Journal of Lie Theory
Volume **28** (2018) 561–575

Heisenberg Algebras from Division Algebras and Parabolic Subalgebras of Simple Lie Algebras

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Communicated by E. B. Vinberg

Abstract. Every real simple Lie algebra which is not compact or isomorphic to $\mathfrak{so}(1,n)$ contains a unique standard parabolic subalgebra whose nilradical is a Heisenberg-like algebra associated to a division algebra. Some geometric consequences are discussed.
Mathematics Subject Classification 2010: 22E25, 58A30, 17C60.
Key Words and Phrases: Heisenberg, parabolic subalgebras, distributions.

1. Introduction

A normed real division algebra $\mathbb{A}$ determines two series of graded nilpotent algebras

$$\mathfrak{h}_n(\mathbb{A}) = (\mathbb{A}^n \oplus \mathbb{A}^n) \oplus \mathbb{A}, \qquad \mathfrak{h}'_{p,q}(\mathbb{A}) = (\mathbb{A}^p \oplus \mathbb{A}^q) \oplus \Im(\mathbb{A})$$

with respective brackets

$$[x+y+t, \hat{x}+\hat{y}+\hat{t}] = \sum x_i\hat{y}_i - \hat{x}_i y_i$$

$$[x+y+t, \hat{x}+\hat{y}+\hat{t}] = -\Im(\sum_j x_j\bar{\hat{x}}_j + \sum_k \bar{\hat{y}}_k y_k).$$

Excluding the $\mathfrak{h}'_{p,q}(\mathbb{R})$'s, which are abelian, and the $\mathfrak{h}_n(\mathbb{O})$, $\mathfrak{h}'_{p,q}(\mathbb{O})$ for $n > 1$ or $p+q > 1$, which are non-prolongable (see 3.2 below), and taking the isomorphism $\mathfrak{h}'_{p,q}(\mathbb{C}) \cong \mathfrak{h}_{p+q}(\mathbb{R})$ into account, the remaining ones are

$$\mathfrak{h}_n(\mathbb{R}) \qquad \mathfrak{h}_n(\mathbb{C}) \qquad \mathfrak{h}_n(\mathbb{H}) \qquad \mathfrak{h}_{p,q}(\mathbb{H}) \qquad \mathfrak{h}_1(\mathbb{O}) \qquad \mathfrak{h}_{1,0}(\mathbb{O}).$$

We call these algebras and associated objects *of type* $\mathfrak{h}(\mathbb{A})$.

In this article we prove that every real simple Lie algebra which is not compact or isomorphic to $\mathfrak{so}(1,n)$ contains a unique standard parabolic subalgebra

*This work was partially supported by grants PICT from CONICET for both authors, and a post-doctoral Fellowship from CONICET and a grant from ANPCyT-FONCyT for the second author.

ISSN 0949–5932 / $2.50

whose nilradical is of this type. Moreover, any 2-step non-singular[1] nilradical of a parabolic subalgebra is of type $\mathfrak{h}(\mathbb{A})$.

This correspondence can be used in both directions. It implies, for example, that among fat distributions of given dimension and codimension, the canonical ones on Lie groups of type $\mathfrak{h}(\mathbb{A})$ are infinitesimally maximally symmetric. In terms of Tanaka theory, $\mathfrak{h}(\mathbb{A})$-algebras are exactly the non-singular algebras whose Tanaka prolongation is not trivial.

On the other hand, it assigns a canonical parabolic geometry to most simple algebras. The geometries obtained are the normal regular parabolic geometries supported on fat distributions. These geometries are also the "conformal infinities" of the Einstein metrics on the corresponding Damek-Ricci spaces $\mathbb{R}_+N$. This allows to resolve the singularity of the distribution on the spherical boundary that appears in the non-hyperbolic, anisotropic case.

It is somewhat remarkable that such basic result had not been noticed before (to our knowledge). It is implicit in Wolf's classification of square-integrable nilradicals [25], and in Howe's list of H-tower groups [15][2], where the $\mathfrak{h}(\mathbb{A})$ appear as H-type nilradicals with center of dimensions 1,2,3,4,7,8, but without an actual proof (references to as [6],[23] and [25] are given instead). Here we give a direct construction, independent of type H and representation theory, where the division algebra structure may explain some of the "high degree of symmetry" observed in the corresponding parabolics.

Some of the statements make sense for Lie algebras and algebraic groups over more general fields, as well as for some infinite-dimensional generalizations.

Part of this article is based on the doctoral thesis of one of us (Subils), directed by E. Hullet. P. Deligne pointed out an error in a previous version. We thank then, as well as J. Baez, J. Huerta, J. Humphreys and J. Wolf, for their advise.

2. Parabolic subalgebras

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra, and $\mathfrak{n} \subset \mathfrak{p}$ the nilradical of $\mathfrak{p}$. If $\mathfrak{g}$ is compact, then $\mathfrak{p}$ is either 0 or $\mathfrak{g}$. If $\mathfrak{p}$ is proper and $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(1,n)$, then $\mathfrak{p}$ is unique up to conjugacy and $\mathfrak{n}$ is abelian. Moreover, the $\mathfrak{so}(1,n)$ are the only simple algebras with these properties. Here we will be interested in the remaining ones, those which contain parabolic subalgebras with non-abelian nilradical, the set of which we which often denote by $\mathfrak{S}$.

For a graded nilpotent Lie algebra $\mathfrak{n} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2} \oplus ...$ to be the nilradical of a parabolic subalgebra of a semisimple algebra is equivalent to asking that it can be "prolonged" to a finite dimensional graded semisimple algebra

$$\mathfrak{g}(\mathfrak{n}, \mathfrak{g}_0) = \mathfrak{g}^k \oplus ... \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1} \oplus ... \oplus \mathfrak{g}^{-k} \tag{1}$$

[1] A 2-step nilpotent Lie algebra $\mathfrak{n}$ with center $\mathfrak{z}$ is *non-singular* if the trilinear form $\phi^* \otimes X \wedge Y \mapsto \phi([X,Y])$ on $\mathfrak{z}^* \otimes \Lambda^2(\mathfrak{n}/\mathfrak{z})$ is non-degenerate. A vector distribution D is *fat*, or *strongly bracket generating*, if its symbol, or nilpotentization, is 2-step nilpotent and non-singular at each point [12, 19].

[2] Found after an earlier version of this paper was written [18]

where $\mathfrak{g}^i = \theta\mathfrak{g}^{-i}$ for some Cartan involution. This already implies that $\mathrm{Aut}(\mathfrak{n})$ must be large enough, so as to contain such $\mathfrak{g}^0$. The associated parabolic subalgebra is $\mathfrak{g}^0 \oplus \mathfrak{n}$.

