Journal of Lie Theory
Volume **28** (2018) 581–607

A Characterization of the L^2-Range of the Poisson Transform with Real and Singular Spectral Parameter on Symmetric Spaces of Noncompact Type

Koichi Kaizuka*

Communicated by G. Ólafsson

Abstract. In the previous work, we proved the Strichartz conjecture concerning an image characterization of the Poisson transform with real and regular spectral parameter. In this paper, by developing weighted L^2-estimates on symmetric spaces of noncompact type, we extend our previous result to the case of real and singular spectral parameter. In the real and singular case, a certain degeneracy appears in the scattering formula for the Poisson transform.
Mathematics Subject Classification 2010: Primary 43A85; Secondary 22E30, 43A90.
Key Words and Phrases: Symmetric space, Poisson transform, joint eigenfunction, scattering theory.

1. Introduction

Let us start with a famous result on a characterization of the joint eigenspaces of invariant differential operators on a symmetric space X of noncompact type. In [10], Helgason conjectured that any joint eigenfunction on symmetric spaces of noncompact type is expressed as the image of the Poisson transform of an analytic functional on the boundary B. This conjecture, now called the Helgason conjecture, was proved by Kashiwara et al. [17]. After that, various kinds of image characterizations of the Poisson transform have been extensively studied by many people in connection with representation theory, harmonic analysis, and spectral theory (see e.g. [5], [20], [24], and [6], [7], [14], [16], [18], [19], [21], [28], [29], [30] dealing with $L^2(B)$ or $L^p(B)$).

From the point of view of the spectral theory, Strichartz [29] conjectured an image characterization of the Poisson transform of $L^2(B)$ when spectral parameter is real and regular (see [29, Conjecture 4.5]). In [16], using the idea of the scattering theory (see e.g. [1], [2]), the author proved the conjecture as follows. (For the

*This work was partially supported by JSPS Grant-in-Aid for Young Scientists (B) #17K14208.

ISSN 0949–5932 / $2.50

details, see [16], Theorem 3.3 and Theorem 6.1.) We assume that $\lambda \in \mathfrak{a}^*_{\mathrm{reg}}$ and $f \in \mathcal{E}_\lambda(X)$. Then the joint eigenfunction f is expressed as $f = \mathcal{P}_\lambda F$ for some $F \in L^2(B)$ (which is in fact unique) if and only if

$$\|f\|_* := \sup_{R>1} \frac{1}{R^{l/2}} \left(\int_{B(o,R)} |f(x)|^2 dx \right)^{1/2} < \infty,$$

where l denotes the rank of X and $B(o, R)$ the open ball in X centered at the origin $o = eK$ with radius R. Moreover, there exists a positive constant C independent of $\lambda \in \mathfrak{a}^*_{\mathrm{reg}}$ such that

$$C^{-1} |\boldsymbol{c}(\lambda)| \|F\|_{L^2(B)} \leqslant \|\mathcal{P}_\lambda F\|_* \leqslant C |\boldsymbol{c}(\lambda)| \|F\|_{L^2(B)}, \tag{1.1}$$

where $\boldsymbol{c}(\lambda)$ denotes Harish-Chandra's $\boldsymbol{c}$-function. In addition, the joint eigenfunction $\mathcal{P}_\lambda F$ satisfies the following scattering formula at infinity.

$$\mathcal{P}_\lambda F(x) \simeq \sum_{w \in W} e^{(iw\lambda - \rho)(A^+(x))} \boldsymbol{c}(w\lambda) [U_{w,\lambda} F](b_x), \tag{1.2}$$

where W denotes the Weyl group, $U_{w,\lambda}$ a unitary intertwining operator between two spherical principal series representations τ_λ and $\tau_{w\lambda}$ (see Lemma 3.4), and $f_1 \simeq f_2$ means that

$$\lim_{R \to \infty} \frac{1}{R^l} \int_{B(o,R)} |f_1(x) - f_2(x)|^2 dx = 0.$$

The right hand side of (1.2) can be viewed as follows. The joint eigenfunction is approximated by oscillating $|W|$-wave functions $e^{(iw\lambda - \rho)(A^+(x))}$, amplitudes $\boldsymbol{c}(w\lambda)$, and boundary values $U_{w,\lambda} F(b)$ when λ is real and regular. As in the case of scattering theory for Laplacians, the mean-L^2-norm $\|\cdot\|_*$ and the equivalence relation $\simeq$ give a precise characterization of oscillating joint eigenfunctions. (cf. [6], [18], [19].)

In this paper, we develop our scattering theory and prove an image characterization of the Poisson transform $\mathcal{P}_{\lambda_0}$ of $L^2(B)$ for a real and singular spectral parameter $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}} = \mathfrak{a}^* \setminus \mathfrak{a}^*_{\mathrm{reg}}$. When a sequence $\{\lambda_j\}_{j=1}^\infty \subset \mathfrak{a}^*_{\mathrm{reg}}$ converges to $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, the term $|\boldsymbol{c}(\lambda_j)|$ in (1.1) diverges to infinity as $j \to \infty$. Hence, we can see that the mean-L^2-norm $\|\cdot\|_*$ is not well-defined in the singular case. We also see that the coefficients $\boldsymbol{c}(w\lambda_j)$ in (1.2) diverges. In fact, it turns out that joint eigenfunctions degenerate at infinity in the real and singular case (cf. van den Ban and Schlichtkrull [5, Corollary 16.4] or Oshima [23, Theorem 3.6])). It is known that a similar phenomenon happens also in the scattering theory for self-adjoint operators (e.g. Schrödinger operators). Roughly speaking, spectral parameters $\lambda \in \mathfrak{a}^*$ and $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$ of $\boldsymbol{D}(X)$ correspond to that of continuous spectrum and of threshold spectrum of self-adjoint operators, respectively.

As a natural consequence of the above observation, in order to characterize the L^2-image of the Poisson transform for $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, we have to replace the norm $\|\cdot\|_*$ with a suitable mean-L^2-norm in accordance with the order of degeneracy of $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$. For $\sigma > 0$, $f \in L^2_{\mathrm{loc}}(X)$, we define a norm $\|\cdot\|_{*,\sigma}$ by

$$\|f\|_{*,\sigma} := \sup_{R>1} \frac{1}{R^\sigma} \left(\int_{B(o,R)} |f(x)|^2 dx \right)^{1/2}.$$

Then $B^*_\sigma(X) := \{f \in L^2_{\mathrm{loc}}(X); \|f\|_{*,\sigma} < \infty\}$ becomes a Banach space equipped with the norm $\|\cdot\|_{*,\sigma}$. We define an equivalence relation $\simeq$ on $B^*_\sigma(X)$ as follows. For $f_1, f_2 \in B^*_\sigma(X)$, we write $f_1 \simeq f_2$ in $B^*_\sigma(X)$ if

$$\lim_{R\to\infty} \frac{1}{R^\sigma} \left(\int_{B(o,R)} |f_1(x) - f_2(x)|^2 dx \right)^{1/2} = 0.$$

For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, we put $W_{\lambda_0} = \{w \in W; w\lambda_0 = \lambda_0\}$, $\Sigma^0_{\lambda_0} = \{\alpha \in \Sigma^+_0; \langle \alpha, \lambda_0 \rangle = 0\}$. We define $\nu_0 \in \mathbb{N}$, the order of degeneracy of λ_0, by

$$\nu_0 = l + 2|\Sigma^0_{\lambda_0}|.$$

Remark 1.1. The positive integer $\nu_X := l + 2|\Sigma^+_0|$ is called the pseudo-dimension of X in [8]. The pseudo-dimension ν_X also appears in the spectral analysis at the threshold for the Laplace-Beltrami operator on X (see e.g. [15, Theorem 1.1]). Hence, ν_0 can be interpreted as an analogous order for ν_X.

We give a characterization and an asymptotic behavior of a joint eigenfunction $f \in \mathcal{E}_{\lambda_0}(X)$ when f belongs to the Banach space $B^*_{\nu_0/2}(X)$. Now let us state our main result (for the details, see Theorem 2.1 and Theorem 6.1).

Main Theorem. *For any $f \in \mathcal{E}_{\lambda_0}(X)$, f is written as $f = \mathcal{P}_{\lambda_0} F$ for some $F \in L^2(B)$ if and only if $f \in B^*_{\nu_0/2}(X)$. Moreover, we have the scattering formula*

$$f(x) \simeq \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0 - \rho)(A^+(x))} \boldsymbol{a}_0(x, \lambda_0; w)[U_{w,\lambda_0} F](b_x) \quad \text{in } B^*_{\nu_0/2}(X),$$

where $\boldsymbol{a}_0$ denotes a certain real-valued function on $X \times \mathfrak{a}^*_{\mathrm{sing}} \times W$.

Unfortunately, in the real and singular case, our method in [16] does not directly work because of singularities of the $\boldsymbol{c}$-function. We therefore modify our previous method by adopting two crucial techniques. The first one is an elimination technique of the singularity of the $\boldsymbol{c}$-function by a certain polynomial function. This technique was developed in [5], [22] and applied to derive asymptotic expnasions or bounds of the elementary spherical function in the general case. The second one is a face decomposition of the positive Weyl chamber near Weyl walls. The estimates of the elementary spherical function near Weyl walls obtained by the first technique are not sharp enough to be immediately applied to the scattering formula. In order to neglect the error factors of the estimates in the scattering formula, we cut the positive Weyl chamber along Weyl walls by a function having logarithmic growth at infinity. By making use of those two techniques, we obtain a global estimate of the elementary spherical function and establish the stationary scattering theory for joint eigenfunctions in the real and singular case.

The purpose of this paper and [16] is to give a time-independent framework of the scattering theory, called the stationary scattering theory, for a system of invariant differential operators. Here, we would like to mention that there is another approach of time-dependent scattering theory. In [26], Semenov-Tjan-Šanskiĭ introduced a wave equation with the multi-time variables $H \in \mathfrak{a}$ for

$\boldsymbol{D}(X)$, called the multitemporal wave equation. The time-dependent scattering theory and related results for the multitemporal wave equation have been further developed by Helgason [11], Phillips and Shahshahani [25], Shahshahani [27].

Finally, this paper is organized as follows. In Section 2, in the first part, we introduce basic notation for harmonic analysis on symmetric spaces of noncompact type. In the last part, we state our main result (Theorem 2.1) in detail. In Section 3, we recall basic properties of the Poisson transform, Radon transform, and Helgason Fourier transform on symmetric spaces of noncompact type. In Section 4, we give a global estimate for the elementary spherical function on the positive Weyl chamber in the real and singular case (Lemma 4.4). In Section 5, we show a Fourier restriction estimate for the Helgason Fourier transform (Lemma 5.1). In Section 6, by Lemma 4.4 and Lemma 5.1, we derive the scattering formula for the Poisson transform (Theorem 6.1). In Section 7, we prove the main result (Theorem 2.1) and our main theorem. In Section 8, as an application of Theorem 2.1, we give an explicit realization of an inner product of the Hilbert space associated with the left-regular representation.

2. Notation and main result

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, real numbers, positive real numbers, and complex numbers, respectively. Let i be the imaginary unit defined by $i = \sqrt{-1}$. Let $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real part and the imaginary part of $z \in \mathbb{C}$, respectively. If $\{X_i\}_i$ is a disjoint collection of sets, we use the symbol $\sqcup_i X_i$ as a union of disjoint sets X_i. For a function, or a distribution f, we denote its support by $\operatorname{supp} f$. For nonnegative functions f_1 and f_2 on a set, we write $f_1 \asymp f_2$ if there exists a positive constant C such that $C^{-1} f_2 \leqslant f_1 \leqslant C f_2$. For a smooth manifold $\mathcal{M}$, we denote the spaces of $\mathbb{C}$-valued smooth functions on $\mathcal{M}$ and that of compact support by $C^\infty(\mathcal{M})$ and $C_0^\infty(\mathcal{M})$, respectively. Let $C(\mathcal{M})$ denote the space of $\mathbb{C}$-valued continuous functions on $\mathcal{M}$.

