

# The Volume of Complete Anti-de Sitter 3-Manifolds

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**Abstract.** Up to a finite cover, closed anti-de Sitter 3-manifolds are quotients of  $\mathrm{SO}_0(2, 1)$  by a discrete subgroup of  $\mathrm{SO}_0(2, 1) \times \mathrm{SO}_0(2, 1)$  of the form  $j \times \rho(\Gamma)$ , where  $\Gamma$  is the fundamental group of a closed oriented surface,  $j$  a Fuchsian representation and  $\rho$  another representation which is “strictly dominated” by  $j$ .

Here we prove that the volume of such a quotient is proportional to the sum of the Euler classes of  $j$  and  $\rho$ . As a consequence, we obtain that this volume is constant under deformation of the anti-de Sitter structure. Our results extend to (not necessarily compact) quotients of  $\mathrm{SO}_0(n, 1)$  by a “geometrically finite” subgroup of  $\mathrm{SO}_0(n, 1) \times \mathrm{SO}_0(n, 1)$ .

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## 1. Introduction

**1.1. Quotients of  $\mathrm{SO}_0(n, 1)$  as non-Riemannian Clifford–Klein forms.** A trend in homogeneous geometry is the study of *Clifford–Klein forms of reductive homogeneous spaces*, namely, quotients of a homogeneous space  $G/H$  (where  $G$  is semi-simple Lie group and  $H$  a reductive subgroup) by a discrete subgroup  $\Gamma$  of  $G$  acting properly discontinuously on  $G/H$  (see [21] for a survey on this topic).

When  $H$  is not compact, the discreteness of  $\Gamma$  is not enough to guarantee a properly discontinuous action. Understanding “big” discrete groups acting properly discontinuously on  $G/H$  (for instance, groups whose action is cocompact) is difficult in general, and some homogeneous spaces simply do not admit such actions (see for instance [3] and [19]).

In this paper, we focus on a family of homogeneous spaces whose Clifford–Klein forms are now reasonably well-understood, thanks to the recent work of Guéritaud and Kassel [13]. Let  $n$  be an integer greater or equal to 2 and let  $\mathrm{SO}_0(n, 1)$  denote the group of orientation-preserving isometries of the hyperbolic space  $\mathbb{H}^n$ . Here, we consider  $\mathrm{SO}_0(n, 1)$  as our homogeneous space  $G/H$ , provided with the action of  $G = \mathrm{SO}_0(n, 1) \times \mathrm{SO}_0(n, 1)$  by left and right multiplication.

Kobayashi [18] and Kassel [14], generalizing the work of Kulkarni–Raymond [22] for  $n = 2$ , proved that subgroups of  $\mathrm{SO}_0(n, 1) \times \mathrm{SO}_0(n, 1)$  acting properly discontinuously on  $\mathrm{SO}_0(n, 1)$  have (up to finite index and up to switching the factors) the form

$$j \times \rho(\Gamma) ,$$

where  $\Gamma$  is a discrete subgroup of  $\mathrm{SO}_0(n, 1)$ ,  $j : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  is the inclusion and  $\rho : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  is another representation (see Theorem 2.2).

Guéritaud and Kassel then gave (under the hypothesis that  $\Gamma$  is *geometrically finite*) a precise necessary and sufficient condition for  $j \times \rho(\Gamma)$  to act properly discontinuously on  $\mathrm{SO}_0(n, 1)$  (see Theorem 2.6). This condition shows in particular that some Clifford–Klein forms of  $\mathrm{SO}_0(n, 1)$  have interesting *deformation spaces* (see for instance [20]).

The case  $n = 2$  is of particular interest because  $\mathrm{SO}_0(2, 1)$  identifies with the *anti-de Sitter space* of dimension 3 (denoted by  $\mathrm{AdS}_3$ ). Guéritaud–Kassel quotients of  $\mathrm{SO}_0(2, 1)$  are thus *complete anti-de Sitter orbifolds*. Quotients of  $\mathrm{AdS}_3$  were studied in [22, 28, 15]. The case  $n = 3$  is also interesting. Indeed,  $\mathrm{SO}_0(3, 1)$  is isomorphic to  $\mathrm{PSL}(2, \mathbb{C})$  and can thus be seen as a holomorphic analog of  $\mathrm{AdS}_3$ . Compact quotients of  $\mathrm{PSL}(2, \mathbb{C})$  have been studied by Ghys in [12].

The purpose of this paper is to describe the *volume* of Guéritaud–Kassel’s quotients of  $\mathrm{SO}_0(n, 1)$  and determine in particular whether this volume is constant under deformations of these quotients.

**1.2. Statement of the results.** The action of  $\mathrm{SO}_0(n, 1) \times \mathrm{SO}_0(n, 1)$  on  $\mathrm{SO}_0(n, 1)$  by left and right multiplication preserves a volume form on  $\mathrm{SO}_0(n, 1)$ . This allows to define the volume of quotients of  $\mathrm{SO}_0(n, 1)$  by discrete subgroups of  $\mathrm{SO}_0(n, 1) \times \mathrm{SO}_0(n, 1)$  acting properly discontinuously.

Recall that  $\mathrm{SO}_0(n, 1)$  is the group of orientation preserving isometries of the real hyperbolic space  $\mathbb{H}^n$ . If  $j$  is discrete and faithful, then  $j(\Gamma)$  acts properly discontinuously on  $\mathbb{H}^n$ . If, moreover, this action is cocompact, then one can define the volume of a representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  by

$$\mathbf{Vol}(\rho) = \int_{j(\Gamma) \backslash \mathbb{H}^n} f^* \mathrm{vol}^{\mathbb{H}^n} ,$$

where  $f$  is any piecewise smooth  $(j, \rho)$ -equivariant map from  $\mathbb{H}^n$  to  $\mathbb{H}^n$ . Here  $\mathrm{vol}^{\mathbb{H}^n}$  denotes the volume form associated to the hyperbolic metric on  $\mathbb{H}^n$  (see Section 2 for precisions on its normalization). In particular,

$$\mathbf{Vol}(j) = \mathbf{Vol}(j(\Gamma) \backslash \mathbb{H}^n) .$$

This definition can be extended (with some technicality) to representations of a non-cocompact hyperbolic lattice (see Definition 4.6 and the discussion in Section 4).

Finally, let us denote by  $\mathbf{V}_n$  the volume of  $\mathrm{SO}(n)$  with respect to its bi-invariant volume form (see Section 2 for the normalization of this volume form). With these definitions and notations, we can now state our main result:

**Theorem 1.** *Let  $\Gamma$  be a lattice in  $\mathrm{SO}_0(n, 1)$ ,  $j : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  the inclusion and  $\rho$  another representation of  $\Gamma$  into  $\mathrm{SO}_0(n, 1)$ , such that  $j \times \rho(\Gamma)$  acts properly discontinuously on  $\mathrm{SO}_0(n, 1)$  with finite covolume.*

*Then*

$$\mathbf{Vol}(j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)) = \mathbf{V}_n (\mathbf{Vol}(j) + (-1)^n \mathbf{Vol}(\rho)) .$$

**Remark 1.1.** We will give later a slightly stronger version of this theorem (Theorem 2.11), where  $j(\Gamma)$  is only assumed to be geometrically finite *a priori*.

The proof will rely on a theorem of Guéritaud–Kassel [13] which gives a necessary and sufficient condition for  $j \times \rho(\Gamma)$  to act properly discontinuously on  $\mathrm{SO}_0(n, 1)$  (Theorem 2.6). This theorem allows a deep understanding of the geometry of the quotient (see Section 5). When  $\Gamma$  is cocompact, Guéritaud–Kassel’s condition is open, meaning that if  $j_0 \times \rho_0(\Gamma)$  acts properly discontinuously on  $\mathrm{SO}_0(n, 1)$ , then  $j \times \rho(\Gamma)$  still acts properly discontinuously for  $(j, \rho)$  in some neighbourhood of  $(j_0, \rho_0)$ . When  $\Gamma$  has cusps, openness is still true in restriction to pairs  $(j, \rho)$  where  $j(\Gamma)$  is a lattice and  $\rho$  is *cuspidal deteriorating* (see Section 4). We will say that the volume of  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  is *locally rigid* if the volume of  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  is constant along such deformations.