Theorem 2.1.

(a) Every simple non-compact Lie algebra not isomorphic to $\mathfrak{so}(1,n)$ has a parabolic subalgebra with non-singular nilradical.

(b) Any two are conjugate by the adjoint group.

(c) The nilradicals that appear are exactly the algebras of type $\mathfrak{h}(\mathbb{A})$.

Proof.
Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the simple Lie algebra $\mathfrak{g}$. Fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and a system of positive restricted roots with respect to $\mathfrak{a}$. The conjugacy classes of parabolic subalgebras of $\mathfrak{g}$ are in bijective correspondence with subsets Σ of simple roots, defined by the formula

$$\mathfrak{p} = \mathfrak{g}_0 + \sum_{\alpha\in\Delta(\Sigma)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$ and $\alpha \in \Delta(\Sigma)$ if and only if the roots of Σ appear in α with non-negative coefficients. We need the following

Lemma 2.2. *Let Δ be an irreducible root system (not necessarily reduced) distinct from A_1, Δ^+ a positive system and Δ^0 the corresponding simple roots. Then there is a subset $\Sigma \subseteq \Delta^0$, unique up to diagram equivalence, such that the maximal root δ of Δ has Σ-height 2 and for any $\alpha \in \Delta^+$ of Σ-height 1, $\delta - \alpha \in \Delta$.*

Proof. Let $\alpha \in \Delta^0$, consider the α-string through δ. Since $\delta + \alpha$ is not a root, we get that $\delta - \alpha \in \Delta$ if and only if $(\delta, \alpha) > 0$. So,

$$\Sigma \subseteq \Sigma_\delta = \{\alpha \in \Delta^0 : (\delta, \alpha) > 0\}$$

Looking at every root system different from A_1 we check that δ has Σ_δ-height two and height less than 2 for any subset. So $\Sigma = \Sigma_\delta$ and is uniquely determined. This proves the Lemma. ■

If a parabolic subalgebra determined by Σ has a non-singular nilradical then Σ verifies the conditions of Lemma 2.2, so uniqueness follows.

To prove existence we construct explicitly this parabolic subalgebra and show that the nilradicals that appear are exactly the $\mathfrak{h}(\mathbb{A})$-algebras.

The complex contact grading given in [4] descends to a real contact grading for every non-compact real form, except $\mathfrak{su}^*(2n) = \mathfrak{sl}(n, \mathbb{H})$, $\mathfrak{so}(n, 1)$, $\mathfrak{sp}(p, q)$, $EIV = \mathfrak{sl}(3, \mathbb{O})$ and FII (p. 312, [8]). This is equivalent to having a parabolic subalgebra with nilradical isomorphic to $\mathfrak{h}_n(\mathbb{R})$, the usual Heisenberg algebra.

The complex case also implies that every simple complex Lie algebra except $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1)$, viewed as real, has a parabolic subalgebra with nilradical isomorphic to $\mathfrak{h}_n(\mathbb{C})$.

For $\mathfrak{su}^*(2n+2)$ with $n \geq 2$ take the grading determined by $\{\alpha_2, \alpha_{2n}\}$; the corresponding parabolic subalgebra has nilradical $\mathfrak{h}_{n-1}(\mathbb{H})$.

For $\mathfrak{sp}(p+1, q+1)$ with $(p,q) \neq (0,0)$, the parabolic subalgebra determined by $\{\alpha_2\}$ has nilradical isomorphic to $\mathfrak{h}'_{p,q}(\mathbb{H})$.

For EIV and FII the nilpotent Iwasawa subalgebras (nilradical of the Borel subalgebra) are isomorphic to $\mathfrak{h}_1(\mathbb{O})$ and $\mathfrak{h}'_{1,0}(\mathbb{O})$, respectively. Since $\mathfrak{su}^*(4) \cong \mathfrak{so}(1,5)$ and $\mathfrak{sp}(1,1) \cong \mathfrak{so}(1,4)$, this covers all the remaining cases. ■

Proposition 2.3. *Let $\mathfrak{n}$ be of type $\mathfrak{h}(\mathbb{A})$, $\mathfrak{p}$ a standard parabolic subalgebra of some simple Lie algebra $\mathfrak{g}$ having $\mathfrak{n}$ as nilradical, and*

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

its Langlands decomposition. Then

$$\mathfrak{m} = \mathfrak{m}_o \oplus \mathfrak{spin}(\mathfrak{n}), \qquad \mathfrak{a} = \mathfrak{a}_o \oplus \mathfrak{a}_\delta, \qquad \mathfrak{g}_0 = \mathfrak{m}_o \oplus \mathfrak{spin}(\mathfrak{n}) \oplus \mathfrak{a}$$

where

$\mathfrak{m}_o$ *is the centralizer of* $\mathfrak{z}$ *in* $\mathfrak{m}$*;*

$\mathfrak{spin}(\mathfrak{n}) \cong \mathfrak{so}(\mathfrak{z})$ *acts on* $\mathfrak{z}$ *by the standard representation and on* $\mathfrak{v}$ *as a sum of spin representations;*

$\mathfrak{a}_o = \mathfrak{a} \cap \mathrm{Der}_o(\mathfrak{n})$*, which is* 0 *if* $\mathfrak{p}$ *is maximal and 1-dimensional otherwise; and*

$\mathfrak{a}_\delta = \mathbb{R}\delta$.

The individual factors of the resulting decomposition

$$\mathfrak{p} = (\mathfrak{m}_o \oplus \mathfrak{a}_o) \oplus \mathfrak{spin}(\mathfrak{n}) \oplus (\mathfrak{a}_\delta \oplus \mathfrak{n})$$

are listed in Table 1.

Proof. All the assertions follow from the Table, which is obtained applying the construction in the proof of Theorem 1 case by case. The description of $\mathfrak{spin}(\mathfrak{n})$ will be explained in 3.1. ■

Given a simple $\mathfrak{g} \in \mathfrak{S}$, denote by $\mathfrak{p}(\mathfrak{g})$ the parabolic subalgebra with nilradical $\mathfrak{n}(\mathfrak{g})$ of type $\mathfrak{h}(\mathbb{A})$, and $\mathfrak{m}(\mathfrak{g})$ its Levi factor.

Corollary 2.4. $\mathfrak{g} \in \mathfrak{S}$ *has a complex or quaternionic structure if and only if* $\mathfrak{n}(\mathfrak{g})$ *is* $\mathfrak{h}_n(\mathbb{C})$ *or* $\mathfrak{h}_n(\mathbb{H})$*, respectively.*

Proposition 2.5. $\mathfrak{p}(\mathfrak{g})$ *is maximal parabolic except for* $\mathfrak{g} \cong \mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{su}^*(2n)$*, and* EIV*. It is minimal iff* $\mathfrak{g} \cong \mathfrak{su}(1,n)$, $\mathfrak{sp}(1,n)$, $\mathfrak{su}^*(6)$, FII, $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{sl}(3,\mathbb{C})$*, or* EIV.

