Harmonic Analysis for an Olshanski Pair Consisting of Stabilizers of the Horocycles of a Homogeneous Tree

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Abstract. The classification of the irreducible unitary representations of the stabilizer of the horocycles of a homogeneous tree of finite degree is well-known. In this article we use these stabilizers to form an Olshanski pair and then find all spherical functions of this pair. Finally we give realizations of the corresponding irreducible unitary representations.
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1. Introduction

Let G be a locally compact group and K be a compact subgroup of G. The space of integrable functions with respect to a Haar measure on G which are bi-invariant under K becomes an algebra if we consider the convolution as the product operation. If this algebra $L^1(K\backslash G/K)$ is commutative, we say that (G, K) is a Gelfand pair. Suppose that we have a Gelfand pair (G, K). A K-bi-invariant continuous complex function φ on G with $\varphi(e) = 1$ is said to be a spherical function for the pair (G, K) if

$$\varphi(x)\varphi(y) = \int_K \varphi(xky) \, dk$$

for every $x, y \in G$ where dk is the normalized Haar measure on K. An irreducible unitary representation (π, V) of G is called spherical for the pair (G, K) if V has a non-zero K-invariant vector. In this case, up to multiplication by a unit complex number, there is a unique $v \in V$ which is invariant under K and has norm 1. Then $\varphi(g) = \langle \pi(g)v, v \rangle$ is a positive definite spherical function on G. Conversely, there is a construction, the so-called Gelfand-Neimark-Segal construction, which gives a spherical representation of (G, K) corresponding to every positive definite spherical function for (G, K). Therefore there is essentially a one-to-one correspondence between positive definite spherical functions and spherical representations

of (G, K). To be more precise, there is a one-to-one correspondence between the positive definite spherical functions and the unitary equivalence classes of spherical representations of (G, K). It is hence a natural procedure to find spherical functions and then to find the realizations of the corresponding spherical representations of a Gelfand pair given by the GNS-construction. For example, if (X, E) is a homogeneous tree of finite degree and G is the automorphism group of (X, E) and K is the stabilizer of a fixed vertex $x_0 \in X$, then (G, K) is a Gelfand pair and in his work [4] Olshanski gave realizations of the spherical representations of G corresponding to the spherical functions which were given in [2]. If (X, E) is such a tree, then one can also fix a point ω on the boundary of this tree and consider the group of stabilizers of the corresponding horocycles which gives rise to certain Gelfand pairs. In [3] Nebbia found all spherical functions of these pairs and described the corresponding spherical representations.

If the groups under discussion are not locally compact, we do not have a Haar measure anymore. Therefore we do not have a convolution, an algebra structure and so on. But, in some cases it is possible that our large groups can be written as the limit in some sense of an increasing sequence of Gelfand pairs. Suppose that $(G_n, K_n)_{n \geq 1}$ is a sequence of Gelfand pairs that satisfies the following properties:

1. $G_n \subset G_{n+1}$ for every n,

2. G_n is a closed subgroup of G_{n+1} for every n,

3. $K_{n+1} \cap G_n = K_n$ for every n.

Now let $G = \cup_1^\infty G_n$ and $K = \cup_1^\infty K_n$. Then we say that (G, K) is an Olshanski spherical pair (or simply an Olshanski pair). Olshanski realized that we could extend some of the notions for Gelfand pairs to this case: An irreducible unitary representation of G is said to be spherical if it has non-zero K-invariant vectors. If v is a K-invariant vector of norm 1 in a spherical representation V, we define $\varphi(g) = \langle \pi(g)v, v \rangle$ for $g \in G$. When we put the inductive limit topology on G, the function φ is continuous, positive definite and bi-invariant under K. It has also the property that

$$\lim_{n \to \infty} \int_{K_n} \varphi(x k_n y) \, dk_n = \varphi(x)\varphi(y)$$

for every $x, y \in G$ where dk_n is the normalized Haar measure on K_n for every n. In other words, φ satisfies some "asymptotic functional equations". Olshanski defined spherical functions for the Olshanski pair (G, K) to be such non-zero functions and he extended the GNS-construction and the one-to-one correspondence between the positive definite spherical functions and the unitary equivalence classes of spherical representations to this case as well. For example, in [1] Axelgaard studied an embedding of the k-homogeneous tree into the $(k + 1)$-homogeneous tree and the embedding of the corresponding tree automorphism groups $G_k \subset G_{k+1}$. This way he obtained an Olshanski pair. He described all spherical functions and the corresponding spherical representations. He mentioned that such an embedding was also possible for the Gelfand pairs studied by Nebbia in [3] and stated the description of spherical functions and representations of that pair as an open

problem in [1]. We complete this picture by describing all spherical functions and the realizations of the corresponding spherical representations in this case.