Using rigidity results of Besson–Courtois–Gallot [5] and Kim–Kim [16] for the volume of representations of a lattice in  $\mathrm{SO}_0(n, 1)$ , we obtain the following corollary:

**Corollary 2.** *Let  $\Gamma$  be a lattice in  $\mathrm{SO}_0(n, 1)$ ,  $j : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  the inclusion and  $\rho : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  a representation of  $\Gamma$  into  $\mathrm{SO}_0(n, 1)$  such that  $j \times \rho(\Gamma)$  acts properly discontinuously on  $\mathrm{SO}_0(n, 1)$  with finite covolume. Then the volume of  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  is locally rigid, unless  $n = 2$  and  $\Gamma$  is not cocompact.*

The case when  $n = 2$  corresponds to closed anti-de Sitter 3-manifolds and thus answers Question 2.3 in [26]. Interestingly, in that case, the volume is not rigid when  $\Gamma$  is not cocompact. We will prove the following:

**Theorem 3.** *Let  $\Gamma$  be a non-cocompact lattice in  $\mathrm{SO}_0(2, 1)$  and  $j : \Gamma \rightarrow \mathrm{SO}_0(2, 1)$  the inclusion. Then there exists a continuous family  $(\rho_t)_{-1 < t < 1}$  of representations of  $\Gamma$  into  $\mathrm{SO}_0(2, 1)$  such that  $j \times \rho_t(\Gamma)$  acts properly discontinuously on  $\mathrm{SO}_0(2, 1)$  and such that*

$$\mathbf{Vol}(j \times \rho_t(\Gamma) \backslash \mathrm{SO}_0(2, 1))$$

*takes all the values in the open interval  $(0, 4\pi \mathbf{Vol}(j))$  when  $t$  varies.*

**1.3. Other approaches to Theorem 1.** Theorem 1 was originally proven in the author’s thesis [29] for  $n = 2$  and for compact quotients. Since then, it attracted some interest, and several people found other approaches to prove it in this setting.

Jean-Marc Schlenker mentioned to us that the volume rigidity could probably be obtained using the anti-de Sitter version of the *Schläfli formula* (see Proposition 4.8). Alessandrini and Li independently found another proof of this theorem [1]. Their computation seems similar to ours but uses an interesting Higgs bundle interpretation of anti-de Sitter 3-manifolds.

Labourie found an interpretation of the volume of anti-de Sitter 3-manifolds as a Chern–Simons invariant [23]. He noticed that the volume could be understood as a secondary characteristic class of the tangent bundle provided with two natural flat connections coming from left and right parallelism on  $\mathrm{SO}_0(2, 1)$ . The general Chern–Simons theory then directly gives the rigidity of the volume. In a forthcoming work, we will use this idea to prove the local rigidity of the volume of any closed manifold locally modelled on a reductive homogeneous space.

The proof of Theorem 1 given here is essentially a straightforward differential geometric computation that relies on a structure theorem of Guéritaud–Kassel. In a companion paper [30], we recover this theorem for compact quotients, via a more conceptual proof giving general results for compact quotients of reductive homogeneous spaces.

**1.4. Organization of the paper.** In the next section, we introduce the background of our work and give more details about our results. In particular, we state a more precise version of Theorem 1 (Theorem 2.11). In the next two sections, we derive consequences of this theorem in terms of local rigidity of the volume. In Section 3, we specialize our results to the case of anti-de Sitter 3-manifolds ( $n = 2$ ) and prove Theorem 3 assuming Theorem 1. Section 4 focuses on the case  $n \geq 3$  and proves Corollary 2 assuming Theorem 1. This involves a discussion on the definition of the volume of a representation of a non-uniform lattice in  $\mathrm{SO}_0(n, 1)$ . The proof of Theorem 1 is postponed to Section 5. Finally, in Section 6, we mention some open questions raised by this work and some partial answers given recently by Lakeland and Leininger [24].

## 2. Guéritaud–Kassel’s quotients and their volume

**2.1. Hausdorff quotients of  $\mathrm{SO}_0(n, 1)$ .** Throughout this paper,  $n$  denotes an integer greater or equal to 2. We are interested here in particular examples of *Clifford–Klein forms*, namely quotients of a homogeneous space by a discrete subgroup of transformations acting properly discontinuously. Here, the homogeneous space is the Lie group  $\mathrm{SO}_0(n, 1)$ , the connected component of the identity in the group of linear transformations of  $\mathbb{R}^{n+1}$  preserving a quadratic form of signature  $(n, 1)$ . Its transformation group is the group  $\mathrm{SO}_0(n, 1) \times \mathrm{SO}_0(n, 1)$  acting by left and right multiplication:

$$(g, h) \cdot x = gxh^{-1} .$$

Recall that the tangent space to  $\mathrm{SO}_0(n, 1)$  at the identity is the Lie algebra  $\mathfrak{so}(n, 1)$  of  $(n + 1) \times (n + 1)$  matrices  $A$  satisfying

$$A^T I_{n,1} + I_{n,1} A = 0 ,$$

where

$$I_{n,1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & -1 \end{pmatrix} .$$

Let  $\kappa$  be the left invariant pseudo-Riemmanian metric on  $SO_0(n, 1)$  defined on  $\mathfrak{so}(n, 1)$  by

$$\kappa(A, B) = \frac{1}{2} \text{Tr}(AB) .$$

Then the adjoint action of  $SO_0(n, 1)$  on  $\mathfrak{so}(n, 1)$  preserves  $\kappa$ , and  $\kappa$  is thus invariant by both left and right multiplication. Actually, the isometry group of  $(SO_0(n, 1), \kappa)$  is an extension of  $SO_0(n, 1) \times SO_0(n, 1)$  by a finite group of order 4.

**Remark 2.1.** Since  $SO_0(n, 1)$  is simple,  $\kappa$  is a multiple of the *Killing metric*.

Let us now review the previous results describing the discrete subgroups of  $SO_0(n, 1) \times SO_0(n, 1)$  acting properly discontinuously on  $SO_0(n, 1)$ . Recall that  $SO_0(n, 1)$  can be seen as the group of orientation preserving isometries of the hyperbolic space  $\mathbb{H}^n$ .

**Theorem 2.2** (Kulkarni–Raymond [22], Kobayashi [18], Kassel [14]). *Let  $\Gamma$  be a discrete and torsion-free subgroup of  $SO_0(n, 1) \times SO_0(n, 1)$ . Denote by  $j$  (resp.  $\rho$ ) the projection of  $\Gamma$  on the first (resp. second) factor. If  $\Gamma$  acts properly discontinuously on  $SO_0(n, 1)$ , then, up to switching the factors,  $j$  is discrete and faithful.*

**Remark 2.3.** This result was initially proven by Kulkarni and Raymond in the case  $n = 2$  [22]. Kobayashi [18] then proved faithfulness and Kassel [14] proved discreteness in a more general setting.

Guéritaud and Kassel then gave a precise condition for  $j \times \rho(\Gamma)$  to act properly discontinuously, under the assumption that  $j(\Gamma)$  is *geometrically finite*, which always holds when the quotient is compact (see [6]). Let us first introduce some terminology.

**Definition 2.4.** Let  $\Gamma$  be a group,  $j : \Gamma \rightarrow SO_0(n, 1)$  a discrete and faithful representation and  $\rho : \Gamma \rightarrow SO_0(n, 1)$  another representation. We say that  $j$  *strictly dominates*  $\rho$  if there exists a  $(j, \rho)$ -equivariant map from  $\mathbb{H}^n$  to  $\mathbb{H}^n$  which is contracting (i.e.  $\lambda$ -Lipschitz for some  $\lambda < 1$ ).

