}$$
$$[\Theta, t] = 2\Phi - (H+2)t - C, \tag{16}$$
$$[\Theta, \Phi] = 2\Theta t + H\Phi, \tag{17}$$
$$\Theta t^2 = \Phi(\Phi - Ht - C). \tag{18}$$

Furthermore, $Z\big(C_A(H)\big) = \mathbb{K}[C,H]$.

Proof. (i) *Generators of $C_A(H)$*: By Lemma 3.3, the algebra $C_A(H)$ is generated by the elements C, H, t, Φ and Θ. It is clear that C and H are central in $C_A(H)$ and the elements satisfy the relations (15)–(18). It remains to show that these are defining relations.

(ii) $\mathrm{GK}\big(C_A(H)\big) = 4$: Let $\mathcal{D}$ be the subalgebra of $C_A(H)$ generated by the elements C, H, t and Φ. Then $\mathcal{D} = \mathbb{K}[C,H] \otimes \mathbb{K}[t][\Phi;\delta]$ is a tensor product of algebras where δ is the $\mathbb{K}$-derivation of the algebra $\mathbb{K}[t]$ defined by $\delta(t) = t^2$. Clearly, $\mathcal{D}$ is a Noetherian domain of Gelfand-Kirillov dimension 4. Now, the inclusions $\mathcal{D} \subseteq C_A(H) \subseteq C_{A_X}(H)$ yield the inequalities $4 = \mathrm{GK}(\mathcal{D}) \leqslant \mathrm{GK}\big(C_A(H)\big) \leqslant \mathrm{GK}\big(C_{A_X}(H)\big) = 4$ (see Lemma 3.1.(2)). Hence, $\mathrm{GK}\big(C_A(H)\big) = 4$. Notice that t is a normal element of the algebra $\mathcal{D}$ and $\mathcal{D}/(t) \simeq \mathbb{K}[C,H,\Phi]$ is a domain. In particular, (t) is a completely prime ideal of the algebra $\mathcal{D}$.

Let $\mathcal{C}$ be the $\mathbb{K}$-algebra generated by the symbols C, H, t, Φ and Θ subject to the defining relations (15)–(18) with C and H central in $\mathcal{C}$.

(iii) $\mathrm{GK}(\mathcal{C}) = 4$:There is a natural epimorphism of algebras $f{:}\mathcal{C} \twoheadrightarrow C_A(H)$. Our aim is to show that f is an algebra isomorphism. Let $\mathcal{C}_t$ be the localization of $\mathcal{C}$ at the powers of the element t. Then by (18), we see that $\mathcal{C}_t \simeq \mathcal{D}_t = \mathbb{K}[C,H] \otimes \mathbb{K}[t^{\pm 1}][\Phi;\delta]$ where $\mathcal{D} = \mathbb{K}[C,H] \otimes \mathbb{K}[t][\Phi;\delta]$ is a subalgebra of $\mathcal{C}$. Hence, $\mathrm{GK}(\mathcal{C}_t) = 4$. Now, the inclusions $\mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{C}_t$ yield that $4 = \mathrm{GK}(\mathcal{D}) \leqslant \mathrm{GK}(\mathcal{C}) \leqslant \mathrm{GK}(\mathcal{C}_t) = 4$. Hence, $\mathrm{GK}(\mathcal{C}) = 4$.

(iv) *The algebra $\mathcal{C}$ is a domain*: Let $\mathcal{E}$ be the algebra generated by the symbols C, H, t, Φ and Θ subject to the defining relations (15)–(17) with C and H central in $\mathcal{E}$. Then $\mathcal{E}$ is an Ore extension

$$\mathcal{E} = \mathbb{K}[C,H,t][\Phi;\delta][\Theta;\sigma,\delta'] = \mathcal{D}[\Theta;\sigma,\delta'] \tag{19}$$

where σ is the automorphism of the algebra $\mathcal{D}$ defined by $\sigma(C) = C, \sigma(H) = H, \sigma(t) = t$ and $\sigma(\Phi) = \Phi + 2t$; δ' is the σ-derivation of the algebra $\mathcal{D}$ given by the rule: $\delta'(C) = \delta'(H) = 0$, $\delta'(t) = 2\Phi - (H+2)t - C$ and $\delta'(\Phi) = (H+4)\Phi - 2(H+2)t - 2C$. In particular, $\mathcal{E}$ is a Noetherian domain. Let

$Z := \Theta t^2 - \Phi(\Phi - Ht - C)$. Then Z is a central element of the algebra $\mathcal{E}$. Clearly, $\mathcal{C} \simeq \mathcal{E}/(Z)$. To prove that $\mathcal{C}$ is a domain, it suffices to show that the ideal (Z) of $\mathcal{E}$ is a completely prime ideal. Let $\mathcal{E}_t$ be the localization of the algebra $\mathcal{E}$ at the powers of the element t. Then $\mathcal{E}_t \simeq \mathbb{K}[C, H, Z, t^{\pm 1}][\Phi; \delta] = \mathbb{K}[C, H, Z] \otimes \mathbb{K}[t^{\pm 1}][\Phi; \delta]$ is a tensor product of algebras where δ is a derivation of the algebra $\mathbb{K}[t^{\pm 1}]$ such that $\delta(t) = t^2$. Hence, $\mathcal{E}_t/(Z)_t \simeq \mathbb{K}[C, H] \otimes \mathbb{K}[t^{\pm 1}][\Phi; \delta]$ is a domain.

Claim 1: If $tu \in (Z)$ for some $u \in \mathcal{E}$, then $u \in (Z)$.

Proof of Claim 1: Recall that $\mathcal{E} = \mathcal{D}[\Theta; \sigma, \delta']$ (see (19)), t is a normal element of the algebra $\mathcal{D}$, $Z = t^2\Theta + \xi$ is a central element of $\mathcal{E}$ where

$$\xi := (H + 4)t\Phi - (H + 2)t^2 - 2Ct + C\Phi - \Phi^2 \in \mathcal{D}.$$

Notice that $tu = Zv$ for some element $v \in \mathcal{E}$. To prove Claim 1, we use induction on the degree $m = \deg_\Theta(u)$ of the element $u \in \mathcal{E}$. Since $\mathcal{E}$ is a domain, $\deg_\Theta(fg) = \deg_\Theta(f) + \deg_\Theta(g)$ for all $f, g \in \mathcal{E}$. The case when $m \leqslant 0$, i.e., $u \in \mathcal{D}$ is obvious. So, we may assume that $m \geqslant 1$. The element u can be written as $u = u_0 + u_1\Theta + \cdots + u_m\Theta^m$ where $u_i \in \mathcal{D}$ and $u_m \neq 0$. The equality $tu = Zv$ implies that $\deg_\Theta(v) = m - 1$, since $\deg_\Theta(Z) = 1$. Therefore, $v = v_0 + v_1\Theta + \cdots + v_{m-1}\Theta^{m-1}$ for some $v_i \in \mathcal{D}$ and $v_{m-1} \neq 0$. Then

$$\begin{aligned} tu_0 + tu_1\Theta + \cdots + tu_m\Theta^m &= (t^2\Theta + \xi)(v_0 + v_1\Theta + \cdots + v_{m-1}\Theta^{m-1}) \\ = t^2\big(\sigma(v_0)\Theta + \delta'(v_0)\big) &+ t^2\big(\sigma(v_1)\Theta + \delta'(v_1)\big)\Theta + \cdots + t^2\big(\sigma(v_{m-1})\Theta + \delta'(v_{m-1})\big)\Theta^{m-1} \\ &+ \xi v_0 + \xi v_1\Theta + \cdots + \xi v_{m-1}\Theta^{m-1} \\ &= t^2\delta'(v_0) + \xi v_0 + \big(t^2\sigma(v_0) + t^2\delta'(v_1) + \xi v_1\big)\Theta + \cdots + t^2\sigma(v_{m-1})\Theta^m. \end{aligned}$$

Comparing the terms of degree zero we have the equality $tu_0 = t^2\delta'(v_0) + \xi v_0$, i.e.,

$$t\big(u_0 - t\delta'(v_0) - (H + 4)\Phi v_0 + (H + 2)tv_0 + 2Cv_0\big) = \Phi(C - \Phi)v_0.$$

All terms in the equality belong to the algebra $\mathcal{D}$. Since t is a normal element of the algebra $\mathcal{D}$ such that $\mathcal{D}/\mathcal{D}t \simeq \mathbb{K}[C, H, \Phi]$ is a domain and the elements Φ and $C - \Phi$ do not belong to the ideal $\mathcal{D}t$, we have $v_0 \in \mathcal{D}t$, i.e., $v_0 = tw_0$ for some $w_0 \in \mathcal{D}$. Now, the element v can be written as $v = tw_0 + v'\Theta$ where $v' = 0$ if $m = 1$, and $\deg_\Theta(v') = m - 2$ if $m \geqslant 2$. Then $tu = Z(tw_0 + v'\Theta)$ and so $t(u - Zw_0) = Zv'\Theta$. Hence, $u - Zw_0 = u'\Theta$ for some $u' \in \mathcal{E}$ with $\deg_\Theta(u') < \deg_\Theta(u)$. Now, $tu'\Theta = Zv'\Theta$, hence $tu' = Zv' \in (Z)$ (by deleting Θ). By induction, $u' \in (Z)$, and then $u = Zw_0 + u'\Theta \in (Z)$. This completes the proof of the Claim 1.

Claim 2: $\mathcal{E} \cap (Z)_t = (Z)$.

Proof of Claim 2: Clearly, $(Z) \subseteq \mathcal{E} \cap (Z)_t$. It remains to establish the reverse inclusion. Let $u \in \mathcal{E} \cap (Z)_t$. Then $t^i u \in (Z)$ for some $i \in \mathbb{N}$. Then by the Claim 1, $u \in (Z)$. Hence, $\mathcal{E} \cap (Z)_t = (Z)$.

By Claim 2, the algebra $\mathcal{E}/(Z)$ is a subalgebra of $\mathcal{E}_t/(Z)_t$. So, $\mathcal{E}/(Z)$ is a domain. In particular, the algebra $\mathcal{C} \simeq \mathcal{E}/(Z)$ is a Noetherian domain.

(v) $\mathcal{C} \simeq C_A(H)$: Since $\mathrm{GK}(\mathcal{C}) = \mathrm{GK}\big(C_A(H)\big) = 4$ and the algebra $\mathcal{C}$ is a domain. The algebra epimorphism $f : \mathcal{C} \twoheadrightarrow C_A(H)$ must be an isomorphism, i.e., $\mathcal{C} \simeq C_A(H)$, by [12, Proposition 3.15]. This means that the relations (15)–(18) are defining relations of the algebra $C_A(H)$ together with the condition that C and H are central elements. By Lemma 3.1.(2), $Z\big(C_A(H)\big) = \mathbb{K}[C,H]$. ■

The Weyl algebra $A_1' = \mathbb{K}[h][t,e;\sigma,a=h]$ is a GWA where $\sigma(h)=h-1$ and $h := et$. So, $A_1' = \oplus_{i\in\mathbb{Z}} A'_{1,i}$ is a $\mathbb{Z}$-graded algebra where $A'_{1,0} = \mathbb{K}[h]$ is a polynomial algebra in h and, for $i \geqslant 1$, $A'_{1,\pm i} = \mathbb{K}[h]v_{\pm i}$ where $v_i = t^i$ and $v_{-i} = e^i$. The algebra $C_{A_X}(H) = \oplus_{i\in\mathbb{Z}} C_{A_X}(H)_i$ is a $\mathbb{Z}$-graded algebra where $C_{A_X}(H)_i = \mathbb{K}[C,H]\otimes A'_{1,i}$.

By Lemma 3.1, the algebra $C_A(H)$ is a subalgebra of $C_{A_X}(H) = \mathbb{K}[C,H]\otimes A_1'$ where

$$\Phi = C + Ht + et^2 = C + (h+H)t, \tag{20}$$

$$\Theta = FE = FX^2\cdot EX^{-2} = \Phi e = Ce + (h+H)(h-1), \tag{21}$$

since $et = h$ and $te = h-1$.

In order to prove Proposition 3.6, we need to change the generators of the algebra (we replace Φ by $\phi = ht$).

Corollary 3.5. *Let $\phi := EY^2$. Then $\phi = et^2 = ht$ and the algebra $C_A(H)$ is generated by the elements C, H, t, ϕ and Θ subject to the defining relations*

$$[\phi, t] = t^2, \tag{22}$$

$$[\Theta, t] = 2\phi + (H-2)t + C, \tag{23}$$

$$[\Theta, \phi] = 2\Theta t + (-\phi + 2t)H, \tag{24}$$

$$\Theta t^2 = (\phi + Ht + C)\phi. \tag{25}$$

Proof. Since $\phi = EX^{-2}X^2Y^2 = et^2 = ht = \Phi - C - Ht$, the algebra $C_A(H)$ is generated by the elements C, H, t, ϕ and Θ. It is routine to check that the defining relations (15)–(18) can be written as (22)–(25), respectively. ■

By (21), for all $n \geqslant 1$, $\Theta^n = \sum_{i=0}^n \Theta_{n,i} e^i$ for some $\Theta_{n,i} \in \mathbb{K}[C,H,h]$ with $\deg_h \Theta_{n,i} = 2(n-i)$. Moreover, $\Theta_{n,n} = C^n$ and $\Theta_{n,0} = (h+H)^n(h-1)^n$. For all $n \geqslant 1$,

$$\phi^n = \phi_n t^n \quad \text{where } \phi_n := h(h-1)\cdots(h-n+1). \tag{26}$$

For all $i \geqslant 1$ and $j \geqslant 0$,

$$\Theta^i\phi^j = \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)e^s t^j = \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)(-s,j)v_{-s+j} = \sum_{s=0}^{i} P_{i,j,s}v_{-s+j}$$

where $(-s,j) = h(h+1)\cdots(h+s-1)$ for $1 \leqslant s \leqslant j$; $(-s,j) = \sigma^{-(s-j)}\big((-j,j)\big) = (h+s-1)\cdots(h+s-j)$ for all $s \geqslant j$; and $(0,j) := 1$; $P_{i,j,s} \in \mathbb{K}[C,H,h]$ with

$$\deg_h P_{i,j,s} = 2(i-s) + j + \min(s,j) = (2i+j) - 2s + \min(s,j) \leqslant 2i+j$$

and $\deg_h P_{i,j,s} = 2i+j$ iff $s=0$. For all $i \geqslant 1$ and $j \geqslant 0$ we have

$$\begin{aligned}\Theta^i\phi^j t &= \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)e^s t^{j+1} = \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)(-s,j+1)v_{-s+j+1} \\ &= \sum_{s=0}^{i} Q_{i,j,s}v_{-s+j+1},\end{aligned}$$

where $Q_{i,j,s} \in \mathbb{K}[C,H,h]$ with

$$\deg_h Q_{i,j,s} = 2(i-s)+j+\min(s,j+1) = 2i+j-2s+\min(s,j+1) \leqslant 2i+j$$

and $\deg_h Q_{i,j,s} = 2i+j$ iff $s=0$.