2. The Olshanski pair (B_ω, B_n)

In this section we give the necessary definitions and notations to construct the Olshanski pair on which we study.

Let (X, E) be a non-directed graph where X is the set of vertices and E, which is a subset of two-element subsets of X, is the set of edges. Two vertices $x, y \in X$ are called neighbours if $\{x, y\} \in E$. The degree of a vertex x is the number of neighbours of x. A path of length n in (X, E) is a sequence $x_0, x_1, ..., x_n$ of vertices such that $\{x_i, x_{i+1}\} \in E$ for all $i \in \{0, 1, ..., n-1\}$. A path $x_0, x_1, ..., x_n$ is called a chain if $x_i \neq x_{i+2}$ for all $i \in \{0, 1, ..., n-2\}$ and a chain $x_0, x_1, ..., x_n$ is called a circuit if $x_0 = x_n$. The graph (X, E) is said to be connected if for all $x, y \in X$ there is a path $x_0, x_1, ..., x_n$ such that $x_0 = x$ and $x_n = y$.

A tree is a connected graph without circuits. If (X, E) is a tree and x, y are two vertices, then there is a unique chain that joins x and y and this chain is called the geodesic between x and y. Hence a natural distance $d(x, y)$ between two vertices x and y can be defined as the length of the geodesic between them.

A tree is said to be homogeneous if every vertex has the same degree. The common degree of all vertices of a homogeneous tree is called the degree of the tree. From now on, let (X, E) be a homogeneous tree of countably infinite degree. An infinite chain starting at x_0 is an infinite sequence $x_0, x_1, x_2, ...$ of vertices such that $x_0, x_1, ..., x_i$ is a chain for every i. Fix a vertex $x_0 \in X$. We define the boundary Ω of the tree (X, E) as the set of all infinite chains starting at x_0. Note that a point of the boundary may be considered as a point at infinity. Fix two distinct elements ω and ω' of the boundary Ω. Let us entitle the vertices of ω and ω' such that $(x_n)_{n=0}^{\infty}$ is the unique infinite chain associated to ω and $(x_n)_{n=0}^{-\infty}$ is the unique infinite chain associated to ω'. Then the sequence $(x_n)_{n \in \mathbb{Z}}$ is the unique doubly infinite chain joining ω and ω'. Let us denote $(x_n)_{n \in \mathbb{Z}}$ by (ω', ω).

For each vertex x in X, we enumerate its neighbours by positive integers: If $x \neq x_n$ for any $n \in \mathbb{Z}$, then we choose a bijection $\tau_x : \mathbb{N} \to \{y \in X \mid d(x, y) = 1\}$ such that $\tau_x(1)$ is the unique neighbour of x with minimal distance to the chain (ω', ω). If $x = x_n$ for some $n \in \mathbb{Z}$, then we choose a bijection $\tau_x : \mathbb{N} \to \{y \in X \mid d(x, y) = 1\}$ such that $\tau_x(1) = x_{n-1}$ and $\tau_x(2) = x_{n+1}$. We fix these bijections $\{\tau_x\}_{x \in X}$.

For each integer $k \geq 2$, we define a subtree (X^k, E^k) of the tree (X, E) as follows: If $x = x_n$ for some $n \in \mathbb{Z}$, we let $x \in X^k$. If $x \neq x_n$ for any $n \in \mathbb{Z}$, then there exists a unique vertex x_{n_0} on the chain (ω', ω) with minimal distance to x. Suppose $y_0, y_1, ..., y_m$ are the vertices on the geodesic from x_{n_0} to x with $y_0 = x_{n_0}$ and $y_m = x$ and that y_{j+1} is a neighbour of y_j. Then we say that $x \in X^k$ if $y_{j+1} \in \tau_{y_j}(\{2, 3, ..., k+1\})$ for all $j \in \{0, 1, ..., m-1\}$. Let $E^k = \{\{x, y\} \in E \mid x \in X^k, y \in X^k\}$. Note that (X^k, E^k) is a locally finite, homogeneous subtree of degree $k + 1 \geq 3$ of the tree (X, E).