**Definition 2.5.** The *translation length* of an element  $g \in SO_0(n, 1)$  is defined by

$$l(g) = \inf_{x \in \mathbb{H}^n} d(x, g \cdot x) .$$

**Theorem 2.6** (Guéritaud–Kassel [13]). *Let  $\Gamma$  be a group,  $j : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  a discrete and faithful representation and  $\rho$  another representation of  $\Gamma$  into  $\mathrm{SO}_0(n, 1)$ . Up to switching  $j$  and  $\rho$ , we assume that there exists  $\gamma \in \Gamma$  such that  $l(j(\gamma)) \geq l(\rho(\gamma))$ . If the group  $j(\Gamma) \subset \mathrm{SO}_0(n, 1)$  is geometrically finite, then the group  $j \times \rho(\Gamma)$  acts (freely and) properly discontinuously on  $\mathrm{SO}_0(n, 1)$  if and only if  $\rho$  is strictly dominated by  $j$ .*

**Remark 2.7.** The fact that the strict domination is a sufficient condition follows from the Benoist–Kobayashi properness criterion ([3, 19]). It was explicitly stated for the first time in [20]. A first version of Theorem 2.6 was proven by Kassel in her thesis [15] for  $n = 2$  and  $\Gamma$  convex-cocompact.

**Remark 2.8.** Guéritaud–Kassel’s theorem does not hold if we remove the hypothesis that  $\Gamma$  is geometrically finite. It is not clear to us whether the fact that  $\Gamma$  acts properly discontinuously on  $\mathrm{SO}_0(n, 1)$  with finite covolume should imply that  $j(\Gamma)$  must be geometrically finite (see Section 6).

From now on, we will call *Guéritaud–Kassel quotient* a quotient of  $\mathrm{SO}_0(n, 1)$  by a discrete group of the form  $j \times \rho(\Gamma)$  where  $\Gamma$  is a geometrically finite subgroup of  $\mathrm{SO}_0(n, 1)$ ,  $j : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  the inclusion and  $\rho : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  a representation which is strictly dominated by  $j$ . As a corollary of Theorem 2.6, we have:

**Corollary 2.9.** *Under the assumptions of Theorem 1, the space  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  is a Guéritaud–Kassel quotient (up to switching  $j$  and  $\rho$  when  $n = 2$ ).*

**Proof.** Assume as in Theorem 1 that  $j(\Gamma)$  is a lattice. In particular,  $j(\Gamma)$  is geometrically finite. If there exists  $\gamma \in \Gamma$  such that  $l(j(\gamma)) \geq l(\rho(\gamma))$  then Theorem 2.6 applies and we deduce that  $j$  strictly dominates  $\rho$ . Otherwise we have  $l(j(\gamma)) < l(\rho(\gamma))$  for all  $\gamma \in \Gamma$ , which implies that  $\rho$  is also discrete and faithful. For  $n \geq 3$  this contradicts Mostow’s rigidity theorem. For  $n = 2$ , since  $\Gamma$  is finitely generated, it implies that  $\rho(\Gamma)$  is also geometrically finite, and we deduce from Theorem 2.6 that  $\rho$  strictly dominates  $j$ . ■

**2.2. Volume.** There is, up to scaling, a unique bi-invariant volume form on  $\mathrm{SO}_0(n, 1)$ . A natural way to normalize it is to decide that we get 1 when we evaluate it on a direct orthonormal basis of  $(\mathfrak{so}(n, 1), \kappa)$  (this requires first a choice of an orientation). We denote this volume form by  $vol^\kappa$ .

Let  $\Gamma$  be a finitely generated group and  $j, \rho$  two representations of  $\Gamma$  into  $\mathrm{SO}_0(n, 1)$  such that  $j$  is discrete and faithful and strictly dominates  $\rho$ . Then the volume form  $vol^\kappa$  on  $\mathrm{SO}_0(n, 1)$  induces a volume form on its quotient by the action of  $\Gamma$ .

**Definition 2.10.** The volume of Guéritaud–Kassel’s quotient

$$j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$$

is defined as

$$\mathbf{Vol}(j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)) = \int_{j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)} \mathrm{vol}^\kappa .$$

Let us reformulate our main theorem in a more precise way. We denote by  $\mathrm{vol}^{\mathbb{H}^n}$  the volume form on  $\mathbb{H}^n$  associated to the metric of constant curvature  $-1$ .

**Theorem 2.11.** *Let  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  be a Guéritaud–Kassel quotient. Let  $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$  be a  $(j, \rho)$ -equivariant contracting map. Then we have:*

$$\mathbf{Vol}((j \times \rho)(\Gamma) \backslash \mathrm{SO}_0(n, 1)) = \mathbf{V}_n \left( \int_{j(\Gamma) \backslash \mathbb{H}^n} \mathrm{vol}^{\mathbb{H}^n} + (-1)^n f^* \mathrm{vol}^{\mathbb{H}^n} \right) ,$$

where  $\mathbf{V}_n$  is the volume of  $\mathrm{SO}(n)$ .

**Corollary 2.12.** *Let  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  be a Guéritaud–Kassel quotient. If  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  has finite volume, then  $j(\Gamma)$  is a lattice in  $\mathrm{SO}_0(n, 1)$  and the integral*

$$\int_{j(\Gamma) \backslash \mathbb{H}^n} f^* \mathrm{vol}^{\mathbb{H}^n}$$

does not depend on the choice of  $f$ .

In that case, one can take  $\int_{j(\Gamma) \backslash \mathbb{H}^n} f^* \mathrm{vol}^{\mathbb{H}^n}$  as a definition of the “volume” of the representation  $\rho$ . This coincides with the classical definition when  $j(\Gamma)$  is cocompact, and with more subtle definitions introduced in [7] and [10] in general (see Section 4). Theorem 1 thus follows from Theorem 2.11 and Corollary 2.9.

**Proof of Corollary 2.12.** Let  $\lambda < 1$  be such that  $f$  is  $\lambda$ -Lipschitz. Then we have

$$|f^* \mathrm{vol}^{\mathbb{H}^n}| \leq \lambda^n \mathrm{vol}^{\mathbb{H}^n}$$

and thus, by Theorem 2.11,

$$\mathbf{Vol}(j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)) \geq (1 - \lambda^n) \mathbf{V}_n \int_{j(\Gamma) \backslash \mathbb{H}^n} \mathrm{vol}^{\mathbb{H}^n} .$$

Therefore, if  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  has finite volume, then  $j(\Gamma) \backslash \mathbb{H}^n$  has finite volume and  $j(\Gamma)$  is thus a lattice. We can then re-write the volume of  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  as

$$\mathbf{V}_n \left( \int_{j(\Gamma) \backslash \mathbb{H}^n} \mathrm{vol}^{\mathbb{H}^n} + (-1)^n \int_{j(\Gamma) \backslash \mathbb{H}^n} f^* \mathrm{vol}^{\mathbb{H}^n} \right) .$$

Since this volume does not depend on the contracting map  $f$ , the second term obviously does not depend on  $f$  either. ■

**2.3. A word on conventions.** These results depend on the normalization of the form  $\kappa$ . The definition we chose is particularly suited to hyperbolic geometry. Indeed, the orthogonal complement of  $\mathfrak{so}(n)$  in  $\mathfrak{so}(n, 1)$  naturally identifies to the tangent space to  $\mathbb{H}^n$  at some base point  $x_0$ . With our convention, the restriction of

the metric  $\kappa$  to  $\mathfrak{so}(n)^\perp$  precisely gives the Riemannian metric of constant curvature  $-1$  on  $\mathbb{H}^n$ .

In Theorem 2.11, The volume  $\mathbf{V}_n$  is the integral over  $\mathrm{SO}(n)$  of the volume form associated to the restriction of the metric  $\kappa$ . With this convention, the volumes  $\mathbf{V}_n$  satisfy the nice recurrence relation:

$$\mathbf{V}_{n+1} = \mathbf{V}_n \times \mathbf{Vol}(\mathbb{S}^n) ,$$

where  $\mathbb{S}^n$  denotes the standard  $n$ -sphere. One obtains the following (not very enlightening) expression for  $\mathbf{V}_n$ :

$$\mathbf{V}_n = \prod_{k=1}^{n-1} \frac{\pi^{(k+1)/2}}{\Gamma((k+1)/2)} ,$$

where  $\Gamma$  is the Gamma function.

Finally, our results seem to depend on the orientation of  $\mathrm{SO}_0(n, 1)$  and  $\mathbb{H}^n$ . The most natural convention to take is to ask for the volume of a manifold to be positive. Note however that, once we fixed an orientation on  $\mathbb{H}^n$  so that  $\mathbf{Vol}(j)$  is positive,  $\mathbf{Vol}(\rho)$  could very well be negative. Interestingly, if  $\rho'$  is conjugate to the representation  $\rho$  by some orientation reversing isometry, then  $\mathbf{Vol}(\rho') = -\mathbf{Vol}(\rho)$ . In that case, both  $\rho$  and  $\rho'$  are strictly dominated by  $j$ , but the quotients  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  and  $j \times \rho'(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  do not have the same volume (unless  $\mathbf{Vol}(\rho) = 0$ ).