Proposition 3.6. *The algebra $C_A(H)$ is a free module over its centre. Furthermore, the set $B(H) := \{\Theta^i\phi^j t^k, \phi^l t^m \mid i \geqslant 1, k=0,1$ and $j,l,m \in \mathbb{N}\}$ is a free basis of the $Z\big(C_A(H)\big)$-module $C_A(H)$.*

Proof. Let M be a free semigroup generated by the symbols Θ and ϕ, i.e., M is the set of all words in letters Θ and ϕ. Let a be an element of $C_A(H)$. By (22) and (23), the element a is a linear combination of the elements $m't^kC^lH^m$ where $m' \in M$ and $k,l,m \in \mathbb{N}$. By (24), the element a is a linear combination of the elements $\phi^i\Theta^j t^k C^l H^m$ where $i,j,k,l,m \in \mathbb{N}$. Using the induction on the degree $\deg_\Theta$ with respect to the variable Θ (i.e., $\deg_\Theta(\Theta)=1$ and $\deg_\Theta(\phi)=\deg_\Theta(t)=\deg_\Theta(C)=\deg_\Theta(H)=0$) and the relation (25), i.e., $\Theta t^2 = (\phi+Ht+C)\phi$, and the relations (22)–(24), it follows that the element a is a linear combination of the elements bC^lH^m where $b \in B(H)$.

To finish the proof of the proposition it suffices to show that the elements of the set $B(H)$ are $\mathbb{K}(C,H)$-linearly independent in the algebra $\mathbb{K}(C,H)\otimes A_1'$ (since $C_A(H) \subseteq \mathbb{K}[C,H]\otimes A_1' \subseteq \mathbb{K}(C,H)\otimes A_1'$ where $\mathbb{K}(C,H)$ is the field of fractions of the polynomial algebra $\mathbb{K}[C,H]$). Let $\mathcal{F} := \mathbb{K}(C,H)$. Then the algebra $\mathcal{F}\otimes A_1'$ is the Weyl algebra $A_1'(\mathcal{F})$ over the field $\mathcal{F}$. By (22) and the equality $\phi = ht$ (Corollary 3.5), the $\mathcal{F}$-subalgebra of $A_1'(\mathcal{F})$ generated by the elements t and ϕ is equal to $\mathcal{F}[t][\phi; t^2\frac{d}{dt}]$. Therefore, the elements $\{\phi^l t^m \mid l,m \in \mathbb{N}\}$ are $\mathcal{F}$-linearly independent.

Suppose that the elements of the set $B(H)$ are linearly dependent over the field $\mathcal{F}$. Fix a non-trivial linear combinations,

$$L := \sum_{i\geqslant 1, j\geqslant 0} \Theta^i\phi^j(\lambda_{ij}+\mu_{ij}t) + \sum_{k,l\geqslant 0} \gamma_{kl}\phi^k t^l$$

where $\lambda_{ij}, \mu_{ij}, \gamma_{kl} \in \mathcal{F}$. Then necessarily one of the elements $\lambda_{ij}+\mu_{ij}t$ is nonzero. We seek a contradiction. Let $N := \max\{2i+j \mid \lambda_{ij}+\mu_{ij}t \neq 0\}$. Then $N \geqslant 2$. Let $j_0 = \min\{j \mid 2i+j=N, \lambda_{ij}+\mu_{ij}t \neq 0\}$. Then either $\lambda_{i_0,j_0} \neq 0$ or $\mu_{i_0,j_0} \neq 0$ (or both) where $i_0 = \frac{1}{2}(N-j_0)$.

Notice that $L = \sum L_i v_i$ for some elements $L_i \in \mathcal{F}[h]$. Suppose that $\lambda_{i_0,j_0} \neq 0$. Then $L_{j_0} = \lambda_{i_0,j_0}P_{i_0,j_0,0}+\alpha$ where $\alpha \in \mathcal{F}[h]$ with $\deg_h \alpha < N$ (since

$\phi^k t^l = h(h-1)\cdots(h-k+1)t^{k+l}$ and $\deg_h h(h-1)\cdots(h-k+1) = k \leqslant k+l$, $\deg_h P_{i_0,j_0,0} = 2i_0 + j_0 = N > j_0$ as $i_0 \geqslant 1$). Therefore, $\lambda_{i_0,j_0} = 0$, a contradiction. Similarly, if $\mu_{i_0,j_0} \neq 0$. Then $L_{j_0+1} = \mu_{i_0,j_0} Q_{i_0,j_0,0} + \beta$ where $\beta \in \mathcal{F}[h]$ with $\deg_h \beta < N$ (since $\deg_h Q_{i_0,j_0,0} = 2i_0 + j_0 = N > j_0 + 1$ as $i_0 \geqslant 1$). Therefore, $\mu_{i_0,j_0} = 0$, a contradiction. The proof of the proposition is complete. ■

The algebras $C^{\lambda,\mu}$. For elements $\lambda, \mu \in \mathbb{K}$, let $C^{\lambda,\mu} := C_A^{\lambda,\mu}(H) := C_A(H)/(C-\lambda, H-\mu)$. By Theorem 3.4 and Corollary 3.5, the algebra $C^{\lambda,\mu}$ is generated by the images of the elements $\{\Phi, \Theta, t\}$ or $\{\phi, \Theta, t\}$ in $C^{\lambda,\mu}$. For simplicity reason, we denote by the same letters their images. By Lemma 3.1.(2),

$$C_{A_X}^{\lambda,\mu} := C_{A_X}^{\lambda,\mu}(H) := C_{A_X}(H)/(C-\lambda, H-\mu) \simeq A_1'.$$

So, there is a natural algebra homomorphism $C^{\lambda,\mu} \to C_{A_X}^{\lambda,\mu} = A_1'$. The following proposition shows that the homomorphism is a monomorphism.

Proposition 3.7. *Let $\lambda, \mu \in \mathbb{K}$. Then*

1. *The algebra $C^{\lambda,\mu}$ is generated by the elements ϕ, Θ and t subject to the defining relations*

$$[\phi, t] = t^2, \tag{27}$$
$$[\Theta, t] = 2\phi + (\mu-2)t + \lambda, \tag{28}$$
$$[\Theta, \phi] = 2\Theta t + (-\phi + 2t)\mu, \tag{29}$$
$$\Theta t^2 = (\phi + \mu t + \lambda)\phi. \tag{30}$$

2. *The set $B^{\lambda,\mu} = \{\Theta^i \phi^j t^k, \phi^l t^m \mid i \geqslant 1, k = 0,1 \text{ and } j,l,m \in \mathbb{N}\}$ is a $\mathbb{K}$-basis for the algebra $C^{\lambda,\mu}$.*

3. *The algebra homomorphism*

$$C^{\lambda,\mu} \longrightarrow C_{A_X}^{\lambda,\mu} = A_1', \quad t \mapsto t,\ \phi \mapsto ht,\ \Theta \mapsto \lambda e + (h+\mu)(h-1),$$

   is a monomorphism.

4. *The ideal $(C-\lambda, H-\mu)$ of the algebra $C_A(H)$ is equal to the intersection of $C_A(H)$ and the ideal $(C-\lambda, H-\mu)$ of the algebra $C_{A_X}(H)$.*

5. $\mathrm{GK}(C^{\lambda,\mu}) = 2$ *and* $Z(C^{\lambda,\mu}) = \mathbb{K}$.

Proof. 1. Statement 1 follows from Corollary 3.5.

2. and 3. By repeating the proof of Proposition 3.6 (where the elements C and H are replaced by λ and μ, respectively), we have that the elements of $B^{\lambda,\mu}$ span the vector space $C^{\lambda,\mu}$. Let $\overline{C}^{\lambda,\mu}$ be the image of the algebra $C^{\lambda,\mu}$ in A_1' and $\overline{B}^{\lambda,\mu}$ be the image of the set $B^{\lambda,\mu}$ in A_1'. The set $\overline{B}^{\lambda,\mu}$ spans $\overline{C}^{\lambda,\mu}$. By repeating the proof of Proposition 3.6 (where the elements C and H are replaced by λ and

μ, respectively), we have that the set $\overline{B}^{\lambda,\mu}$ is a $\mathbb{K}$-basis for the algebra $\overline{C}^{\lambda,\mu}$. Now, statements 2 and 3 follows.

4. Statement 4 follows from statement 3.

5. By statement 3, the subalgebra J of $C^{\lambda,\mu}$ generated by the elements t and ϕ is isomorphic to the algebra $\mathbb{K}[t][\phi; t^2\frac{d}{dt}]$. The inclusions $J \subseteq C^{\lambda,\mu} \subseteq A_1'$ yield the inequalities $2 = \mathrm{GK}(J) \leqslant \mathrm{GK}(C^{\lambda,\mu}) \leqslant \mathrm{GK}(A_1') = 2$, i.e., $\mathrm{GK}(C^{\lambda,\mu}) = 2$. Notice that the centralizers of the elements t and $\phi = ht$ in the Weyl algebra A_1' are $\mathbb{K}[t]$ and $\mathbb{K}[\phi]$, respectively. Therefore, $\mathbb{K} \subseteq Z(C^{\lambda,\mu}) = \mathbb{K}[t] \cap \mathbb{K}[\phi] = \mathbb{K}$, i.e., $Z(C^{\lambda,\mu}) = \mathbb{K}$. ∎

By Proposition 3.7.(1,3), we have the inclusions of algebras

$$C^{\lambda,\mu} \subset A_1' \subset A_{1,t}' = C_t^{\lambda,\mu} \tag{31}$$

where $A_{1,t}'$ and $C_t^{\lambda,\mu}$ are localizations of the algebras A_1' and $C^{\lambda,\mu}$ at the powers of the element t.

The Weyl algebra A_1' has a standard ascending filtration $\{A_{1,i}'\}_{i\in\mathbb{N}}$ by the total degree of the variables e and t ($\deg(e^it^j) = i + j$ for all $i, j \geqslant 0$). The associated graded algebra $\mathrm{gr}A_1'$ is a polynomial algebra $\mathbb{K}[e, t]$, by abusing the notation. The subalgebra $C^{\lambda,\mu}$ of A_1' has the induced filtration $\{C^{\lambda,\mu} \cap A_{1,i}'\}_{i\in\mathbb{N}}$. Therefore, the associated graded algebra $\mathrm{gr}(C^{\lambda,\mu})$ is a subalgebra of the polynomial algebra $\mathrm{gr}(A_1')$. The elements t, ϕ and Θ have total degrees $1, 3$ and 4, respectively; and their images in $\mathrm{gr}(C^{\lambda,\mu})$ are t, et^2 and e^2t^2, respectively.

Now, let us consider $C^{\lambda,\mu}$ as an abstract algebra and equip it with the degree filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ where $\deg(t) = 1, \deg(\phi) = 3$ and $\deg(\Theta) = 4$. By (27)–(30), the associated graded algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is a commutative algebra which is an epimorphic image of the factor algebra $\mathbb{K}[t, \phi, \Theta]/(\Theta t^2 - \phi^2)$. So, by abusing the notation, the algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is generated by (the images of) the elements t, ϕ and Θ that commute (see (27)–(29)) and satisfy the relation $\Theta t^2 = \phi^2$, see (30).

Lemma 3.8. *1. For all $i \in \mathbb{N}$, $\mathcal{F}_i = C^{\lambda,\mu} \cap A_{1,i}'$.*

2. $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu}) = \mathbb{K}[t, \phi, \Theta]/(\Theta t^2 - \phi^2)$, $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu}) = \mathrm{gr}(C^{\lambda,\mu}) \subset \mathrm{gr}(A_1') = \mathbb{K}[t, e]$ where $\phi = et^2$ and $\Theta = e^2t^2$ as elements of $\mathbb{K}[e, t]$.

3. The algebra $\mathrm{gr}(A_1')$ is not a finitely generated $\mathrm{gr}(C^{\lambda,\mu})$-module.

4. The algebra A_1' is not a finitely generated left/right $C^{\lambda,\mu}$-module.

Proof. 1. By Proposition 3.7.(2), the set $B^{\lambda,\mu}$ is a $\mathbb{K}$-basis of the algebra $C^{\lambda,\mu}$. We keep the notation as above. Since $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is a commutative algebra, each vector space $\mathcal{F}_n$ is a linear span of elements of $B^{\lambda,\mu}$ with degrees $\leqslant n$ ($\deg(\Theta^i\phi^jt^k) = 4i + 3j + k$ and $\deg(\phi^lt^m) = 3l + m$). Then, also each vector space $C^{\lambda,\mu} \cap A_{1,n}'$ is a linear space of elements of $B^{\lambda,\mu}$ with total degree $\leqslant n$ ($\deg(\Theta^i\phi^jt^k) = 4i + 3j + k$ and $\deg(\phi^lt^m) = 3l + m$). Therefore, $\mathcal{F}_n = C^{\lambda,\mu} \cap A_{1,n}'$ for all $n \geqslant 0$.

2. The set $B^{\lambda,\mu}$ is a $\mathbb{K}$-basis of the factor algebra $\Lambda = \mathbb{K}[t,\phi,\Theta]/(\Theta t^2-\phi^2)$. Therefore, the algebra epimorphism $\Lambda \to \mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is an isomorphism. Now, statement 2 follows from statement 1.

3. By statement 2, $\mathrm{gr}(C^{\lambda,\mu}) \subseteq D = \mathbb{K}[t,et]$. Since the polynomial algebra $\mathbb{K}[t,e]$ is not a finitely generated D-module, it is not a finitely generated $\mathrm{gr}(C^{\lambda,\mu})$-module.

4. Statement 4 follows from statement 3. ∎

Corollary 3.9. *Let M be a finitely generated $C^{\lambda,\mu}$-module. Then* $\mathrm{GK}(M) \in \{0,1,2\}$.