Even if $\mathfrak{p}(\mathfrak{g})$ is not minimal, it contains the following distinguished minimal ones. First note that any reductive Lie algebra can be uniquely decomposed as

$\mathfrak{r} = \mathfrak{r}' \oplus \mathfrak{r}''$ where $\mathfrak{r}'$ is semisimple with simple factors in $\mathfrak{S}$, and $\mathfrak{r}''$ is reductive with simple factors not in $\mathfrak{S}$.

Proposition 2.6.
(a) If $\mathfrak{g} \in \mathfrak{S}$, then $\mathfrak{m}(\mathfrak{g})' \in \mathfrak{S}$.
(b) If $\mathfrak{g} \in \mathfrak{S}$ is classical, then $\mathfrak{m}(\mathfrak{g})'$ is classical and of the same type as $\mathfrak{g}$.
(c) If $\mathfrak{n}$ is $\mathfrak{h}(\mathbb{A})$, then $\mathrm{Der}_o(\mathfrak{n})' \in \mathfrak{S}$.

Proof. By inspection of Table 1 ■

One obtains a filtration of Lie subalgebras

$$\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^{-1} \supset \cdots \supset \mathfrak{g}^{-k}$$

with $\mathfrak{g}^{-i-1} = \mathfrak{m}(\mathfrak{g}^{-i})'$, all of class $\mathfrak{S}$, with corresponding $\mathfrak{h}(\mathbb{A})$-nilradicals $\mathfrak{n}(\mathfrak{g}^{-i})$, such that

Proposition 2.7. *$\mathfrak{p}(\mathfrak{g}^{-k})$ is a minimal parabolic subalgebra of $\mathfrak{g}$, and $\bigoplus_{i=0}^{k} \mathfrak{n}(\mathfrak{g}^{-i})$ is a maximal nilpotent one.*

It follows that every classical $\mathfrak{g} \in \mathfrak{S}$ fits into a strictly increasing filtration of algebras $0 \subset \mathfrak{g}^{-k} \subset \ldots \subset \mathfrak{g}^{-1} \subset \mathfrak{g} \subset \mathfrak{g}^1 \subset \cdots$ of the same simple type as $\mathfrak{g}$ and satisfying $\mathfrak{g}^{i-1} = (\mathfrak{g}^i)'$.

Remark 2.8. The $\bigoplus_{i=0}^{j} \mathfrak{n}(\mathfrak{g}^{-i})$ are essentially Howe's H-tower algebras.

3. Symmetries

Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a 2-graded nilpotent Lie algebra with center $\mathfrak{z}$ and let $m = \dim \mathfrak{z}$ and $n = \dim \mathfrak{v}$. Since $\mathrm{Der}(\mathfrak{n}) = \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) \oplus \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$,

$$\dim \mathrm{Der}(\mathfrak{n}) = \dim \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) + mn.$$

Generically, $\dim \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) = 1$. Here we will see that if $\mathfrak{n}$ is of type H,

$$\dim \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) \geq \frac{1}{2} m(m+1).$$

Now let N be the csc Lie group with Lie algebra $\mathfrak{n}$, $\mathcal{V}$ the left-invariant distribution on N determined by $\mathfrak{v}$, and $\mathrm{Inf}(\mathfrak{n})$ the algebra of infinitesimal automorphisms of $\mathcal{V}$ at e, that is, germs of vector fields X on N near e such that $L_X(\mathcal{V}) \subset \mathcal{V}$. Clearly, $\mathrm{Inf}(\mathfrak{n}) \supset \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n})$. Then, generically, even among type H, $\dim \mathrm{Inf}(\mathfrak{n})/\mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) = \dim \mathfrak{n}$. For type $\mathfrak{h}(\mathbb{A})$ instead,

$$\dim \mathrm{Inf}(\mathfrak{n})/\mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) \geq 2 \dim \mathfrak{n}.$$

3.1. Algebras of type H. These are the graded 2-step nilpotent Lie algebras

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$$

admitting a graded inner product such that $\mathfrak{v}$ is a real unitary module over the Clifford algebra $C(\mathfrak{z})$, with bracket

$$< [V,V'], z >_{\mathfrak{z}} = < z \cdot V, V' >_{\mathfrak{v}},$$

$V, V' \in \mathfrak{v}$, $z \in \mathfrak{z}$. It is clear that type $\mathfrak{h}(\mathbb{A}) \Rightarrow$ type H. For the rest of this subsection $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ will denote an algebra of type H.

For $0 \neq z \in \mathfrak{z}$ let

$$J_z V = z \cdot V.$$

These operators on $\mathfrak{v}$ satisfy the Canonical Anticommutation Relations

$$J_z^2 = -|z|^2 I, \qquad J_z J_w + J_w J_z = - < z, w > I.$$

Let $r_z : \mathfrak{z} \to \mathfrak{z}$ denote the reflection with respect to the hyperplane orthogonal to z. Then the graded endomorphism of $\mathfrak{n}$ defined by

$$s_z(X + w) = J_z X - |z|^2 r_z w$$

is an automorphism. Let $\mathrm{Cl}(\mathfrak{n})$ be the subgroup of $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{n})$ generated by the s_z, and $\mathrm{Aut}_o(\mathfrak{n})$ the subgroup of $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{n})$ acting as the identity on $\mathfrak{z}$. Let $\mathfrak{cl}(\mathfrak{n}) = \mathrm{Lie}(\mathrm{Cl}(\mathfrak{n}))$ and $\mathrm{Der}_{\mathrm{o}}(\mathfrak{n}) = \mathrm{Lie}(\mathrm{Aut}_{\mathrm{o}}(\mathfrak{n}))$. There is a the semidirect decomposition [22]

$$\mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) = \mathfrak{cl}(\mathfrak{n}) \oplus \mathrm{Der}_{\mathrm{o}}(\mathfrak{n}).$$

Since $\dim \mathfrak{cl}(\mathfrak{n}) = \dim \mathfrak{so}(\mathfrak{z}) + 1$, this implies $\dim \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}) \geq m(m+1)/2$, as claimed before.

Lemma 3.1. *If two Lie algebras of type H are isomorphic then there exists an isometric isomorphism between them. In particular, the inner product on the center is unique up to multiple.*

Proof. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be of type H and $(\ ,\ )$, $\langle\ ,\ \rangle$ two graded inner products satisfying the defining condition. Let $P : \mathfrak{n} \to \mathfrak{n}$ be a graded linear map, symmetric and positive definite with respect to both inner products, that satisfies $\langle x, y \rangle = (Px, Py)$ for all $x, y \in \mathfrak{n}$. We will show that P is an automorphism of $\mathfrak{n}$.