In the next two sections, we discuss the consequences of Theorem 2.11 concerning the volume rigidity. The proof of Theorem 2.11 is postponed to Section 5.

### 3. Volume of complete anti-de Sitter 3-manifolds

The initial motivation for this work is the case when  $n = 2$ . Indeed, the Killing metric on the Lie group  $\mathrm{SO}_0(2, 1) = \mathrm{PSL}(2, \mathbb{R})$  is a Lorentz metric of constant negative sectional curvature, and Guéritaud–Kassel quotients in that particular case are thus 3-dimensional *anti-de Sitter orbifolds*.

Conversely, as a consequence of a theorem of Klingler [17], every closed anti-de Sitter 3-manifold is a quotient of the universal cover of  $\mathrm{SO}_0(2, 1)$  by a discrete subgroup of transformations preserving the Killing metric. Kulkarni and Raymond proved that these manifolds are actually quotients of a finite cover of  $\mathrm{SO}_0(2, 1)$  [22]. Therefore, every closed anti-de Sitter 3-manifold is, up to a finite cover, a Guéritaud–Kassel quotient.

In a recent survey by specialists of this domain [26], it is asked whether the volume of these manifolds is rigid (Question 2.3). Theorem 1 provides an answer. Let us reformulate it in this particular case. Recall that a torsion-free cocompact lattice of  $\mathrm{SO}_0(2, 1)$  is isomorphic to the fundamental group of some closed orientable surface  $S$  of genus  $\geq 2$ . In that case, the volume of a representation  $\rho : \pi_1(S) \rightarrow \mathrm{SO}_0(2, 1)$  is  $2\pi$  times the *Euler class* of the representation, denoted  $\mathbf{eu}(\rho)$ .

**Theorem 3.1** (Theorem 1 in the  $\text{AdS}_3$  case). *Let  $S$  be a closed surface of genus at least 2 and  $j$  and  $\rho$  two representations of  $\pi_1(S)$  into  $\text{SO}_0(2,1)$  such that  $j$  is discrete and faithful and such that there exists a  $(j, \rho)$ -equivariant contracting map from  $\mathbb{H}^2$  to  $\mathbb{H}^2$ . Then*

$$\text{Vol}(j \times \rho(\pi_1(S)) \backslash \text{SO}_0(2,1)) = 4\pi^2(\mathbf{eu}(j) + \mathbf{eu}(\rho)) .$$

*In particular, this volume is an integral multiple of  $4\pi^2$  and is therefore constant under continuous deformations of  $j$  and  $\rho$ .*

**Remark 3.2.** This theorem was initially proven in the author’s thesis ([29, ch. 4]). There, more details are given about the various possible values for the volume of closed anti-de Sitter 3-manifolds. In particular, the volume is expressed in terms of the topology of the quotient (which is a Seifert bundle over a hyperbolic orbifold) and an invariant that we called the *length of the fiber*. Though this expression follows from Theorem 3.1, it requires a discussion about how to lift an action on  $\text{SO}_0(2,1)$  to a finite cover of  $\text{SO}_0(2,1)$ . For simplicity, we chose here to restrict to the setting of surface group actions on  $\text{SO}_0(2,1)$ .

Note that the expression of the volume given in Theorem 3.1 differs from the author’s thesis by a factor 8. This is due to the fact that, here, we normalized the Killing metric on  $\text{SO}_0(2,1)$  so that it induces a metric of curvature  $-1$  on  $\mathbb{H}^2$  while, in [29], the Killing metric on  $\text{SO}_0(2,1)$  is normalized so that it has itself sectional curvature  $-1$ .

We now turn to non-compact quotients of  $\text{AdS}_3$  and prove that their volume is not rigid (Theorem 3).

**Proof of Theorem 3 assuming Theorem 2.11.** Let  $\Gamma$  be a non-cocompact lattice in  $\text{SO}_0(2,1)$  and  $j : \Gamma \rightarrow \text{SO}_0(2,1)$  the inclusion. Then  $j(\Gamma) \backslash \mathbb{H}^2$  is a finite volume hyperbolic orbifold with at least one cusp. It is thus isometric to  $S \setminus \{p_1, \dots, p_k\}$ , where  $S$  is a closed Riemann surface and  $\{p_1, \dots, p_k\}$  a non-empty finite set of points, with a complete conformal hyperbolic metric  $g_1$  having  $l$  conical singularities  $q_1, \dots, q_l$  with respective angle  $\frac{2\pi}{d_1}, \dots, \frac{2\pi}{d_l}$  for some non-zero integers  $d_1, \dots, d_l$ .

For every  $0 < t < 1$ , a famous theorem of Troyanov [31] guarantees the existence of a (unique) metric  $g_t$  conformal to  $g_1$ , of constant curvature  $-1$  on  $S$ , having a conical singularity of angle

$$\alpha(t) = (1 - t) \frac{\text{Vol}(j(\Gamma) \backslash \mathbb{H}^2)}{k}$$

at each  $p_i$ , and a conical singularity of angle  $\frac{2\pi}{d_i}$  at each  $q_i$ .

For each  $0 < t < 1$ , the metric  $g_t$  induces a holonomy representation

$$\rho_t : \Gamma \rightarrow \text{SO}_0(2,1)$$

and a developing map  $f_t : \mathbb{H}^2 \rightarrow \mathbb{H}^2$  which is  $(j, \rho_t)$ -equivariant. Moreover, since  $g_t$  is conformal to  $g_1$ , the developing map  $f_t$  is holomorphic. By Schwarz’s lemma, it is either isometric or locally contracting at every point.

Since  $g_t$  has a conical singularity at each point  $p_i$  while  $g_1$  has a cusp, the ratio  $g_t/g_1$  goes to 0 at each  $p_i$ . Therefore  $f_t$  is not an isometry and, by compactness of  $S$ , it is actually a global contraction. Hence  $j \times \rho_t(\Gamma)$  satisfies Guéritaud–Kassel’s criterion and we have

$$\mathbf{Vol}(j \times \rho_t(\Gamma) \backslash \mathrm{SO}_0(2, 1)) = 2\pi (\mathbf{Vol}(j) + \mathbf{Vol}(\rho_t)) .$$

Since the volume of  $\rho_t$  is nothing but the volume of  $(S, g_t)$ , the Gauss–Bonnet formula gives

$$\mathbf{Vol}(\rho_t) = \mathbf{Vol}(j(\Gamma) \backslash \mathbb{H}^2) - k\alpha(t) = t\mathbf{Vol}(j(\Gamma) \backslash \mathbb{H}^2) ,$$

and therefore

$$\mathbf{Vol}(j \times \rho_t(\Gamma) \backslash \mathrm{SO}_0(2, 1)) = 2\pi(1 + t)\mathbf{Vol}(j(\Gamma) \backslash \mathbb{H}^2) .$$

When  $t$  goes to 0,  $\rho_t$  converges (up to conjugation) to an elliptic representation  $\rho_0$  (i.e. a representation fixing a point in  $\mathbb{H}^2$ ). Finally, one can define  $\rho_{-t}$  as the representation  $\rho_t$  conjugated by an orientation reversing isometry. This way, we have

$$\mathbf{Vol}(\rho_t) = t\mathbf{Vol}(j(\Gamma) \backslash \mathbb{H}^2)$$

for all  $-1 < t < 1$ . Each  $j \times \rho_t(\Gamma)$  satisfies Guéritaud–Kassel’s criterion and the volume of  $j \times \rho_t(\Gamma) \backslash \mathrm{SO}_0(2, 1)$  varies linearly between 0 and  $4\pi\mathbf{Vol}(j(\Gamma) \backslash \mathbb{H}^2)$ . ■

#### 4. Rigidity of the volume in higher dimension

The purpose of this section is to show that the non-rigidity phenomenon that we just exhibited cannot happen in higher dimension. By Mostow’s rigidity theorem, the volume of  $j(\Gamma) \backslash \mathbb{H}^n$  is rigid. According to Theorem 1 (the proof of which is postponed to the next section), it is thus enough to prove the following:

**Lemma 4.1.** *Let  $\Gamma$  be a (not necessarily cocompact) lattice in  $\mathrm{SO}_0(n, 1)$ ,  $n \geq 3$ . Denote by  $j$  the inclusion of  $\Gamma$  into  $\mathrm{SO}_0(n, 1)$  and let  $\rho_t$  be a continuous family of representations from  $\Gamma$  to  $\mathrm{SO}_0(n, 1)$  that are strictly dominated by  $j$ . Then*

$$t \mapsto \mathbf{Vol}(\rho_t)$$

*is constant.*

When  $\Gamma$  is cocompact, this result is due to Besson–Courtois–Gallot [5]. It was extended by Kim and Kim [16] for representations of non-cocompact lattices of  $\mathrm{SO}_0(n, 1)$  when  $n \geq 4$ . In these cases, the assumption that  $\rho_t$  is strictly dominated by  $j$  is unnecessary.