Proof. The statement follows from Lemma 3.8.(2). ∎

4. Classification of simple $C_A(H)$-modules

In this section, $\mathbb{K}$ is an algebraically closed field. In this section, a classification of simple $C_A(H)$-modules is given. This classification is used in a classification of simple weight A-modules which is obtained in Section 6. Two cases where the element C acts as zero or nonzero are very different cases, they are dealt with separately with different techniques. For an algebra A, we denote by $\widehat{A}$ the set of isomorphism classes of simple A-modules. If $\mathcal{P}$ is a property of simple modules which is invariant under isomorphisms of modules then $\widehat{A}(\mathcal{P})$ stands for the set of all isomorphism classes of simple A-modules that satisfy $\mathcal{P}$. Clearly,

$$\widehat{C_A(H)} = \bigsqcup_{\lambda,\mu\in\mathbb{K}} \widehat{C^{\lambda,\mu}}. \tag{32}$$

The simple $C^{\lambda,\mu}$-module $M^{\lambda,\mu}$ (where $\lambda \neq 0$). Suppose that $\lambda \neq 0$ and μ is arbitrary. By Proposition 3.7.(3), $C^{\lambda,\mu}$ is a subalgebra of the Weyl algebra A_1' where $\phi = ht$ and $\Theta = \lambda e + (h+\mu)(h-1)$. The A_1'-module $M := A_1'/A_1't = \mathbb{K}[e]\bar{1}$ is a free $\mathbb{K}[e]$-module of rank 1 where $\bar{1} := 1 + A_1't$. The A_1'-module M is simple and can be identified with the algebra $\mathbb{K}[e]$ as a vector space. Then the element t acts on M as $-\frac{d}{de}$. The concept of $\deg_e$ is well-defined for $M \simeq \mathbb{K}[e]$. Since $\Theta \cdot e^i\bar{1} = \lambda e^{i+1}\bar{1} + \cdots$ for all $i \geqslant 0$ (where the three dots denote a polynomial of degree $< i+1$) and t acts on M as $-\frac{d}{de}$, the $C^{\lambda,\mu}$-module M is *simple*. We denote it by $M^{\lambda,\mu}$.

Lemma 4.1. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then*

1. *The $C^{\lambda,\mu}$-module $M^{\lambda,\mu}$ is a simple module of* GK *dimension 1, $M^{\lambda,\mu} \simeq C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi)$ and the map $t : M^{\lambda,\mu} \to M^{\lambda,\mu}$, $m \mapsto tm$, is a surjection.*
2. *$C^{\lambda,\mu} = \mathbb{K}[\Theta] \oplus C^{\lambda,\mu}(t,\phi)$ and $C^{\lambda,\mu}(t,\phi) = \mathbb{K}[\Theta]\phi \oplus C^{\lambda,\mu}t$.*

Proof. It follows from Proposition 3.7.(3) that

$$1 = \mathrm{GK}_{\mathbb{K}[\Theta]}(M^{\lambda,\mu}) \leqslant \mathrm{GK}_{C^{\lambda,\mu}}(M^{\lambda,\mu}) \leqslant \mathrm{GK}_{A_1'}(M^{\lambda,\mu}) = 1,$$

i.e., $\mathrm{GK}_{C^{\lambda,\mu}}(M^{\lambda,\mu}) = 1$. The map $t\cdot = -\frac{d}{de} : M^{\lambda,\mu} \simeq \mathbb{K}[e] \to M^{\lambda,\mu} \simeq \mathbb{K}[e]$ is a surjection. Since $t\bar{1} = 0$ and $\phi\bar{1} = ht \cdot \bar{1} = 0$, there is a natural $C^{\lambda,\mu}$-module epimorphism $C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi) \longrightarrow M^{\lambda,\mu}$ which is necessarily an isomorphism, by Proposition 3.6. In particular, $C^{\lambda,\mu} = \mathbb{K}[\Theta] \oplus C^{\lambda,\mu}(t,\phi)$. Then $C^{\lambda,\mu}(t,\phi) = \mathbb{K}[\Theta]\phi \oplus C^{\lambda,\mu}t$ (by Proposition 3.6, (27) and (30)). ∎

The simple $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ (where $\lambda \neq 0$). By Lemma 4.1, there is a short exact sequence of $C^{\lambda,\mu}$-modules

$$0 \longrightarrow N^{\lambda,\mu} \longrightarrow C^{\lambda,\mu}/C^{\lambda,\mu}t \longrightarrow M^{\lambda,\mu} \longrightarrow 0 \tag{33}$$

where $N^{\lambda,\mu} := C^{\lambda,\mu}(t,\phi)/C^{\lambda,\mu}t = \mathbb{K}[\Theta]\phi\tilde{1}$ and $\tilde{1} = 1 + C^{\lambda,\mu}t$. Clearly, ${}_{\mathbb{K}[\Theta]}N^{\lambda,\mu} \simeq \mathbb{K}[\Theta]$ (Lemma 4.1.(2)), $t\phi\tilde{1} = 0$ and $(\phi+\lambda)\phi\tilde{1} = 0$ (by (30)).

Lemma 4.2. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then the $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ is a simple module of* GK *dimension 1, $N^{\lambda,\mu} \simeq C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi+\lambda)$ and the map $t\cdot : N^{\lambda,\mu} \to N^{\lambda,\mu}$, $n \mapsto tn$, is a surjection.*

Proof. Since ${}_{\mathbb{K}[\Theta]}N^{\lambda,\mu} \simeq \mathbb{K}[\Theta]$, the concept of $\deg_\Theta$ of the elements of $N^{\lambda,\mu}$ is well-defined ($\deg_\Theta(\Theta^i\phi\tilde{1}) := i$ for all $i \geqslant 0$). Let us show that, for all $n \geqslant 0$,

$$t \cdot \Theta^n\phi\tilde{1} = \lambda n\Theta^{n-1}\phi\tilde{1} + \cdots, \tag{34}$$

$$(\phi+\lambda) \cdot \Theta^n\phi\tilde{1} = -\lambda n(\mu+n-1)\Theta^{n-1}\phi\tilde{1} + \cdots, \tag{35}$$

where the three dots means a term of $\deg_\Theta < n-1$. We use induction on n. The case $n = 0$ was proved above ($t\phi\tilde{1} = 0$ and $(\phi+\lambda)\phi\tilde{1} = 0$). Suppose that $n > 0$ and the equalities are true for all $n' < n$. Then

$$\begin{aligned}
t \cdot \Theta^{n+1}\phi\tilde{1} &= \big([t,\Theta] + \Theta t\big)\Theta^n\phi\tilde{1} \\
&= -\big(2\phi + (\mu-2)t + \lambda\big)\Theta^n\phi\tilde{1} + \lambda n\Theta^n\phi\tilde{1} + \cdots \\
&= -\big(-\lambda + 2(\phi+\lambda)\big)\Theta^n\phi\tilde{1} + \lambda n\Theta^n\phi\tilde{1} + \cdots \\
&= \lambda(n+1)\Theta^n\phi\tilde{1} + \cdots, \\
(\phi+\lambda) \cdot \Theta^{n+1}\phi\tilde{1} &= \big([\phi+\lambda,\Theta] + \Theta(\phi+\lambda)\big)\Theta^n\phi\tilde{1} \\
&= -\big(2\Theta t + (-\phi+2t)\mu\big)\Theta^n\phi\tilde{1} - \lambda n(\mu+n-1)\Theta^n\phi\tilde{1} + \cdots \\
&= -\big(2\lambda n + \lambda\mu\big)\Theta^n\phi\tilde{1} - \lambda n(\mu+n-1)\Theta^n\phi\tilde{1} + \cdots \\
&= -\lambda(n+1)(\mu+n)\Theta^n\phi\tilde{1} + \cdots.
\end{aligned}$$

By (34), the $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ is simple. By (34) and (35), $\mathrm{GK}(N^{\lambda,\mu}) = 1$. By (34), the map $t\cdot : N^{\lambda,\mu} \to N^{\lambda,\mu}$ is a surjection. Finally, by Lemma 4.1.(2),

$$C^{\lambda,\mu} = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]\phi \oplus C^{\lambda,\mu}t = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta](\phi+\lambda) \oplus C^{\lambda,\mu}t.$$

Therefore, the canonical $C^{\lambda,\mu}$-module epimorphism $C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi+\lambda) \to N^{\lambda,\mu}$ must be an isomorphism. ∎

Corollary 4.3. *The map* $t\cdot : C^{\lambda,\mu}/C^{\lambda,\mu}t \to C^{\lambda,\mu}/C^{\lambda,\mu}t$, $a+C^{\lambda,\mu}t \mapsto ta+C^{\lambda,\mu}t$, *is a surjection provided* $\lambda \neq 0$.

Proof. By Lemma 4.1.(1) and Lemma 4.2, the maps $t\cdot : N^{\lambda,\mu} \to N^{\lambda,\mu}$ and $t\cdot : M^{\lambda,\mu} \to M^{\lambda,\mu}$ are surjections, hence so is the map $t\cdot$ in the lemma, in view of the short exact sequence (33). ■

Lemma 4.4. *Let* R *be a ring,* $s, r \in R$ *and*

$$s_{m,n} : R/Rr^n \to R/Rr^n,\ a + Rr^n \mapsto s^m a + Rr^n,\ \text{for } m, n \geqslant 1.$$

If the map $s_{1,1}$ *is a surjection then all the maps* $s_{m,n}$ *are surjections and* $R = s^m R + Rr^n$ *for all* $m, n \geqslant 1$.

Proof. If the map $s_{m,n}$ is a surjection then $R = s^m R + Rr^n$. For each $i \geqslant 1$, consider the map $s_i := s\cdot : Rr^i/Rr^{i+1} \to Rr^i/Rr^{i+1}$, $ar^i + Rr^{i+1} \mapsto sar^i + Rr^{i+1}$. In the commutative diagram

$$\begin{array}{ccc} R/Rr & \xrightarrow{s_{1,1}} & R/Rr \\ \downarrow{\cdot r^i} & & \downarrow{\cdot r^i} \\ Rr^i/Rr^{i+1} & \xrightarrow[s_i]{} & Rr^i/Rr^{i+1} \end{array}$$

the vertical maps are surjection. Since the map $s_{1,1}$ is surjective, the map s_i is also surjective. By considering the finite filtration of the abelian group R/Rr^n,

$$0 \subseteq Rr^{n-1}/Rr^n \subseteq Rr^{n-2}/Rr^n \subseteq \ldots \subseteq R/Rr^n,$$

we see that the map $s_{1,n}$ is a surjection. Then so is its powers $(s_{1,n})^m = s_{m,n}$. ■

Theorem 4.5. *For all* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$*, the algebra* $C^{\lambda,\mu}$ *is a central simple algebra of Gelfand-Kirillov dimension 2.*

Proof. In view of Proposition 3.7.(5), it remains to show that the algebra $C^{\lambda,\mu}$ is simple (where $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$). By Corollary 4.3 and Lemma 4.4 (where $s = r = t$), $C^{\lambda,\mu} = t^n C^{\lambda,\mu} + C^{\lambda,\mu} t^n$. In particular, $C^{\lambda,\mu} = (t^n)$ for all $n \geqslant 1$. Let $\mathfrak{a}$ be a nonzero ideal of the algebra $C^{\lambda,\mu}$. We have to show that $\mathfrak{a} = C^{\lambda,\mu}$. By (31), the algebra $C_t^{\lambda,\mu} = A'_{1,t}$ is a simple Noetherian algebra. Therefore, $t^n \in \mathfrak{a}$ for some $n \geqslant 1$, and so $\mathfrak{a} = C^{\lambda,\mu}$, as required. ■

Proposition 4.6. *Let* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$*. Then, for all nonzero elements* $a \in A'_1$*, the* $C^{\lambda,\mu}$*-module* $A'_1/A'_1 a$ *has finite length but the* $C^{\lambda,\mu}$*-module* A'_1 *has infinite length.*

Proof. The A'_1-module $M = A'_1/A'_1 a = A'_1 \bar{1}$ (where $\bar{1} = 1 + A'_1 a$) admits the standard filtration $\{M_i := A'_{1,i}\bar{1}\}$. Then $\dim(M_i) = e(M)i + s$ for all $i \gg 0$ where $e(M) \in \mathbb{N} \setminus \{0\}$ is the multiplicity of the A'_1-module M and $s \in \mathbb{Z}$. The algebra $C^{\lambda,\mu}$ is simple (since $\lambda \neq 0$). Hence, every simple $C^{\lambda,\mu}$-module has GK dimension

1. Then using a concept of multiplicity of a finitely generated $C^{\lambda,\mu}$-module (see Lemma 3.8.(2)), we must have that the $C^{\lambda,\mu}$-module M has finite length. By Lemma 3.8.(4), the $C^{\lambda,\mu}$-module A_1' has infinite length. ∎

Classification of simple $C^{\lambda,\mu}$-modules where $\lambda \neq 0$. The Weyl algebra A_1' is a subalgebra of the skew Laurent polynomial algebra $B = \mathbb{K}(h)[t, t^{-1}; \sigma]$ where $\sigma(h) = h - 1$. The algebra B is the localization $S^{-1}A_1'$ of the Weyl algebra A_1' at $S := \mathbb{K}[h] \setminus \{0\}$. The algebra B is a Euclidean ring with left and right division algorithms. The algebra B is a principle left and right ideal domain. Each simple B-module is isomorphic to B/Bb where b is an irreducible (indecomposable) element of B. B-modules B/Bb and B/Bc are isomorphic iff the elements b and c are *similar*, i.e., there exists an element $d \in B$ such that 1 is the greatest common right divisor of c and d, and bd is the least common left multiple.

Let $\alpha, \beta \in S = \mathbb{K}[h] \setminus \{0\}$. We write $\alpha < \beta$ if there are no roots λ and μ of the polynomials α and β, respectively, such that $\lambda - \mu \in \mathbb{N}$.

Definition. [4]. An element $b = e^m\beta_m + e^{m-1}\beta_{m-1} + \cdots + \beta_0$, where $m > 0$, $\beta_i \in \mathbb{K}[h]$ and $\beta_0, \beta_m \neq 0$, is called *normal* if $\beta_0 < \beta_m$ and $\beta_0 < h$.