For $Z \in \mathfrak{z}$, let J_Z and K_Z be the endomorphisms of $\mathfrak{v}$ defined by:

$$(J_Z X, Y) = (Z, [X,Y]\ ), \quad \langle K_Z X, Y \rangle = \langle Z, [X,Y] \rangle$$

Then $(P^2 K_Z X, Y) = \langle K_Z X, Y \rangle = \langle Z, [X,Y] \rangle = (P^2 Z, [X,Y]) = (J_{P^2 Z} X, Y)$ for $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. We conclude that $P^2 K_Z = J_{P^2 Z}$. Squaring both sides and multiplying by K_Z on the right we obtain $P^2 K_Z P^2 = \frac{(P^2 Z, P^2 Z)}{(PZ, PZ)} K_z$. Let $\{Z_i\}_{i=1}^n$ be a unitary basis of $\mathfrak{z}$ such that $PZ_i = \lambda_i Z_i$ with $\lambda_i > 0$, and set $W = Z_1 + \ldots + Z_m$. Then $P^2 K_{Z_i} P^2 = \frac{(P^2 Z_i, P^2 Z_i)}{(PZ_i, PZ_i)} K_{Z_i} = \lambda_i^2 K_{Z_i}$ and, therefore,

$$K_{\sum_i \lambda_i^2 Z_i} = \sum_i \lambda_i^2 K_{Z_i} = P^2 K_W P^2 = \frac{(P^2 W, P^2 W)}{(PW, PW)} K_W.$$

We conclude that $\frac{(P^2W,P^2W)}{(PW,PW)}W = \sum_i \lambda_i^2 Z_i$ and, therefore,

$$\lambda_i^2 = \frac{(P^2W, P^2W)}{(PW, PW)} =: \lambda^2$$

for all i. One has $P|_{\mathfrak{z}} = \lambda Id$ and on $\mathfrak{v}$, $P^2 K_Z P^2 = K_{\lambda^2 Z}$. Therefore K_Z interchanges the eigenspaces with eigenvalues μ^2 and λ^2/μ^2 of P^2, which are also the eigenspaces of P with eigenvalues μ and λ/μ, respectively. So, $PK_ZP = \lambda K_Z = K_{PZ}$. Since P is symmetric, it must be an automorphism. ■

Proposition 3.2. *Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an algebra of type H. Then $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{n})$ acts irreducibly in each summand.*

Proof. If $\mathfrak{v}$ is irreducible the assertion is true because $s_z \in \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{n})$.

Suppose now $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2 \oplus \mathfrak{u}$ where the $\mathfrak{v}_i$ are proper irreducible subrepresentations of $\mathfrak{v}$ and the sum is orthogonal. Then the type H subalgebras $\mathfrak{v}_1 \oplus \mathfrak{z}$ and $\mathfrak{v}_2 \oplus \mathfrak{z}$ are irreducible and therefore isomorphic. Let $\theta : \mathfrak{v}_1 \oplus \mathfrak{z} \to \mathfrak{v}_2 \oplus \mathfrak{z}$ be an isometric isomorphism. Then the orthogonal automorphism $\Theta : \mathfrak{n} \to \mathfrak{n}$ defined by $\Theta|_{\mathfrak{z}} = \theta|_{\mathfrak{z}}$, $\Theta|_{\mathfrak{v}_1} = \theta|_{\mathfrak{v}_1}$, $\Theta|_{\mathfrak{v}_2} = \theta|_{\mathfrak{v}_1}^{-1}$ and $\Theta|_{\mathfrak{u}} = Id$, interchanges the $\mathfrak{v}_i$. Since $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are arbitrary, this proves the assertion. ■

The 1-Iwasawa algebras are characterized among type H by the "J^2 condition" [9]. The following is a generalization of it to algebras of type $\mathfrak{h}(\mathbb{A})$.

Recall the decomposition determined by any non-zero $X \in \mathfrak{v}$:

$$\mathfrak{v} = \mathbb{R}X \oplus J_{\mathfrak{z}}X \oplus \mathfrak{k}(X)$$

where $J_{\mathfrak{z}}X = \{J_z X : \ z \in \mathfrak{z}\}$ and

$$\mathfrak{k}(X) = \{Y \in \mathfrak{v} : \ [X, Y] = 0, \ < X, Y >= 0\}.$$

Given a unitary pair (z_1, z_2) or triple (z_1, z_2, z_3) in $\mathfrak{z}$ and $X \neq 0$ in $\mathfrak{v}$, define

$$\zeta_2(z_1, z_2; X), \ \zeta_3(z_1, z_2, z_3; X) \in \mathfrak{z} \qquad \kappa_2(z_1, z_2; X), \ \kappa_3(z_1, z_2, z_3; X) \in \mathfrak{k}(X)$$

by

$$J_{z_1} J_{z_2} X = J_{\zeta_2(z_1, z_2; X)} X + \kappa_2(z_1, z_2; X)$$
$$J_{z_1} J_{z_2} J_{z_3} X = J_{\zeta_3(z_1, z_2, z_3; X)} X + \kappa_3(z_1, z_2, z_3; X).$$

and define subsets of $\mathfrak{v}$ by

$$K_2(z_1, z_2) = \{X \in \mathfrak{v} : \ \kappa_2(z_1, z_2; X) = 0\},$$

$$K_3(z_1, z_2, z_3) = \{X \in \mathfrak{v} : \ \kappa_3(z_1, z_2, z_3; X) = 0\}.$$

Theorem 3.3. *Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an algebra of type H. Then $\mathfrak{n}$ is of type $\mathfrak{h}(\mathbb{A})$ if and only if there is a linear decomposition*

$$\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$$

such that

$$\mathfrak{v}_1 \cup \mathfrak{v}_2 = \begin{cases} K_2(z_1, z_2) \quad \forall (z_1, z_2), \text{ if } \dim \mathfrak{z} \text{ is odd} & (2) \\ \displaystyle\bigcap_{z_1, z_2, z_3} K_3(z_1, z_2, z_3), \text{ if } \dim \mathfrak{z} \text{ is even} & (3) \end{cases}$$

Proof. Let $m = \dim \mathfrak{z}$. If $\mathfrak{n}$ is of type $\mathfrak{h}(\mathbb{A})$ with $m = 1, 3, 7$ and, in the case $m = 3$, $\mathfrak{v}$ is isotypic as a $C(3)$-module, the algebra satisfies the J^2 condition, which implies that the trivial decomposition with $\mathfrak{v}_2 = \{0\}$ satisfies (2).