On the contrary, when  $n = 3$ , Lemma 4.1 is not true anymore without the domination hypothesis. However, Besson–Courtois–Gallot’s proof can be applied when we restrict to representations that are *cusp deteriorating*, and in particular for representations strictly dominated by  $j$ .

In the next subsection, we relate our definition of  $\mathbf{Vol}(\rho)$  to other classical (and more general) definitions, in particular that of Kim and Kim [16]. This allows more flexibility in the proof of Lemma 4.1, which is given in Subsection 3.2.

**4.1. Volume of representations of a lattice.** Let  $\Gamma$  be a lattice in  $\mathrm{SO}_0(n, 1)$ , with  $n \geq 3$ . Denote by  $j$  the inclusion of  $\Gamma$  into  $\mathrm{SO}_0(n, 1)$  and let  $\rho : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  be a representation strictly dominated by  $j$ .

So far, we have been working with the *ad hoc* definition of  $\mathbf{Vol}(\rho)$  as

$$\int_{j(\Gamma)\backslash\mathbb{H}^n} f^* \mathrm{vol}^{\mathbb{H}^n},$$

where  $f$  is a  $(j, \rho)$ -equivariant contracting map from  $\mathbb{H}^n$  to  $\mathbb{H}^n$ . According to Corollary 2.12, this value does not depend on the choice of a contracting map. The purpose of this section is to show that this definition coincides with a more general definition due to Kim and Kim [16], which itself coincides with other classical definitions (see [9, 10, 7]).

Following [13], we call a representation  $\rho : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  *cuspidal preserving* (resp. *cuspidal deteriorating*) if for every parabolic element  $\gamma \in \Gamma$ ,  $\rho(\gamma)$  is parabolic or elliptic (resp. elliptic). Note that, ultimately, the only representations we are interested in here are cuspidal deteriorating, according to the following proposition:

**Proposition 4.2** ([13], Lemma 2.6). *If  $\rho$  is strictly dominated by  $j$ , then  $\rho$  is cuspidal deteriorating.*

Let us start by proving the following:

**Proposition 4.3.** *If  $f_0$  and  $f_1$  are two  $(j, \rho)$ -equivariant Lipschitz maps, then*

$$\int_{j(\Gamma)\backslash\mathbb{H}^n} f_0^* \mathrm{vol}^{\mathbb{H}^n} = \int_{j(\Gamma)\backslash\mathbb{H}^n} f_1^* \mathrm{vol}^{\mathbb{H}^n}.$$

**Remark 4.4.** The existence of a  $(j, \rho)$ -equivariant Lipschitz map is equivalent to  $\rho$  being cuspidal-preserving (see [13, Lemma 4.9]).

**Proof.** When  $\Gamma$  is cocompact, the fact that  $\int_{j(\Gamma)\backslash\mathbb{H}^n} f^* \mathrm{vol}^{\mathbb{H}^n}$  does not depend on  $f$  is a classical consequence of Stokes’s formula. In the non-compact case we simply apply Stokes’s formula to the complement of some cusped regions. A boundary term appears, and we must prove that this term goes to 0 when these cusped regions are chosen smaller and smaller.

Let  $C > 0$  be such that  $f_0$  and  $f_1$  are  $C$ -Lipschitz. For all  $t \in [0, 1]$ , let  $f_t$  be the map from  $\mathbb{H}^n$  to  $\mathbb{H}^n$  sending a point  $x$  to the barycenter

$$\mathbf{Bar} \{(f_0(x), 1 - t); (f_1(x), t)\}$$

(i.e. the point on the segment  $[f_0(x), f_1(x)]$  satisfying

$$d(f_t(x), f_0(x)) = t d(f_1(x), f_0(x)).$$

By convexity of the hyperbolic distance,  $f_t$  is  $C$ -Lipschitz for all  $t$ . By equivariance of  $f_0$  and  $f_1$ , each  $f_t$  is  $(j, \rho)$ -equivariant. Putting all these maps together, we obtain a  $(j, \rho)$ -equivariant map  $F : \mathbb{H}^n \times [0, 1] \rightarrow \mathbb{H}^n$ .

Denote  $M = j(\Gamma) \backslash \mathbb{H}^n$ . A *cusps* in  $M$  is an open connected domain whose lift to  $\mathbb{H}^n$  is a disjoint union of horoballs. Let  $K_0$  be a compact set in  $M$  whose complement is a disjoint union of cusps. Let  $K_p$  be the set of points of  $M$  at distance at most  $p$  from  $K_0$ . Then the complement of  $K_p$  in  $M$  is still a disjoint union of cusps.

Since  $vol^{\mathbb{H}^n}$  is closed, by Stokes's formula, we have

$$\int_{\partial(K_p \times [0,1])} F^* vol^{\mathbb{H}^n} = 0 ,$$

hence

$$\int_{K_p} f_0^* vol^{\mathbb{H}^n} - \int_{K_p} f_1^* vol^{\mathbb{H}^n} = \int_{[0,1] \times \partial K_p} F^* vol^{\mathbb{H}^n} . \tag{1}$$

We need to prove that the right hand term goes to 0 when  $p$  goes to infinity.

Since each  $f_t$  is  $C$ -Lipschitz and since  $F(x, \cdot)$  maps  $[0, 1]$  to the segment  $[f_0(x), f_1(x)]$ , it follows that

$$\left| \int_{[0,1] \times \partial K_p} F^* vol^{\mathbb{H}^n} \right| \leq C^{n-1} \times area(\partial K_p) \times \max_{x \in \partial \tilde{K}_p} d(f_0(x), f_1(x)) .$$

By definition of  $K_p$ , we have

$$\max_{x \in \partial \tilde{K}_p} d(f_0(x), f_1(x)) \leq \max_{x \in \partial \tilde{K}_0} d(f_0(x), f_1(x)) + 2Cp .$$

On the other side, it is a well-known fact in hyperbolic geometry that

$$area(\partial K_p) = e^{-(n-1)p} area(\partial K_0)$$

(see for instance [2, Proposition D.3.12]).

Putting these two facts together we obtain

$$\left| \int_{[0,1] \times \partial K_p} F^* vol^{\mathbb{H}^n} \right| \leq (Cst \cdot p + Cst') e^{-(n-1)p} \xrightarrow{p \rightarrow +\infty} 0 .$$

By Equation (1), we finally obtain

$$\int_M f_0^* vol^{\mathbb{H}^n} = \lim_{p \rightarrow +\infty} \int_{K_p} f_0^* vol^{\mathbb{H}^n} = \lim_{p \rightarrow +\infty} \int_{K_p} f_1^* vol^{\mathbb{H}^n} = \int_M f_1^* vol^{\mathbb{H}^n} . \quad \blacksquare$$

Proposition 4.3 allows us to define the volume of a cusps preserving representation  $\rho$  as the integral of the pull-back volume form by any  $(j, \rho)$ -equivariant Lipschitz map. We will show that this is a particular case of a more general definition.