The simple modules over the (first) Weyl algebra was classified by Block [8] and later using a different approach with a short proof by Bavula [2, 4]. For a simple A_1'-module M there are two options either $S^{-1}M = 0$ or $S^{-1}M \neq 0$. Accordingly, we say that the simple module is $\mathbb{K}[h]$-*torsion* or $\mathbb{K}[h]$-*torsionfree*, respectively.

Theorem 4.7. *[2, 4].* $\widehat{A_1'} = \widehat{A_1'}\,(\mathbb{K}[h]\text{-torsion}) \sqcup \widehat{A_1'}\,(\mathbb{K}[h]\text{-torsionfree})$ *where*

1. $\widehat{A_1'}\,(\mathbb{K}[h]\text{-torsion}) = \{A_1'/A_1't,\ A_1'/A_1'e,\ A_1'/A_1'(h - \lambda_{\mathcal{O}}) \,|\, \mathcal{O} \in \mathbb{K}/\mathbb{Z} \setminus \{\mathbb{Z}\}\}$ *where $\lambda_{\mathcal{O}}$ is any fixed element of $\mathcal{O} = \lambda_{\mathcal{O}} + \mathbb{Z}$.*

2. *Each simple $\mathbb{K}[h]$-torsionfree A_1'-module is isomorphic to $M_b := A_1'/A_1' \cap Bb$ for a normal, irreducible element b. Simple A_1'-modules M_b and $M_{b'}$ are isomorphic iff the elements b and b' are similar.*

The following theorem gives a classification of simple $C^{\lambda,\mu}$-modules where $\lambda \neq 0$. It shows that there is a tight connection between the sets of simple $C^{\lambda,\mu}$-modules and A_1'-modules. The theorem gives an explicit construction for each simple $C^{\lambda,\mu}$-module as a factor module $C^{\lambda,\mu}/I$ where I is a left maximal ideal of $C^{\lambda,\mu}$. Let M be an A-module. The sum of all simple submodules of the A-module M is called the *socle* of M, denoted by $\mathrm{soc}_A(M)$. A submodule M' of M is called *essential* if its intersection with any nonzero submodule of M is nonzero. For $C^{\lambda,\mu}$-module M, we denote by $l_{C^{\lambda,\mu}}(M)$ its *length*.

Theorem 4.8. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then*

1. *The map* $\mathrm{soc} = \mathrm{soc}_{C^{\lambda,\mu}} : \widehat{A_1'} \longrightarrow \widehat{C^{\lambda,\mu}},\ [M] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(M)]$, *is an injection, and* $\widehat{C^{\lambda,\mu}} = \mathrm{soc}(\widehat{A_1'}) \sqcup \{N^{\lambda,\mu}\}$. *Furthermore,*

(a) the map $\mathrm{soc}^{tf} : \widehat{A_1'}\,(t\text{-torsionfree}) \longrightarrow \widehat{C^{\lambda,\mu}}\,(t\text{-torsionfree})$, $[M] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(M)]$*, is a bijection, but*

(b) the map $\mathrm{soc}^{tt} : \widehat{A_1'}\,(t\text{-torsion}) = \{A_1'/A_1't\} \longrightarrow \widehat{C^{\lambda,\mu}}\,(t\text{-torsion}) = \{M^{\lambda,\mu}, N^{\lambda,\mu}\}$, $[A_1'/A_1't] \mapsto [M^{\lambda,\mu}]$*, is an injection which is not a bijection. In particular, the simple* $C^{\lambda,\mu}$*-modules* $M^{\lambda,\mu}$ *and* $N^{\lambda,\mu}$ *are not isomorphic and the short exact sequence (33) splits.*

2. *For each* $[M] \in \widehat{A_1'}\,(\mathbb{K}[h]\text{-torsion})$*, the* $C^{\lambda,\mu}$*-module* M *is simple, i.e.,* $\mathrm{soc}_{C^{\lambda,\mu}}(M) = M$*.*

3. *For each* $[M] \in \widehat{A_1'}\,(\mathbb{K}[h]\text{-torsionfree})$*, i.e.,* $M{=}M_b{=}A_1'/A_1' \cap Bb$ *where* $b{\in}B$ *is as in Theorem 4.7.(2),* $N_b := C^{\lambda,\mu}/C^{\lambda,\mu} \cap Bb \subseteq M_b$ *and* $\mathrm{soc}_{C^{\lambda,\mu}}(M_b) = \mathrm{soc}_{C^{\lambda,\mu}}(N_b) \simeq N_{bt^{-n}}$ *for all* $n \gg 0$*.*

Proof. 1. Let M be a simple A_1'-module. By Proposition 4.6, the $C^{\lambda,\mu}$-module M has finite length. In particular, $\mathrm{soc}_{C^{\lambda,\mu}}(M) \neq 0$. *Let us show that* $\mathrm{soc}_{C^{\lambda,\mu}}(M)$ *is a simple* $C^{\lambda,\mu}$*-module.* Let M_t be the localization of the A_1'-module M at the powers of the element t. If $M_t = 0$, i.e., $M \simeq A_1'/A_1't$, then $\mathrm{soc}_{C^{\lambda,\mu}}(A_1'/A_1't) = A_1'/A_1't$ since the $C^{\lambda,\mu}$-module $A_1'/A_1't = M^{\lambda,\mu}$ is simple, as we have seen above. If $M_t \neq 0$, then the A_1'-module M is an essential submodule of M_t. By (31), the $C^{\lambda,\mu}$-module M is also an essential $C^{\lambda,\mu}$-submodule of M_t. Therefore, $\mathrm{soc}_{C^{\lambda,\mu}}(M)$ is a simple $C^{\lambda,\mu}$-module. This implies that the map soc is an injection (if simple A_1'-modules M and M' are isomorphic they are also isomorphic as $C^{\lambda,\mu}$-module, and so $\mathrm{soc}_{C^{\lambda,\mu}}(M) \simeq \mathrm{soc}_{C^{\lambda,\mu}}(M')$).

(a) *The map* soc^{tf} *is a bijection*: It remains to show that the map soc^{tf} is a surjection. Let N be a simple t-torsionfree $C^{\lambda,\mu}$-module. Then N_t is a simple $C_t^{\lambda,\mu}$-module which is automatically a simple $A_{1,t}'$-module, by (31). Then $N = \mathrm{soc}_{C^{\lambda,\mu}}(N_t) \subseteq M := \mathrm{soc}_{A_1'}(N_t) \subseteq N_t$ and $M \neq 0$ (by Proposition 4.6), and therefore M is a simple t-torsionfree A_1'-module (since M is an essential A_1'-submodule of $M_t = N_t$). Now, $N = \mathrm{soc}_{C^{\lambda,\mu}}(M)$, as we have seen above. So, the map soc^{tf} is a bijection.

(b) The A_1'-module $A_1'/A_1't$ is a simple t-torsion A_1'-module. Hence, $\widehat{A_1'}\,(t\text{-torsion}) = \{A_1'/A_1't\}$. Let us show that $\widehat{C^{\lambda,\mu}}\,(t\text{-torsion}) = \{M^{\lambda,\mu}, N^{\lambda,\mu}\}$. By Lemma 4.1.(1), the $C^{\lambda,\mu}$-module $M^{\lambda,\mu} = A_1'/A_1't$ is simple and t-torsion. By Lemma 4.2 and (34), the $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ is simple and t-torsion. The $C^{\lambda,\mu}$-modules $M^{\lambda,\mu}$ and $N^{\lambda,\mu}$ are not isomorphic since $M^{\lambda,\mu} = \cup_{n\geqslant 1}\ker(\phi^n\cdot)$ and $N^{\lambda,\mu} = \cup_{n\geqslant 1}\ker((\phi+\lambda)^n\cdot)$ (by (35)). Let M be a simple t-torsion $C^{\lambda,\mu}$-module. It remains to show that either $M \simeq M^{\lambda,\mu}$ or $M \simeq N^{\lambda,\mu}$. The $C^{\lambda,\mu}$-module M is an epimorphic image of the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/C^{\lambda,\mu}t$. By (33), either $M \simeq M^{\lambda,\mu}$ or, otherwise, $M \simeq N^{\lambda,\mu}$. Since both cases do occur the short exact sequence (33) splits otherwise the only first case ($M \simeq M^{\lambda,\mu}$) would occur. Now, the statement (b) is obvious. Then, $\widehat{C^{\lambda,\mu}} = \mathrm{soc}(\widehat{A_1'}) \sqcup \{N^{\lambda,\mu}\}$.

2. By Theorem 4.7.(1), there are 3 cases to consider. The first case, i.e., $A_1'/A_1't = M^{\lambda,\mu}$, is obvious. Let $M = A_1'/A_1'e$. Then $M = \mathbb{K}[t]\bar{1}$ where $\bar{1} := 1 + A_1'e$. If N is a nonzero $C^{\lambda,\mu}$-module then necessarily $N = f\mathbb{K}[t]\bar{1}$ for

some nonzero polynomial $f \in \mathbb{K}[t]$. Since $\dim_{\mathbb{K}}(M/N) < \infty$ and the algebra $C^{\lambda,\mu}$ is a simple infinite dimensional algebra (Theorem 4.5) we must have $N = M$, i.e., the $C^{\lambda,\mu}$-module $A_1'/A_1'e$ is simple.

Finally, let $M = A_1'/A_1'(h-\nu)$ where $\nu = \nu_{\mathcal{O}} \notin \mathbb{Z}$. Then $M = \oplus_{i\in\mathbb{Z}}\mathbb{K}v_i\bar{1}$ where $\bar{1} := 1 + A_1'(h-\nu)$, $v_0 := 1$ and, for all $i \geqslant 1$, $v_i = t^i$ and $v_{-i} = e^i$. For all $i \geqslant 1$ and $j \in \mathbb{Z}$, $t^i v_j \bar{1} = \lambda_{ij} v_{i+j}$ for some $\lambda_{ij} \in \mathbb{K}^*$. Therefore, the element t acts as a bijection on the module M and $M = \mathbb{K}[t,t^{-1}]\bar{1} \simeq \mathbb{K}[t,t^{-1}]$, as $\mathbb{K}[t,t^{-1}]$-modules. If N is a nonzero submodule of M then $N = gM$ for some nonzero element g of $\mathbb{K}[t,t^{-1}]$. Since $\dim_{\mathbb{K}}(M/N) < \infty$ and the algebra $C^{\lambda,\mu}$ is a simple infinite dimensional algebra (Theorem 4.5) we must have $N = M$. This means that the $C^{\lambda,\mu}$-module M is simple.

3. Let $M = M_b = A_1'\bar{1}$ where $\bar{1} = 1 + A_1' \cap Bb$. Recall that the $C^{\lambda,\mu}$-module M has finite length and let N be a simple $C^{\lambda,\mu}$-submodule of M (statement 1). Since $h = et$, the A_1'-module M is t-torsionfree, and so $0 \neq N_t \subseteq M_t$. The $C_t^{\lambda,\mu}$-module N_t is also an $A_{1,t}'$-module since $C_t^{\lambda,\mu} = A_{1,t}'$ (see (31)). Therefore, $N_t = M_t$, since the $A_{1,t}'$-module M_t is simple and N_t is its nonzero $A_{1,t}'$-submodule. Notice that $N_t = M_t = A_{1,t}'/A_{1,t}' \cap Bb = C_t^{\lambda,\mu}/C_t^{\lambda,\mu} \cap Bb \supseteq C^{\lambda,\mu}/C^{\lambda,\mu} \cap Bb = N_b \neq 0$ and $(N_b)_t = M_t$. Hence, $N = \mathrm{soc}_{C^{\lambda,\mu}}(N_b)$. Clearly, $N_b = C^{\lambda,\mu}\tilde{1}$ where $\tilde{1} := 1 + C^{\lambda,\mu} \cap Bb$. For each $n \in \mathbb{N}$, let $N_n = C^{\lambda,\mu}t^n\tilde{1}$. Then $N_n \neq 0$ since $(N_n)_t = (N_b)_t \neq 0$. Since the $C^{\lambda,\mu}$-module N_b has finite length the descending chain of $C^{\lambda,\mu}$-submodules of N_b, $N_b = N_0 \supseteq N_1 \supseteq \cdots$, stabilizes, say, at m'th step, i.e., $N_b = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_m = N_{m+1} = \ldots$ and $N_m \neq 0$. Since $(M/N)_t = M_t/N_t = M_t/M_t = 0$, we must have $t^n\bar{1} \in N$, i.e., $N = N_n$ for some n. Then necessarily $n \geqslant m$ and $N_m = N$. Now, for all $n \geqslant m$,

$$N = C^{\lambda,\mu}t^n\tilde{1} \simeq \frac{C^{\lambda,\mu}t^n + C^{\lambda,\mu} \cap Bb}{C^{\lambda,\mu} \cap Bb} \simeq \frac{C^{\lambda,\mu}t^n}{C^{\lambda,\mu}t^n \cap Bb} \simeq C^{\lambda,\mu}/C^{\lambda,\mu} \cap Bbt^{-n}. \qquad \blacksquare$$

The algebras $C^{0,\mu}$. The subalgebra R of the Weyl algebra A_1' which is generated by the elements t and $h = et$ is a skew polynomial algebra $R = \mathbb{K}[h][t;\sigma]$ where $\sigma(h) = h - 1$. The algebra R is a homogeneous subalgebra of the $\mathbb{Z}$-graded algebra A_1', it is the non-negative part of the $\mathbb{Z}$-grading of A_1'. By Proposition 3.7.(3), for all $\mu \in \mathbb{K}$, $C^{0,\mu} \subset R \subset A_1'$ and the subalgebra $C^{0,\mu}$ of R is generated by the elements $t, \phi = ht$ and $\Theta = (h+\mu)(h-1)$. Clearly, $\mathbb{K}[\Theta] \subseteq \mathbb{K}[h]$ and $\mathbb{K}[h] = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]h$. The element t is a normal element of the algebra R and $(t) = \oplus_{i\geqslant 1}\mathbb{K}[h]t^i$.