We shall now define the horocycle H_n of the tree (X, E) associated to the boundary point ω, for $n \in \mathbb{Z}$. For $x \in X$, let x_{n_0} be the unique vertex on the

chain (ω', ω) with minimal distance to x. Then we say that x belongs to the horocycle H_n if the equation $d(x, x_{n_0}) = d(x_n, x_{n_0})$ holds. The family $\{H_n\}_{n \in \mathbb{Z}}$ is a partition of X. Given $n \in \mathbb{Z}$ and an integer $k \geq 2$, set the horocycle H_n^k of the subtree (X^k, E^k) by $H_n^k = H_n \cap X^k$.

Given an integer $k \geq 2$, let $Aut(X^k)$ be the group of all automorphisms of the tree (X^k, E^k), i.e. the group of all bijections from X^k onto itself which preserve the edges. The group $Aut(X^k)$ is a locally compact topological group with the topology of compact convergence. The collection of sets $V_F(g) = \{h \in Aut(X^k) : h(x) = g(x) \text{ for all } x \in F\}$ forms a base for this topology where $g \in Aut(X^k)$ and $F \subset X^k$ is finite. One can also see a fundamental system of compact open neighbourhoods of the identity in [4]. Given a vertex $x \in X^k$, let K_x^k be the subgroup of $Aut(X^k)$ consisting of all automorphisms fixing x. Then K_x^k is an open compact subgroup of $Aut(X^k)$ and the pair $(Aut(X^k), K_x^k)$ is a Gelfand pair by [[1], Proposition 2.3.2]. In particular, the pair $(Aut(X^k), K_{x_n}^k)$ is a Gelfand pair for all $n \in \mathbb{Z}$.

Let
$$B_\omega^k = \{g \in Aut(X^k) \mid g(H_n^k) = H_n^k \text{ for all } n \in \mathbb{Z}\}.$$
It can be observed that
$$B_\omega^k = \{g \in Aut(X^k) \mid g(x_n) = x_n \text{ for n sufficiently large}\}.$$

Indeed, if $g \in B_\omega^k$ and $g(x_0) = x_0$, then $g(x_n) = x_n$ for all $n \geq 0$. If $g \in B_\omega^k$ and $g(x_0) = x$ for some $x \in H_n^k \backslash \{x_0\}$ and if x_m is the unique vertex on (ω', ω) with minimal distance to x, then $g(x_n) = x_n$ for all $n \geq m$.

On the group B_ω^k we consider the topology induced from that of $Aut(X^k)$. The group B_ω^k is a closed subgroup of $Aut(X^k)$, hence B_ω^k is itself a locally compact topological group.

For all $n \in \mathbb{Z}$, set $B_n^k = B_\omega^k \cap K_{x_n}^k$. Then B_n^k is an open compact subgroup of B_ω^k for which we have the following proposition.

Proposition 2.1 ([3]). *The pair (B_ω^k, B_n^k) is a Gelfand pair for all integers $k \geq 2$ and for all $n \in \mathbb{Z}$.*

By using these Gelfand pairs as building blocks, we shall construct an Olshanski pair. Let $Aut(X)$ be the group of all automorphisms of the tree (X, E) and let
$$B_\omega^\infty = \{g \in Aut(X) \mid g(H_n) = H_n \text{ for all } n \in \mathbb{Z}\}.$$
The group $Aut(X)$ is a topological group with the compact-open topology. The collection of sets $U_F(g) = \{h \in Aut(X) : h(x) = g(x) \text{ for all } x \in F\}$ forms a base for this topology where $g \in Aut(X)$ and $F \subset X$ is finite. Then B_ω^∞ is also a topological group with the topology induced from that of $Aut(X)$.