Let  $\widehat{\mathbb{H}}_{\Gamma}^n$  denote the union of  $\mathbb{H}^n$  with all the fixed points in  $\partial_{\infty}\mathbb{H}^n$  of parabolic elements in  $\Gamma$ . We provide  $\widehat{\mathbb{H}}_{\Gamma}^n$  with the topology extending the topology of  $\mathbb{H}^n$  and such that for any point  $x \in \widehat{\mathbb{H}}_{\Gamma}^n \cap \partial_{\infty}\mathbb{H}^n$ , a neighbourhood basis of  $x$  is given by the family of horoballs tangent to  $x$ . With this topology,  $j(\Gamma)$  acts properly discontinuously and cocompactly on  $\widehat{\mathbb{H}}_{\Gamma}^n$ , and the quotient is the compactification of  $j(\Gamma)\backslash\mathbb{H}^n$  where one point as been “added” at the end of each cusp.

**Definition 4.5.** A piecewise smooth  $(j, \rho)$ -equivariant map  $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$  is *nicely ending* if it extends continuously to a map from  $\widehat{\mathbb{H}}_{\Gamma}^n$  to  $\overline{\mathbb{H}}^n$ .

**Proposition-Definition 4.6** (Kim–Kim, [16]). Let  $\rho : \Gamma \rightarrow \text{SO}_0(n, 1)$  be a representation. Let  $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$  be a  $(j, \rho)$ -equivariant nicely ending map. Then

$$\int_{j(\Gamma)\backslash\mathbb{H}^n} f^* \text{vol}^{\mathbb{H}^n}$$

is finite and does not depend on the choice of  $f$ . This number is called the *volume* of the representation  $\rho$ .

This notion of “nicely ending map” is slightly more general than a so-called *pseudo-developing* or *properly ending* map ([10], [11]). Therefore, this definition of  $\mathbf{Vol}(\rho)$  coincides with the definition of the volume of  $\rho$  by Dunfield [9] and Francaviglia [10]. Kim and Kim show that one can always construct a nicely ending *simplicial* map  $f$  and deduce that this definition of  $\mathbf{Vol}(\rho)$  also coincides with the definition by Bucher–Burger–Iozzi using relative cohomology [7]. Here we will prove that it also coincides with our definition:

**Proposition 4.7.** Let  $\rho$  be a cusp preserving representation of  $\Gamma$  into  $\text{SO}_0(n, 1)$ . Then for any  $(j, \rho)$ -equivariant Lipschitz map  $f$ ,

$$\int_{j(\Gamma)\backslash\mathbb{H}^n} f^* \text{vol}^{\mathbb{H}^n}$$

is equal to the volume of  $\rho$  (in the sense of Kim and Kim).

**Proof.** By Propostion 4.3 and Proposition 4.6, it is enough to find a nicely ending Lipschitz map. We start by defining this map in the (lifts of the) cusps. Since the complement is compact, it is then easy to extend it as a global Lipschitz map.

Let us choose a compact core in  $M = j(\Gamma)\backslash\mathbb{H}^n$  whose lift to  $\mathbb{H}^n$  is the complement of a disjoint union of horoballs. The action of  $j(\Gamma)$  on the space of these horoballs has finitely many orbits. Let us choose  $B_1, \dots, B_k$  some representatives in each orbit, and denote by  $\Gamma_i$  the subgroup of  $\Gamma$  whose image by  $j$  stabilizes  $B_i$ .

The group  $j(\Gamma_i)$  contains no hyperbolic element. Since  $\rho$  is cusp preserving, this is also true for  $\rho(\Gamma_i)$ . It is then a classical fact that one of the following holds (see [25, Theorem III.3.1]):

- either  $\rho(\Gamma_i)$  fixes a point  $y_i$  in  $\mathbb{H}^n$
- or  $\rho(\Gamma_i)$  fixes a point  $y_i$  in  $\partial_\infty\mathbb{H}^n$  and preserves a Buseman function  $\beta_{y_i}$  associated to  $y_i$ .

In the first case, we say that  $\rho(\Gamma_i)$  is of elliptic type. In the second case, we say that  $\rho(\Gamma_i)$  is of parabolic type.

If  $\rho(\Gamma_i)$  is of parabolic type, then it commutes with the flow  $F_{y_i}$  such that  $F_{y_i}(x, t)$  is at distance  $t$  from  $x$  on the geodesic ray from  $x$  to  $y_i$ . Let  $x_i$  be the point in  $\partial_\infty\mathbb{H}^n$  fixed by  $j(\Gamma_i)$ . One can find a  $(j|_{\Gamma_i}, \rho|_{\Gamma_i})$ -equivariant  $C$ -Lipschitz map  $f$  from the horosphere  $\partial B_i$  to the horosphere  $\{\beta_{y_i} = 0\}$ , and then extend it to  $B_i$  by setting

$$f(F_{x_i}(x, t)) = F_{y_i}(f(x), t)$$

for all  $x \in \partial B_i$  and all  $t \in \mathbb{R}_+$ . The fact that  $f$  is still  $C$ -Lipschitz can easily be deduced from the fact that  $F_{x_i}(\cdot, t)$  (resp.  $F_{y_i}$ ) is a homothety of ratio  $e^{-t}$  from  $\partial B_i$  (resp. the level set  $\{\beta_{y_i} = 0\}$ ) to its image and is isometric in restriction to geodesic rays going to  $x_i$  (resp.  $y_i$ ).

If  $\rho(\Gamma_i)$  is of elliptic type, one can simply set  $f(B_i) = \{y_i\}$ . There is now a unique way of extending  $f$  equivariantly to

$$\bigsqcup_{i=1}^k \bigsqcup_{\gamma \in \Gamma} j(\gamma) \cdot B_i .$$

Finally, one can extend  $f$  smoothly and equivariantly to  $\mathbb{H}^n$ . Since  $f$  is locally  $C$ -Lipschitz on  $B_i$  (for some constant  $C$ ), and since  $j(\Gamma) \setminus \left( \mathbb{H}^n - \bigsqcup_{i=1}^k \Gamma \cdot B_i \right)$  is compact, the resulting map  $f$  is globally Lipschitz.

Finally, by construction,  $f$  extends naturally to a map from  $\widehat{\mathbb{H}}_T^n$  to  $\overline{\mathbb{H}}^n$  by setting

$$f(x_i) = y_i .$$

Therefore,  $f$  is a nicely ending Lipschitz map. ■

**4.2. Proof of Lemma 4.1.** The main ideas of the proof are due to Besson–Courtois–Gallot [5] and Kim–Kim [16]. Here we will only sketch the general strategy and explain how to overcome the technical difficulties arising when working with non-compact lattices.

Besson–Courtois–Gallot’s idea to prove the volume rigidity of representations in  $\mathrm{SO}_0(n, 1)$  is to use the Schläfli formula. Let us start by recalling this formula.

Let  $x_0, \dots, x_n$  be  $n + 1$  points in  $\mathbb{H}^n$ . We say that  $(x_0, \dots, x_n)$  are (projectively) independent if their convex hull has non-empty interior. In that case, the convex hull is an  $n$ -dimensional tetrahedron denoted  $[x_0, \dots, x_n]$ . For any pairwise distinct  $i_0, \dots, i_k$  in  $\{0, \dots, n\}$ , the convex hull of  $x_{i_0}, \dots, x_{i_k}$  forms a  $k$ -face of  $[x_0, \dots, x_n]$ , denoted  $[x_{i_0}, \dots, x_{i_k}]$ .

**Proposition 4.8** (Schläfli formula). *Let  $x_0(t), \dots, x_n(t)$  be  $\mathcal{C}^1$  curves in  $\mathbb{H}^n$  that are projectively independent at each time  $t$ . Then*

$$\frac{d}{dt} \mathbf{Vol}[x_0, \dots, x_n] = \sum_{i < j} \text{area}[x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n] \cdot \frac{d}{dt} \theta(x_i, x_j) ,$$

where  $\theta(x_i, x_j)$  is the dihedral angle at the  $(n - 2)$ -face  $[x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n]$ .

Besson–Courtois–Gallot use this to prove the rigidity of the volume of a representation of a cocompact lattice. Indeed, in the cocompact case, one can fix a triangulation of  $M = j(\Gamma) \backslash \mathbb{H}^n$  and choose equivariant maps that are simplicial (i.e. cells of the triangulation are mapped to hyperbolic tetrahedra). The dihedral angles around a given  $(n - 2)$ -cell then add up to a multiple of  $2\pi$ . This implies a cancellation when computing the derivative of the volume along a  $\mathcal{C}^1$  family of representations.