Proposition 4.9. *Let $\mu \in \mathbb{K}$.*

1. $C^{0,\mu} = \mathbb{K}[\Theta] \oplus \bigoplus_{i\geqslant 1}\mathbb{K}[h]t^i$ *and* $C^{0,\mu} \cap Rt = Rt = \bigoplus_{i\geqslant 1}\mathbb{K}[h]t^i = (t,\phi) = (t)$ *where* (t,ϕ) *is the ideal of* $C^{0,\mu}$ *generated by the elements* t *and* ϕ. *Furthermore,* $(t,\phi) = C^{0,\mu}t + C^{0,\mu}\phi$.
2. $C^{0,\mu} \subset R \subset R_t = C_t^{0,\mu} = A_{1,t}'$.
3. $\mathrm{Spec}(C^{0,\mu}) = \{0, (t), (t,\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\mathbb{K}[\Theta])\}$, $C^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$, *and* $C^{0,\mu}/(t,\mathfrak{m}) \simeq \mathbb{K}[\Theta]/\mathfrak{m}$. *In particular, all prime ideals of* $C^{0,\mu}$ *are completely prime.*
4. $\mathrm{Max}(C^{0,\mu}) = \{(t,\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\mathbb{K}[\Theta])\}$.

Proof. 1. The equality $(t,\phi)=(t)$ follows from (28). Multiplying the equality $\mathbb{K}[h]=\mathbb{K}[\Theta]\oplus\mathbb{K}[\Theta]h$ by the element t on the right yields

$$\mathbb{K}[h]t=\mathbb{K}[\Theta]t\oplus\mathbb{K}[\Theta]\phi\subseteq C^{0,\mu}.$$

For all $i\geqslant 1$, $C^{0,\mu}\supseteq(\mathbb{K}[h]t)^i=\mathbb{K}[h]t^i$, and so $C^{0,\mu}=\mathbb{K}[\Theta]\oplus\bigoplus\mathbb{K}[h]t^i$ (since $C^{0,\mu}\subseteq R$). Then $C^{0,\mu}\cap Rt=Rt=\bigoplus_{i\geqslant 1}\mathbb{K}[h]t^i=(t,\phi)$. By Proposition 3.6, $(t,\phi)=C^{0,\mu}t+C^{0,\mu}\phi$.

2. Statement 2 follows from statement 1 and (31).

3. and 4. The ideal (t) of $C^{0,\mu}$ is a completely prime ideal since $C^{0,\mu}/(t)\simeq\mathbb{K}[\Theta]$, by statement 1. Therefore, the set of prime ideals that properly contain the ideal (t) is $\{(t,\mathfrak{m})\,|\,\mathfrak{m}\in\operatorname{Max}(\mathbb{K}[\Theta])\}$. Each such an ideal is a completely prime, maximal ideal of $C^{0,\mu}$. The algebra $C^{0,\mu}$ is a domain, so 0 is the completely prime ideal of $C^{0,\mu}$. To finish the proof of statements 3 and 4 it suffices to show that if $\mathfrak{p}$ is a nonzero prime ideal of $C^{0,\mu}$ then $(t)\subseteq\mathfrak{p}$. Recall that $Rt=(t)$, by statement 1. By statement 2, $C^{0,\mu}_t=A'_{1,t}$ is a simple Noetherian domain. Therefore, $t^n\in\mathfrak{p}$. Hence, $\mathfrak{p}\supseteq Rt^n\cdot t=Rt^{n+1}=(Rt)^{n+1}$, and so $\mathfrak{p}\supseteq Rt$ since $\mathfrak{p}$ is a prime ideal and $Rt=(t)$. ■

Classification of simple $C^{0,\mu}$-modules. The set $S=\mathbb{K}[h]\setminus\{0\}$ is a (left and right) Ore set of the domain $C^{0,\mu}$ and $B:=S^{-1}C^{0,\mu}=\mathbb{K}(h)[t;\sigma]$ is a skew polynomial algebra where $\sigma(h)=h-1$. The algebra B is a (left and right) principle ideal domain. Let $\operatorname{Irr}(B)$ be the set of irreducible elements of B.

In [5], simple modules for an arbitrary Ore extension $D[X;\sigma,\delta]$ are classified where D is a commutative Dedekind domain, σ is an automorphism of D and δ is a σ-derivation of D. The ring $R=\mathbb{K}[h][t;\sigma]$ is a very special case of such an Ore extension.

Theorem 4.10.

1. $\widehat{R}\,(\mathbb{K}[h]\text{-torsion})=\widehat{R}\,(t\text{-torsion})=\widehat{R/(t)}=\{[R/R(h-\nu,t)]\,|\,\nu\in\mathbb{K}\}$.
2. $\widehat{R}\,(\mathbb{K}[h]\text{-torsionfree})=\widehat{R}\,(t\text{-torsionfree})=\{[M_b]\,|\ b\in\operatorname{Irr}(B),\ R=Rt+R\cap Bb\}$, *where* $M_b:=R/R\cap Bb$; $M_b\simeq M_{b'}$ *iff the elements* b *and* b' *are similar (iff* $B/Bb\simeq B/Bb'$ *as* B*-modules).*

Proof. 1. The last two equalities in statement 1 follow from the fact that t is a normal element of R. Then, clearly, $\widehat{R}\,(\mathbb{K}[h]\text{-torsion})\supseteq\widehat{R/(t)}$. It remains to show that the reverse inclusion holds. Let M be a simple $\mathbb{K}[h]$-torsion R-module. The filed $\mathbb{K}$ is algebraically closed, so the R-module M is an epimorphic image of the R-module $R/R(h-\nu)=\mathbb{K}[t]\overline{1}$ for some $\nu\in\mathbb{K}$ where $\overline{1}=1+R(h-\nu)$. It follows from the equalities $ht^i\overline{1}=(\nu+i)t^i\overline{1}$ for all $i\geqslant 0$ that $t\mathbb{K}[t]\overline{1}$ is the only maximal R-submodule of $R/R(h-\nu)$. So, $M\simeq R/R(h-\nu,t)\in\widehat{R/(t)}$, as required.

2. The first equality in statement 1 implies (in fact, is equivalent to) the first equality in statement 2. By [5, Theorem 1.3]

$$\widehat{R}\,(\mathbb{K}[h]\text{-torsionfree})=\{[M_b]\,|\,b\in\operatorname{Irr}(B),\ R=Rt+R\cap Bb\}$$

(the condition (LO) of [5, Theorem 1.3] is equivalent to the condition $R=Rt+R\cap Bb$). ■

Theorem 4.11 gives a classification of simple $C^{0,\mu}$-modules. It shows a close connection between the sets $\widehat{C^{0,\mu}}$ and $\widehat{R}$, they are almost identical, i.e., $\widehat{C^{0,\mu}}\,(t\text{-torsionfree}) = \widehat{R}\,(t\text{-torsionfree})$.

Theorem 4.11.

1. $\widehat{C^{0,\mu}}\,(t\text{-torsion}) = \{[M] \in \widehat{C^{0,\mu}} \mid (t)M = 0\} = \widehat{C^{0,\mu}/(t)}$
$= \{[C^{0,\mu}/C^{0,\mu}(\Theta - \nu, t, \phi)] \mid \nu \in \mathbb{K}\}$.
2. $\widehat{C^{0,\mu}}\,(t\text{-torsionfree}) = \widehat{R}\,(t\text{-torsionfree}) = \widehat{R}\,(\mathbb{K}[h]\text{-torsionfree})$
$= \{[M_b = R/R \cap Bb] \mid b \in \mathrm{Irr}(B),\ R = Rt + R \cap Bb\}$ *(see Theorem 4.10).*

Proof. 1. The last two equalities are obvious. Clearly, $\widehat{C^{0,\mu}}\,(t\text{-torsion}) \supseteq \widehat{C^{0,\mu}/(t)}$. It remains to show that the reverse inclusion holds. If M is a simple t-torsion $C^{0,\mu}$-module that either $(t)M = 0$ or, otherwise, $(t)M = M$. The second case is impossible since otherwise, $M = (t)M = RtM \in \widehat{R}\,(t\text{-torsionfree})$, a contradiction ($t$ is a normal element of R). So, $(t)M = 0$, as required.
2. In view of Theorem 4.10.(2), it remains to show that the first equality holds. Let $[M] \in \widehat{C^{0,\mu}}\,(t\text{-torsionfree})$. By statement 1, $M = (t)M = RtM \in \widehat{R}\,(t\text{-torsionfree})$. Given $[N] \in \widehat{R}\,(t\text{-torsionfree})$. To finish the proof of statement 2, it suffices to show that N is a simple $C^{0,\mu}$-module. If L is a nonzero $C^{0,\mu}$-submodule of N then $N \supseteq L \supseteq (t)L \neq 0$, since N is t-torsionfree. Then $(t)L = RtL = N$, since N is a simple R-module. Hence, $L = N$, i.e., N is a simple $C^{0,\mu}$-module, as required. ∎

5. The prime spectrum of the algebra $C_A(H)$

In this section, $\mathbb{K}$ is an algebraically closed field. In this section, the prime, completely prime, maximal and primitive spectra of the algebra $C_A(H)$ are described together with inclusions of primes. (Theorem 5.3, Corollary 5.4, Corollary 5.5 and Theorem 5.6).

Proposition 5.1. *1. For all* $\mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[C, H])$,
$$C_A(H)\mathfrak{p} = C_A(H) \cap C_{A_X}(H)\mathfrak{p}$$
is a completely prime ideal of $C_A(H)$ *and* $\mathfrak{p} = \mathbb{K}[C, H] \cap C_A(H)\mathfrak{p}$.

2. The map $\mathrm{Spec}\,(\mathbb{K}[C, H]) \to \mathrm{Spec}\,\big(C_A(H)\big)$, $\mathfrak{p} \mapsto C_A(H)\mathfrak{p}$*, is an injection.*

Proof. 1. Let $\mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[C, H])$. By Proposition 3.6, $\mathfrak{p} = \mathbb{K}[C, H] \cap C_A(H)\mathfrak{p}$. Clearly, $C_A(H)\mathfrak{p} \subseteq I_\mathfrak{p} := C_A(H) \cap C_{A_X}(H)\mathfrak{p}$ and $I_\mathfrak{p}$ is a completely prime ideal of the algebra $C_A(H)$ since $C_A(H)/I_\mathfrak{p} \subseteq C_{A_X}(H)/C_{A_X}(H)\mathfrak{p} \simeq \mathbb{K}[C, H]/\mathfrak{p} \otimes A_1'$, a domain. The field $\mathbb{K}$ is an algebraically closed field, so
$\mathrm{Max}(\mathbb{K}[C, H]) = \{(C - \lambda, H - \mu) \mid \lambda, \mu \in \mathbb{K}\}$.

Claim: $C_A(H)\mathfrak{m} = I_\mathfrak{m}$ *for all* $\mathfrak{m} \in \mathrm{Max}(\mathbb{K}[C, H])$: Notice that $\mathfrak{m} = (C - \lambda, H - \mu)$ where $\lambda, \mu \in \mathbb{K}$. Suppose that $\lambda \neq 0$. By Theorem 4.5, $C_A(H)\mathfrak{m}$ is a maximal ideal of $C_A(H)$. Since $C_A(H)\mathfrak{m} \subseteq I_\mathfrak{m} \neq C_A(H)$, the inclusion must be the equality. Suppose that $\lambda = 0$, i.e., $\mathfrak{m} = (C, H - \mu)$. Since

$C_{A_X}(H) = \mathbb{K}[C,H] \otimes A'_1$ (Lemma 3.1.(2)) and $C^{0,\mu} \subseteq C_{A_X}(H)/C_{A_X}(H)\mathfrak{m} \simeq A'_1$ (a domain), the ideal $\mathfrak{q} := I_{\mathfrak{m}}/C_A(H)\mathfrak{m}$ of $C^{0,\mu}$ is a prime ideal. By Proposition 4.9.(3), either $\mathfrak{q} = 0$ or, otherwise, $t \in \mathfrak{q}$. The later case is impossible since $C_{A_X}(H)/C_{A_X}(H)\mathfrak{m} \simeq A'_1 \ni t \neq 0$. Therefore, $\mathfrak{q} = 0$. The proof of the claim is complete.

Let $V(\mathfrak{p}) := \{\mathfrak{m} \in \mathrm{Max}(\mathbb{K}[C,H]) \,|\, \mathfrak{p} \subseteq \mathfrak{m}\}$. Then $\mathfrak{p} \subseteq \bigcap_{\mathfrak{m}\in V(\mathfrak{p})} \mathfrak{m} = \mathrm{rad}(\mathfrak{p}) = \mathfrak{p}$, i.e., $\mathfrak{p} = \bigcap_{\mathfrak{m}\in V(\mathfrak{p})} \mathfrak{m}$. Then, by Proposition 3.6 and the claim, $C_A(H)\mathfrak{p} = \bigcap_{\mathfrak{m}\in V(\mathfrak{p})} C_A(H)\mathfrak{m} = \bigcap_{\mathfrak{m}\in V(\mathfrak{p})} C_A(H) \cap C_{A_X}(H)\mathfrak{m} = C_A(H) \cap \bigcap_{\mathfrak{m}\in V(\mathfrak{p})} C_{A_X}(H)\mathfrak{m} = C_A(H) \cap (\bigcap_{\mathfrak{m}\in V(\mathfrak{p})} \mathfrak{m}) \otimes A'_1 = C_A(H) \cap \mathfrak{p} \otimes A'_1 = C_A(H) \cap C_{A_X}(H)\mathfrak{p}$.