If $m = 3$ but $\mathfrak{v}$ is not isotypic, $\mathfrak{n}$ does not satisfy the J^2 condition. Let $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ be the decomposition into isotypes. Then each of the algebras of type H $\mathfrak{v}_1 \oplus \mathfrak{z}$, $\mathfrak{v}_2 \oplus \mathfrak{z}$ does satisfy the J^2 condition, so that $\kappa_2(z, w, X) = 0$ if $X \in \mathfrak{v}_1 \cup \mathfrak{v}_2$. On the other hand, if $X_1 \in \mathfrak{v}_1$, $X_2 \in \mathfrak{v}_2$ are both non-zero,

$$J_{z_1} J_{z_2} (X_1 + X_2) = J_{z_3} (X_1 - X_2)$$

where (z_1, z_2, z_3) is an orthonormal basis of $\mathfrak{z}$. In particular $\kappa_2(z_1, z_2, X_1 + X_2) \neq 0$.

If $m = 2$, letting $\mathfrak{v}_2 = \{0\}$, (3) is trivially satisfied.

If $m = 4$, $\mathfrak{v}$ decomposes as two eigenspaces of $J_{z_1} J_{z_2} J_{z_3} J_{z_4}$, $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_{-1}$. For this decomposition $\kappa_3(z, w, v, X) = 0$ and $\kappa_3(z, w, v, Y) = 0$ for all unitary triple (z, w, v) in $\mathfrak{z}$, $X \in \mathfrak{v}_1$ and $Y \in \mathfrak{v}_{-1}$. If X, $Y \neq 0$, then

$$J_{z_1} J_{z_2} J_{z_3} (X + Y) = J_{z_4} (X - Y)$$

Therefore $\kappa_3(z_1, z_2, z_3, X + Y) \neq 0$.

If $m = 8$ and $\mathfrak{n}$ is irreducible, $\mathfrak{v}$ decomposes as two eigenspaces of $J_{z_1} \ldots J_{z_8}$, $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_{-1}$. For any $w \in \mathfrak{z}$, $J_w(\mathfrak{v}_1) = \mathfrak{v}_{-1}$ and $J_w(\mathfrak{v}_{-1}) = \mathfrak{v}_1$. If $X \in \mathfrak{v}_1$, since $\dim \mathfrak{v}_{-1} = 8$, $J_{\mathfrak{z}}(X) = \mathfrak{v}_{-1}$. Then $J_z J_w J_v X \in \mathfrak{v}_{-1} = J_{\mathfrak{z}}(X)$ and $\kappa_3(z, w, v, X) = 0$. Analogously, $\kappa_3(z, w, v, Y) = 0$ for $Y \in \mathfrak{v}_{-1}$. If X, $Y \neq 0$, there exist $v \in \mathfrak{z}$ such that $Y = J_v X$. Take z, $w \in \mathfrak{z}$ such that v, z and w are orthonormal. Then, $J_z J_w J_v (X + Y) = J_z J_w J_v (X + J_v X) = J_u X + J_v J_u X = J_u X - J_u J_v X = J_u (X - Y)$. We conclude that $\kappa_3(z, w, v, X + Y) \neq 0$.

Suppose that $K_2(z, w) = \mathfrak{v}_1 \cup \mathfrak{v}_2$ for every unitary pair z, w and $\mathfrak{v}_1 \neq \{0\}$. Fix $0 \neq X \in \mathfrak{v}_1$. Then the condition $J_z J_w X = J_u X$ defines a bilinear product such that $\mathfrak{z} \oplus \langle 1 \rangle$ is a division algebra and therefore $m = 1, 3, 7$.

Let see now that non irreducible Lie algebras of type H with $m = 7$ do not verify (2). Suppose that there is a decomposition $\mathfrak{v}_1 \oplus \mathfrak{v}_2$. Consider $J_{z_1} J_{z_2} J_{z_3}$ for orthonormal vector z_1, z_2, z_3. The eigenspaces of $J_{z_1} J_{z_2} J_{z_3}$ must be contained in $K_2(z_2, z_3) = \mathfrak{v}_1 \cup \mathfrak{v}_2$, so the decomposition $\mathfrak{v}_1 \oplus \mathfrak{v}_2$ must be the decomposition in eigenspaces of $J_{z_1} J_{z_2} J_{z_3}$. Then all triple products must have the same eigenspaces, hence commute. But $J_{z_1} J_{z_2} J_{z_3} J_{z_1} J_{z_2} J_{z_4} = -J_{z_1} J_{z_2} J_{z_4} J_{z_1} J_{z_2} J_{z_3}$, which is a contradiction.

Suppose now that $\bigcap_{z,w,v} K_3(z, w, v) = \mathfrak{v}_1 \cup \mathfrak{v}_2$ and $\mathfrak{v}_1 \neq \{0\}$. Let $0 \neq X \in \mathfrak{v}_1$. Fixing w we get a bilinear product on $\mathfrak{z}$ given by $z * v = -u$ where $J_z J_w J_v X = J_u X$. With this product $\mathfrak{z}$ is a division algebra with identity w, then $m = 1, 2, 4, 8$. Let see that the condition does not hold for $m = 8$ and $\mathfrak{n}$ non irreducible. First we observe that a decomposition $\mathfrak{v} = \mathfrak{n}_1 \oplus \mathfrak{n}_1^{\perp}$ in $C(8)$-modules is compatible with the decomposition $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$. In fact, if $X =$

$X_1 + X_2$ with $X_1 \in \mathfrak{n}_1$, $X_2 \in \mathfrak{n}_1^\perp$ then $\kappa_3(z_1, z_2, z_3; X) = 0$ if and only if $\kappa_3(z_1, z_2, z_3; X_i) = 0$ for $i = 1, 2$; for every unitary triple (z_1, z_2, z_3). Also we get, for $X \in \mathfrak{v}_1 \cup \mathfrak{v}_2$, $\zeta_3(z_1, z_2, z_3; X) = \zeta_3(z_1, z_2, z_3; X_1) = \zeta_3(z_1, z_2, z_3; X_2)$. This implies that $\zeta_3(z_1, z_2, z_3; X)$ is constant on $\mathfrak{v}_1$ for any fixed unitary triple (z_1, z_2, z_3). Let $z_4 := \zeta_3(z_1, z_2, z_3; X)$ for some $X \in \mathfrak{v}_1$ and suppose without loss of generality that $\dim \mathfrak{v}_1 \geq \dim \mathfrak{v}_2$, then $\mathfrak{v}_1$ must be the eigenspace of $J_{z_1} J_{z_2} J_{z_3} J_{z_4}$ corresponding to the eigenvalue 1. Let $z_5 \in \mathfrak{z}$ be orthonormal to z_1, z_2, z_3 and z_4; $z_6 := \zeta_3(z_1, z_2, z_5; X)$ for some (any) $X \in \mathfrak{v}_1$ and $z_7 \in \mathfrak{z}$ orthonormal to all z_i, $i = 1, \ldots, 6$. Then $J_{z_5} J_{z_7} X \in \mathfrak{v}_1$ because is an eigenvector of $J_{z_1} J_{z_2} J_{z_3} J_{z_4}$ but $\zeta_3(z_1, z_2, z_5; J_{z_5} J_{z_7} X) = -z_6 \neq \zeta_3(z_1, z_2, z_5; X)$, which is a contradiction. ■