Kim and Kim have adapted this proof to the non-cocompact case by considering a triangulation of  $M$  with an “ideal” vertex at the end of each cusp and allowing to map these ideal vertices to the boundary of  $\mathbb{H}^n$ . This gives the rigidity in dimension  $n \geq 4$ , where the Schläfli formula naturally extends to tetrahedra with ideal vertices, but it partly fails in dimension 3 because, in that case, the Schläfli formula does not make sense when some vertex is at infinity (indeed, the area of an  $(n - 2)$ -face is the length of an edge and becomes infinite when some vertex is in the boundary). However, using a variation of the Schläfli formula in that case, they were able to prove the rigidity of  $\mathbf{Vol}(\rho)$  in restriction to representations sending parabolic elements to parabolic elements.

The situation is simpler when we restrict to representations  $\rho$  that are strictly dominated by  $j$ . Indeed, such representations are cusp deteriorating and, therefore, the image of each cusp group is of elliptic type and fixes a point in  $\mathbb{H}^n$ . One can choose a simplicial map sending the vertex at the end of a cusp to such a fixed point. This way, the image of each cell is a compact tetrahedron in  $\mathbb{H}^n$ . One can hence use the Schläfli formula and Besson–Courtois–Gallot’s argument to prove that the volume is constant along every  $\mathcal{C}^1$  path in the space  $\text{Dom}(\Gamma, \text{SO}_0(n, 1))$  of representations of  $\Gamma$  into  $\text{SO}_0(n, 1)$  that are strictly dominated by  $j$ .

Now the space of cusp deteriorating representations is a real semi-algebraic variety in which  $\text{Dom}(\Gamma, \text{SO}_0(n, 1))$  forms an open set. Therefore, the space of smooth points is dense in  $\text{Dom}(\Gamma, \text{SO}_0(n, 1))$  (for the metric topology) and we conclude that the volume function is locally constant on  $\text{Dom}(\Gamma, \text{SO}_0(n, 1))$ .

### 5. Computation of the volume

This section is devoted to the proof of Theorem 1, and more precisely of Theorem 2.11. From now on, we fix an integer  $n \geq 2$ , a finitely generated group  $\Gamma$ , a discrete and faithful representation  $j : \Gamma \rightarrow \text{SO}_0(n, 1)$  and another representation  $\rho : \Gamma \rightarrow \text{SO}_0(n, 1)$  such that there is a map  $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$  which is  $(j, \rho)$ -equivariant and contracting. Since Theorem 2.11 is consistent with replacing  $\Gamma$  by a finite index subgroup, Selberg’s Lemma allows us to assume that  $\Gamma$  is torsion-free.

**5.1. Bundle structure of Guéritaud–Kassel’s quotients.** The map  $f$  allows to describe the quotient  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  as a bundle over  $j(\Gamma) \backslash \mathbb{H}^n$  whose fibers have the form  $Kg$ , with  $g$  an element of  $\mathrm{SO}_0(n, 1)$  and  $K$  a maximal compact subgroup of  $\mathrm{SO}_0(n, 1)$  (i.e. a subgroup conjugate to  $\mathrm{SO}(n)$ ). The way we do this construction here is a differentiable version of the construction given in [13, Proposition 7.2].

There are classical ways of smoothening the map  $f$  without changing too much its Lipschitz constant,<sup>1</sup> and we can thus assume without loss of generality that  $f$  is smooth. Let us introduce the map

$$\pi : \mathrm{SO}_0(n, 1) \rightarrow \mathbb{H}^n$$

such that  $\pi(g)$  is the unique fixed point of  $g \circ f$ . This map is well defined because  $f$  (and hence  $g \circ f$ ) is a contracting map.

The following propositions follow easily from the definition.

**Proposition 5.1.** *The map  $\pi$  is onto and for all  $x \in \mathbb{H}^n$ , one has*

$$\pi^{-1}(x) = K_x g ,$$

where  $g$  is some element in  $\pi^{-1}(x)$  and  $K_x$  is the stabilizer of  $x$  in  $\mathrm{SO}_0(n, 1)$ .

**Proposition 5.2.** *The map  $\pi$  is  $\Gamma$ -equivariant, i.e. for all  $\gamma \in \Gamma$  and all  $g \in \mathrm{SO}_0(n, 1)$ ,*

$$\pi(j(\gamma)g\rho(\gamma)^{-1}) = j(\gamma) \cdot \pi(g) .$$

**Proof of Proposition 5.1.** By transitivity of the action of  $\mathrm{SO}_0(n, 1)$  on  $\mathbb{H}^n$ , there exists, for all  $x \in \mathbb{H}^n$ , some  $g \in \mathrm{SO}_0(n, 1)$  such that  $g \cdot f(x) = x$ . We thus have  $\pi(g) = x$ . Hence  $\pi$  is onto. Let  $h$  be another element in  $\pi^{-1}(x)$ . Then

$$hg^{-1} \cdot x = hg^{-1}g \cdot f(x) = h \cdot f(x) = x .$$

Hence  $h \in K_x g$ . The same computation shows that, conversely, if  $h \in K_x g$  then  $\pi(h) = x$ . ■

**Proof of Proposition 5.2.** Fix some  $\gamma \in \Gamma$  and some  $g \in \mathrm{SO}_0(n, 1)$ . One easily verifies that  $j(\gamma)g\rho(\gamma)^{-1} \circ f$  fixes  $j(\gamma) \cdot \pi(g)$ . Indeed,

$$\begin{aligned} j(\gamma)g\rho(\gamma)^{-1} \cdot f(j(\gamma) \cdot \pi(g)) &= j(\gamma)g \cdot f(\pi(g)) \quad (\text{by equivariance of } f) \\ &= j(\gamma) \cdot \pi(g) \quad (\text{by definition of } \pi). \end{aligned}$$

Therefore  $\pi(j(\gamma)g\rho(\gamma)^{-1}) = j(\gamma) \cdot \pi(g)$ . ■

Let us now compute the first derivative of  $\pi$ . This will show that  $\pi$  is a submersion. We will then show that  $\pi$  is proper, hence a fiber bundle, according to Ehresmann’s theorem.

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<sup>1</sup>For instance, one can replace  $f(x)$  by the barycenter of the images  $f(y)$  with respect to the measure at time  $\varepsilon$  of a Brownian motion started at  $x$ .

**Lemma 5.3.** *Let  $g$  be a point in  $\text{SO}_0(n, 1)$ . Identify  $T_g\text{SO}_0(n, 1)$  with the Lie algebra of right invariant vector fields on  $\text{SO}_0(n, 1)$ . Given  $u$  such a vector field, denote by  $X_u$  the induced vector field on  $\mathbb{H}^n$ . Then we have*

$$d_g\pi(u) = (\text{Id} - d_{\pi(g)}(g \circ f))^{-1} (X_u(\pi(g))) .$$

The proof will rely on the following application of the implicit function theorem:

**Proposition 5.4.** *Let  $U$  be a neighborhood of 0 in  $\mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}^n$  a smooth map such that  $f(0) = 0$  and such that  $\|df_0\| < 1$  for some operator norm on  $\mathbb{R}^n$ . Let  $X$  be a smooth vector field on  $U$ , with flow  $\Phi_t$  ( $t \in (-\varepsilon, \varepsilon)$ ) defined on some neighbourhood  $V$  of 0.*

*Then, up to taking some smaller  $V$  and  $\varepsilon$ ,  $\Phi_t \circ f$  has a unique fixed point  $x(t)$  for all  $t \in (-\varepsilon, \varepsilon)$ . Moreover, the function  $t \mapsto x(t)$  is smooth at 0 and we have*

$$\dot{x}(0) = (\text{Id} - df_0)^{-1} (X_0) .$$

**Proof of Proposition 5.4.** Define

$$\begin{aligned} F : (-\varepsilon, \varepsilon) \times V &\rightarrow \mathbb{R}^n \\ (t, x) &\mapsto \Phi_t \circ f(x) - x . \end{aligned}$$