We can embed $Aut(X^k)$ in $Aut(X)$ for each $k \geq 2$ as follows: Let $g \in Aut(X^k)$. We will extend g to an automorphism $\tilde{g}$ in $Aut(X)$ which we will call the natural extension of g. So let $x \in X$ and let y be the unique vertex of X^k with minimal distance to x. Assume that $y_0, y_1, ..., y_m$ are the vertices on the geodesic from y to x with $y_0 = y$, $y_m = x$ and $d(y_i, y_{i+1}) = 1$. Since $y_1, ..., y_m$

are not in X^k, for each $i \in \{0, 1, ..., m-1\}$ there exists an integer $k_i > k+1$ such that $y_{i+1} = \tau_{y_i}(k_i)$. We now define $\tilde{g}(x) = \tilde{g}(y_m)$ where $\tilde{g}(y_m)$ is defined inductively by $\tilde{g}(y_0) = g(y)$ and $\tilde{g}(y_{i+1}) = \tau_{\tilde{g}(y_i)}(k_i)$ for all $i \in \{0, 1, ..., m-1\}$. Then $\tilde{g}$ is an automorphism of the tree (X, E) which extends the automorphism g of the subtree (X^k, E^k).

Lemma 2.2 ([1]).   *The map $\varphi_k : Aut(X^k) \to Aut(X)$ defined by $\varphi_k(g) = \tilde{g}$ is an injective group homomorphism, which is also a homeomorphism onto its image.*

Proof.   Let $g, h \in Aut(X^k)$ and $x \in X$. Let $y = y_0, y_1, ..., y_m = x$ and $k_0, k_1, ..., k_{m-1}$ be given as above. So $\tilde{g}(y_m) = \tau_{\tilde{g}(y_{m-1})}(k_{m-1})$ where $\tilde{g}(y_i)$ is defined inductively by $\tilde{g}(y_0) = g(y)$ and $\tilde{g}(y_{i+1}) = \tau_{\tilde{g}(y_i)}(k_i)$ for $i \in \{0, 1, ..., m-1\}$. Let $z_i = \tilde{g}(y_i)$ for all $i \in \{0, 1, ..., m\}$. Then given any $i \in \{1, ..., m\}$, $z_i = \tau_{\tilde{g}(y_{i-1})}(k_{i-1}) = \tau_{z_{i-1}}(k_{i-1})$ and since $k_{i-1} > k+1$, the unique vertex of X^k with minimal distance to z_m is z_0. Hence $\tilde{h}(z_m) = \tau_{\tilde{h}(z_{m-1})}(k_{m-1})$ where $\tilde{h}(z_i)$ is defined inductively by $\tilde{h}(z_0) = h(z_0)$ and $\tilde{h}(z_{i+1}) = \tau_{\tilde{h}(z_i)}(k_i)$ for $i \in \{0, 1, ..., m-1\}$. On the other hand, $\widetilde{hg}(y_m) = \tau_{\widetilde{hg}(y_{m-1})}(k_{m-1})$ where $\widetilde{hg}(y_i)$ is defined inductively by $\widetilde{hg}(y_0) = h(g(y))$ and $\widetilde{hg}(y_{i+1}) = \tau_{\widetilde{hg}(y_i)}(k_i)$ for $i \in \{0, 1, ..., m-1\}$. Observe that $\widetilde{hg}(y_0) = h(z_0)$, $\widetilde{hg}(y_1) = \tau_{h(z_0)}(k_0) = \tilde{h}(z_1)$ and by induction $\widetilde{hg}(y_m) = \tilde{h}(z_m)$. Hence $\widetilde{hg}(x) = \widetilde{hg}(y_m) = \tilde{h}(z_m) = \tilde{h}(\tilde{g}(y_m)) = \tilde{h}(\tilde{g}(x))$ and this shows that φ_k is a group homomorphism. If $\tilde{g}$ is the identity automorphism on X, then g which is the restriction of $\tilde{g}$ on X^k is the identity automorphism on X^k. So the kernel of φ_k is trivial and φ_k is one to one.