2. By statement 1, the map is well-defined and it is an injection (since $\mathfrak{p} = \mathbb{K}[C,H] \cap C_A(H)\mathfrak{p}$, see statement 1). ■

By Proposition 5.1.(2), we identify the poset $\big(\mathrm{Spec}\,(\mathbb{K}[C,H]), \subseteq\big)$ with its isomorphic copy in $\big(\mathrm{Spec}\,(C_A(H)), \subseteq\big)$. Let A_t and $C_A(H)_t$ be the (left and right) localizations of the algebras A and $C_A(H)$ at the powers of the element t, respectively. The inclusions of algebras $A \subset A_X \subset A_t$ yield the inclusions of the centralizers $C_A(H) \subset C_{A_X}(H) \subset C_{A_t}(H)$. By Lemma 3.1.(3), $A_t = (A_X)_t = C_{A_X}(H)_t[X^{\pm 1};\sigma]$. Then by Lemma 3.1.(2), $C_{A_t}(H) = C_{A_X}(H)_t = \mathbb{K}[C,H] \otimes A'_{1,t}$. By Corollary 3.5, $C_A(H)_t = \mathbb{K}[C,H] \otimes A'_{1,t}$. Therefore,

$$C_{A_t}(H) = C_A(H)_t = \mathbb{K}[C,H] \otimes A'_{1,t}. \tag{36}$$

The factor algebra $\overline{C} := C_A(H)/(C)$. Since $(C) = C_A(H)C \overset{\mathrm{Pr}.5.1.(1)}{=} C_A(H) \cap C_{A_X}(H)C = C_A(H) \cap C_{A_X}(H) \cap C_{A_t}(H)C = C_A(H) \cap C_{A_t}(H)C$, the algebra $\overline{C} := C_A(H)/(C)$ is a subalgebra of the factor algebra $C_{A_t}(H)/C_{A_t}(H)C \simeq \mathbb{K}[H] \otimes A'_{1,t}$ that is generated by the elements $t, \phi = ht, H$ and $\Theta = (h+H)(h-1)$ (see (21)). In particular, $\overline{C}$ is a Noetherian domain with $\mathrm{GK}(\overline{C}) = 3$. Denote by $\overline{R}$ the subalgebra $\mathbb{K}[H] \otimes R$ of $\mathbb{K}[H] \otimes A'_{1,t}$ where $R = \mathbb{K}[h][t;\sigma]$ and $\sigma(h) = h-1$. Clearly, $\overline{R} = \mathbb{K}[H,h][t;\sigma]$ where $\sigma(H) = H$ and $\sigma(h) = h-1$, and $\overline{C} \subseteq \overline{R}$. The element t is normal element of the algebra $\overline{R}$ and $\overline{R}t = \bigoplus_{i\geqslant 1} \mathbb{K}[H,h]t^i$. Clearly, $\mathbb{K}[\Theta,H] \subseteq \mathbb{K}[H,h]$ and $\mathbb{K}[H,h] = \mathbb{K}[\Theta,H] \oplus \mathbb{K}[\Theta,H]h$ since $\Theta = (h+H)(h-1)$ in $\overline{R}$.

Lemma 5.2.

1. *$\overline{C} = \mathbb{K}[\Theta,H] \oplus \bigoplus_{i\geqslant 1} \mathbb{K}[H,h]t^i = \mathbb{K}[\Theta,H] \oplus \overline{R}t$ is a Noetherian domain with $\mathrm{GK}(\overline{C}) = 3$ and $\overline{C} \cap \overline{R}t = \overline{R}t = \bigoplus_{i\geqslant 1} \mathbb{K}[H,h]t^i = (t,\phi) = (t)$.*
2. *$\overline{C}/(t) \simeq \mathbb{K}[\Theta,H]$, a polynomial algebra in Θ and H.*
3. *For all $n \geqslant 1$, $[\Theta, t^n] = n(2\phi + C)t^{n-1} + n(H-n-1)t^n$.*

Proof. 1. The equality $(t,\phi) = (t)$ follows from (23). Multiplying the equality $\mathbb{K}[H,h] = \mathbb{K}[\Theta,H] \oplus \mathbb{K}[\Theta,H]h$ by the element t on the right yields the equality $\mathbb{K}[H,h]t = \mathbb{K}[\Theta,H]t \oplus \mathbb{K}[\Theta,H]\phi \subseteq \overline{C}$. For all $i \geqslant 1$, $\overline{C} \supseteq (\mathbb{K}[H,h]t)^i = \mathbb{K}[H,h]t^i$, and so $\overline{C} = \mathbb{K}[\Theta,H] \oplus \overline{R}t$.

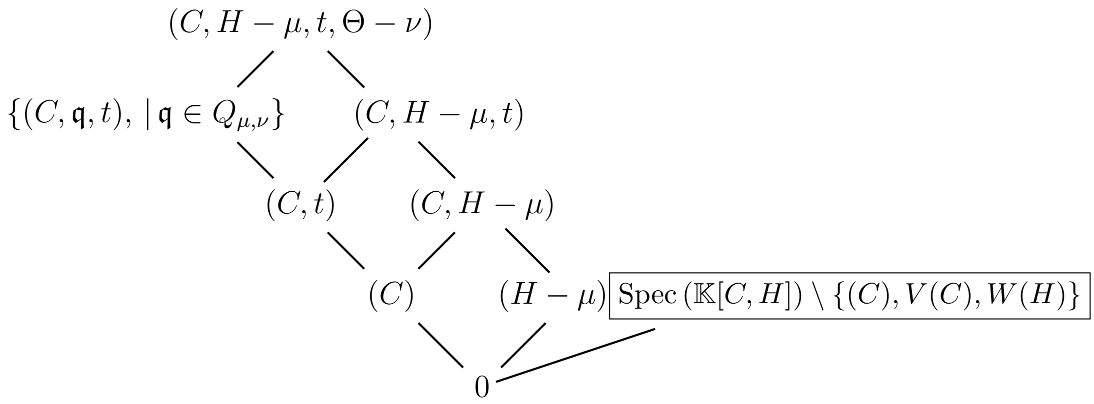
2. Statement 2 follows from statement 1.

3. The case $n = 1$ is obvious, see (23). Suppose that $n \geqslant 2$ and the equality holds for all $n' < n$. Then

$$\begin{aligned}[\Theta, t^n] &= [\Theta, t]t^{n-1} + t[\Theta, t^{n-1}] \\ &= (2\phi + C)t^{n-1} + (H-2)t^n + (n-1)t\big((2\phi + C)t^{n-2} + (H-n)t^{n-1}\big)\end{aligned}$$

and using the equalities $t(2\phi + C) = (2\phi + C)t - 2t^2$ and $tH = Ht$ we obtain the result. ■

Theorem 5.3. $\mathrm{Spec}\,\big(C_A(H)\big) = \mathrm{Spec}\,(\mathbb{K}[C,H]) \sqcup \{(C,t), (C, H-\mu, t), (C, H-\mu, t, \Theta-\nu) \,|\, \mu, \nu \in \mathbb{K}\}$ *and the inclusions between the prime ideals of* $C_A(H)$ *are given in the following diagram:*



(37)

where $Q_{\mu,\nu} := \{\mathfrak{q} \in \mathrm{Spec}\,(\mathbb{K}[H,\Theta]) \,|\, \mathrm{ht}\mathfrak{q} = 1, \mathfrak{q} \subseteq (H-\mu, \Theta-\nu)\}$, $V(C) := \{(C, H-\mu) \,|\, \mu \in \mathbb{K}\}$ *and* $W(H) := \{(H-\mu) \,|\, \mu \in \mathbb{K}\}$.

Proof. By Proposition 5.1.(2) and (36), each prime ideal P of the algebra $C_A(H)$ is either equal to $C_A(H)\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of $\mathbb{K}[C,H]$ or, otherwise, $t^n \in P$ for some $n \geqslant 1$. From now on, we assume that the second case holds ($t^n \in P$).

(i) *If* P *is a prime ideal of* $C_A(H)$ *such that* $t^n \in P$ *for some* $n \geqslant 1$ *then* $C \in P$. To prove the statement we use induction on n. Let $n = 1$, i.e., $(t) \subseteq P$. By (23), $2\phi \equiv -C \mod P$. Then, by (25), $C^2 \equiv 0 \mod P$. Hence, $(C)^2 \subseteq P$ and so $C \in P$.

Suppose that $n > 1$ and the statement is true for all $n' < n$. We assume that $t^{n-1} \notin P$. By Lemma 5.2.(3), $(2\phi + C)t^{n-1} \in P$. Then, multiplying the equality (25) on the right by t^{n-1}, we obtain that $0 \equiv (\phi + C)\phi t^{n-1} \equiv -\frac{1}{4}C^2 t^{n-1} \mod P$, i.e., $(C)^2(t^{n-1}) \subseteq P$. Therefore, $(C)^2 \subseteq P$ since $t^{n-1} \notin P$, and finally $C \in P$, as required.

(ii) $(C,t) \subseteq P$: The ideal $\overline{P} := P/(C)$ of the algebra $\overline{C}$ (see Lemma 5.2) is a prime ideal that contains the element t^n where $n \geqslant 1$. Using the notation of Lemma 5.2, it suffices to show that the ideal $(t) = \overline{R}t$ of $\overline{C}$ is contained in $\overline{P}$. Since $t^n \in \overline{P}$, we have that $\overline{P} \supseteq \overline{R}t^n \cdot t = \overline{R}t^{n+1} = (\overline{R}t)^{n+1}$, and so $\overline{P} \supseteq \overline{R}t$, as required.

By Lemma 5.2.(2), $\overline{C}/(C,t) \simeq \overline{C}/(t) \simeq \mathbb{K}[\Theta, H]$. Now, the description of Spec $(C_A(H))$ of the theorem is obvious. The inclusions in (37) are obvious. By Lemma 5.2.(2) and the statement (ii), these are the only possible inclusions. ■

Remark. Using Corollary 3.5 and Theorem 5.3, for each prime ideal P, we can easily write down the defining relations of the algebra $C_A(H)/P$. Using Proposition 3.6, Proposition 4.9 and Lemma 5.2, we can also give a $\mathbb{K}$-basis for the algebra $C_A(H)/P$.

Corollary 5.4. *Every prime ideal of $C_A(H)$ is completely prime.*

Proof. The result follows from Theorem 5.3, Proposition 5.1.(1) and Proposition 4.9.(3). ■

The next corollary describes the set $\mathrm{Max}(C_A(H))$ of maximal ideals of the algebra $C_A(H)$.

Corollary 5.5.
$\mathrm{Max}(C_A(H)) = \{(C-\lambda, H-\mu), (C, H-\mu, t, \Theta-\nu) \mid \lambda \in \mathbb{K}^*; \mu, \nu \in \mathbb{K}\}$.

The following theorem is a description of the set

$$\mathrm{Prim}\,(C_A(H)) := \{\mathrm{ann}_{C_A(H)}(M) \mid M \in \widehat{C_A(H)}\}$$

of primitive ideals of the algebra $C_A(H)$.

Theorem 5.6. $\mathrm{Prim}\,\big(C_A(H)\big) =$

$$\{(C-\lambda, H-\mu) \mid \lambda, \mu \in \mathbb{K}\} \sqcup \{(C, H-\mu, t, \Theta-\nu) \mid \mu, \nu \in \mathbb{K}\}.$$

Proof. Let $\mathcal{R}$ be the RHS of the equality. Since the elements C and H are central in $C_A(H)$, each primitive ideal P must contain an ideal $(C-\lambda, H-\mu)$ for some $\lambda, \mu \in \mathbb{K}$. By Theorem 4.11 and Proposition 4.9.(3), $\mathrm{ann}_{C^{0,\mu}}(M_b) = 0$ for all $[M_b] \in \widehat{C^{0,\mu}}$ (t-torsionfree) since the algebra $C_t^{0,\mu}$ is simple, i.e., $(C, H-\mu)$ is a primitive ideal. Every ideal which is not of the type $(C, H-\mu)$ is maximal (Corollary 5.5), hence primitive. So, all ideals in $\mathcal{R}$ are primitive. In view of (37), it suffices to show that the ideals $(C, H-\mu, t)$ are not primitive. This is obvious since $C_A(H)/(C, H-\mu, t) \simeq C^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$ is a commutative algebra.

■

6. A classification of simple weight *A*-modules

The aim of this section is to give a classification of simple weight A-modules. They are partitioned into several classes of modules which are classified separately using different techniques. The key idea is to use the classification of simple $C_A(H)$-modules. In this section, we assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. For each coset $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$, we fix a representative $\mu_{\mathcal{O}} \in \mathcal{O} = \mu_{\mathcal{O}} + \mathbb{Z}$. An A-module M is called a *weight module* provided that $M = \bigoplus_{\mu\in\mathbb{K}} M_\mu$ where $M_\mu = \{m \in M \mid Hm = \mu m\}$. An element $\mu \in \mathbb{K}$ such that $M_\mu \neq 0$ is called a *weight* of M. Let $\mathrm{Wt}(M)$ be the set of all weights of the A-module M. For an algebra A, let $\widehat{A}$ be the set of isomorphism classes of simple A-modules and $\widehat{A}\,(\text{weight})$ be the set of isomorphism classes of simple weight A-modules. Let M be an A-module and $x \in A$. We say that M is *x-torsion* provided that for each element $m \in M$ there exists a natural number $i \in \mathbb{N}$ such that $x^i m = 0$, and that M is *x-torsionfree* if the only element of M annihilated by the element x is 0. Since the set $\{X^i \mid i \in \mathbb{N}\}$ is an Ore set in A,

$$\widehat{A}\,(\text{weight}) = \widehat{A}\,(\text{weight}, X\text{-torsion}) \;\sqcup\; \widehat{A}\,(\text{weight}, X\text{-torsionfree}). \tag{38}$$

Description of the set $\widehat{A}\,(\text{weight}, X\text{-torsion})$. An explicit description of the set $\widehat{A}\,(\text{weight}, X\text{-torsion})$ is given in Theorem 6.5. Clearly,

$$\begin{aligned}\widehat{A}\,(\text{weight}, X\text{-torsion}) = {} & \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion}) \\ & \sqcup\; \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree}).\end{aligned} \tag{39}$$

Lemma 6.1. *Let $M \in \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion})$. Then $XM = YM = 0$, i.e.,*

$$\widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion}) = \widehat{U}\,(\text{weight}).$$

Proof. Let $M \in \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion})$. There exists a nonzero weight vector $m \in M$ such that $Xm = 0$ and $Ym = 0$, since $XY = YX$. Notice that $M = Am$ (since M is a simple A-module). So, the A-module M is an epimorphic image of the A-module $A/(AX + AY) = A/(X, Y) = U$, by Lemma 2.3. ∎

Lemma 6.2. *1. Let $M \in \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree})$. Then the central element C acts on M as a nonzero scalar C_M.*

2. Let $M \in \widehat{A}\,(\text{weight}, Y\text{-torsion}, X\text{-torsionfree})$. Then the central element C acts on M as a nonzero scalar C_M.

Proof. 1. Since M is a simple A-module, the central element C acts on M as a scalar C_M. It remains to show that $C_M \neq 0$. Suppose this is not the case, then there is a nonzero weight vector $m \in M$ such that $Xm = 0$ and $Cm = 0$. Since $C = FX^2 - (H+2)YX - Y^2E$, we have $Y^2Em = 0$ and so $Em = 0$, since M is Y-torsionfree. Let $m' = Ym$ then $m' \neq 0$ and $Xm' = Em' = 0$. So, the A-module $M' := Am' = \sum_{i,j\geqslant 0} \mathbb{K} F^i Y^j m'$ is a proper submodule of the A-module M (since $m \notin M'$). This contradicts to the fact that M is a simple module.