3.2. Tanaka prolongation.

In this section we compare the semisimple prolongations (1) with Tanaka's prolongations. Recall that given a subalgebra $\mathfrak{g}_0 \subset \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n})$, $\mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0)$ is the smallest $\mathbb{Z}$-graded Lie algebra of the form $\bigoplus \mathfrak{g}_{i>0} \oplus \mathfrak{g}_0 \oplus \mathfrak{v} \oplus \mathfrak{z}$ such that for all $X \in \mathfrak{g}^i$, $[X, \mathfrak{v}] = 0$ implies $X = 0$. One writes $\mathrm{Prol}(\mathfrak{n}) = \mathrm{Prol}(\mathfrak{n}, \mathrm{Der}_{\mathrm{gr}}(\mathfrak{n}))$.

These prolongations are related to the infinitesimal automorphisms of the canonical distribution on N, as follows. $\mathrm{Inf}(\mathcal{V})$ has a natural filtration F such that [24]

$$\mathrm{Gr}^{\mathrm{F}} \mathrm{Inf}(\mathcal{V}) = \mathrm{Prol}(\mathfrak{n}).$$

More generally, if $\mathcal{V}$ carries an additional N-invariant geometric structure $\mathcal{G}$, and $\mathfrak{g}_0 \subset \mathrm{Der}_{gr}(\mathfrak{n})$ is the subalgebra of derivations preserving $\mathcal{G}_e$, then $\mathrm{Gr}^{\mathrm{F}} \mathrm{Inf}(\mathcal{V}, \mathcal{G}) = \mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0)$.

Examples 3.4.

(a) $\dim \mathrm{Prol}(\mathfrak{h}_n(\mathbb{R})) = \infty = \dim \mathrm{Prol}(\mathfrak{h}_n(\mathbb{C}))$

(b) If $\mathfrak{g}_0$ is compact semisimple, then $\mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0) = \mathfrak{n} \oplus \mathfrak{g}_0$, i.e., the prolongation is *trivial*.

(c) If $\mathfrak{g}_0$ is the subalgebra of derivations that preserves a conformal structure, then $\mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0)$ is trivial except for $\mathfrak{n} = \mathfrak{h}_n(\mathbb{C})$, $\mathfrak{h}'_{n,0}(\mathbb{H})$ and $\mathfrak{h}'_{1,0}(\mathbb{O})$, whose prolongations are $\mathfrak{su}(1, n+1)$, $\mathfrak{sp}(1, n+1)$ and FII, respectively [10, 17, 27].

The H-type algebras with center of dimension ≥ 3 have finite-dimensional prolongations [20]. Here we show that, more precisely

Theorem 3.5.

(a) An algebra of type H is of type $\mathfrak{h}(\mathbb{A})$ if and only $\mathrm{Prol}(\mathfrak{n})$ is not trivial.

(b) The semisimple prolongation $\mathfrak{g}(\mathfrak{n}, \mathfrak{g}_0)$ of (1) coincides with $\mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0)$ except for $\mathfrak{g}(\mathfrak{n}, \mathfrak{g}_0)$ isomorphic to $\mathfrak{sp}(n, \mathbb{R})$ or $\mathfrak{sp}(n, \mathbb{C})$.

Proof. From Proposition 3.2 and Lemma 5.8 in [26], its prolongation must be trivial, semisimple or infinite. The only Lie algebras of type H with infinite prolongation are the real and complex Heisenberg algebras, $\mathfrak{h}_n(\mathbb{R})$ and $\mathfrak{h}_n(\mathbb{C})$ (cf. [20]). If the prolongation of $\mathfrak{n}$ is semisimple, by the non-singularity condition it must be simple and $\mathfrak{n}$ must be the nilradical of a parabolic subalgebra of this simple Lie algebra. By Theorem 2.1, $\mathfrak{n} = \mathfrak{h}'_{p,q}(\mathbb{H}), \mathfrak{h}_n(\mathbb{H}), \mathfrak{h}'_{1,0}(\mathbb{O})$ or $\mathfrak{h}_1(\mathbb{O})$. The prolongation of these algebras are $\mathfrak{sp}(p+1, q+1)$, $\mathfrak{su}^*(2n+4)$, FII and EIV, respectively. ■

One consequence of the above discussion is that among the distributions with symbol of type H, *those with symbol of type* $\mathfrak{h}(\mathbb{A})$ *have compact Klein models.* More precisley, let $\mathcal{V}$ be the canonical distribution on a group N of type $\mathfrak{h}(\mathbb{A})$, and choose a simple G and a parabolic P with N as nilradical. The tangent space to G/P at the origin can be identified with $\bar{\mathfrak{n}} = \bar{\mathfrak{v}} \oplus \bar{\mathfrak{z}}$, and P respects this grading. Let $\bar{\mathcal{V}}$ be the G-invariant distribution on G/P determined by $\bar{\mathfrak{v}}$. Therefore

Proposition 3.6. *G/P carries a G-invariant distribution locally equivalent to $\mathcal{V}$.*

4. Associated Geometries

4.1. Parabolic geometries.

Consider now a parabolic geometry of type $\mathfrak{h}(\mathbb{A})$ on a manifold M, i.e., a Cartan geometry of type (G, P) where $\mathfrak{g} = Lie(G) \in \mathfrak{S}$ and $P \subset G$ is a parabolic subgroup with unipotent radical of type $\mathfrak{h}(\mathbb{A})$. Let ω be its Cartan connection and κ its curvature. Together with the gradings

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \qquad \mathfrak{n} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},$$

ω determines a distribution $\mathcal{D}$ on M together with a principal G_0-bundle, where G_0 is the subgroup of P that preserves the grading of $\mathfrak{g}$. Then G_0 is isomorphic to a subgroup of $\mathrm{Aut}_{gr}(\mathfrak{n})$ and $Lie(G_0) = \mathfrak{g}_0$.