We basically follow the notation in the book [13] of Helgason for harmonic analysis on symmetric spaces. Let G be a noncompact, connected, semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, and $X = G/K$ the associated symmetric space. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let θ be the Cartan involution associated with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. Let $B(\cdot,\cdot)$ denote the Killing form on $\mathfrak{g}$ given by $B(X_1, X_2) = \operatorname{Tr}(\operatorname{ad}X_1 \circ \operatorname{ad}X_2)$ for $X_1, X_2 \in \mathfrak{g}$. We define the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ by $\langle X_1, X_2\rangle = -B(X_1, \theta(X_2))$ for $X_1, X_2 \in \mathfrak{g}$. Then the Killing form metric induces the G-invariant metric on $X = G/K$. Let L_X be the Laplace-Beltrami operator on X. Let $d(\cdot,\cdot)$ denote the distance function on X. We put $r(x) = d(x, o)$ for $x \in X$. Let $B(x, R)$ denote the open ball in X centered at $x \in X$ with radius R. Let dg be a suitable normalized Haar measure on G. Let dx be a left-G-invariant measure on X. We put $L^2(X) = L^2(X, dx)$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ its dual space. Let $l = \dim_{\mathbb{R}} \mathfrak{a}$ be the rank of X. For $\alpha \in \mathfrak{a}^*$, we put $\mathfrak{g}_\alpha = \{Y \in \mathfrak{g}; [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}\}$. We set $\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\}; \mathfrak{g}_\alpha \neq \{0\}\}$, the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then we have the following root space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \{\oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha\}$ of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $\mathfrak{a}_{\mathrm{reg}} = \{H \in \mathfrak{a}; \alpha(H) \neq 0 \text{ for all } \alpha \in \Sigma\}$ be the set of regular elements in $\mathfrak{a}$. We choose a connected component $\mathfrak{a}_{\mathrm{reg}}$ in $\mathfrak{a}$. Let $\mathfrak{a}^+$ denote the connected component, called a positive Weyl chamber. We also define the sets of indivisible roots, positive roots, and positive indivisible roots by $\Sigma_0 = \{\alpha \in \Sigma; \alpha/2 \notin \Sigma\}$, $\Sigma^+ = \{\alpha \in \Sigma; \alpha > 0 \text{ on } \mathfrak{a}^+\}$, and $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$, respectively. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ denote the set of positive simple roots. We put $\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ and $\overline{\mathfrak{n}} = \theta \mathfrak{n}$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ be the half sum of positive roots, counted with multiplicity $m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$. We set $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$, and $\overline{N} = \exp \overline{\mathfrak{n}}$. Then we have the Iwasawa decomposition $G = KAN$ of G. Each $g \in G$ is uniquely written as $g = k(g) \exp(H(g)) n(g)$, where $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. Those mappings $g \mapsto k(g)$, $g \mapsto H(g)$, and $g \mapsto n(g)$ are smooth. Let M denote the centralizer and M' the normalizer of A in K. Let W be the factor group M'/M, called the Weyl group of X. The group W acts as a group of linear transformations on $\mathfrak{a}^*$ by $(s\lambda)(H) = \lambda(s^{-1} \cdot H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}^*$ and $s \in W$, where $g \cdot X = \mathrm{Ad}(g) X$ for $g \in G$, $X \in \mathfrak{g}$. We put $B = K/M$, B is the boundary of X. Let db be the normalized left-K-invariant measure on B. For simplicity, we write $L^2(B) = L^2(B, db)$. We set $A(x,b) = -H(g^{-1}k)$ for $(x,b) = (gK, kM) \in X \times B$. Let $\mathfrak{a}^{\mathbb{C}}$ (resp. $\mathfrak{a}^*_{\mathbb{C}}$) denote the complexification of $\mathfrak{a}$ (resp. $\mathfrak{a}^*$). We extend the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}$ to the $\mathbb{C}$-bilinear form on $\mathfrak{a}^{\mathbb{C}}$ in the standard way. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we define $A_\lambda \in \mathfrak{a}^{\mathbb{C}}$ so that $\langle A_\lambda, H \rangle = \lambda(H)$ for $H \in \mathfrak{a}$. Let $\langle \cdot, \cdot \rangle$ denote the $\mathbb{C}$-bilinear form on $\mathfrak{a}^*_{\mathbb{C}}$ given by $\langle \lambda, \mu \rangle = \langle A_\lambda, A_\mu \rangle$ for $\lambda, \mu \in \mathfrak{a}^*_{\mathbb{C}}$. We also put $\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^*; A_\lambda \in \mathfrak{a}^+\}$, $\mathfrak{a}^*_{\mathrm{reg}} = \{\lambda \in \mathfrak{a}^*; A_\lambda \in \mathfrak{a}_{\mathrm{reg}}\}$, and $\mathfrak{a}^*_{\mathrm{sing}} = \mathfrak{a}^* \setminus \mathfrak{a}^*_{\mathrm{reg}}$. We set $\lambda_\alpha = \langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $\alpha \in \Sigma$. The Killing form metric on $\mathfrak{g}$ induces the Euclidean measure on $\mathfrak{a}$ and $\mathfrak{a}^*$. We multiply these measures by the factor $(2\pi)^{-l/2}$ and obtain invariant measures dH and $d\lambda$ on $\mathfrak{a}$ and $\mathfrak{a}^*$, respectively. We often identify K-invariant functions on X with W-invariant functions on $\mathfrak{a}$ or A via the mapping $A \ni a \mapsto a \cdot o \in X$. For simplicity, we write $f(e^H)$ for a K-invariant function $f(x)$ on X. For a complex vector space V, let $S(V)$ denote the symmetric algebra over V considered as the space of polynomial functions on V^*. For $p \in S(\mathfrak{a}^*_{\mathbb{C}})$, we define a constant coefficient differential operator $\partial(p)$ on $\mathfrak{a}^*_{\mathbb{C}}$ by $\partial(p) e^{i\lambda(H)} = p(iH) e^{i\lambda(H)}$. When there is no fear of confusion, we identify $S(\mathfrak{a}^*_{\mathbb{C}})$ with $S(\mathfrak{a}^{\mathbb{C}})$ via the map $\mathfrak{a}^*_{\mathbb{C}} \ni \lambda \to A_\lambda \in \mathfrak{a}^{\mathbb{C}}$.

Let $\boldsymbol{D}(X)$ denote the algebra of differential operators on X which are invariant under the action of G. Let $\boldsymbol{D}(A)$ be the algebra of differential operators which are invariant under all translation actions and $\boldsymbol{D}_W(A)$ its subalgebra consisting of W-invariant elements. For $D \in \boldsymbol{D}(X)$, let $\Delta_N(D)$ denote the N-radial part of D. Then we have the following Harish-Chandra's isomorphism (see e.g. Helgason [13], Chapter II, §2, Theorem 2.1, p. 70).

$$\Gamma : \boldsymbol{D}(X) \ni D \mapsto e^{-\rho} \Delta_N(D) \circ e^{\rho} \in \boldsymbol{D}_W(A),$$

where e^ρ denotes the smooth function on A defined by $e^\rho(a) = e^{\rho(\log a)}$. For $D \in \boldsymbol{D}(A)$, we define a polynomial function $D(\lambda)$ on $\mathfrak{a}^*_{\mathbb{C}}$ by the following equation.

$$D e^{i\lambda(\log a)} = D(i\lambda) e^{i\lambda(\log a)} \quad \text{on } A.$$

For example, for the (modified) Laplace-Beltrami operator we have $\Gamma(L_X+|\rho|^2)=L_A$ and $\Gamma(L_X+|\rho|^2)(\lambda)=\langle\lambda,\lambda\rangle$, where L_A denotes the Euclidean Laplacian on A with respect to the Killing form metric. Hence $\lambda_0=0\in\mathfrak{a}^*_{\mathrm{sing}}$ corresponds to the threshold spectrum 0 of $L_X+|\rho|^2$.

Let $\boldsymbol{c}(\lambda)$ denote Harish-Chandra's $\boldsymbol{c}$-function defined by the integral

$$\boldsymbol{c}(\lambda)=\int_{\overline{N}}e^{-(i\lambda+\rho)(H(\overline{n}))}d\overline{n},$$

where dn and $d\overline{n}=\theta(dn)$ denote the Haar measures on N and $\overline{N}$ respectively normalized so that $\boldsymbol{c}(-i\rho)=1$. More precisely, Harish-Chandra's $\boldsymbol{c}$-function is written in terms of the beta function.

$$\begin{aligned}\boldsymbol{c}(\lambda)&=\prod_{\alpha\in\Sigma_0^+}c_\alpha(\lambda),\\ c_\alpha(\lambda)&=B(\rho_\alpha,\frac{m_\alpha}{2})^{-1}B(\frac{\rho_\alpha}{2}+\frac{m_\alpha}{4},\frac{m_{2\alpha}}{2})^{-1}\\ &\quad\times B(i\lambda_\alpha,\frac{m_\alpha}{2})B(\frac{i\lambda_\alpha}{2}+\frac{m_\alpha}{4},\frac{m_{2\alpha}}{2}),\end{aligned}$$

where $B(z_1,z_2)$ denotes the beta function defined by $B(z_1,z_2)=\Gamma(z_1)\Gamma(z_2)/\Gamma(z_1+z_2)$. The above formula is called the Gindikin-Karpelvič formula (see e.g. [9], Chapter 4, §4.7, Theorem 4.7.5). Now we put

$$\boldsymbol{\pi}(\lambda)=\prod_{\alpha\in\Sigma_0^+}\langle\alpha,\lambda\rangle,\quad \boldsymbol{b}(\lambda)=\boldsymbol{\pi}(i\lambda)\boldsymbol{c}(\lambda).$$

For $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$, we consider the following joint eigenspace.

$$\mathcal{E}_\lambda(X)=\{f\in C^\infty(X);Df(x)=\Gamma(D)(i\lambda)f(x)\ \text{for}\ D\in\boldsymbol{D}(X)\}.$$

It is known that for any homomorphism $\chi:\boldsymbol{D}(X)\to\mathbb{C}$, there exists $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ such that $\chi(D)=\Gamma(D)(i\lambda)$ (see e.g. Helgason [12], Chapter III, §3, Lemma 3.11, p. 364). Hence the eigenspaces of the above type exhaust all joint eigenspaces. For $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ and an integrable function F on B, the Poisson transform $\mathcal{P}_\lambda$ of F is defined by

$$\mathcal{P}_\lambda F(x)=\int_B e^{(i\lambda+\rho)(A(x,b))}F(b)db.$$

It follows immediately from the equation

$$De^{(i\lambda+\rho)(A(x,b))}=\Gamma(D)(i\lambda)e^{(i\lambda+\rho)(A(x,b))},\quad D\in\boldsymbol{D}(X)$$

that $\mathcal{P}_\lambda F\in\mathcal{E}_\lambda(X)$ for $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$. For $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$, the elementary spherical function $\varphi_\lambda(x)$ on X is given by

$$\varphi_\lambda(x)=\int_B e^{(i\lambda+\rho)(A(x,b))}db.$$

For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, we set $\boldsymbol{\pi}_0(\lambda) = \prod_{\alpha \in \Sigma^0_{\lambda_0}} \langle \alpha, \lambda \rangle$. We define the meromorphic function $\boldsymbol{b}_0(\lambda)$ on $\mathfrak{a}^*_{\mathbb{C}}$ by $\boldsymbol{b}_0(\lambda) = \boldsymbol{\pi}_0(i\lambda)\boldsymbol{c}(\lambda)$. Let $(\mathcal{E}^2_{\lambda_0}(X), \|\cdot\|_{*,\nu_0/2})$ denote the Banach space defined by

$$\mathcal{E}^2_{\lambda_0}(X) = \left\{ f \in \mathcal{E}_{\lambda_0}(X); \|f\|_{*,\nu_0/2} < \infty \right\}.$$

We have $\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0) > 0$ (see e.g. [32], part I, Corollary 7, p. 59) and set

$$\gamma_0 := \frac{|W_{\lambda_0}|^{1/2}}{\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)} \left(\int_{|H|<1} |\boldsymbol{\pi}_0(H)|^2 dH \right)^{1/2}. \tag{2.1}$$

Then our first main result is as follows.

Theorem 2.1. *We assume that* $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$.

(i) *There exists a positive constant C independent of λ_0 such that for any $F \in L^2(B)$, we have*

$$C^{-1}|\boldsymbol{b}_0(\lambda_0)|\|F\|_{L^2(B)} \leqslant \|\mathcal{P}_{\lambda_0}F\|_{*,\nu_0/2} \leqslant C|\boldsymbol{b}_0(\lambda_0)|\|F\|_{L^2(B)}. \tag{2.2}$$

Moreover, we have

$$\lim_{R\to\infty} \frac{1}{R^{\nu_0}} \int_{B(o,R)} |\mathcal{P}_{\lambda_0}F(x)|^2 dx = \gamma_0^2 |\boldsymbol{b}_0(\lambda_0)|^2 \|F\|^2_{L^2(B)}, \tag{2.3}$$

where γ_0 denotes the positive constant defined by (2.1).

(ii) *The Poisson transform $\mathcal{P}_{\lambda_0}$ is a topological isomorphism from $L^2(B)$ onto $\mathcal{E}^2_{\lambda_0}(X)$.*

(iii) *The inverse image $F \in L^2(B)$ of $f \in \mathcal{E}^2_{\lambda_0}(X)$ is given by the following formula.*

$$F(b) = \lim_{R\to\infty} \gamma_0^{-2} |\boldsymbol{b}_0(\lambda_0)|^{-2} \frac{1}{R^{\nu_0}} \int_{B(o,R)} e^{(-i\lambda_0+\rho)(A(x,b))} f(x)dx \quad \text{in } L^2(B).$$

We will prove Theorem 2.1 in a similar way to Theorem 3.3 in [16]. A uniform Fourier restriction estimate for the Helgason Fourier transform and the scattering formula for the Poisson transform lead us naturally to conclude Theorem 2.1. However, there is an essential difference between the proof of the singular case and that of the regular case. In our argument of the singular case, we have to take into account two difficulties; the singularity of Harish-Chandra's $\boldsymbol{c}$-function $\boldsymbol{c}(\lambda)$ at Weyl walls on $\mathfrak{a}^*$, and degeneracy of joint eigenfunctions at infinity on X.