2. Statement 2 follows from statement 1 by applying the automorphism S of A, see (2). ∎

For $\lambda, \mu \in \mathbb{K}$, we define the left A-modules $\mathscr{X}^{\mu} := A/A(H-\mu, X)$ and $\mathbb{X}^{\lambda,\mu} := A/A(C-\lambda, H-\mu, X)$. Clearly, $\mathbb{X}^{\lambda,\mu} \simeq \mathscr{X}^{\mu}/(C-\lambda)\mathscr{X}^{\mu}$. Since $XH = (H-2)X$, using the PBW Theorem we see that $\mathscr{X}^{\mu} = \oplus_{i,j,k\geqslant 0}\mathbb{K}F^iY^jE^k\tilde{1} = \mathbb{K}[F] \otimes \mathcal{V}\tilde{1}$ where $\tilde{1} := 1 + A(H-\mu, X)$ and $\mathcal{V} = \oplus_{j,k\geqslant 0}\mathbb{K}Y^jE^k$. It follows from the equalities $[E, Y] = X$ and $X\tilde{1} = 0$ and the fact that the element X commutes with E and Y that $Y^jE^k\tilde{1} = E^kY^j\tilde{1}$. Hence, abusing the notation we can write $\mathcal{V}\tilde{1} = \mathbb{K}[Y, E]\tilde{1}$ where $\mathbb{K}[Y, E]$ is a *polynomial* algebra in letters Y and E. Therefore, $\mathcal{V}\tilde{1} = \Sigma \otimes \mathbb{K}[EY^2]\tilde{1}$ where $\Sigma := \mathbb{K}[Y]Y^2 \oplus \mathbb{K}[E] \oplus Y\mathbb{K}[E]$ and $\mathbb{K}[EY^2]$ is a polynomial in EY^2. Now,

$$\mathscr{X}^{\mu} = \mathbb{K}[F] \otimes \Sigma \otimes \mathbb{K}[EY^2]\tilde{1} \simeq \mathbb{K}[F] \otimes \Sigma \otimes \mathbb{K}[EY^2]$$

is an isomorphism of vector spaces. Since $C = FX^2 - HYX - EY^2, (C-\lambda)\tilde{1} = -(EY^2+\lambda)\tilde{1}$,

$$(C-\lambda)\mathscr{X}^{\mu} = \mathbb{K}[F] \otimes \Sigma \otimes \mathbb{K}[EY^2](-EY^2-\lambda)\tilde{1}.$$

Therefore,

$$\mathbb{X}^{\lambda,\mu} \simeq \mathscr{X}^{\mu}/(C-\lambda)\mathscr{X}^{\mu} \simeq \mathbb{K}[F] \otimes \Sigma\bar{1}$$

where $\bar{1} = 1 + A(C-\lambda, H-\mu, X)$, and the equality of statement 1 of the following proposition follows. Furthermore, the proposition shows that for all $\lambda \in \mathbb{K}^*$, the modules $\mathbb{X}^{\lambda,\mu}$ are simple, weight, X-torsion, Y-torsionfree A-modules. Later in Proposition 6.4.(1), we will see that the set $\widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree})$ consists precisely of the modules $\mathbb{X}^{\lambda,\mu}$. Moreover, the $\mathbb{K}$-bases, weight space decompositions and annihilators of $\mathbb{X}^{\lambda,\mu}$ are given.

Proposition 6.3. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then*

1. *The A-module $\mathbb{X}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2}\mathbb{K}F^iY^j\bar{1} \oplus \bigoplus_{i,k\geqslant 0}\mathbb{K}F^iE^k\bar{1} \oplus \bigoplus_{i,k\geqslant 0}\mathbb{K}YF^iE^k\bar{1}$ is a simple module where $\bar{1} = 1 + A(C-\lambda, H-\mu, X)$.*

2. *Recall that $\Theta = FE$. Then*

$$\mathbb{X}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2}\mathbb{K}F^iY^j\bar{1} \oplus \Big(\bigoplus_{i\geqslant 1, k\geqslant 0}\mathbb{K}F^i\Theta^k\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\bar{1} \oplus \bigoplus_{i\geqslant 1, k\geqslant 0}\mathbb{K}E^i\Theta^k\bar{1}\Big)$$
$$\oplus \Big(\bigoplus_{i\geqslant 1, k\geqslant 0}\mathbb{K}YF^i\Theta^k\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}Y\Theta^k\bar{1} \oplus \bigoplus_{i\geqslant 1, k\geqslant 0}\mathbb{K}YE^i\Theta^k\bar{1}\Big).$$

3. *The weight space $\left(\mathbb{X}^{\lambda,\mu}\right)_{\mu+i}$ of $\mathbb{X}^{\lambda,\mu}$ that corresponds to the weight $\mu + i$ (where $i \in \mathbb{Z}$) is*

$$\left(\mathbb{X}^{\lambda,\mu}\right)_{\mu+i} = \begin{cases} \mathbb{K}[\Theta]\bar{1}, & i = 0, \\ E^r\mathbb{K}[\Theta]\bar{1}, & i = 2r,\ r \geqslant 1, \\ YE^r\mathbb{K}[\Theta]\bar{1}, & i = 2r-1,\ r \geqslant 1, \\ F^r\mathbb{K}[\Theta]\bar{1} \oplus \bigoplus_{j=0}^{r-1}\mathbb{K}F^jY^{2(r-j)}\bar{1}, & i = -2r,\ r \geqslant 1, \\ Y\mathbb{K}[\Theta]\bar{1}, & i = -1, \\ YF^{r-1}\mathbb{K}[\Theta]\bar{1} \oplus \bigoplus_{j=0}^{r-2}\mathbb{K}F^jY^{2(r-j)-1}\bar{1}, & i = -2(r-1)-1,\ r \geqslant 2. \end{cases}$$

In particular, $\mathrm{Wt}(\mathbb{X}^{\lambda,\mu}) = \{\mu + i \,|\, i \in \mathbb{Z}\}$ *and each weight space is infinite dimensional.*

4. $\mathrm{ann}_A\big(\mathbb{X}^{\lambda,\mu}\big) = (C - \lambda)$.

5. $\mathbb{X}^{\lambda,\mu}$ *is an* X*-torsion,* Y*-torsionfree weight* A*-module.*

Proof. 1. It remains to show that the A-module $\mathbb{X}^{\lambda,\mu}$ is simple. We use notation as above. Using the definition of C, we have the equality $EY^2\bar{1} = -\lambda\bar{1}$. Then, for all $k \geqslant 1$, $Y^{2k}E^k\bar{1} = (EY^2)^k\bar{1} = (-\lambda)^k\bar{1}$ (since $\mathcal{V}\tilde{1} = \mathbb{K}[Y, E]\tilde{1}$). Since $EY^2\bar{1} = -\lambda\bar{1} \neq 0$, the map $Y\cdot : \Sigma\bar{1} \to \Sigma\bar{1}$, $s\bar{1} \mapsto Ys\bar{1}$, is an injection. Let u be a nonzero element of $\mathbb{X}^{\lambda,\mu}$. To prove that the A-module $\mathbb{X}^{\lambda,\mu}$ is simple it suffices to show that $au = \bar{1}$ for some $a \in A$. It follows from the equalities $XF^i = F^iX - iF^{i-1}Y$, $X\bar{1}=0$ and $\mathbb{X}^{\lambda,\mu}=\mathbb{K}[F]\otimes\Sigma\bar{1}$, that the map $X\cdot\colon \mathbb{X}^{\lambda,\mu}\to\mathbb{X}^{\lambda,\mu}$, $u \mapsto Xu$, acts as $\frac{d}{dF} \otimes (-Y\cdot)_\Sigma$. So, we can assume that $u = s\bar{1}$ where $0 \neq s \in \Sigma$.

Notice that $s = pY^2 + \sum_{i=0}^m(\lambda_i + \mu_iY)E^i$ for some $p \in \mathbb{K}[Y]$ and $\lambda_i, \mu_i \in \mathbb{K}$. Then

$$\begin{aligned} Y^{2m}u = Y^{2m}s\bar{1} &= \big(pY^{2m+2} + \sum_{i=0}^m(\lambda_i + \mu_iY)Y^{2(m-i)}Y^{2i}E^i\big)\bar{1} \\ &= \big(pY^{2m+2} + \sum_{i=0}^m(\lambda_i + \mu_iY)Y^{2(m-i)}(-\lambda)^i\big)\bar{1} = f\bar{1} \end{aligned}$$

where $f \in \mathbb{K}[Y] \setminus \{0\}$ (since $s \neq 0$). So, we may assume that $u = f\bar{1}$ where $0 \neq f \in \mathbb{K}[Y]$. Let $f = \sum_{i=0}^l \gamma_iY^i$ where $\gamma_i \in \mathbb{K}$ and $\gamma_l \neq 0$. Since $HY^i\bar{1} = (\mu - i)Y^i\bar{1}$ for all i and all the eigenvalues $\{\mu - i \,|\, i \geqslant 0\}$ are distinct, there is a polynomial $g \in \mathbb{K}[H]$ such that $gf\bar{1} = Y^l\bar{1}$. If $l = 0$, we are done. We may assume that $l \geqslant 1$. Multiplying (if necessary) the equality above by Y we may assume that $l = 2k$ for some natural number $k \in \mathbb{N}$. Then $(-\lambda)^{-k}E^kY^{2k}\bar{1} = \bar{1}$, as required.

2. Using the fact that the algebra U is a generalized Weyl algebra

$$U = \mathbb{K}[\Theta, H][E, F; \sigma, a = \Theta]$$

where $\sigma(\Theta) = \Theta + H$, $\sigma(H) = H - 2$ and the equality $F^iE^i = \Theta\sigma^{-1}(\Theta)\cdots\sigma^{-i+1}(\Theta)$, we see that

$$\bigoplus_{i,k\geqslant 0}\mathbb{K}F^iE^k\bar{1} = \bigoplus_{i\geqslant 1,k\geqslant 0}\mathbb{K}F^i\Theta^k\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0}\mathbb{K}E^i\Theta^k\bar{1}.$$

Then statement 2 follows from statement 1.

3. Statement 3 follows from statement 2.

4. It is clear that $(C - \lambda) \subseteq \mathrm{ann}_A\big(\mathbb{X}^{\lambda,\mu}\big)$. By Proposition 2.5.(1), the ideal $(C - \lambda)$ of A is a maximal ideal, hence $(C - \lambda) = \mathrm{ann}_A\big(\mathbb{X}^{\lambda,\mu}\big)$.

5. Clearly, $\mathbb{X}^{\lambda,\mu}$ is an X-torsion, weight A-module. By statement 1, $\mathbb{X}^{\lambda,\mu}$ is a simple module, it must be Y-torsionfree (since, otherwise, by Lemma 6.1, $C\mathbb{X}^{\lambda,\mu} = 0$, a contradiction). ■

The sets $\widehat{A}\,(\text{weight}, X\text{-torsion})$ and $\widehat{A}\,(\text{weight}, Y\text{-torsion})$. For $\lambda, \mu \in \mathbb{K}$, let us consider the A-module $\mathbb{Y}^{\lambda,\mu} := A/A(C-\lambda, H-\mu, Y)$. Then $\mathbb{Y}^{\lambda,\mu} \simeq {}^S\mathbb{X}^{-\lambda,-\mu}$ where ${}^S\mathbb{X}^{-\lambda,-\mu}$ is the A-module $\mathbb{X}^{-\lambda,-\mu}$ twisted by the automorphism S of the algebra A ($S(H) = -H$, $S(C) = -C$, $S(Y) = -X$). The subgroup $\mathbb{Z}$ of $(\mathbb{K}, +)$ acts on $\mathbb{K}$ in the obvious way. For each $\lambda \in \mathbb{K}$, $\mathcal{O}(\lambda) := \lambda + \mathbb{Z}$ is the orbit of λ under the action of $\mathbb{Z}$. The set of all $\mathbb{Z}$-orbits can be identified with the elements of the factor group $\mathbb{K}/\mathbb{Z}$. For each orbit $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$, we fix an element $\mu_{\mathcal{O}} \in \mathcal{O}$.

Proposition 6.4.

1. $\widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree}) = \{[\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda \in \mathbb{K}^*, \mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$ *and the A-modules $\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}$ and $\mathbb{X}^{\lambda',\mu_{\mathcal{O}'}}$ are isomorphic iff $\lambda = \lambda'$ and $\mathcal{O} = \mathcal{O}'$.*

2. $\widehat{A}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsion}) = \{[\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda \in \mathbb{K}^*, \mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$ *and the A-modules $\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}$ and $\mathbb{Y}^{\lambda',\mu_{\mathcal{O}'}}$ are isomorphic iff $\lambda = \lambda'$ and $\mathcal{O} = \mathcal{O}'$.*

Proof. 1. Let $M \in \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree})$. By Lemma 6.2, $C_M = \lambda \neq 0$ for some $\lambda \in \mathbb{K}^*$. Then M is a factor module of $\mathbb{X}^{\lambda,\mu}$ for some $\mu \in \mathbb{K}$. By Proposition 6.3.(1), the module $\mathbb{X}^{\lambda,\mu}$ is a simple module, hence $M \simeq \mathbb{X}^{\lambda,\mu}$. Clearly, $\mathbb{X}^{\lambda,\mu} \simeq \mathbb{X}^{\lambda',\mu'}$ iff $\lambda = \lambda'$ and $\mu = \mu' + i$ for some $i \in \mathbb{Z}$.

2. Since $\mathbb{Y}^{\lambda,\mu} \simeq {}^S\mathbb{X}^{-\lambda,-\mu}$, statement 2 follows from statement 1. ∎

The next theorem gives a complete description of simple, weight, X-torsion A-modules and of simple, weight, Y-torsion A-modules.

Theorem 6.5.

1. $\widehat{A}\,(\text{weight}, X\text{-torsion}) = \widehat{U}\,(\text{weight}) \sqcup \{[\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda \in \mathbb{K}^*, \mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$.

2. $\widehat{A}\,(\text{weight}, Y\text{-torsion}) = \widehat{U}\,(\text{weight}) \sqcup \{[\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda \in \mathbb{K}^*, \mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$.

Proof. 1. The theorem follows from the equality (39), Lemma 6.1 and Proposition 6.4.