We shall next prove the continuity of φ_k. So let $F = \{x_1, ..., x_n\}$ be a finite subset of X and $g \in Aut(X^k)$. Consider the basic open subset $U_F(\tilde{g}) = \{h \in Aut(X) : h(x) = \tilde{g}(x) \text{ for all } x \in F\}$ of $Aut(X)$. For each $x_i \in F$, let x_i' be the unique vertex of X^k with minimal distance to x_i. Let $F' = \{x_1', ..., x_n'\}$. If $f \in Aut(X^k)$ and $f = g$ on F', then $\tilde{f} = \tilde{g}$ on F. In other words, if f is an element of the basic open subset $V_{F'}(g) = \{h \in Aut(X^k) : h(x) = g(x) \text{ for all } x \in F'\}$ of $Aut(X^k)$, then $\tilde{f} \in U_F(\tilde{g})$. Hence $\varphi_k(V_{F'}(g)) \subset U_F(\tilde{g})$ and φ_k is continuous at g.

Now, let F be a finite subset of X^k. Then for all $g \in Aut(X^k)$, we have $\varphi_k(V_F(g)) = U_F(\tilde{g}) \cap \varphi_k(Aut(X^k))$ where $V_F(g)$ and $U_F(\tilde{g})$ are basic open sets of $Aut(X^k)$ and $Aut(X)$, respectively. Hence φ_k is a homeomorphism onto its image $\varphi_k(Aut(X^k))$.   ∎

For $g \in B_\omega^k$, we have $\tilde{g} \in B_\omega^\infty$. Hence if we let ψ_k to be the restriction of the map φ_k to B_ω^k, then we have the following corollary of Lemma 2.2.

Corollary 2.3.   *The map $\psi_k : B_\omega^k \to B_\omega^\infty$ defined by $\psi_k(g) = \tilde{g}$ is an injective group homomorphism, which is a homeomorphism onto its image.*

Given $n \in \mathbb{Z}$, we identify B_ω^k and B_n^k with their images $\psi_k(B_\omega^k)$ and $\psi_k(B_n^k)$ in B_ω^∞, respectively, and we define

$$B_\omega = \cup_{k=2}^\infty B_\omega^k \quad \text{and} \quad B_n = \cup_{k=2}^\infty B_n^k.$$

We endow B_ω with the inductive limit topology.

Proposition 2.4.   *The pair* (B_ω, B_n) *is an Olshanski spherical pair for all* $n \in \mathbb{Z}$.

Proof.   By Proposition 2.1 and Corollary 2.3, we have a sequence of Gelfand pairs $(B_\omega^k, B_n^k)_{k \geq 2}$. Given $g \in B_\omega^k$, the automorphism g is the natural extension of its restriction onto the finite degree subtree X^m for each $m \geq k$. Hence $B_\omega^k \subset B_\omega^{k+1}$, $B_n^k \subset B_n^{k+1}$ and $B_n^k = B_n^{k+1} \cap B_\omega^k$ for all $k \geq 2$.

What is rest is to show that B_ω^k is a closed subgroup of B_ω^{k+1}. For this, we identify each $Aut(X^k)$ with its image in $Aut(X)$ under φ_k. Then $Aut(X^k)$ is a closed subgroup of $Aut(X^{k+1})$ by [[1],Proposition 3.1.2]. The topologies on B_ω^{k+1} and $Aut(X^{k+1})$ are both induced from the same topology of $Aut(X)$ and $B_\omega^{k+1} \subset Aut(X^{k+1})$, so the topology of B_ω^{k+1} is the one induced from the topology of $Aut(X^{k+1})$.

Hence $B_\omega^{k+1} \backslash B_\omega^k = (Aut(X^{k+1}) \backslash Aut(X^k)) \cap B_\omega^{k+1}$ is open in B_ω^{k+1} as desired. ∎

Within the rest of the text, we fix $n \in \mathbb{Z}$ and consider the Olshanski pair (B_ω, B_n). It is a natural question to find all positive definite spherical functions and the corresponding spherical representations of this pair.