In order to deal with the singularity of the $\boldsymbol{c}$-function, we employ the elimination technique by the polynomial function $\boldsymbol{\pi}_0$. By multiplying the polynomial function $\boldsymbol{\pi}_0(\lambda)$ and acting the differential operator $\partial(\boldsymbol{\pi}_0)$, we can eliminate the singularity at λ_0 and recover an original function (cf. [5, §9], [22, §2]). For example, we have a somewhat tautological but essential identity

$$\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)\varphi_{\lambda_0} = \partial(\boldsymbol{\pi}_0)\left[\boldsymbol{\pi}_0(\lambda)\varphi_\lambda\right]|_{\lambda=\lambda_0}. \tag{2.4}$$

The above identity enables us to eliminate the singularity of each coefficient in the Harish-Chandra expansion (see Theorem 4.1) and obtain a global estimate of the elementary spherical function away from Weyl walls (see Lemma 4.4 (i)). We must also handle an asymptotic behavior of the elementary spherical function near Weyl walls. By combining identity (2.4), the method of the Trombi-Varadarajan expansion (cf. [9], [31]), and a face decomposition of $\mathfrak{a}^+$, we get a global estimate of φ_{λ_0} near Weyl walls (see Lemma 4.4 (ii)). Then those two estimates yield the scattering formula. Moreover, a similar argument works for a uniform Fourier restriction estimate.

Once we obtain a uniform Fourier restriction estimate and the scattering formula, the rest of the proof of Theorem 2.1 proceeds as in [16]. For this reason, we sometimes omit the detailed proof in this paper.

3. Preliminaries

In this section, we give a brief summary on well-known results related to the Poisson, Radon, and Helgason Fourier transform on symmetric spaces of noncompact type.

First, we recall basic properties of the Poisson transform on symmetric spaces of noncompact type, to be used in later sections. The element $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is said to be *simple* if the mapping

$$F \in C^\infty(B) \to \mathcal{P}_\lambda F \in \mathcal{E}_\lambda(X)$$

is injective. (For an equivalent definition, see [13], Chapter II, §5, No.4, Definition, p. 151.) For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we define a linear subspace $\mathcal{L}^2_\lambda(B)$ of $L^2(B)$ by

$$\mathcal{L}^2_\lambda(B) = \left\{ \sum_{j=1}^{r} a_j e^{(-i\lambda+\rho)(A(g_j \cdot o, b))}; a_j \in \mathbb{C}, g_j \in G, r \in \mathbb{N} \right\}.$$

Lemma 3.1 ([13], Chapter III, §2, Lemma 2.2, p. 228)**.** *If $-\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is simple, then $\mathcal{L}^2_\lambda(B)$ is a dense subspace of $L^2(B)$.*

Here, we set

$$\boldsymbol{e}(\lambda)^{-1} = \prod_{\alpha \in \Sigma_0^+} \Gamma(\frac{1}{2}(\frac{m_\alpha}{2} + 1 + \langle i\lambda, \alpha_0 \rangle))\Gamma(\frac{1}{2}(\frac{m_\alpha}{2} + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle)),$$

where $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$. Then the following simplicity criterion is well-known.

Theorem 3.2 ([13], Chapter III, §4, Theorem 4.4 and Theorem 4.5, p. 260)**.** *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then λ is simple if and only if $\boldsymbol{e}(\lambda) \neq 0$.*

By Lemma 3.1 and Theorem 3.2, if $\lambda \in \mathfrak{a}^*$, then λ is simple and $\mathcal{L}^2_\lambda(B)$ is dense in $L^2(B)$. By the simplicity, for $\lambda \in \mathfrak{a}^*$ we define a Hilbert space $\mathcal{H}_\lambda(X)$ equipped with an inner product $(\cdot, \cdot)_\lambda$ by

$$\mathcal{H}_\lambda(X) = \{ f = \mathcal{P}_\lambda F; F \in L^2(B) \}$$
$$(f_1, f_2)_\lambda = (F_1, F_2)_{L^2(B)} \quad \text{for } f_j = \mathcal{P}_\lambda F_j \ (j = 1, 2).$$

(For the details, see e.g. [13], Chapter III, § 7, No. 2 and the proof of Theorem 6.2 therein.) We recall that the elementary spherical function satisfies the symmetric identity

$$\varphi_\lambda(h^{-1}g\cdot o)=\int_B e^{(i\lambda+\rho)(A(g\cdot o,b))}e^{(-i\lambda+\rho)(A(h\cdot o,b))}db \tag{3.1}$$

for $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$, $g,h\in G$ (see e.g. [13], Chapter III, §1, Theorem 1.1, p. 200). Then the Hilbert space $\mathcal{H}_\lambda(X)$ coincides with the closed hull (with respect to $(\,\cdot\,,\cdot\,)_\lambda$) of the subspace consisting of finite linear combinations of the G-translates of φ_λ. For $\lambda\in\mathfrak{a}^*$, let us define two unitary representations of G, $(T_\lambda,\mathcal{H}_\lambda(X))$ and $(\tau_\lambda,L^2(B))$, as follows.

$$\begin{aligned}(T_\lambda(g)f)(x)&=f(g^{-1}\cdot x), && f\in\mathcal{H}_\lambda(X),\\ (\tau_\lambda(g)F)(b)&=e^{(-i\lambda+\rho)(A(g\cdot o,b))}F(g^{-1}\cdot b), && F\in L^2(B).\end{aligned}$$

It is well-known that the unitary representation τ_λ is irreducible and unitary equivalent to $\tau_{w\lambda}$ ($w\in W$) for all $\lambda\in\mathfrak{a}^*$ (see e.g. [13], Chapter VI, §3). By the definition of τ_λ, T_λ, and (3.1), we have the following intertwining property.

Proposition 3.3. *For $\lambda\in\mathfrak{a}^*$, the Poisson transform $\mathcal{P}_\lambda$ is a unitary intertwining operator from $(\tau_\lambda,L^2(B))$ to $(T_\lambda,\mathcal{H}_\lambda(X))$.*

Schur's lemma also implies an existence of the intertwining operator $U_{\lambda,w}$ appeared in (1.2) and main theorem.

Lemma 3.4. *For $w\in W$, $\lambda\in\mathfrak{a}^*$, there exists a unique unitary intertwining operator $U_{w,\lambda}$ from $(\tau_\lambda,L^2(B))$ to $(\tau_{w\lambda},L^2(B))$ such that $U_{w,\lambda}[e^{(-i\lambda+\rho)(A(g\cdot o,\,\cdot\,))}](b)=e^{(-iw\lambda+\rho)(A(g\cdot o,b))}$ for $g\in G$.*

Next, we introduce the Helgason Fourier transform $\mathcal{F}$ and the modified Radon transform $\mathcal{R}$ respectively as follows.

$$\begin{aligned}\mathcal{F}f(\lambda,b)&=\int_X e^{(-i\lambda+\rho)(A(x,b))}f(x)dx, \quad (\lambda,b)\in\mathfrak{a}^*\times B,\\ \mathcal{R}f(H,b)&=e^{\rho(H)}\int_N f(ke^Hn\cdot o)dn, \quad (H,b)=(H,kM)\in\mathfrak{a}\times B.\end{aligned}$$

We also define a standard Fourier transform $\mathcal{F}_\mathfrak{a}$ on $\mathfrak{a}$ by

$$\mathcal{F}_\mathfrak{a}f(\lambda)=\int_\mathfrak{a} e^{-i\lambda(H)}f(H)dH, \quad \lambda\in\mathfrak{a}^*.$$

Then a simple change of variables yields the Fourier slice theorem:

$$\mathcal{F}f(\lambda,b)=\mathcal{F}_\mathfrak{a}\left[\mathcal{R}f(\,\cdot\,,b)\right](\lambda). \tag{3.2}$$

We have also the Plancherel theorem for the Helgason Fourier transform.

Theorem 3.5 (Plancherel theorem). *The Helgason Fourier transform can be uniquely extended to the unitary isomorphism*

$$\mathcal{F} : L^2(X) \to L^2_W(\mathfrak{a}^* \times B, |W|^{-1}|\boldsymbol{c}(\lambda)|^{-2}d\lambda db),$$

where the Hilbert space $L^2_W(\mathfrak{a}^* \times B, |W|^{-1}|\boldsymbol{c}(\lambda)|^{-2}d\lambda db)$ *consisting of the elements of the Hilbert space* $L^2(\mathfrak{a}^* \times B, |W|^{-1}|\boldsymbol{c}(\lambda)|^{-2}d\lambda db)$ *satisfying the following* W*-invariant condition.*

$$\int_B e^{(iw\lambda+\rho)(A(x,b))}\psi(w\lambda, b)db = \int_B e^{(i\lambda+\rho)(A(x,b))}\psi(\lambda, b)db.$$

Then we have the Plancherel formula

$$\int_X |f(x)|^2 dx = |W|^{-1} \int_{\mathfrak{a}^* \times B} |\mathcal{F}f(\lambda, b)|^2 |\boldsymbol{c}(\lambda)|^{-2} d\lambda db.$$

(*For the details, see e.g. [13], Chapter III, §1, Theorem 1.5, pp. 202–203.*)

4. A global estimate of the elementary spherical function

First, we recall the Harish-Chandra expansion of the elementary spherical function. We put $\Lambda = \sum_{\alpha\in\Pi} \mathbb{N}_0\alpha$, $\widetilde{\Lambda} = \sum_{\alpha\in\Pi} \mathbb{Z}\alpha$, and $\Lambda^+ = \Lambda \setminus \{o\}$. We set

$$\sigma_\mu = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}}; \langle\mu, \mu\rangle = 2i\langle\lambda, \mu\rangle\}, \quad \mu \in \Lambda,$$
$$\mathcal{T}^\dagger = \mathfrak{a}^*_{\mathbb{C}} \setminus \cup_{\mu\in\Lambda^+}\sigma_\mu.$$

Theorem 4.1 ([12], Chapter IV, §5, Theorem 5.5, p. 430). *Suppose that* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ *satisfies* $i(w_1\lambda - w_2\lambda) \notin \widetilde{\Lambda}$ *for* $w_1, w_2 \in W$ *with* $w_1 \neq w_2$ *and* $w\lambda \in \mathcal{T}^\dagger$ *for all* $w \in W$. *Then we have*

$$\varphi_\lambda(e^H) = \sum_{w\in W} \boldsymbol{c}(w\lambda)e^{(iw\lambda-\rho)(H)} \sum_{\mu\in\Lambda} \Gamma_\mu(w\lambda)e^{-\mu(H)} \quad \text{on} \quad A^+,$$

where the rational function Γ_μ *on* $\mathfrak{a}^*_{\mathbb{C}}$ *is given by* $\Gamma_o \equiv 1$ *and the following recursion formula for* $\mu \in \Lambda^+$.

$$\begin{aligned}&\{\langle\mu, \mu\rangle - 2i\langle\mu, \lambda\rangle\}\Gamma_\mu(\lambda)\\ &= 2\sum_{\alpha\in\Sigma^+} m_\alpha \sum_{j\in\mathbb{N}} \Gamma_{\mu-2j\alpha}(\lambda)\{\langle\mu + \rho - 2j\alpha, \alpha\rangle - i\langle\alpha, \lambda\rangle\}.\end{aligned}$$

Moreover, the above series converges absolutely at each point in A^+ *and uniformly on each subchamber* $\{e^H \in A^+; \alpha(H) > c > 0, \alpha \in \Pi\}$.

We also recall a uniform estimate for the rational functions $\Gamma_\mu(\lambda)$ $(\mu \in \Lambda^+)$ in [9]. We put

$$\mathcal{T}_\eta = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}}; \min_{\alpha\in\Pi}\langle \operatorname{Im}\lambda, \alpha\rangle > -\eta\}, \quad \eta > 0,$$
$$\mathcal{T}_{\mathrm{Im}}(\varepsilon) = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}}; |\operatorname{Im}\lambda| < \varepsilon\}, \qquad \varepsilon > 0.$$

We define a weight function m on the nonnegative root lattice Λ by $m(\mu) = \sum_{j=1}^l m_j$ for $\mu = \sum_{j=1}^l m_j\alpha_j \in \Lambda$.

Theorem 4.2 ([9, Chapter 4, §4.5, Theorem 4.5.4]). *We can find $\eta_0 > 0$ with the following property: $\mathcal{T}_{\eta_0} \subset \mathcal{T}^\dagger$; and there are constants $C > 0$, $s_0 \geqslant 0$ such that*

$$|\Gamma_\mu(\lambda)| \leqslant Cm(\mu)^{s_0}$$

for all $\mu \in \Lambda^+$, $\lambda \in \mathcal{T}_{\eta_0}$.

We note that $\mathcal{T}_{\mathrm{Im}}(\varepsilon_0) \subset \mathcal{T}_{\eta_0}$ for sufficiently small $\varepsilon_0 > 0$. Then Cauchy's integral formula implies the following uniform derivative estimates for Γ_μ.

