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A Simple Proof of Lie's Theorem

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Abstract. We give a new proof of Lie's theorem on solvable Lie algebras.
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Lie's theorem on representations of solvable algebras is one of most basic results in the theory of Lie algebras. There are two ways to prove this theorem. The "standard" one, that is the one contained in almost all textbooks, uses Dynkin's lemma, which is nontrivial by itself. See e.g. [3], [5]. (Recall that Dynkin's lemma reads as follows: if L is a Lie algebra, V is an L-module, I is an ideal of L, and $\alpha \in I^*$ is a linear function on I, then

$$U = \{v \in V \mid hv = \alpha(h)v, \ \forall\, h \in I\}$$

is an L-submodule.) The other way uses some simple structure theory of Lie algebras and their representations, see e.g. [1], [4], [2]. In this approach, one needs to prove first Engel's theorem, and Lie's theorem is then derived from that of Engel.

In the present note we give another proof, which is not more difficult than the traditional ones, at least in the author's opinion. Thus, the aim of this note is mostly methodical or pedagogical.

We suppose that the reader is familiar with the most elementary concepts related to Lie algebras: algebra, subalgebra, ideal, solvable algebra, and representation (see [3], §1–2). All the algebras and representations or modules that we consider are supposed to be of finite dimension.

Let L be a Lie algebra over a field K and $\rho : L \longrightarrow \mathfrak{gl}(V)$ be a representation of L by operators on a space V. Then ρ is said to be *triangulizable* if there exists a basis of V such that the matrices of all $\rho(x)$, $x \in L$, in this basis are upper triangular. In other words, ρ is triangulizable if there exists a flag of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

where $n = \dim V$ and $\dim V_i/V_{i-1} = 1$ for all i, such that all V_i are invariant under $\rho(L)$.

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Note that if L is a Lie algebra, then the condition that any representation of L is triangulizable is equivalent to the condition that any irreducible representation/module is of dimension 1.

Lie's Theorem. *If K is an algebraically closed field of characteristic 0 and L is a solvable Lie algebra over K, then any (finite-dimensional) representation of L is triangulizable.*

To prove Lie's theorem, we need several well-known simple statements. Let x be an operator acting on a space V. For $\lambda \in K$ let

$$V(x, \lambda) = \{v \in V \mid xv = \lambda v\}$$

be the subspace of all eigenvectors of x corresponding to eigenvalue λ. Clearly, $V(x, \lambda) \neq 0$ if and only if λ is an eigenvalue of x.

Let $\rho : L \longrightarrow \mathfrak{gl}(V)$ be a representation of a Lie algebra. The following two statements hold:

(1) *If an element $z \in L$ is in the center of L, then $V(\rho(z), \lambda)$ is an L-invariant subspace for any $\lambda \in K$.*

(2) *Let $I \lhd L$ be an ideal. Then its "zero subspace"*

$$N = \{v \in V \mid \rho(x)v = 0, \ \forall\, x \in I\}$$

is invariant under L.

These statements are easy and standard and we refrain from proving them.

We need a lemma.

Lemma. *Suppose K is algebraically closed and of characteristic 0. Let x and y be linear operators on V such that $[x, y] = xy - yx = \lambda x$, where $\lambda \neq 0$. Then $V(x, 0)$ $(= \operatorname{Ker} x) \neq 0$.*

Proof. We show first that if $v \in V(y, \mu)$ is an eigenvector for y, then xv is also an eigenvector with eigenvalue $\mu - \lambda$ or $xv = 0$. Indeed, $[x, y] = \lambda x$ implies $x(yv) - y(xv) = \lambda(xv)$, whence $x(\mu v) - y(xv) = \lambda(xv)$, whence $y(xv) = (\mu - \lambda)(xv)$.

Now let μ be an eigenvalue for y and v be a non-zero eigenvector. Then induction on k shows that always $x^k v \in V(y, \mu - k\lambda)$. If $x^k v \neq 0$ for all k, then all $\mu - k\lambda$ are eigenvalues for x, a contradiction. So $x^k v = 0$ for large k, whence the lemma follows. ■

Now we prove the theorem. It is sufficient to prove that any L-module V of dimension > 1 contains a proper submodule. We argue by induction on $\dim L$.

If $\dim L = 1$, then $L = \langle x \rangle$, and it suffices to note that $\rho(x)$ has an eigenvector.

Let $\dim L > 1$. Obviously, we may assume that the representation of L on V is faithful. Thus, L is identified with a subalgebra of $\mathfrak{gl}(V)$.

Let us show that L has an ideal of dimension 1. Since L is solvable, there exists a nontrivial ideal $I \lhd L$ such that $[I, I] = 0$. Then we see, in a usual way, that I is a module over L/I. Since $\dim L/I < \dim L$, we may apply the induction hypothesis and see that there exists an L/I-invariant subspace $J \subseteq I$ of dimension 1. Then, clearly, J is an ideal of L.

Thus, let $\langle x \rangle$ be a one-dimensional ideal of L. There are three possibilities: (i) x is a non-central element of L, (ii) x is central, but not a scalar transformation of V, and (iii) x is scalar.

In case (i) there exists $y \in L$ such that $[x, y] = \lambda x$, $\lambda \neq 0$. By the lemma, $V(x, 0)$ is a proper nontrivial subspace of V. But $V(x, 0)$ is an L-submodule, by observation (2) above.

In case (ii), let λ be an eigenvalue of x. Then $V(x, \lambda) \neq 0, V$, and, moreover, $V(x, \lambda)$ is a submodule by observation (1).

Finally, suppose that case (iii) holds. Consider the subspace

$$L_0 = \{y \in L \mid \operatorname{tr} y = 0\}.$$

As $\operatorname{tr} x \neq 0$, it is easy to see that $L = \langle x \rangle \oplus L_0$, the direct sum of subspaces. Moreover, $[a, b] \in L_0$ for all $a, b \in L$, because $\operatorname{tr}[a, b] = 0$ for any $a, b \in \mathfrak{gl}(V)$. In particular, L_0 is a subalgebra. But $\dim L_0 = \dim L - 1$, so V contains an L_0-submodule $\langle v \rangle$ of dimension 1, by the induction hypothesis. Obviously, this submodule is invariant under x as well, which finishes the proof. ■

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