3.   Spherical Functions For (B_ω, B_n)

According to Theorem 22.10 in [5], every positive definite spherical function for an Olshanski spherical pair is the uniform limit on compact sets of positive definite spherical functions of the underlying Gelfand pairs. In [3], Nebbia determines the spherical functions for the Gelfand pairs (B_ω^k, B_n^k) as follows:

Proposition 3.1.   *The non-trivial spherical functions for the Gelfand pair* $\left(B_\omega^k, B_n^k\right)$ *are the following:*

$$\varphi_{m,k} = 1_{B_m^k} + \frac{1}{1-k} 1_{B_{m+1}^k \backslash B_m^k} \qquad (1)$$

for every $m \geq n$. (1_E *is the characteristic function of the set* E.)

Since compactly supported spherical functions for a Gelfand pair are positive definite, all spherical functions are positive definite.

Hence, for each $m \geq n$ we consider the following pointwise limits

$$\lim_{k \to \infty} \varphi_{m,k} = 1_{B_m}$$

as candidates of positive definite spherical functions for the Olshanski pair (B_ω, B_n). Next we observe that the set $\{1_{B_m} : m \geq n\}$ consists of all non-trivial spherical functions for (B_ω, B_n).

Theorem 3.2.   *The non-trivial spherical functions for the pair* (B_ω, B_n) *are the characteristic functions* 1_{B_m} *with* $m \geq n$. *The spherical functions are all positive definite.*

Proof.   Let $\varphi : B_\omega \to \mathbb{C}$ be a non-trivial spherical function. If $g, h \in B_m \backslash B_{m-1}$ for some integer $m > n$, then the equality $d(x_n, g(x_n)) = d(x_n, h(x_n))$ holds. Hence there exists $k \in B_n$ satisfying $k(g(x_n)) = h(x_n)$. This shows that $h^{-1}kg \in B_n$, i.e. $g \in B_n h B_n$ and that $\varphi(g) = \varphi(h)$ by the B_n-bi-invariance of φ. So φ is constant on B_n and $B_m \backslash B_{m-1}$ for all $m > n$. Let us denote by $\varphi(n)$ the value of φ on B_n and by $\varphi(m)$ the value of φ on $B_m \backslash B_{m-1}$ given $m > n$.

Now fix two integers m and p such that $m > p \geq n$. Let $g \in B_p \backslash B_{p-1}$ and $h \in B_m \backslash B_{m-1}$. Choose L such that $g, h \in B_m^L$. If $k > L$ and $l \in B_n^k$, then $glh(x_m) = gl(x_m) = g(x_m) = x_m$ whereas $glh(x_{m-1}) \neq x_{m-1}$ because otherwise $h(x_{m-1}) = l^{-1}g^{-1}(x_{m-1}) = l^{-1}(x_{m-1}) = x_{m-1}$ which is impossible as $h \in B_m \backslash B_{m-1}$. Hence $glh \in B_m \backslash B_{m-1}$ and $\varphi(glh) = \varphi(m)$ for all $l \in B_n^k$. Since φ is spherical, we then get

$$\varphi(p)\varphi(m) = \varphi(g)\varphi(h) = \lim_{k \to \infty} \int_{B_n^k} \varphi(glh) \, dl = \lim_{k \to \infty} \int_{B_n^k} \varphi(m) \, dl = \varphi(m) \qquad (2)$$

where dl is the normalized Haar measure on B_n^k.

From the equation (2) we conclude that if $\varphi(p) = 0$ for $p \geq n$, then $\varphi(m) = 0$ for all $m > p$ and that if $\varphi(m) \neq 0$ for some $m > n$, then $\varphi(p) = 1$ for all $p < m$. Since $\varphi \neq 0$ and $\varphi \neq 1$, then either $\varphi = \alpha 1_{B_n}$ for some $\alpha \in \mathbb{C} \backslash \{0\}$ or $\varphi = 1_{B_{m-1}} + \alpha 1_{B_m \backslash B_{m-1}}$ for some $\alpha \in \mathbb{C} \backslash \{0\}$ and $m > n$.

To compute α in case $\varphi = \alpha 1_{B_n}$, take $g \in B_n$. Since φ is spherical,

$$\alpha^2 = \varphi(g)^2 = \lim_{k \to \infty} \int_{B_n^k} \varphi(glg) \, dl = \lim_{k \to \infty} \int_{B_n^k} \varphi(n) \, dl = \varphi(n) = \alpha$$

so that $\alpha = 1$ and $\varphi = 1_{B_n}$.