Corollary 4.3. *For $p \in S(\mathfrak{a}_{\mathbb{C}}^*)$, there exists a positive constant $C = C(p)$ such that for any $\mu \in \Lambda^+$, $\lambda \in \mathfrak{a}^*$, we have*

$$|\partial(p)\Gamma_\mu(\lambda)| \leqslant Cm(\mu)^{s_0}.$$

Next, in order to consider an asymptotic behavior φ_{λ_0} on $\mathfrak{a}^+$, we introduce a face decomposition of $\mathfrak{a}^+$. Let I be a subset of Π. Let Σ_I, Σ_I^+, and $\Sigma_{0,I}^+$ denote the root system, the positive root system, and positive indivisible root system generated by I, respectively. Let $\mathfrak{a}_I$ be the subspace of $\mathfrak{a}$ spanned by $\{A_\alpha\}_{\alpha \in I}$ and $\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}^I$ the orthogonal decomposition of $\mathfrak{a}$ with respect to the Killing form metric. Then we have also $\mathfrak{a}^* = (\mathfrak{a}_I)^* \oplus (\mathfrak{a}^I)^*$. Let $H = H_I + H^I$ and $\lambda = \lambda_I + \lambda^I$ be the orthogonal decompositions of $H \in \mathfrak{a}$ and $\lambda \in \mathfrak{a}^*$ corresponding to $\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}^I$ and $\mathfrak{a}^* = (\mathfrak{a}_I)^* \oplus (\mathfrak{a}^I)^*$, respectively. For $H \in \mathfrak{a}$, we set

$$\tau(H) = \tau^\emptyset(H) := \min_{\alpha \in \Pi} \alpha(H),\ \tau^I(H) := \min_{\alpha \in \Pi \setminus I} \alpha(H).$$

Then we have a stratification

$$\overline{\mathfrak{a}^+} = \bigsqcup_{I \subset \Pi} \left\{H \in \overline{\mathfrak{a}^+}; \tau^I(H) > 0 \text{ and } \alpha(H) = 0 \text{ for all } \alpha \in I\right\}.$$

We split the positive Weyl chamber $\mathfrak{a}^+$ along Weyl walls as follows.

$$\begin{aligned} \mathcal{D}_1 &= \{H \in \mathfrak{a}^+; \alpha(H) \geqslant 1 \text{ for all } \alpha \in \Pi\}, \\ \mathcal{D}_I &= \{H \in \mathfrak{a}^+; \alpha(H) \geqslant s\log(e + |H|) \text{ for all } \alpha \in \Pi \setminus I, \\ &\qquad \alpha(H) < s\log(e + |H|) \text{ for all } \alpha \in I\} \setminus \mathcal{D}_1, \end{aligned} \tag{4.1}$$

where $s \geqslant 1$ is a constant to be determined later. Then we have the face decomposition

$$\mathfrak{a}^+ = \mathcal{D}_1 \sqcup [\sqcup_{\emptyset \subsetneq I \subsetneq \Pi} \mathcal{D}_I] \sqcup \mathcal{D}_\Pi.$$

We note that $\mathcal{D}_\Pi$ is a bounded subset of $\mathfrak{a}^+$.

For a nonempty subset $I \subsetneq \Pi$, let P_I denote the standard parabolic subgroup P_I of G. P_I has Langlands decomposition $P_I = M_I \exp(\mathfrak{a}^I)N^I$. We put $K_I = M_I \cap K$. Then K_I is a maximal compact subgroup of M_I and the coset space $X_I = M_I/K_I$ a symmetric space of noncompact type. X_I is called a boundary symmetric space associated to P_I. Let W_I be a fixed subgroup of W at $\mathfrak{a}^I$, i.e. $W_I = \{w \in W; w \cdot H^I = H^I \text{ for all } H^I \in \mathfrak{a}^I\}$. Let $\varphi^I_{\lambda_I}$ denote the elementary spherical function on X_I (for the details, see e.g. [9], Chapter 5, §5.8, and (5.8.26) therein):

$$\varphi^I_{\lambda_I}(m) = \int_{K_I} e^{-(i\lambda_I+\rho_I)(H(m^{-1}k_I))} dk_I, \quad m \in M_I,$$

where dk_I denotes the normalized Haar measure on K_I. We write the function $\boldsymbol{c}(\lambda)$ in the form

$$\boldsymbol{c}(\lambda) = \boldsymbol{c}_I(\lambda)\boldsymbol{c}^I(\lambda),$$

where $\boldsymbol{c}_I(\lambda)$ and $\boldsymbol{c}^I(\lambda)$ denote the products over $\Sigma^+_{0,I}$ and $\Sigma^+_0 \setminus \Sigma^+_{0,I}$ of the factor $c_\alpha(\lambda)$, respectively. We define $\Lambda_I = \sum_{\alpha\in I} \mathbb{N}_0\alpha$, $\Lambda^I = \sum_{\alpha\in\Pi\setminus I} \mathbb{N}_0\alpha$ so that the decomposition $\Lambda = \Lambda_I + \Lambda^I$ holds. We also set $\Lambda^I_+ = \Lambda^I \setminus \{o\}$. For $w \in W$, we put

$$\boldsymbol{\pi}_{0,w}(H) = \boldsymbol{\pi}_0(w^{-1}\cdot H),\ \boldsymbol{b}_{0,w}(\lambda) = \boldsymbol{\pi}_0(i\lambda)\boldsymbol{c}(w\lambda).$$

The main objective of this section is to show a global estimate for the elementary spherical function in the real and singular case. By adopting the elimination technique and the face decomposition, we prove the following auxiliary estimates to be used in the proof of Lemma 6.3.

Lemma 4.4. (i) (*Estimate away from Weyl walls*) *For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, there exists a positive constant $C = C(\lambda_0)$ such that*

$$\left| \partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)\varphi_{\lambda_0}(e^H) - |W_{\lambda_0}| \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(H)}\boldsymbol{\pi}_{0,w}(H)\boldsymbol{b}_{0,w}(\lambda_0) \right|$$
$$\leqslant C\left\{(1+|H|)^{-1} + e^{-2\tau(H)}\right\}(1+|H|)^{|\Sigma^0_{\lambda_0}|} e^{-\rho(H)}, \quad H \in \mathcal{D}_1.$$

(ii) (*Estimate along Weyl walls*) *For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, $I \subsetneq \Pi, \neq \emptyset$, there exist positive constants $s \geqslant 1$ (which appeared in (4.1)) and $C = C(s, \lambda_0, I)$ such that*

$$|\varphi_{\lambda_0}(e^H)| \leqslant C(1+|H_I|)^{2|\Sigma^+_0|}(1+|H|)^{|\Sigma^0_{\lambda_0}|}\Delta(H)^{-1/2}, \quad H \in \mathcal{D}_I,$$

where $\Delta(H)$ denotes the Weyl denominator defined by

$$\Delta(H) = \prod_{\alpha\in\Sigma^+} \left(e^{\alpha(H)} - e^{-\alpha(H)}\right)^{m_\alpha}.$$

In the general case, van den Ban and Schlichtkrull [5], and Narayanan et al. [22] dealt with asymptotic expansions of the elementary spherical function away from Weyl walls by the elimination technique. Thus the inequality in Lemma 4.4 (i) is essentially not new (see e.g. [22, Theorem 3.1]). However, in order to compare (ii) with (i), we give a proof for all inequalities in Lemma 4.4 in the paper. It seems that the inequality in Lemma 4.4 (ii) is new, but not sharp because of the singular term $\Delta(H)^{-1/2}$. On the other hand, in the real and regular case, the Trombi-Varadarajan expansion (see [31, Theorem 2.11.2], [9, Theorem 7.6.2], and [4, Theorem 2.2.8]) gives a very precise information about an asymptotic expansion for the elementary spherical function along faces of the Weyl chamber. In the singular case, it may be a future work to obtain more precise estimate along faces which is similar to the Trombi-Varadarajan expansion. For the present, the inequality in Lemma 4.4 (ii) is enough to construct our weighted L^2-theory.

In order to prove Lemma 4.4, we first give derivative estimates for remainder terms in the Harish-Chandra expansion and the Trombi-Varadarajan expansion. Here, we note that $\Gamma_\mu \equiv 0$ for $\mu \in \Lambda \setminus 2\Lambda$ in Theorem 4.1. For $H \in \mathfrak{a}^+$, $\lambda \in \mathfrak{a}^*$, we put

$$\begin{aligned} \Gamma_1^+(H,\lambda) &= \sum_{\mu \in 2\Lambda^+} \Gamma_\mu(\lambda) e^{-\mu(H)}, \\ \Gamma_I^+(H,\lambda) &= \sum_{\substack{\mu_1 \in 2\Lambda_I \\ \mu_2 \in 2\Lambda_+^I}} \Gamma_{\mu_1+\mu_2}(\lambda) e^{-(\mu_1+\mu_2)(H)}. \end{aligned} \tag{4.2}$$

We also define

$$\Delta'(H) = \prod_{\alpha \in \Sigma^+} \left(\alpha(H)^{-1}+1\right)^{-m_\alpha}, \quad H \in \mathfrak{a}^+$$

so that $\Delta(H) \asymp e^{2\rho(H)}\Delta'(H)$ holds on $\mathfrak{a}^+$.

Lemma 4.5. (i) (*Estimate away from Weyl walls*) *For $p \in S(\mathfrak{a}_{\mathbb{C}}^*)$, there exists a positive constant $C = C(p)$ such that*

$$|\partial(p)\Gamma_1^+(H,\lambda)| \leqslant Ce^{-2\tau(H)}$$

for $\lambda \in \mathfrak{a}^*$, $H \in \mathfrak{a}^+$ with $\tau(H) \geqslant 1$.

(ii) (*Estimate along Weyl walls*) *For $p \in S(\mathfrak{a}_{\mathbb{C}}^*)$, $I \subsetneq \Pi, \neq \emptyset$, there exists a positive constant $C = C(p,I)$ such that*

$$|\partial(p)\Gamma_I^+(H,\lambda)| \leqslant C\Delta'(H)^{-(s_0+1)}e^{-2\tau^I(H)}$$

for $\lambda \in \mathfrak{a}^*$, $H \in \mathfrak{a}^+$ with $\tau^I(H) \geqslant 1$.

Proof. (i) By Corollary 4.3 and the inequality $\sum_{m=1}^\infty m^{s_0}e^{-mt} \leqslant Ct^{-(s_0+1)}$ for $t \in \mathbb{R}_+$, we have

$$\sum_{\mu \in 2\Lambda^+} |\partial(p)\Gamma_\mu(\lambda)|\, e^{-\mu(H)} \leqslant Ce^{-2\tau(H)} \prod_{j=1}^{l} \alpha_j(H)^{-(s_0+1)} \tag{4.3}$$

for $H \in \mathfrak{a}^+$. Therefore, we obtain the assertion for $H \in \mathfrak{a}^+$ with $\tau(H) \geqslant 1$.

(ii) In a similar manner as in the proof of (i), by Corollary 4.3, we get

$$\begin{aligned} \sum_{\substack{\mu_1 \in 2\Lambda_I \\ \mu_2 \in 2\Lambda_+^I}} |\partial(p)\Gamma_{\mu_1+\mu_2}(\lambda)|\, e^{-(\mu_1+\mu_2)(H)} &\leqslant C \sum_{\substack{\mu_1 \in 2\Lambda_I \\ \mu_2 \in 2\Lambda_+^I}} m(\mu_1+\mu_2)^{s_0} e^{-(\mu_1+\mu_2)(H)} \\ &\leqslant Ce^{-2\tau^I(H)} \left\{ \prod_{j=1}^{l} (\alpha_j(H)^{-1}+1) \right\}^{(s_0+1)}. \end{aligned}$$

Thus, we complete the proof. ■

Now, by making use of Lemma 4.5, we prove Lemma 4.4.