ω is called *regular* when $\mathcal{D}$ has constant symbol isomorphic to $\mathfrak{n}$ and the principal G_0-bundle is a reduction of the canonical $\mathrm{Aut}_{gr}(\mathfrak{n})$-bundle. When $G_0 = \mathrm{Aut}_{gr}(\mathfrak{n})$, we have just a distribution of constant symbol. ω is called *normal* if it satisfies $\partial^*\kappa = 0$ where ∂^* is the Kostant codifferential. This condition assures the uniqueness of the Cartan connection.

Since a distribution is fat if and only if its symbol is non-singular, Theorem 1 implies

Theorem 4.1. *The regular normal parabolic geometries supported on fat distributions are exactly those of $\mathfrak{h}(\mathbb{A})$ type.*

Remark 4.2.

(a) Distributions with symbol $\mathfrak{h}_n(\mathbb{R})$ are associated to contact parabolic geometries: Lagangean, partially integrable almost CR, Lie contact, contact projective, and exotic contact structures [8].

(b) Distributions with symbol $\mathfrak{h}_n(\mathbb{C})$ are associated to the complex contact structures of Boothby [4] and, more generally, to partially integrable almost CR-structures of CR-codimension 2 with additional structure. These have not received much attention except for special cases [7].

(c) For the cases $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{g}$ is not the prolongation of $(\mathfrak{n}, \mathfrak{g}_0)$, so the underlying structure they determine on the manifold is just a real or complex contact structure with the canonical $\mathrm{Aut}_{gr}(\mathfrak{n})$-bundle. To characterize this parabolic geometries we have to consider finer underlying structures, in this case are a contact projective structure, i.e. a contact projective equivalence class of partial contact connections [8].

(d) For all the other $\mathfrak{h}(\mathbb{A})$ algebras the parabolic geometry is determined by the distribution alone, with no additional structure (Proposition 4.3.1 in [8]). Quaternionic and octonionic contact structures associated to $\mathfrak{h}'_{p,q}(\mathbb{H})$ and $\mathfrak{h}'_{1,0}(\mathbb{O})$ have been the subject of interest [2, 8].

(e) Distributions whose symbol is $\mathfrak{h}_1(\mathbb{O})$ or $\mathfrak{h}'_{1,0}(\mathbb{O})$ are locally isomorphic to the flat model. This is a consequence of the fact that the second generalized Spencer cohomology groups vanish in these cases [26].

4.2. On the infinity of Damek-Ricci spaces.

Let N be a group of type H, $A = \exp(\mathbb{R}\delta)$ the group of dilations and $S = AN$ their semidirect product. Endowing $\mathfrak{n}$ with a compatible metric induces a left-invariant riemannian metric g on S, called a Damek-Ricci space [21]. S is harmonic - in fact *every* harmonic space which is homogeneous and non-compact is isometric to S for some N of type H [14], and therefore Einstein. One is interested in the asymptotic behavior of the metric g.

If S is hyperbolic the metric satisfies

$$g = dt^2 + e^t\gamma + e^{2t}\delta + o(e^{-t}) \tag{4}$$

for $t \to \infty$, where $(\gamma, \mathcal{D}^\delta)$ is a generalized G-conformal structure in the sense of Biquard-Mazzeo [3] on the geodesic boundary of S.

For the general S no such formula seems to exist (cf. [5]) - unless $\mathfrak{n}$ is of type $\mathfrak{h}(\mathbb{A})$. For the first statement consider the Poincaré-like realization of S in euclidean unit ball $\mathbb{B}$ of the same dimension, as well as the Siegel-like one on

$$\mathcal{U} = \{(X, Z, t) \in \mathfrak{v} \times \mathfrak{z} \times \mathbb{R} : \quad t > \frac{1}{4}|X|^2\}.$$

The Cayley transform $\mathcal{C} : \mathcal{U} \to \mathbb{B}$

$$\mathcal{C}(X, Z, t) = \frac{1}{(1+t)^2 + |Z|^2}\big((1 + t - J_Z)X, 2Z, -1 + t^2 - |Z|^2\big)$$

is a diffeomorphism. It extends to the boundary of $\mathcal{U}$ in $v \times z \times \mathbb{R}$, $\partial\mathcal{U} = \{(X, Z, \frac{1}{4}|X|^2) : X \in \mathfrak{v},\ Z \in \mathfrak{z}\}$ giving a diffeomorphism

$$\mathcal{C}_\partial : \partial\mathcal{U} \to \mathbb{S}^*$$

onto the punctured sphere. N acts simply transitively on $\partial\mathcal{U}$, hence on $\mathbb{S}^*$, and its canonical distribution induces invariant distributions on these boundaries. Writing

$$T_{(X,Z,\frac{1}{4}|X|^2)}(\partial\mathcal{U}) = \{(2Y, W, < X, Y >) : Y \in \mathfrak{v},\ W \in \mathfrak{z}\},$$

the distribution is given by

$$\mathcal{D}^{\partial\mathcal{U}}_{(X,Z,\frac{1}{4}|X|^2)} = \{(Y, \frac{1}{2}[X, Y], \frac{1}{2} < X, Y >) : Y \in \mathfrak{v}\}.$$

Let now $\mathcal{D}^{\mathbb{S}^*} = d\mathcal{C}_\partial(\mathcal{D}^{\partial\mathcal{U}})$ and let ∞ denote the puncture of $\mathbb{S}^*$.

Proposition 4.3. *$\mathcal{D}^{\mathbb{S}^*}$ extends smoothly over ∞ if and only if S is a hyperbolic space.*

Proof. If S is a hyperbolic space G/K, K is transitive on $\mathbb{S}$ and leaves invariant the distribution, hence it can have no singularities.

Otherwise, N does not satisfy the J^2 condition of [9]. This implies that there is a unitary triple $X \in \mathfrak{v}$, $Z, W \in \mathfrak{z}$ such that $[X, J_Z J_W X] = 0$. The vector fields on $\partial\mathcal{U} \cong \mathfrak{v} \times \mathfrak{z}$

$$(v_1)_{(X,Z)} = (X, 0, \frac{1}{2}|X|), \qquad (v_2)_{(X,Z)} = (J_Z X, \frac{1}{2}|X|Z, 0)$$

correspond to the copy of $\mathfrak{h}_1(\mathbb{R})$ spanned by the triple $X, J_Z X, Z$. On $\mathbb{S}^*$ and along the orbit $\exp(\mathfrak{h}_1(\mathbb{R})) \cdot (-\infty)$, the plane spanned by $d\mathcal{C}(v_1)$, $d\mathcal{C}(v_2)$, is horizontal and has a limit as $|X|, |Z| \to \infty$, namely the plane $(\mathbb{R}X \oplus \mathbb{R}J_Z X, 0, 0)$. Doing the same with the copy of $\mathfrak{h}_1(\mathbb{R})$ spanned by $J_Z J_W X, J_W X, Z$, the corresponding limiting plane is $(\mathbb{R}J_W X \oplus \mathbb{R}J_Z J_W X, 0, 0)$. Therefore, if the distribution extends, its value at ∞ must be $(\mathfrak{v}, 0, 0)$. On the other hand, the vector

$$((1 + |Z|^2 - \frac{1}{16}|X|^4)J_W X, (1 + \frac{1}{4}|X|^2)|X|^2 W, 0)$$

is horizontal along the curve $1 + |Z|^2 = \frac{1}{16}|X|^4$, where it spans line $(0, \mathbb{R}W, 0)$, which is a contradiction. ∎