In case $\varphi = 1_{B_{m-1}} + \alpha 1_{B_m \backslash B_{m-1}}$, take $g \in B_m \backslash B_{m-1}$. There exists L such that $g \in B_m^L \backslash B_{m-1}^L$. Given $k > L$, let $y_1, y_2, ..., y_{k-1}$ be the neighbours of x_m in X^k except from x_{m-1} and x_{m+1}. We may assume that $y_1 = g^{-1}(x_{m-1})$. For $l \in B_n^k$, $l(g(x_{m-1}))$ may take one of the $k - 1$ values $y_1, y_2, ..., y_{k-1}$. Therefore, B_n^k is the disjoint union of the sets $A_1, A_2, ..., A_{k-1}$ where

$$A_i = \left\{ l \in B_n^k : l(g(x_{m-1})) = y_i \right\}.$$

Choosing an automorphism $k_{i,j}$ in B_n^k such that $k_{i,j}(y_i) = y_j$, we observe that $k_{i,j}(A_i) = A_j$. It follows that each A_i has the same measure $\frac{1}{k-1}$ with respect to the normalized Haar measure on B_n^k. Then,

$$\int_{B_n^k} \varphi(glg) \, dl = \int_{A_1} \varphi(glg) \, dl + \int_{B_n^k \backslash A_1} \varphi(glg) \, dl$$

$$= \int_{A_1} 1 \, dl + \int_{B_n^k \backslash A_1} \varphi(m) \, dl$$

$$= \frac{1}{k-1} + \frac{k-2}{k-1}\alpha$$

and together with the fact that φ is spherical we get

$$\alpha^2 = (\varphi(m))^2 = (\varphi(g))^2 = \lim_{k \to \infty} \int_{B_n^k} \varphi(glg) \, dl = \lim_{k \to \infty} (\frac{1}{k-1} + \frac{k-2}{k-1}\alpha) = \alpha.$$

So $\alpha = 1$ and $\varphi = 1_{B_{m-1}} + 1_{B_m \setminus B_{m-1}} = 1_{B_m}$.

Conversely, let $\varphi = 1_{B_m}$ for $m \geq n$. Since B_m is an open subgroup of B_ω, the function 1_{B_m} is continuous and since $B_n \subseteq B_m$, the function 1_{B_m} is B_n-bi-invariant.

Now let $g, h \in B_\omega$ and $\epsilon > 0$. There exists L such that $g, h \in B_\omega^L$. Since $\varphi_{m,k}$ defined in (1) is spherical for all m and k, we get

$$\left| \int_{B_n^k} 1_{B_m}(glh) - 1_{B_m}(g)1_{B_m}(h) \; dl \right|$$

$$\leq \int_{B_n^k} |1_{B_m}(glh) - \varphi_{m,k}(glh)| \; dl + |\varphi_{m,k}(g)\varphi_{m,k}(h) - 1_{B_m}(g)1_{B_m}(h)|$$

(3)

for all $k > L$. But $|1_{B_m}(glh) - \varphi_{m,k}(glh)|$ is either 0 or $\frac{1}{k-1}$ for any $l \in B_\omega$ and we can choose $N_1 > L$ so that for all $k > N_1$ we have $\frac{1}{k-1} < \frac{\epsilon}{2}$. Since $\varphi_{m,k} \to 1_{B_m}$ as $k \to \infty$, we can choose $N_2 > N_1$ so that for all $k > N_2$, both $|\varphi_{m,k}(g) - 1_{B_m}(g)| < \frac{\epsilon}{4}$ and $|\varphi_{m,k}(h) - 1_{B_m}(h)| < \frac{\epsilon}{4}$. Hence by (3) for all $k > N_2$, $\left| \int_{B_n^k} 1_{B_m}(glh) - 1_{B_m}(g)1_{B_m}(h) \; dl \right| < \epsilon$ and this shows 1_{B_m} is spherical for the Olshanski pair (B_ω, B_n).

Every spherical function 1_{B_m} is positive definite because given an arbitrary group G and a subgroup H of G, the characteristic function 1_H of H is always positive definite. ∎

4. Spherical Representations For (B_ω, B_n)

In this section we will make concrete realizations of the spherical representations of the Olshanski pair (B_ω, B_n) which correspond to the positive definite spherical functions for this pair by the Gelfand-Naimark-Segal construction.