Proof of Lemma 4.4. (i) For $\lambda \in \mathfrak{a}^*_{\mathrm{reg}}$, $H \in \mathcal{D}_1$, by taking the summation for Λ^+ in the Harish-Chandra expansion, we write

$$\varphi_\lambda(e^H) = \sum_{w \in W} e^{(iw\lambda-\rho)(H)} \boldsymbol{c}(w\lambda) \left\{1 + \Gamma_1^+(H, w\lambda)\right\}. \tag{4.4}$$

Here, for $S \subset \Sigma$, we put $\pi_S(\lambda) = \Pi_{\alpha \in S}\langle \lambda, \alpha\rangle$. Then (2.4), (4.4), and the Leibniz rule demonstrate that

$$\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)\varphi_{\lambda_0}(e^H) = M_0(H, \lambda_0) + R_1(H, \lambda_0) + R_2(H, \lambda_0), \tag{4.5}$$

where

$$\begin{aligned}
M_0(H, \lambda_0) &= \sum_{w \in W} e^{(iw\lambda_0-\rho)(H)} \boldsymbol{\pi}_{0,w}(H) \boldsymbol{b}_{0,w}(\lambda_0),\\
R_1(H, \lambda_0) &= \sum_{w \in W} \sum_{\substack{S_1 \sqcup S_2 = \Sigma^0_{\lambda_0}\\ S_1 \neq \Sigma^0_{\lambda_0}}} e^{(iw\lambda_0-\rho)(H)} i^{-|\Sigma^0_{\lambda_0}|} \pi_{S_1}(iw^{-1}H) \left[\partial(\pi_{S_2})\boldsymbol{b}_{0,w}\right](\lambda_0),\\
R_2(H, \lambda_0) &= \sum_{w \in W} \sum_{S_1 \sqcup S_2 \sqcup S_3 = \Sigma^0_{\lambda_0}} e^{(iw\lambda_0-\rho)(H)} i^{-|\Sigma^0_{\lambda_0}|} \pi_{S_1}(iw^{-1}H)\\
&\qquad \times \left[\partial(\pi_{S_2})\boldsymbol{b}_{0,w}\right](\lambda_0) \left[\partial(\pi_{S_3})\Gamma_1^+(H, w\,\cdot)\right]\Big|_{\lambda=\lambda_0}.
\end{aligned}$$

For $w \in W, w_0 \in W_{\lambda_0}$, we see that

$$\boldsymbol{\pi}_{0,ww_0}(H) = (\det w_0)\boldsymbol{\pi}_{0,w}(H), \ \boldsymbol{b}_{0,ww_0}(\lambda) = (\det w_0)^{-1}\boldsymbol{b}_{0,w}(\lambda).$$

Hence, for the main term $M_0(H, \lambda_0)$, we have

$$M_0(H, \lambda_0) = |W_{\lambda_0}| \sum_{[w] \in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(H)} \boldsymbol{\pi}_{0,w}(H) \boldsymbol{b}_{0,w}(\lambda_0). \tag{4.6}$$

For the first remainder term $R_1(H, \lambda_0)$, an elementary computation shows that

$$|R_1(H, \lambda_0)| \leqslant C_{\lambda_0}(1 + |H|)^{|\Sigma^0_{\lambda_0}|-1} e^{-\rho(H)}. \tag{4.7}$$

For the second remainder term $R_2(H, \lambda_0)$, Lemma 4.5 (i) yields that

$$|R_2(H, \lambda_0)| \leqslant C_{\lambda_0}(1 + |H|)^{|\Sigma^0_{\lambda_0}|} e^{-\rho(H)-2\tau(H)}. \tag{4.8}$$

By combining (4.5), (4.6), (4.7), and (4.8), we obtain the assertion.

(ii) First, by rearranging the terms in the Harish-Chandra expansion, or by taking the leading term in the Trombi-Varadarajan expansion, we write

$$\begin{aligned}
\varphi_\lambda(e^H) = &\sum_{[w] \in W_I \backslash W} e^{(iw\lambda-\rho)(H^I)} \boldsymbol{c}^I(w\lambda) \varphi^I_{(w\lambda)_I}(e^{H_I})\\
&+ \sum_{w \in W} e^{(iw\lambda-\rho)(H)} \boldsymbol{c}(w\lambda) \Gamma_I^+(H, w\lambda),
\end{aligned} \tag{4.9}$$

where Γ_I^+ is defined by (4.2). For each $w \in W$, $\boldsymbol{\pi}_0(\lambda)\boldsymbol{c}^I(w\lambda)$ and $\boldsymbol{\pi}_0(\lambda)\boldsymbol{c}(w\lambda)$ are smooth near λ_0. In a small neighborhood of λ_0, we set

$$R_{I,1}(H,\lambda) = \partial(\boldsymbol{\pi}_0)\left\{\sum_{[w]\in W_I\backslash W} e^{(iw\lambda-\rho)(H^I)}\boldsymbol{\pi}_0(\lambda)\boldsymbol{c}^I(w\lambda)\varphi^I_{(w\lambda)_I}(e^{H_I})\right\},$$

$$R_{I,2}(H,\lambda) = \partial(\boldsymbol{\pi}_0)\left\{\sum_{w\in W} e^{(iw\lambda-\rho)(H)}\boldsymbol{\pi}_0(\lambda)\boldsymbol{c}(w\lambda)\Gamma_I^+(H,w\lambda)\right\}.$$

Then (2.4) and (4.9) show that

$$\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)\varphi_{\lambda_0}(e^H) = R_{I,1}(H,\lambda_0) + R_{I,2}(H,\lambda_0). \tag{4.10}$$

For the first term $R_{I,1}(H,\lambda_0)$ in (4.10), we have

$$\begin{aligned} R_{I,1}(H,\lambda_0) = &\sum_{[w]\in W_I\backslash W}\ \sum_{\substack{S_1\sqcup S_2\sqcup S_3\\ =\Sigma^0_{\lambda_0}}} e^{(iw\lambda-\rho)(H^I)}\pi_{S_1}(iw^{-1}H^I)\\ &\times \partial(\pi_{S_2})\left\{\boldsymbol{\pi}_0(\lambda)\boldsymbol{c}^I(w\lambda)\right\}\big|_{\lambda=\lambda_0}\partial(\pi_{S_3})\left\{\varphi^I_{(w\lambda)_I}(e^{H_I})\right\}\big|_{\lambda=\lambda_0}. \end{aligned}$$

A direct computation yields that

$$|\partial(p)\varphi^I_{\lambda_I}(e^{H_I})| \leqslant C_{\lambda_I}\,(1+|H_I|)^{(|\Sigma^+_{0,I}|+\deg p)}\,e^{-\rho_I(H_I)},\quad H_I\in(\mathfrak{a}_I)^+,\lambda_I\in\mathfrak{a}_I^*.$$

Since $e^{-\rho(H)} \leqslant C\Delta(H)^{-1/2}$ on $\mathfrak{a}^+$, and $\Sigma^+_{0,I}, \Sigma^0_{\lambda_0} \subset \Sigma^+_0$, we obtain

$$|R_{I,1}(H,\lambda_0)| \leqslant C_{\lambda_0}\,(1+|H_I|)^{2|\Sigma_0^+|}\,(1+|H|)^{|\Sigma^0_{\lambda_0}|}\,\Delta(H)^{-1/2}. \tag{4.11}$$

On the other hand, for the second term $R_{I,2}(H,\lambda_0)$ in (4.10), we first prove a rough inequality without singularities on Weyl walls. It follows from (4.10) that

$$\begin{aligned} |R_{I,2}(H,\lambda_0)| &\leqslant \partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)|\varphi_{\lambda_0}(e^H)| + |R_{I,1}(H,\lambda_0)|\\ &\leqslant C_{\lambda_0}(1+|H|)^{3|\Sigma_0^+|}e^{-\rho(H)}. \end{aligned} \tag{4.12}$$

Next, we consider a fine inequality involving singularities on Weyl walls. We see that

$$\begin{aligned} R_{I,2}(H,\lambda_0) = &\sum_{w\in W}\ \sum_{\substack{S_1\sqcup S_2\sqcup S_3\\ =\Sigma^0_{\lambda_0}}} e^{(iw\lambda-\rho)(H^I)}\pi_{S_1}(iw^{-1}H^I)\\ &\times \partial(\pi_{S_2})\left\{\boldsymbol{\pi}_0(\lambda)\boldsymbol{c}(w\lambda)\right\}\big|_{\lambda=\lambda_0}\partial(\pi_{S_3})\left\{\Gamma_I^+(H,w\lambda)\right\}\big|_{\lambda=\lambda_0}. \end{aligned}$$

By Lemma 4.5 (ii), we deduce that

$$|R_{I,2}(H,\lambda_0)| \leqslant C\,(1+|H|)^{|\Sigma^0_{\lambda_0}|}\,\Delta'(H)^{-(s_0+1)}e^{-\rho(H)-2\tau^I(H)} \tag{4.13}$$

for $H\in\mathfrak{a}^+$ with $\tau^I(H)\geqslant 1$. By interpolating (4.12) and (4.13), we get

$$|R_{I,2}(H,\lambda_0)| \leqslant C(1+|H|)^{3|\Sigma_0^+|}e^{-(s_0+1)^{-1}\tau^I(H)}e^{-\rho(H)}\Delta'(H)^{-1/2}. \tag{4.14}$$

For $H \in \mathcal{D}^I$, we have $\tau^I(H) \geqslant s \log(e + |H|)$. We therefore take a constant s which appeared in (4.1) so that $s \geqslant 3|\Sigma_0^+|(s_0 + 1)$, and hence

$$e^{-(s_0+1)^{-1}\tau^I(H)} \leqslant (1 + |H|)^{-3|\Sigma_0^+|}. \tag{4.15}$$

By (4.14), (4.15), and $\Delta(H)^{-1/2} \asymp e^{-\rho(H)}\Delta'(H)^{-1/2}$, we obtain

$$|R_{I,2}(H, \lambda_0)| \leqslant C\Delta(H)^{-1/2}, \quad H \in \mathcal{D}^I. \tag{4.16}$$

By combining (4.10), (4.11), and (4.16), we obtain the assertion. ■

5. A uniform Fourier restriction estimate

In this section, we prove a uniform estimate of a Fourier restriction operator for the Helgason Fourier transform.

First, we introduce three Banach spaces $B_\sigma(X)$, $B_\sigma^*(X)$, and $\mathring{B}_\sigma^*(X)$ for $\sigma \in \mathbb{R}_+$. We put $\Omega_0 = \{x \in X; r(x) < 1\}$, $\Omega_j = \{x \in X; 2^{j-1} \leqslant r(x) < 2^j\}$ for $j \in \mathbb{N}$. Let $\chi_{\Omega_j}(x)$ denote the characteristic function of the set Ω_j. For $f \in L^2_{\mathrm{loc}}(X)$, we define the norm $\|\cdot\|_{B_\sigma(X)}$ by

$$\|u\|_{B_\sigma(X)} = \sum_{j=0}^{\infty} 2^{\sigma j} \left\|\chi_{\Omega_j} u\right\|_{L^2(X)}.$$

We define the Banach space $B_\sigma(X)$ by

$$B_\sigma(X) = \left\{u \in L^2_{\mathrm{loc}}(X); \|u\|_{B_\sigma(X)} < \infty\right\}.$$

Then the dual space $B_\sigma^*(X)$ of $B_\sigma(X)$ is realized as follows.

$$\|f\|_{B_\sigma^*(X)} = \sup_{j \in \mathbb{N}_0} \left\{2^{-\sigma j} \left\|\chi_{\Omega_j} f\right\|_{L^2(X)}\right\},$$
$$B_\sigma^*(X) = \left\{f \in L^2_{\mathrm{loc}}(X); \|f\|_{B_\sigma^*(X)} < \infty\right\}.$$

An elementary computation yields the equivalence of the two norms $\|\cdot\|_{B_\sigma^*(X)}$ and $\|\cdot\|_{*,\sigma}$:

$$\|f\|_{B_\sigma^*(X)} \leqslant \sup_{R>1} \frac{1}{R^\sigma} \left(\int_{B(o,R)} |f(x)|^2 dx\right)^{1/2} \leqslant \frac{2^\sigma}{2^\sigma - 1} \|f\|_{B_\sigma^*(X)}.$$

We also define $\mathring{B}_\sigma^*(X) = \{f \in B_\sigma^*(X); f \simeq 0 \text{ in } B_\sigma^*(X)\}$. Then $\mathring{B}_\sigma^*(X)$ is a closed subspace of $B_\sigma^*(X)$.

For $\lambda \in \mathfrak{a}^*$, we define a Fourier restriction operator $\mathcal{F}_\lambda$ from $C_0^\infty(X)$ into $L^2(B)$ by

$$[\mathcal{F}_\lambda f](b) = \mathcal{F}f(\lambda, b), \quad f \in C_0^\infty(X).$$

Then we have the following uniform Fourier restriction estimate.

Lemma 5.1. *For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, the Fourier restriction operator $\mathcal{F}_{\lambda_0}$ can be uniquely extended to a linear continuous operator from $B_{\nu_0/2}(X)$ into $L^2(B)$. Moreover, there exists a positive constant C independent of λ_0 such that*

$$\|\mathcal{F}_{\lambda_0} u\|_{L^2(B)} \leqslant C|\boldsymbol{b}_0(\lambda_0)|\|u\|_{B_{\nu_0/2}(X)}, \tag{5.1}$$

or equivalently, for $R > 1$, and $u \in L^2(X)$ with $\operatorname{supp} u \subset B(o,R)$, we have

$$\|\mathcal{F}_{\lambda_0} u\|_{L^2(B)} \leqslant CR^{\nu_0/2}|\boldsymbol{b}_0(\lambda_0)|\|u\|_{L^2(X)}. \tag{5.2}$$

Proof. The equivalence of (5.1) and (5.2) immediately follows from the definition of $B_{\nu_0/2}(X)$. Hence it is sufficient to prove (5.2) for $R > 1$, $u \in C_0^\infty(X)$ with $\operatorname{supp} u \subset B(o,R)$.