2. Statement 2 follows from statement 1. ∎

Description of the set $\widehat{A}\,(\text{weight}, X\text{-torsionfree})$. Since the element C belongs to the centre of the algebra A and the field $\mathbb{K}$ is algebraically closed,

$$\widehat{A}\,(\text{weight}) = \bigsqcup_{\lambda \in \mathbb{K}} \widehat{A(\lambda)}\,(\text{weight}) \tag{40}$$

where $A(\lambda) := A/(C-\lambda)$. Moreover,

$$\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}) = \widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree}) \bigsqcup \widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsion}). \tag{41}$$

The simple modules in the set $\widehat{A(\lambda)}$ (weight, X-torsionfree, Y-torsion) are classified by Proposition 6.4.(2). So, in order to finish the classification of simple weight A-modules it remains to describe the set $\widehat{A(\lambda)}$ (weight, X-torsionfree, Y-torsionfree).

The set $\widehat{A(0)}$ (weight, X-torsionfree, Y-torsionfree)**.** We denote by C_t the algebra in (36). Let $[M] \in \widehat{C^{0,\mu}}$ (t-torsionfree). By Theorem 4.11.(2), the element t acts *bijectively* on M (since t is a normal element of R and $(t) = Rt$). Therefore, the $C_A(H)$-module M is also a C_t-module. Using the equality $A_t = C_t[X^{\pm 1}; \sigma]$, let us define an A_t-module

$$\widetilde{M} := A_t \otimes_{C_t} M = \bigoplus_{i \in \mathbb{Z}} X^i \otimes M = \bigoplus_{i \geqslant 1} Y^i \otimes M \oplus \bigoplus_{i \geqslant 0} X^i \otimes M. \quad (42)$$

Clearly, $\widetilde{M}$ is a weight A-module with $\mathrm{Wt}(\widetilde{M}) = \mathcal{O}(\mu) = \mu + \mathbb{Z}$ and $\widetilde{M}_{\mu+i} = X^i \otimes M$ for all $i \in \mathbb{Z}$. The A-module $\widetilde{M}$ is X- and Y-torsionfree. Moreover, *the A-module $\widetilde{M}$ is simple* since if N is a nonzero submodule of $\widetilde{M}$ then it contains a nonzero element $X^i \otimes m$ for some $i \in \mathbb{Z}$ and $m \in M$. If $i = 0$ then $N = Am = \widetilde{M}$. If $i < 0$ then $N \ni X^{|i|} X^i \otimes m = 1 \otimes m$, and so $N = \widetilde{M}$. If $i > 0$ then $N \ni Y^i X^i \otimes m = 1 \otimes t^i m \neq 0$, and so $N = \widetilde{M}$. If $M' \in \widehat{C^{0,\mu'}}$ (t-torsionfree) then the A-modules $\widetilde{M}$ and $\widetilde{M'}$ are isomorphic iff $\mathcal{O}(\mu) = \mathcal{O}(\mu')$ and the $C^{0,\mu}$-modules M and $X^i \otimes M'$ are isomorphic where $\mu = \mu' + i$ for a unique $i \in \mathbb{Z}$. Clearly, $\mathrm{GK}(\widetilde{M}) = 2$. The following theorem is an explicit description of the set $\widehat{A(0)}$ (weight, X-torsionfree, Y-torsionfree).

Theorem 6.6. $\widehat{A(0)}$ (weight, X-torsionfree, Y-torsionfree) $= \{[\widetilde{M}] \mid [M] \in \widehat{C^{0,\mu_{\mathcal{O}}}}$ (t-torsionfree), $\mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$ *and* $\mathrm{GK}(\widetilde{M}) = 2$ *for all* $\widetilde{M}$.

Proof. It suffices to show that if $\mathcal{M} \in \widehat{A(0)}$ (weight, X-torsionfree, Y-torsionfree) then $\mathcal{M} \simeq \widetilde{M}$ for some $[M] \in \widehat{C^{0,\mu_{\mathcal{O}}}}$ (t-torsionfree). Fix $\mu \in \mathrm{Wt}(\mathcal{M})$. Since the elements X and Y act injectively on $\mathcal{M}$, we have that $\mathcal{O}(\mu) \subseteq \mathrm{Wt}(\mathcal{M})$. So, we may assume that $\mu = \mu_{\mathcal{O}}$ where $\mathcal{O} = \mathcal{O}(\mu)$. Then $M := \mathcal{M}_\mu \in \widehat{C^{0,\mu_{\mathcal{O}}}}$ (t-torsionfree), and so $\mathcal{M} \supseteq \bigoplus_{i \geqslant 1} Y^i M \oplus \bigoplus_{i \geqslant 0} X^i M = \widetilde{M}$, by (42) and simplicity of $\widetilde{M}$. So, $\mathcal{M} = \widetilde{M}$, as required. ∎

The set $\widehat{A(\lambda)}$ (weight, X-torsionfree, Y-torsionfree) **where** $\lambda \neq 0$**.** Recall that $A_t = C_t[X^{\pm 1}; \sigma]$. Let $[M] \in \widehat{C^{\lambda,\mu}}$ (t-torsionfree). Then $[M_t] \in \widehat{C_t^{\lambda,\mu}}$. The A_t-module

$$M^\diamond := A_t \otimes_{C_t} M_t = \bigoplus_{i \in \mathbb{Z}} X^i \otimes M_t \quad (43)$$

is a simple, weight A_t-module with $\mathrm{Wt}(M^\diamond) = \mathcal{O}(\mu) = \mu + \mathbb{Z}$ and $M^\diamond_{\mu+i} = X^i \otimes M_t$ for all $i \in \mathbb{Z}$ (if N is a nonzero submodule of $M^\diamond$ then it contains a nonzero element $X^i \otimes m$ for some $i \in \mathbb{Z}$ and $m \in M_t$. Then $N \ni X^{-i} X^i \otimes m = 1 \otimes m$, and so $N = M^\diamond$). For all $i \in \mathbb{Z}$,

$$M^\diamond_i = X^i \otimes M_t \simeq M_t^{\sigma^{-i}} \quad (44)$$

where $M_t^{\sigma^{-i}}$ is the C_t-module twisted by the automorphism σ^{-i} of the algebra C_t. (Recall that $A_t = C_t[X^{\pm 1};\sigma]$). By Theorem 4.8 and Theorem 4.7,

$$\widehat{C^{\lambda,\mu}}\,(t\text{-torsionfree}) = \Big\{[\frac{A_1'}{A_1'e}],\,[\frac{A_1'}{A_1'(h-\nu_{\mathcal{O}})}]\,|\,\mathcal{O}\in\mathbb{K}/\mathbb{Z}\setminus\{\mathbb{Z}\}\Big\}$$
$$\sqcup\ \Big\{[\mathrm{soc}(N_b)]\,|\,b\in \mathrm{Irr}(B)/\sim,\ b \text{ is normal}\Big\}$$

where $\mathrm{Irr}(B)/\sim$ is the set of equivalence classes of irreducible elements of the algebra $B = \mathbb{K}(h)[t^{\pm 1};\sigma]$ and $N_b = C^{\lambda,\mu}/C^{\lambda,\mu}\cap Bb$. Moreover, by Theorem 4.8.(3), $\mathrm{soc}(N_b)\simeq N_{bt^{-n}}$ for all $n\gg 0$.

For all $\lambda\in\mathbb{K}^*$ and $\mu\in\mathbb{K}$, the module $\mathfrak{m}^{\lambda,\mu} := C_t^{\lambda,\mu}/C_t^{\lambda,\mu}e$ is a simple $C_A(H)$-module. Hence, $\mathrm{soc}_{C_A(H)}(\mathfrak{m}^{\lambda,\mu}) = \mathfrak{m}^{\lambda,\mu}$. Notice that

$$(\frac{A_1'}{A_1'e})^{\diamond} = \bigoplus_{i\in\mathbb{Z}} X^i\otimes\frac{A_{1,t}'}{A_{1,t}'e}\simeq\bigoplus_{i\in\mathbb{Z}} X^i\otimes\frac{C_t^{\lambda,\mu}}{C_t^{\lambda,\mu}e} = \bigoplus_{i\in\mathbb{Z}} X^i\otimes\mathfrak{m}^{\lambda,\mu}$$

and $X^i\otimes\mathfrak{m}^{\lambda,\mu}\simeq\mathfrak{m}^{\lambda,\mu+i}$ as $C_A(H)$-modules. Then there are equalities of A-modules

$$\mathrm{soc}_A\left(\left(\frac{A_1'}{A_1'e}\right)^{\diamond}\right) = \bigoplus_{i\in\mathbb{Z}}\mathrm{soc}_{C_A(H)}(X^i\otimes\mathfrak{m}^{\lambda,\mu}) = \bigoplus_{i\in\mathbb{Z}}\mathfrak{m}^{\lambda,\mu+i}. \quad (45)$$

For all $\lambda\in\mathbb{K}^*$, $\mu\in\mathbb{K}$ and $\mathcal{O}\in\mathbb{K}/\mathbb{Z}\setminus\{\mathbb{Z}\}$, the module $\mathsf{M}^{\lambda,\mu,\mathcal{O}} := C_t^{\lambda,\mu}/C_t^{\lambda,\mu}(h-\nu_{\mathcal{O}})$ is a simple $C_A(H)$-module. Hence, $\mathrm{soc}_{C_A(H)}(\mathsf{M}^{\lambda,\mu,\mathcal{O}}) = \mathsf{M}^{\lambda,\mu,\mathcal{O}}$. Since $\left(\frac{A_1'}{A_1'(h-\nu_{\mathcal{O}})}\right)^{\diamond}$

$$= \bigoplus_{i\in\mathbb{Z}} X^i\otimes\frac{A_{1,t}'}{A_{1,t}'(h-\nu_{\mathcal{O}})}\simeq\bigoplus_{i\in\mathbb{Z}} X^i\otimes\frac{C_t^{\lambda,\mu}}{C_t^{\lambda,\mu}(h-\nu_{\mathcal{O}})} = \bigoplus_{i\in\mathbb{Z}} X^i\otimes\mathsf{M}^{\lambda,\mu,\mathcal{O}}$$

and $X^i\otimes\mathsf{M}^{\lambda,\mu,\mathcal{O}}\simeq\mathsf{M}^{\lambda,\mu+i,\mathcal{O}}$ as $C_A(H)$-modules. Then there are equalities of A-modules

$$\mathrm{soc}_A\left(\left(\frac{A_1'}{A_1'(h-\nu_{\mathcal{O}})}\right)^{\diamond}\right) = \bigoplus_{i\in\mathbb{Z}}\mathrm{soc}_{C_A(H)}(X^i\otimes\mathsf{M}^{\lambda,\mu,\mathcal{O}}) = \bigoplus_{i\in\mathbb{Z}}\mathsf{M}^{\lambda,\mu+i,\mathcal{O}}. \quad (46)$$

If $M\simeq N_b = C^{\lambda,\mu}/C^{\lambda,\mu}\cap Bb$ for an irreducible element b of $B = \mathbb{K}(h)[t^{\pm 1};\sigma]$. For all $i\in\mathbb{Z}$,

$$M_t^{\sigma^{-i}}\supseteq\frac{C_t^{\lambda,\mu+i}}{C_t^{\lambda,\mu+i}\cap B\sigma^i(b)} =: N_{\sigma^i(b)}.$$

By Theorem 4.8.(3),

$$\mathrm{soc}_{C_A(H)}(M_t^{\sigma^{-i}}) = \mathrm{soc}_{C_A(H)}(N_{\sigma^i(b)}) = N_{\sigma^i(b)t^{-n_i}}$$

for all $n_i\gg 0$. Then the A-module

$$\mathrm{soc}_A(M^{\diamond}) = \bigoplus_{i\in\mathbb{Z}}\mathrm{soc}_{C_A(H)}(X^i\otimes M_t)\simeq\bigoplus_{i\in\mathbb{Z}} N_{\sigma^i(b)t^{-n_i}} \quad (47)$$

belongs to the set $\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$. The next theorem shows that all elements of the set $\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$ are precisely of this kind.

Theorem 6.7. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then*

$$\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree}) = \{[\mathrm{soc}_A(M^\diamond)] \mid [M] \in \widehat{C^{\lambda,\mu_{\mathcal{O}}}}\,(t\text{-torsionfree}),\ \mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$$

and $\mathrm{soc}_A(M^\diamond)$ *is explicitly described in (45), (46) and (47), the* A*- modules* $\mathrm{soc}_A(M^\diamond)$ *and* $\mathrm{soc}_A(M'^\diamond)$ *are isomorphic iff* $\lambda = \lambda'$*,* $\mathcal{O} = \mathcal{O}'$ *and* $M \simeq M'$*;* $\mathrm{GK}(\mathrm{soc}_A(M^\diamond)) = 2$.

Proof. Let $[\mathcal{M}] \in \widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$. Then $\mathrm{Wt}(\mathcal{M}) = \mathcal{O} \in \mathbb{K}/\mathbb{Z}$. Let $\mu = \mu_{\mathcal{O}}$. Then $M := \mathcal{M}_\mu \in \widehat{C^{\lambda,\mu}}\,(t\text{-torsionfree})$ and $M_t \in \widehat{C_t^{\lambda,\mu}}$. Clearly, $\mathcal{M} \subseteq \mathcal{M}_t = M^\diamond$, and so $\mathcal{M} = \mathrm{soc}_A(M^\diamond)$. Given $[M'] \in \widehat{C^{\lambda',\mu_{\mathcal{O}'}}}\,(t\text{-torsionfree})$. If $\mathrm{soc}_A(M^\diamond) \simeq \mathrm{soc}_A(M'^\diamond)$ then $M^\diamond = \mathrm{soc}_A(M^\diamond)_t \simeq \mathrm{soc}_A(M'^\diamond)_t = M'^\diamond$ as A_t-modules, and so $\lambda = \lambda'$, $\mathcal{O} = \mathcal{O}'$ and $M_t \simeq M'_t$ as $C_t^{\lambda,\mu_{\mathcal{O}}}$-modules. Then $M = \mathrm{soc}_{C_A(H)}(M_t) \simeq \mathrm{soc}_{C_A(H)}(M'_t) = M'$ as $C_A(H)$-modules. Clearly, $\mathrm{GK}(\mathrm{soc}_A(M^\diamond)) = 2$. ∎

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