The proof has the following

Corollary 4.4. *AN is not hyperbolic if and only if $\mathfrak{n}$ contains a copy of $\mathfrak{h}_1(\mathbb{C})$.*

If N is of type $\mathfrak{h}(\mathbb{A})$ however, the S-orbit of any point gives an isometric embedding into the associated symmetric space

$$S \hookrightarrow G/K.$$

Table 1: Langlands factors of $\mathfrak{h}(\mathbb{A})$ parabolics

$\mathfrak{g}$	$\mathfrak{m}$	$\dim\mathfrak{a}$	$\mathfrak{n}$	Σ
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(n-2,\mathbb{R})$	2	$\mathfrak{h}_{n-2}(\mathbb{R})$	$\{\alpha_1,\alpha_{n-1}\}$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n-2,\mathbb{C})\oplus\mathbb{R}^2$	2	$\mathfrak{h}_{n-2}(\mathbb{C})$	$\{\alpha_1,\alpha_{n-1}\}$
$\mathfrak{su}^*(2n)$	$\mathfrak{su}(2)^2\oplus\mathfrak{su}^*(2n-4)$	2	$\mathfrak{h}_{n-2}(\mathbb{H})$	$\{\alpha_2,\alpha_{n-2}\}$
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p-1,q-1)\oplus\mathbb{R}$	1	$\mathfrak{h}_{p+q-2}(\mathbb{R})$	$\{\alpha_1,\alpha_{p+q-1}\}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(n-1,\mathbb{R})$	1	$\mathfrak{h}_{n-1}(\mathbb{R})$	$\{\alpha_1\}$
$\mathfrak{sp}(p,q)$	$\mathfrak{su}(2)\oplus\mathfrak{sp}(p-1,q-1)$	1	$\mathfrak{h}'_{p-1,q-1}(\mathbb{H})$	$\{\alpha_2\}$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n-1,\mathbb{C})\oplus\mathbb{R}$	1	$\mathfrak{h}_{n-1}(\mathbb{C})$	$\{\alpha_1\}$
$\mathfrak{so}(p,q)$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(p-2,q-2)$	1	$\mathfrak{h}_{p+q-4}(\mathbb{R})$	$\{\alpha_2\}$
$\mathfrak{so}^*(2n)$	$\mathfrak{su}(2)\oplus\mathfrak{so}^*(2n-4)$	1	$\mathfrak{h}_{2n-4}(\mathbb{R})$	$\{\alpha_2\}$
$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{so}(n-4,\mathbb{C})\oplus\mathbb{R}$	1	$\mathfrak{h}_{n-4}(\mathbb{C})$	$\{\alpha_2\}$
EI	$\mathfrak{sl}(6,\mathbb{R})$	1	$\mathfrak{h}_{10}(\mathbb{R})$	$\{\alpha_6\}$
EII	$\mathfrak{su}(3,3)$	1	$\mathfrak{h}_{10}(\mathbb{R})$	$\{\alpha_6\}$
$EIII$	$\mathfrak{su}(1,5)$	1	$\mathfrak{h}_{10}(\mathbb{R})$	$\{\alpha_6\}$
EIV	$\mathfrak{so}(8)$	2	$\mathfrak{h}_1(\mathbb{O})$	$\{\alpha_1,\alpha_5\}$
E_6	$\mathfrak{sl}(6,\mathbb{C})\oplus\mathbb{R}$	1	$\mathfrak{h}_{10}(\mathbb{C})$	$\{\alpha_6\}$
EV	$\mathfrak{so}(6,6)$	1	$\mathfrak{h}_{16}(\mathbb{R})$	$\{\alpha_6\}$
EVI	$\mathfrak{so}^*(12)$	1	$\mathfrak{h}_{16}(\mathbb{R})$	$\{\alpha_6\}$
$EVII$	$\mathfrak{so}(2,10)$	1	$\mathfrak{h}_{16}(\mathbb{R})$	$\{\alpha_6\}$
E_7	$\mathfrak{so}(12,\mathbb{C})\oplus\mathbb{R}$	1	$\mathfrak{h}_{16}(\mathbb{C})$	$\{\alpha_6\}$
$EVIII$	EV	1	$\mathfrak{h}_{28}(\mathbb{R})$	$\{\alpha_1\}$
EIX	$EVII$	1	$\mathfrak{h}_{28}(\mathbb{R})$	$\{\alpha_1\}$
E_8	$E_7\oplus\mathbb{R}$	1	$\mathfrak{h}_{28}(\mathbb{C})$	$\{\alpha_1\}$
FI	$\mathfrak{sp}(3,\mathbb{R})$	1	$\mathfrak{h}_7(\mathbb{R})$	$\{\alpha_4\}$
FII	$\mathfrak{so}(7)$	1	$\mathfrak{h}'_{1,0}(\mathbb{O})$	$\{\alpha_1\}$
F_4	$\mathfrak{sp}(3,\mathbb{C})\oplus\mathbb{R}$	1	$\mathfrak{h}_7(\mathbb{C})$	$\{\alpha_4\}$
G	$\mathfrak{sl}(2,\mathbb{R})$	1	$\mathfrak{h}_2(\mathbb{R})$	$\{\alpha_2\}$
G_2	$\mathfrak{sl}(2,\mathbb{C})\oplus\mathbb{R}$	1	$\mathfrak{h}_2(\mathbb{C})$	$\{\alpha_2\}$

Denoting by ∂S the boundary of S in a compactification of G/K as a manifold with corners, the natural projection

$$\pi : \partial S \to \mathbb{S}$$

onto the geodesic spherical boundary resolves the singularities of $\mathcal{D}^{\mathbb{S}^*}$. As a consequence, (4) holds. Such formula seems to characterize the $\mathfrak{h}(\mathbb{A})$ among all H-types and, more importantly, to yield new examples of anisotropic Einstein metrics admitting non-regular deformations, as in [3]. Details are left for a sequel, where the boundary structures (γ,δ) will be described for each $\mathfrak{h}(\mathbb{A})$ type.

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Received June 15, 2017
and in final form December 10, 2017