We introduce a certain modified $\boldsymbol{c}$-function $\widetilde{\boldsymbol{c}}(\lambda)$ on $\mathfrak{a}^*$ as in the proof of [16, Proposition 6.3] (see Anker [3] for the original construction of $\widetilde{\boldsymbol{c}}(\lambda)$). We can choose $\widetilde{\boldsymbol{b}}(\lambda) \in C^\infty(\mathfrak{a}^*)$ so that

$$|\widetilde{\boldsymbol{b}}(\lambda)| \asymp |\boldsymbol{b}(\lambda)| \tag{5.3}$$

and the tempered distribution $\mathcal{F}_{\mathfrak{a}}^{-1}[\widetilde{\boldsymbol{b}}^{-1}]$ has compact support in $\mathfrak{a}$. We define a modified $\boldsymbol{c}$-function $\widetilde{\boldsymbol{c}}(\lambda)$ by

$$\widetilde{\boldsymbol{c}}(\lambda) := \boldsymbol{\pi}(i\lambda)^{-1}\widetilde{\boldsymbol{b}}(\lambda).$$

Let E denote the compactly supported distribution on $\mathfrak{a}$ defined by $E := \mathcal{F}_{\mathfrak{a}}^{-1}[\widetilde{\boldsymbol{c}}^{-1}]$. We take $R_0 > 0$ so that

$$\operatorname{supp} E \subset \{H \in \mathfrak{a}; |H| \leqslant R_0\}. \tag{5.4}$$

We define the Fourier multipliers J_0, J_1 on $\mathfrak{a}$ by $J_0 = \mathcal{F}_{\mathfrak{a}}^{-1}[\boldsymbol{c}^{-1}\mathcal{F}_{\mathfrak{a}}]$, $J_1 = \mathcal{F}_{\mathfrak{a}}^{-1}[\widetilde{\boldsymbol{c}}^{-1}\mathcal{F}_{\mathfrak{a}}]$. By using $|\boldsymbol{c}(\lambda)|^{-1} \asymp |\widetilde{\boldsymbol{c}}(\lambda)|^{-1}$, we get

$$C^{-1}\|J_1\mathcal{R}u\|_{L^2(\mathfrak{a}\times B, dHdb)} \leqslant \|J_0\mathcal{R}u\|_{L^2(\mathfrak{a}\times B, dHdb)} \leqslant C\|J_1\mathcal{R}u\|_{L^2(\mathfrak{a}\times B, dHdb)} \tag{5.5}$$

for $u \in L^2(X)$. For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, we also define a modified function $\widetilde{\boldsymbol{b}}_0(\lambda)$ of $\boldsymbol{b}_0(\lambda)$ by

$$\widetilde{\boldsymbol{b}}_0(\lambda) = \boldsymbol{\pi}_0(i\lambda)\widetilde{\boldsymbol{c}}(\lambda).$$

Then the function $\widetilde{\boldsymbol{b}}_0(\lambda)$ is smooth in a small neighborhood of λ_0 and $\widetilde{\boldsymbol{b}}_0(\lambda_0) \neq 0$. For any $\lambda \in \mathfrak{a}^*_{\mathrm{reg}}$, $R > 1$, and $u \in C_0^\infty(X)$ with $\operatorname{supp} u \subset B(o,R)$, the Fourier slice theorem (3.2) shows that

$$\boldsymbol{\pi}_0(i\lambda)\widetilde{\boldsymbol{b}}_0(\lambda)^{-1}\mathcal{F}_\lambda u(b) = \int_{\mathfrak{a}} e^{-i\lambda(H)} J_1\mathcal{R}u(H,b)dH.$$

By differentiating the above equality, we obtain

$$\partial(\boldsymbol{\pi}_0)\left\{\boldsymbol{\pi}_0(\lambda)\widetilde{\boldsymbol{b}}_0(\lambda)^{-1}\mathcal{F}_\lambda u(b)\right\} = \int_{\mathfrak{a}} e^{-i\lambda(H)}\boldsymbol{\pi}_0(-H)J_1\mathcal{R}u(H,b)dH.$$

Then it follows from the equality

$$\partial(\boldsymbol{\pi}_0)\left\{\boldsymbol{\pi}_0(\lambda)\widetilde{\boldsymbol{b}}_0(\lambda)^{-1}\mathcal{F}_\lambda u(b)\right\}\Big|_{\lambda=\lambda_0} = \partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)\widetilde{\boldsymbol{b}}_0(\lambda_0)^{-1}\mathcal{F}_{\lambda_0}u(b)$$

that

$$\mathcal{F}_{\lambda_0}u(b) = [\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)]^{-1}\widetilde{\boldsymbol{b}}_0(\lambda_0)\int_{\mathfrak{a}} e^{-i\lambda_0(H)}\boldsymbol{\pi}_0(-H)J_1\mathcal{R}u(H,b)dH. \tag{5.6}$$

Since $E = \mathcal{F}_{\mathfrak{a}}^{-1}[\widetilde{\boldsymbol{c}}^{-1}]$, we have $J_1 f = E * f$ for $f \in C_0^\infty(\mathfrak{a})$, where $*$ denotes the convolution on $\mathfrak{a}$. Hence we see that

$$J_1\mathcal{R}u(H,b) = [E * \mathcal{R}u(\cdot,b)](H).$$

By using the assumption $\operatorname{supp} u \subset B(o,R)$ and (5.4), we get

$$\operatorname{supp} J_1\mathcal{R}u(\cdot,b) \subset \{H \in \mathfrak{a}; |H| \leqslant R + R_0\}, \quad b \in B.$$

Then the Schwartz inequality yields that

$$\left|\int_{\mathfrak{a}} e^{-i\lambda_0(H)}\boldsymbol{\pi}_0(-iH)J_1\mathcal{R}(H,b)dH\right| \leqslant CR^{\nu_0/2}\left\|J_1\mathcal{R}u(\cdot,b)\right\|_{L^2(\mathfrak{a},dH)}.$$

Therefore, by taking the L^2-norm on B of the both hand side of (5.6) and applying the above inequality, we obtain

$$\left\|\mathcal{F}_{\lambda_0}u\right\|_{L^2(B)} \leqslant CR^{\nu_0/2}|\widetilde{\boldsymbol{b}}_0(\lambda_0)|\left\|J_1\mathcal{R}u\right\|_{L^2(\mathfrak{a}\times B,dHdb)}. \tag{5.7}$$

By (5.3), (5.5), (5.7), and the Plancherel formula, we deduce that

$$\left\|\mathcal{F}_{\lambda_0}u\right\|_{L^2(B)} \leqslant CR^{\nu_0/2}|\boldsymbol{b}_0(\lambda_0)|\left\|u\right\|_{L^2(X)}. \qquad \blacksquare$$

Since $\mathcal{P}_{\lambda_0}$ is the adjoint operator of $\mathcal{F}_{\lambda_0}$, Lemma 5.1 immediately yields the following corollary.

Corollary 5.2. *We assume that $\lambda_0 \in \mathfrak{a}^*_{\text{sing}}$. Then we have the linear continuous map*

$$\mathcal{P}_{\lambda_0} : L^2(B) \to B^*_{\nu_0/2}(X).$$

Moreover, there exists a positive constant C independent of λ_0 such that

$$\left\|\mathcal{P}_{\lambda_0}F\right\|_{B^*_{\nu_0/2}(X)} \leqslant C|\boldsymbol{b}_0(\lambda_0)|\left\|F\right\|_{L^2(B)}, \quad F \in L^2(B). \tag{5.8}$$

6. Scattering formula for the Poisson transform

This section is devoted to prove the scattering formula for the Poisson transform in the singular case. Let $X_{\text{reg}} = K\cdot(A^+\cdot o)$ be the regular set in X. Then the mapping $\iota : (kM,a) \mapsto ka\cdot o$ is a diffeomorphism from $K/M \times A^+$ onto X_{reg}. For

$x \in X_{\mathrm{reg}}$, we associate the radial part $A^+(x) \in \mathfrak{a}^+$ and the boundary point $b_x \in B$ by $(b_x, \exp(A^+(x))) = \iota^{-1}(x)$, i.e.

$$\begin{aligned} x &= k_x \exp(A^+(x)) \cdot o, \\ b_x &= k_x M. \end{aligned}$$

Then we have the Weyl integration formula on X:

$$\int_X f(x)dx = \int_{K/M \times A^+} f(ka \cdot o)\Delta(H)dH db(kM).$$

For $x \in X_{\mathrm{reg}}$, $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, $w \in W$, we define an amplitude function $\boldsymbol{a}_0(x, \lambda_0; w)$ by

$$\boldsymbol{a}_0(x, \lambda_0; w) = \frac{|W_{\lambda_0}|}{\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)} \boldsymbol{\pi}_{0,w}(A^+(x))\boldsymbol{b}_{0,w}(\lambda_0).$$

Then our second main result is as follows.

Theorem 6.1. *For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, $F \in L^2(B)$, we have the following asymptotic expansion in $B^*_{\nu_0/2}(X)$ for the Poisson transform.*

$$\begin{aligned} \mathcal{P}_{\lambda_0}F(x) &\simeq \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(A^+(x))}\boldsymbol{a}_0(x,\lambda_0;w)[U_{w,\lambda_0}F](b_x) && (6.1)\\ &\simeq \Delta^{-1/2}(A^+(x)) \sum_{[w]\in W/W_{\lambda_0}} e^{iw\lambda_0(A^+(x))}\boldsymbol{a}_0(x,\lambda_0;w)[U_{w,\lambda_0}F](b_x). && (6.2) \end{aligned}$$

Remark 6.2. In the singular case, degenerate terms $\boldsymbol{a}_0(\,\cdot\,, \lambda_0; w)$, which have polynomial growth at infinity, appear in the scattering formula. The amplitude $\boldsymbol{a}_0(x, \lambda_0; w)$ satisfies a K-invariance in the x-variable, and a homogeneous property; $\boldsymbol{a}_0(e^{tH} \cdot o, \lambda_0; w) = t^{|\Sigma^0_{\lambda_0}|}\boldsymbol{a}_0(e^H \cdot o, \lambda_0; w)$ for $H \in \mathfrak{a}^+_{\mathrm{reg}}$, $t \in \mathbb{R}_+$. Then we can see that the amplitudes $\boldsymbol{a}_0(x, \lambda_0; w)$ are unique among functions satisfying the K-invariance, the homogeneous property, and the scattering formula (6.1). We also note that since $e^{-\rho(tH)} = \Delta(tH)^{-1/2}(1 + o(t^{-1}))$ as $t \to \infty$ for $H \in \mathfrak{a}^+$, the right hand side of (6.1) is a priori equivalent to that of (6.2) in $B^*_{\nu_0/2}(X)$.

We prove the scattering formula (6.1) in the following three steps: (1) $F(b) = 1$ and hence $\mathcal{P}_{\lambda_0}[1] = \varphi_{\lambda_0}$; (2) $F(b) = e^{(-i\lambda_0+\rho)(A(g\cdot o,b))}$, $g \in G$; (3) $F \in L^2(B)$.

Lemma 6.3 (Step 1). *For $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, we have*

$$\varphi_{\lambda_0}(x) \simeq \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(A^+(x))}\boldsymbol{a}_0(x,\lambda_0;w) \quad \textit{in } B^*_{\nu_0/2}(X).$$

Proof. Let $\chi_1(x)$ and $\chi_I(x)$ be the characteristic functions of the sets $K\exp(\mathcal{D}_1)\cdot o$ and $K\exp(\mathcal{D}_I)\cdot o$ on X, respectively. Then Lemma 4.4 (i) yields that

$$\chi_1(x)\varphi_{\lambda_0}(x) \simeq \chi_1(x) \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(A^+(x))}\boldsymbol{a}_0(x,\lambda_0;w) \quad \text{in } B^*_{\nu_0/2}(X). \qquad (6.3)$$

By applying Lemma 4.4 (ii) for $I \subsetneq \Pi, \neq \emptyset$, we get

$$\frac{1}{R^{\nu_0}} \int_{B(o,R)} |\chi_I(x)\varphi_{\lambda_0}(x)|^2 \, dx \leqslant C L_I(R),$$

where

$$L_I(R) = \frac{1}{R^{\nu_0}} \int_{\substack{|H|<R \\ H\in\mathcal{D}^I}} (1+|H_I|)^{4|\Sigma_0^+|} (1+|H|)^{2|\Sigma^0_{\lambda_0}|} \, dH.$$

For $H \in \mathcal{D}_I$, $\alpha \in I$, we have $\alpha(H) < s\log(e+|H|)$. Hence, there exists a positive constant C such that $|H_I| \leqslant C\log(e+|H|)$ for $H \in \mathcal{D}_I$. Then, since $\nu_0 = l + 2|\Sigma^0_{\lambda_0}|$, it follows that

$$L_I(R) \leqslant C \left(\log(e+R)\right)^{4|\Sigma_0^+|} \left(\int_{\substack{|H|<1 \\ RH\in\mathcal{D}^I}} 1 \, dH \right).$$

By the inequality $\alpha(H) < s(\log(e+R))/R$ for $\alpha \in I$ and $RH \in \mathcal{D}_I$ with $|H| \leqslant 1$, we get

$$\left(\int_{\substack{|H|<1 \\ RH\in\mathcal{D}^I}} 1 \, dH \right) \leqslant C \left(\frac{\log(e+R)}{R} \right)^{|I|}.$$

Then the above inequality yields that

$$L_I(R) \leqslant C R^{-|I|} \left(\log(e+R)\right)^{4|\Sigma_0^+|+|I|} \to 0 \quad \text{as } R \to \infty.$$

Therefore, we obtain

$$\chi_I(x)\varphi_{\lambda_0}(x) \simeq 0 \quad \text{in } B^*_{\nu_0/2}(X). \tag{6.4}$$

Since $|\boldsymbol{a}_0(x,\lambda_0;w)| \leqslant C(1+r(x))^{|\Sigma^0_{\lambda_0}|}$, it is obvious that

$$(1-\chi_1(x)) \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(A^+(x))} \boldsymbol{a}_0(x,\lambda_0;w) \simeq 0 \quad \text{in } B^*_{\nu_0/2}(X). \tag{6.5}$$

By combining (6.3), (6.4), and (6.5), we complete the proof. ■

In the proof of Step 2, we need the following lemma.