Given $m \in \mathbb{Z}$, consider the horocycle H_m containing the vertex x_m. The group B_ω acts transitively on the horocycle H_m. Indeed, if $x, y \in H_m$, let $m_1 \geq m$ be such that x_{m_1} is the unique vertex on the chain (ω', ω) with minimal distance to x and $m_2 \geq m$ be such that x_{m_2} is the unique vertex on the chain (ω', ω) with minimal distance to y. Assume $m_1 \geq m_2$ and $x, y \in X^k$ for $k \in \mathbb{Z}$. Then we can find an automorphism $g \in Aut(X^k)$ such that $g(x) = y$ and $g(x_n) = x_n$ for all $n \geq m_1$. Then the natural extension $\tilde{g}$ of g is an automorphism in B_w which sends x to y.

The spherical function $\varphi = 1$ corresponds to the one dimensional trivial representation. For the spherical function $\varphi_m = 1_{B_m}$ with $m \geq n$, consider the Hilbert space $l^2(H_m) = L^2(H_m, \lambda_m)$ where λ_m is the counting measure on H_m. By the action of B_ω on the horocycle H_m, we get a representation π_m of B_ω on the Hilbert space $l^2(H_m)$ if we define $(\pi_m(g)f)(x) = f(g^{-1}(x))$ where $g \in B_\omega$, $f \in l^2(H_m)$ and $x \in H_m$. Indeed, π_m is a group homomorphism from B_ω to the group $U(l^2(H_m))$ of unitary operators on the Hilbert space $l^2(H_m)$. The representation π_m is moreover continuous where we put the strong operator topology on $U(l^2(H_m))$: Since $\{1_{\{x\}} : x \in H_m\}$ forms an orthonormal basis for $l^2(H_m)$, it suffices to prove that the map $g \longmapsto \pi_m(g)1_{\{x\}}$ is continuous for each

$x \in H_m$. Given $x \in H_m$ and $g_0 \in B_\omega$, the set $U_0 = \{g \in B_\omega : g(x) = g_0(x)\}$ is open in the topology induced from $Aut(X)$. The inductive limit topology on B_ω is stronger than the topology induced from $Aut(X)$. So, the set U_0 is also open in the inductive limit topology. The map $g \longmapsto \pi_m(g)1_{\{x\}}$ is constant on the open set U_0. So the map $g \longmapsto \pi_m(g)1_{\{x\}}$ is locally constant, hence continuous as desired. Therefore, π_m defines a unitary representation of B_ω.

$1_{\{x_m\}}$ is a B_n-bi-invariant unit vector in $l^2(H_m)$. By the transitive action of B_ω on $l^2(H_m)$, we have

$$\overline{span\left\{\pi_m(g)1_{\{x_m\}} : g \in B_\omega\right\}} = \overline{span\left\{1_{\{g(x_m)\}} : g \in B_\omega\right\}}$$
$$= \overline{span\left\{1_{\{y\}} : y \in H_m\right\}}$$
$$= l^2(H_m)$$

so that $1_{\{x_m\}}$ is also a cyclic vector.

Moreover,

$$< 1_{\{x_m\}}, \pi_m(g)1_{\{x_m\}} > = \sum_{y \in H_m} 1_{\{x_m\}}(y).\overline{\pi_m(g)1_{\{x_m\}}(y)}$$
$$= \sum_{y \in H_m} 1_{\{x_m\}}(y).1_{\{x_m\}}(g^{-1}y)$$
$$= 1_{\{x_m\}}(g^{-1}x_m)$$
$$= 1_{B_m}(g).$$

Hence we have proved the following theorem:

Theorem 4.1. *The spherical representation of the Olshanski pair (B_ω, B_n) corresponding to the spherical function $\varphi = 1$ by the Gelfand-Naimark-Segal construction is the one dimensional trivial representation.*

The spherical representation of the Olshanski pair (B_ω, B_n) corresponding to the spherical function $\varphi_m = 1_{B_m}$, $m \geq n$, by the Gelfand-Naimark-Segal construction is the left regular representation π_m on $l^2(H_m)$.

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