Lemma 6.4 ([16, Lemma 5.3]). *For $y = h \cdot o \in X$ with $h \in G$, and a regular point $x \in X_{\mathrm{reg}} \cap (h \cdot X_{\mathrm{reg}})$, we put*

$$R_+(x,y) = A^+(h^{-1}\cdot x) - A^+(x) + A(y, b_x).$$

Then there exists a positive constant $C(y)$ bounded on each compact subset in X such that

$$|R_+(x,y)| \leqslant C(y) e^{-2\tau(A^+(x))}, \quad x \in X_{\mathrm{reg}} \cap (h \cdot X_{\mathrm{reg}}). \tag{6.6}$$

For $\lambda \in \mathfrak{a}^*$, $g \in G$, we put $F_{\lambda,g}(b) = e^{(-i\lambda+\rho)(A(g\cdot o,b))}$. Then the symmetric identity (3.1) is written as $\varphi_{\lambda_0}(g^{-1}\cdot x) = \mathcal{P}_{\lambda_0} F_{\lambda_0,g}(x)$ and leads us to the following lemma.

Lemma 6.5 (Step 2). *For $\lambda_0 \in \mathfrak{a}_{\mathrm{sing}}$, $g \in G$, we have the following asymptotic expansion in $B^*_{\nu_0/2}(X)$.*

$$\mathcal{P}_{\lambda_0} F_{\lambda_0,g}(x) \simeq \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(A^+(x))} \boldsymbol{a}_0(x,\lambda_0;w) F_{w\lambda_0,g}(b_x).$$

Proof. We put $x_g = g^{-1}\cdot x$ for $x \in X$, $g \in G$. Since $B^*_{\nu_0/2}(X)$ and $\mathring{B}^*_{\nu_0/2}(X)$ are invariant under the action of G, Lemma 6.3 shows that

$$\varphi_{\lambda_0}(x_g) \simeq \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(A^+(x_g))} \boldsymbol{a}_0(x_g,\lambda_0;w) \quad \text{in } B^*_{\nu_0/2}(X).$$

For each term in the right hand side of the above asymptotic expansion, inequality (6.6) in Lemma 6.4 yields that

$$\left| e^{(iw\lambda_0-\rho)(A^+(x_g))} - e^{(iw\lambda_0-\rho)(A^+(x))} F_{w\lambda_0,g}(b_x) \right| \leqslant C e^{-\rho(A^+(x))-2\tau(A^+(x))}$$

and

$$|\boldsymbol{a}_0(x_g,\lambda_0;w) - \boldsymbol{a}_0(x,\lambda_0;w)| \leqslant C(1+r(x))^{|\Sigma^0_{\lambda_0}|-1}.$$

Then, as in the same manner of the proof of Theorem 6.1 (Step I) in [16], we obtain the assertion. ∎

Proof of Theorem 6.1. Corollary 5.2, Lemma 6.5, and Lemma 3.1 imply that the scattering formula (6.1) holds for a general $F \in L^2(B)$. The proof is completely parallel to the one of Theorem 6.1 (Step II) in [16]. Thus we omit the proof. ∎

7. Proof of the main result

In this section, we prove Theorem 2.1. As in the regular case, the uniform estimate (Corollary 5.2) and the scattering formula (Theorem 6.1) for the Poisson transform yield a series of claims in Theorem 2.1.

First, we prove Theorem 2.1 (i).

Proof of Theorem 2.1 (i). The inequality of the right hand side of (2.2) follows from inequality (5.8) in Corollary 5.2. Hence it suffices to show (2.3). We define a linear continuous map $\mathcal{M}_{\lambda_0} : L^2(B) \to B^*_{\nu_0/2}(X)$ by

$$\mathcal{M}_{\lambda_0} F(x) = \Delta^{-1/2}(A^+(x)) \sum_{[w]\in W/W_{\lambda_0}} e^{iw\lambda_0(A^+(x))} \boldsymbol{a}_0(x,\lambda_0;w)[U_{w,\lambda_0}F](b_x).$$

Then we have

$$\frac{1}{R^{\nu_0}} \int_{B(o,R)} |\mathcal{P}_{\lambda_0} F(x)|^2 \, dx = \frac{1}{R^{\nu_0}} \int_{B(o,R)} |\mathcal{M}_{\lambda_0} F(x)|^2 \, dx + I_1(R) + I_2(R), \tag{7.1}$$

where

$$I_1(R) = \frac{1}{R^{\nu_0}} \int_{B(o,R)} |\mathcal{P}_{\lambda_0} F(x) - \mathcal{M}_{\lambda_0} F(x)|^2 \, dx,$$
$$I_2(R) = \frac{2}{R^{\nu_0}} \int_{B(o,R)} \mathrm{Re} \left[\{\mathcal{P}_{\lambda_0} F(x) - \mathcal{M}_{\lambda_0} F(x)\} \overline{\mathcal{P}_{\lambda_0} F(x)} \right] dx.$$

For the first term in the right hand side of (7.1), we see that

$$\begin{aligned} &\frac{1}{R^{\nu_0}} \int_{B(o,R)} |\mathcal{M}_{\lambda_0} F(x)|^2 \, dx \\ &= \frac{|W_{\lambda_0}|^2}{\{\partial(\boldsymbol{\pi}_0)(\boldsymbol{\pi}_0)\}^2} \sum_{\substack{[w_1],[w_2] \\ \in W/W_{\lambda_0}}} \int_{\substack{|H|<1 \\ H \in \mathfrak{a}^+}} e^{(iw_1\lambda_0 - iw_2\lambda_0)(RH)} \boldsymbol{\pi}_{0,w_1}(H) \boldsymbol{\pi}_{0,w_2}(H) dH \\ &\times \boldsymbol{b}_{0,w_1}(\lambda_0) \overline{\boldsymbol{b}_{0,w_2}(\lambda_0)} \left(U_{w_1,\lambda_0} F, U_{w_2,\lambda_0} F \right)_{L^2(B)}. \end{aligned} \tag{7.2}$$

Here we note that $|\boldsymbol{b}_{0,w}(\lambda_0)| = |\boldsymbol{b}_0(\lambda_0)|$, $\|U_{w,\lambda_0} F\|_{L^2(B)} = \|F\|_{L^2(B)}$, and $w_1\lambda_0 = w_2\lambda_0$ if and only if $[w_1] = [w_2]$ in W/W_{λ_0}. Then the Riemann-Lebesgue lemma yields that the sum over $[w_1] \neq [w_2]$ in (7.2) converges to 0 as $R \to \infty$ and hence

$$\lim_{R\to\infty} \frac{1}{R^{\nu_0}} \int_{B(o,R)} |\mathcal{M}_{\lambda_0} F(x)|^2 \, dx = \gamma_0^2 \, |\boldsymbol{b}_0(\lambda_0)|^2 \, \|F\|_{L^2(B)}^2 .$$

On the other hand, Theorem 6.1 means that $\mathcal{P}_{\lambda_0} F - \mathcal{M}_{\lambda_0} F \in \mathring{B}^*_{\nu_0/2}(X)$ and $I_1(R) \to 0$ as $R \to \infty$. The Hölder inequality yields that

$$|I_2(R)| \leqslant 2 I_1(R)^{1/2} \, \|\mathcal{P}_{\lambda_0} F\|_{*,\nu_0/2} .$$

Thus we obtain $I_2(R) \to 0$ as $R \to \infty$. ■

Here, we recall generalized spherical functions on X. Let $\hat{K}$ denote the set of equivalence class of unitary representations of K. For $\delta \in \hat{K}$, let V_δ be a representation space of the class δ. Let V_δ^M denote the subspace consisting of $\delta(M)$-invariant vectors in V_δ. We define $\hat{K}_M = \{\delta \in \hat{K}; V_\delta^M \neq \{0\}\}$. We put $d(\delta) = \dim V_\delta$, $l(\delta) = \dim V_\delta^M$. For $\delta \in \hat{K}_M$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the generalized spherical function $\Phi_{\lambda,\delta}(x)$ is defined by

$$\Phi_{\lambda,\delta}(x) = \int_K e^{(i\lambda+\rho)(A(x,kM))} \delta(k) dk.$$

As a direct consequence of Theorem 2.1 (i) and Theorem 6.1, we obtain the following two corollaries. The proof of each corollary is almost parallel to that of Corollary 7.1 in [16]. We therefore omit the proof.

Corollary 7.1. *We assume that $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$. Then there exists a positive constant C independent of λ_0 such that*

$$C^{-1} \, |\boldsymbol{b}_0(\lambda_0)| \, \frac{\|v_1\| \, \|v_2\|}{\sqrt{d(\delta)}} \leqslant \|\langle \Phi_{\lambda_0,\delta}(\,\cdot\,) v_1, v_2 \rangle\|_{B^*_{\nu_0/2}} \leqslant C \, |\boldsymbol{b}_0(\lambda_0)| \, \frac{\|v_1\| \, \|v_2\|}{\sqrt{d(\delta)}}$$

for $\delta \in \hat{K}_M$, $v_1 \in V_\delta^M$, *and* $v_2 \in V_\delta$. *Moreover, we have*

$$\lim_{R\to\infty} \frac{1}{R^{\nu_0}} \int_{B(o,R)} |\langle \Phi_{\lambda_0,\delta}(x)v_1, v_2\rangle|^2 \, dx = \gamma_0^2 \, |\boldsymbol{b}_0(\lambda_0)|^2 \, \frac{\|v_1\|^2 \, \|v_2\|^2}{d(\delta)}$$

$$\lim_{R\to\infty} \frac{1}{R^{\nu_0}} \int_{B(o,R)} \|\Phi_{\lambda_0,\delta}(x)\|_{\mathrm{HS}}^2 \, dx = \gamma_0^2 \, |\boldsymbol{b}_0(\lambda_0)|^2 \, l(\delta),$$

where $\|\cdot\|_{\mathrm{HS}}$ *denotes the Hilbert-Schmidt norm.*

Corollary 7.2. *We assume that* $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$. *Then for* $\delta \in \hat{K}_M$, $v_1 \in V_\delta^M$, *and* $v_2 \in V_\delta$, *we have*

$$\langle \Phi_{\lambda_0,\delta}(x)v_1, v_2\rangle \simeq \sum_{[w]\in W/W_{\lambda_0}} e^{(iw\lambda_0-\rho)(A^+(x))} \boldsymbol{a}_0(x, \lambda_0; w) \langle \delta(k_x) \boldsymbol{\Gamma}_{w^{-1},w\lambda_0} v_1, v_2\rangle$$

in $B^*_{\nu_0/2}(X)$, *where* $\boldsymbol{\Gamma}_{w,\lambda}$ *denotes a certain meromorphic function on* $\mathfrak{a}^*_{\mathbb{C}}$ *with values in* $\mathrm{Hom}(V_\delta^M, V_\delta^M)$. (*For the definition of* $\boldsymbol{\Gamma}_{w,\lambda}$, *see [13], Chapter III, § 2, and Theorem 2.10 therein.*)

Let $\mathcal{E}_\infty(B)$ denote the space of K-finite smooth functions on B. In the sequel, a *functional* T on B means that T is a linear form on a domain $\mathcal{D}(T)$ satisfying $\mathcal{E}_\infty(B) \subset \mathcal{D}(T) \subset C(B)$. For a functional T on B and $F \in \mathcal{D}(T)$ we write

$$T(F) = \int_B F(b)dT(b).$$

Let $\mathcal{E}(X)$ denote the topological vector space consisting of smooth functions on X with the usual Fréchet topology. Then, as is well-known, we have the following representation theorem for the Poisson transform.

Theorem 7.3 (Helgason [13], Chapter V, §6, Theorem 6.2, p.528)**.** *Let* f *be an arbitrary joint eigenfunction of* $\boldsymbol{D}(X)$. *Then there exist* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ *and a functional* T *on* B *such that*

$$f(x) = \int_B e^{(i\lambda+\rho)(A(x,b))} dT(b).$$

Moreover, let us denote by $T \sim \sum_{\delta\in\widehat{K}_M} d(\delta)\,\mathrm{Tr}(A_\delta \delta(k))$ *a formal Fourier series expansion of* T. *Here* A_δ *is given by*

$$A_\delta = \int_K \delta(k^{-1}) d\widetilde{T}(k) \in \mathrm{Hom}(V_\delta, V_\delta^M), \tag{7.3}$$

where $\widetilde{T}$ *denotes the lift of* T *to* K. *Then* $f(x)$ *has the convergent series expansion in* $\mathcal{E}(X)$ *of the form*

$$f(x) = \sum_{\delta\in\widehat{K}_M} d(\delta)\,\mathrm{Tr}(A_\delta \Phi_{\lambda,\delta}(x)).$$

Next, we consider Theorem 2.1 (ii), (iii). Since the proof is a repetition of that of Theorem 3.3 (ii), (iii) in [16], we give only a brief proof.

Proof of Theorem 2.1 (ii), (iii). (ii) It suffices to prove that the linear operator $\mathcal{P}_{\lambda_0} : L^2(B) \to \mathcal{E}^2_{\lambda_0}(X)$ is surjective. By Theorem 7.3, there exists a functional T on B which has the Fourier series expansion $T \sim \sum_{\delta \in \widehat{K}_M} d(\delta)\operatorname{Tr}(A_\delta \delta(k))$, where $A_\delta \in \operatorname{Hom}(V_\delta, V_\delta^M)$ is given by (7.3), such that

$$f(x) = \sum_{\delta \in \widehat{K}_M} d(\delta)\operatorname{Tr}(A_\delta \Phi_{\lambda,\delta}(x)) \quad \text{in } \mathcal{E}(X).$$

The Schur orthogonality relations imply that

$$\frac{1}{R^{\nu_0}} \int_{B(o,R)} |f(x)|^2 dx = \sum_{\delta \in \widehat{K}_M} d(\delta)^2 \frac{1}{R^{\nu_0}} \int_{B(o,R)} \|\Phi_{\lambda,\delta}(x) A_\delta\|^2_{\mathrm{HS}} dx. \tag{7.4}$$

Then, in the same manner as the proof of Theorem 3.3 (ii) in [16], equality (7.4) and Corollary 7.1 yield that

$$\gamma_0^2 |\boldsymbol{b}_0(\lambda_0)|^2 \sum_{\delta \in \widehat{K}_M} d(\delta)^2 \|A_\delta\|^2_{\mathrm{HS}} \leqslant \|f\|_{*,\nu_0/2} < \infty.$$

Hence the Fourier series $\sum_{\delta \in \widehat{K}_M} d(\delta)\operatorname{Tr}(A_\delta \delta(k))$ converges in $L^2(B)$. Thus there exists a unique $F \in L^2(B)$ such that $T = F$ as a functional on B. Moreover, we obtain $f = \mathcal{P}_{\lambda_0} F$.

(iii) For $f = \mathcal{P}_{\lambda_0} F \in \mathcal{E}^2_{\lambda_0}(X)$ with $F \in L^2(B)$, we put

$$F_{0,R}(b) = \gamma_0^{-2} |\boldsymbol{b}_0(\lambda_0)|^{-2} \frac{1}{R^{\nu_0}} \int_{B(o,R)} e^{(-i\lambda_0 + \rho)(A(x,b))} f(x) dx.$$

Then (2.3) implies that $F_{0,R}$ converges to F weakly in $L^2(B)$ as $R \to \infty$. Therefore it suffices to show that the norm convergence holds. As in the same manner in the proof of Theorem 3.3 (iii) in [16], Lemma 7.1 and the Schur orthogonality relation show that $\lim_{R\to\infty} \|F_{0,R}\|_{L^2(B)} = \|F\|_{L^2(B)}$. ■

8. Concluding remarks

As an application of Theorem 2.1, we give an explicit realization of an inner product of the Hilbert space associated with the left-regular representation.

First, we observe that the left-regular action of G induces a representation on the Banach space $B^*_\sigma(X)$. For $g \in G$, $f \in B^*_\sigma(X)$, we define

$$[L(g)f](x) := f(g^{-1} \cdot x).$$

Lemma 8.1. *$(L, B^*_\sigma(X))$ is a (strongly) continuous representation of G.*

Proof. The inclusion $B(g \cdot o, R) \subset B(o, R + r(g \cdot o))$ and G-invariantness of dx imply the inequality $\|L(g)f\|_{*,\sigma} \leqslant (1 + r(g \cdot o))^\sigma \|f\|_{*,\sigma}$. Since $\|f\|_{*,\sigma} \leqslant \|f\|_{L^2(X)}$ and $C_0^\infty(X)$ (or $L^2(X)$) is dense in $B^*_\sigma(X)$, the strong continuity of L on $B^*_\sigma(X)$ immediately follows from that of L on $L^2(X)$. ■

It follows immediately from Theorem 2.1 that $\mathcal{E}^2_{\lambda_0}(X) = \mathcal{H}_{\lambda_0}(X)$ as a set. Moreover, the polarization form yields the following realization of the inner product $(\cdot,\cdot)_{\lambda_0}$ of the Hilbert space $\mathcal{H}_{\lambda_0}(X)$.

Corollary 8.2. *For* $\lambda_0 \in \mathfrak{a}^*_{\mathrm{sing}}$, $f_1, f_2 \in \mathcal{H}_{\lambda_0}(X)$, *we have*

$$(f_1,f_2)_{\lambda_0} = \lim_{R\to\infty} \gamma_0^{-2} |\boldsymbol{b}_0(\lambda_0)|^{-2} \frac{1}{R^{\nu_0}} \int_{B(o,R)} f_1(x)\overline{f_2(x)}dx.$$

Finally, we observe that the unitary representation T_{λ_0} on $\mathcal{H}_{\lambda_0}(X)$ can be regarded as a uniform limit of a family of representations on Banach spaces. For $f \in \mathcal{E}^2_{\lambda_0}(X)$, $R_0 \geqslant 1$, we define

$$\|f\|_{R_0} = \gamma_0^{-1} |\boldsymbol{b}_0(\lambda_0)|^{-1} \sup_{R>R_0} \frac{1}{R^{\nu_0/2}} \left(\int_{B(o,R)} |f(x)|^2 dx \right)^{1/2}.$$

Then the normed space $(\mathcal{E}^2_{\lambda_0}(X), \|\cdot\|_{R_0})$ is a Banach space. In a similar manner as Lemma 8.1, we can also consider the left-regular representation on $(\mathcal{E}^2_{\lambda_0}(X), \|\cdot\|_{R_0})$. Furthermore, by (2.3), we obtain

$$C^{-1} \|f\|_{\mathcal{H}_{\lambda_0}} \leqslant \|f\|_{R_0} \leqslant C \|f\|_{\mathcal{H}_{\lambda_0}},$$
$$\lim_{R_0\to\infty} \|f\|_{R_0} = \|f\|_{\mathcal{H}_{\lambda_0}}.$$

Acknowledgments. The author would like to thank Professor Tomoyuki Kakehi for his kind advice and helpful comments for the paper. The author would also like to express his gratitude to Professor Sigurdur Helgason, Professor Angela Pasquale, and Professor David Vogan for leading him to this subject and for their valuable suggestions. This work was partially done while the author visited Tufts University in March, 2015. The author is very grateful to Professor Fulton Gonzalez for his hospitality and kindness during the author's stay. The author is also grateful to the referee for careful reading and valuable suggestions for improvement of the paper.

References

[1] Agmon, S., *Spectral theory of Schrödinger operators on Euclidean and on non-Euclidean spaces*, Comm. Pure Appl. Math. **39** (1986), S3–S16.

[2] Agmon, S., and L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Analyse Math. **30** (1976), 1–38.

[3] Anker, J.-P., *A basic inequality for scattering theory on Riemannian symmetric spaces of the noncompact type*, Amer. J. Math. **113** (1991), 391–398.

[4] Anker, J.-P., and L. Ji, *Heat kernel and Green function estimates on noncompact symmetric spaces*, Geom. Funct. Anal. **9** (1999), 1035–1091.

[5] van den Ban, E. P., and H. Schlichtkrull, *Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces*, J. Reine Angew. Math. **380** (1987), 108–165.

[6] Ben Saïd, S., T. Oshima, and N. Shimeno, *Fatou's theorems and Hardy-type spaces for eigenfunctions of the invariant differential operators on symmetric spaces*, Int. Math. Res. Not. **16** (2003), 915–931.

[7] Boussejra, A., and H. Sami, *Characterization of the L^p-range of the Poisson transform in hyperbolic spaces $B(\mathbb{F}^n)$*, J. Lie Theory **12** (2002), 1–14.

[8] Cowling, M., and S. Giulini, and S. Meda, *L^p-L^q estimates for functions of the Laplace-Beltrami operator on noncompact symmetric spaces. I*, Duke Math. J. **72** (1993), 109–150.

[9] Gangolli, R., and V. S. Varadarajan, "Harmonic analysis of spherical functions on real reductive groups," Springer-Verlag, 1988.

[10] Helgason, S., *A duality for symmetric spaces with applications to group representations*, Adv. Math. **5** (1970),1–154.

[11] —, *Integral geometry and multitemporal wave equations*, Comm. Pure Appl. Math. **51** (1998), 1035–1071.

[12] —, "Groups and geometric analysis," American Mathematical Society, Providence, R.I., 2000.

[13] —, "Geometric analysis on symmetric spaces," second edition, American Mathematical Society, Providence, R.I., 2008.

[14] Ionescu, A. D., *On the Poisson transform on symmetric spaces of real rank one*, J. Funct. Anal. **174** (2000), 513–523.

[15] Kaizuka, K., *Resolvent estimates on symmetric spaces of noncompact type*, J. Math. Soc. Japan **66** (2014), 895–926.

[16] —, *A characterization of the L^2-range of the Poisson transform related to Strichartz conjecture on symmetric spaces of noncompact type*, Adv. Math. **303** (2016), 464–501.

[17] Kashiwara, M., A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. **107** (1978), 1–39.

[18] Kumar, P., *Fourier restriction theorem and characterization of weak L^2 eigenfunctions of the Laplace-Beltrami operator*, J. Funct. Anal. **266** (2014), 5584–5597.

[19] Kumar, P., S. K. Ray, and R. P. Sarkar, *Characterization of almost L^p-eigenfunctions of the Laplace-Beltrami operator*, Trans. Amer. Math. Soc. **366** (2014), 3191–3225.

[20] Lewis, J. B., *Eigenfunctions on symmetric spaces with distribution-valued boundary forms*, J. Funct. Anal. **29** (1978), 287–307.

[21] Lohoué, N., and Th. Rychener, *Some function spaces on symmetric spaces related to convolution operators*, J. Funct. Anal. **55** (1984), 200–219.

[22] Narayanan, E. K., A. Pasquale, and S. Pusti, *Asymptotics of Harish-Chandra expansions, bounded hypergeometric functions associated with root systems, and applications*, Adv. Math. **252** (2014), 227–259.

[23] Oshima, T., *Asymptotic behavior of spherical functions on semisimple symmetric spaces*, Adv. Stud. Pure Math. **14** (1988), 561–601.

[24] Oshima, T., and J. Sekiguchi, *Eigenspaces of invariant differential operators on an affine symmetric space*, Invent. Math. **57** (1980), 1–81.

[25] Phillips, R. S., and M. M. Shahshahani, *Scattering theory for symmetric spaces of noncompact type*, Duke Math. J. **72** (1993), 1–29.

[26] Semenov-Tjan-Šanskiĭ, M. A., *Harmonic analysis on Riemannian symmetric spaces of negative curvature and scattering theory*, Math. USSR, Izvestija **10** (1976), 535–563.

[27] Shahshahani, M. M, *Invariant hyperbolic systems on symmetric spaces*, Progr. Math. **32** (1983), 203–233.

[28] Sjögren, P., *Characterizations of Poisson integrals on symmetric spaces*, Math. Scand. **49** (1981), 229–249.

[29] Strichartz, R. S., *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. **87** (1989), 51–148.

[30] —, *Corrigendum to: "Harmonic analysis as spectral theory of Laplacians"*, J. Funct. Anal. **109** (1992), 457–460.

[31] Trombi, P. C., and V. S. Varadarajan, *Spherical transforms of semisimple Lie groups*, Ann. of Math. (2) **94** (1971), 246–303.

[32] Varadarajan, V. S., "Harmonic analysis on real reductive groups", Lecture Notes in Mathematics **576**, Springer-Verlag, Berlin-New York, 1977.

Koichi Kaizuka
Department of Mathematics
Gakushuin University
1-5-1 Mejiro, Toshima-ku
Tokyo 171-8588, Japan
Current address:
Department of Mathematics
Nippon Medical School
1-7-1 Kyounan-cho, Musashino-shi
Tokyo, 180-0023, Japan
k-kaizu@nms.ac.jp

Received October 26, 2016
and in final form December 14, 2017