We have

$$d_{(0,0)}F(s, u) = sX_0 + d_0f(u) - u .$$

Since  $\|df_0\| < 1$ , the linear map  $dF_{(0,0)}$  restricted to  $\{0\} \times \mathbb{R}^n$  is bijective. Therefore  $F$  is a submersion in a neighbourhood of  $(0, 0)$  and  $F^{-1}(0)$  is thus a submanifold of dimension 1, transverse to the fibers  $t = \text{Cst}$ . The locus  $F^{-1}(0)$  can thus be written locally as  $\{(t, x(t)), t \in (-\varepsilon, \varepsilon)\}$ , where  $x(t)$  is smooth,  $x(0) = 0$  and  $\dot{x}(0)$  satisfies

$$X_0 + df_0(\dot{x}(0)) - \dot{x}(0) = 0 .$$

The conclusion follows. ■

**Proof of Lemma 5.3.** Let us now compute the differential of  $\pi$ . Fix  $g \in \text{SO}_0(n, 1)$  and  $u$  a vector in the Lie algebra of  $\text{SO}_0(n, 1)$ . We identify  $u$  with a tangent vector at  $g$  by right translation, i.e. we also denote  $u$  the vector

$$\frac{d}{dt}\Big|_{t=0} \exp(tu)g .$$

Let  $X_u$  be the vector field on  $\mathbb{H}^n$  associated to  $u$ , i.e. the generator of the flow

$$\Phi_t(x) = \exp(tu) \cdot x .$$

By definition,  $\pi(g)$  is fixed by  $g \circ f$  and, since  $f$  is contracting, one has  $\|d_{\pi(g)}(g \circ f)\| < 1$ . Applying Lemma 5.4 to  $g \circ f$  in a neighbourhood of  $\pi(g)$  and to the vector field  $X_u$ , we deduce that  $t \mapsto \pi(\exp(tu)g)$  is smooth at 0 and

$$d_g\pi(u) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tu)g) = (\text{Id} - d_{\pi(g)}(g \circ f))^{-1} (X_u(\pi(g))) . \quad \blacksquare$$

Note that the map  $u \mapsto X_u(\pi(g))$  is a surjective map from  $\mathfrak{so}(n, 1)$  to  $T_{\pi(g)}\mathbb{H}^n$  since the action of  $\mathrm{SO}_0(n, 1)$  on  $\mathbb{H}^n$  is transitive. Moreover, the linear map  $(\mathrm{Id} - d_{\pi(g)}(g \circ f))^{-1}$  is bijective. Hence  $\pi$  is a submersion at  $g$ . If  $(g_k)_{k \in \mathbb{N}}$  is such that  $\pi(g_k)$  is bounded in  $\mathbb{H}^n$ , then the sequence  $(g_k \circ f(x))$  is also bounded in  $\mathbb{H}^n$ . Therefore, the sequence  $(g_k)$  is relatively compact in  $\mathrm{SO}_0(n, 1)$  and  $\pi$  is thus proper.

In conclusion, the map  $\pi$  induces a smooth fiber bundle from  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$  to  $j(\Gamma) \backslash \mathbb{H}^n$ , whose fibers are images of  $\mathrm{SO}(n)$  by left and right multiplication. With a slight abuse of notation, we will still denote by  $\pi$  the induced fiber bundle.

**5.2. Integration of the volume along the fibers.** In order to compute the volume of  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(n, 1)$ , we will first integrate the volume form along the fibers of  $\pi$ .

Let  $g$  be a point in  $\mathrm{SO}_0(n, 1)$ . The Lie algebra  $\mathfrak{so}(n, 1)$  decomposes as  $\mathfrak{k}_g \oplus \mathfrak{k}_g^\perp$ , where  $\mathfrak{k}_g$  is the Lie algebra of the stabilizer of  $\pi(g)$  and  $\mathfrak{k}_g^\perp$  its orthogonal complement with respect to the Killing form. Recall that this Lie algebra is identified to the tangent space  $T_g\mathrm{SO}_0(n, 1)$  by right multiplication.

By Lemma 5.3, the kernel of the differential of  $\pi$  at  $g$  is precisely  $\mathfrak{k}_g$ , and  $d_g\pi$  thus induces an isomorphism from  $\mathfrak{k}_g^\perp$  to  $T_{\pi(g)}\mathbb{H}^n$ . One can thus define a volume form  $vol^\pi$  on  $T_g\mathrm{SO}_0(n, 1)$  by providing  $\mathfrak{k}_g$  with the volume associated to the restricted Killing metric and  $\mathfrak{k}_g^\perp$  with the pull-back of the volume form  $vol^{\mathbb{H}^n}$  by  $\pi$ .

From Lemma 5.3, one easily deduces the following:

**Proposition 5.5.** *The volume form  $vol^\pi$  defined above and the volume form  $vol^\kappa$  associated to the Killing metric are related by*

$$vol_g^\kappa = \det(\mathrm{Id} - d_{\pi(g)}(g \circ f)) vol_g^\pi .$$

In order to integrate this identity along the fibers, we will need the following linear algebra lemma:

**Lemma 5.6.** *Let  $vol^K$  denote the volume form of  $\mathrm{SO}(n)$ . Then for any matrix  $A \in \mathcal{M}_n(\mathbb{R})$ , we have*

$$\int_{\mathrm{SO}(n)} \det(\mathrm{I}_n - UA) dvol^K(U) = \mathbf{V}_n (1 + (-1)^n \det(A)) .$$

**Proof.** If  $B$  is some  $n \times n$  matrix and  $P$  a subset of  $\{1, \dots, n\}$ , let us denote by  $B_P$  the square matrix of size  $n - |P|$  obtained from  $B$  by removing all lines and columns whose indices are in  $P$ . Then one has

$$\det(\lambda \mathrm{I}_n - B) = \sum_{k=0}^n \left( \sum_{|P|=k} \det(B_P) \right) (-1)^{n-k} \lambda^k .$$

Therefore,

$$\int_{\text{SO}(n)} \det(\mathbf{I}_n - UA) d\text{vol}^K(U) = \sum_{k=0}^n (-1)^{n-k} \left( \sum_{|P|=k} \int_{\text{SO}(n)} \det((UA)_P) d\text{vol}^K(U) \right). \quad (2)$$

Now, for any  $P \subset \{1, \dots, n\}$  with  $1 \leq |P| \leq n - 1$ , let  $D$  be the diagonal matrix with coefficients  $(d_1, \dots, d_n)$  where  $d_i$  is  $-1$  if  $i = \inf P$  or  $i = \inf\{1, \dots, n\} \setminus P$  and  $d_i = 1$  otherwise. Then  $D$  is in  $\text{SO}(n)$  and, by construction,

$$\det((DB)_P) = -\det(B_P)$$

for every matrix  $B$ . (Indeed, the multiplication by  $D$  multiplies the first line of  $B_P$  by  $-1$ .)

We thus have

$$\begin{aligned} \int_{\text{SO}(n)} \det((UA)_P) d\text{vol}^K(U) &= \int_{\text{SO}(n)} \det((DUA)_P) d\text{vol}^K(U) \\ &= \int_{\text{SO}(n)} -\det((UA)_P) d\text{vol}^K(U), \end{aligned}$$

hence

$$\int_{\text{SO}(n)} \det((UA)_P) d\text{vol}^K(U) = 0.$$

The only non vanishing terms in the sum (2) thus appear when  $P = \emptyset$  or  $P = \{1, \dots, n\}$ . Therefore,

$$\begin{aligned} \int_{\text{SO}(n)} \det(\mathbf{I}_n - UA) d\text{vol}^K(U) &= \int_{\text{SO}(n)} d\text{vol}^K(U) \\ &\quad + (-1)^n \int_{\text{SO}(n)} \det(UA) d\text{vol}^K(U) \\ &= (1 + (-1)^n \det(A)) \int_{\text{SO}(n)} d\text{vol}^K(U) \\ &= \mathbf{V}_n (1 + (-1)^n \det(A)). \quad \blacksquare \end{aligned}$$

We can now complete the proof of Theorem 2.11. Let us denote by  $B$  the manifold  $j(\Gamma) \backslash \mathbb{H}^n$  and by  $E$  the manifold  $j \times \rho(\Gamma) \backslash \text{SO}_0(n, 1)$ . Let  $U$  be an open set of  $B$  over which the fiber bundle  $\pi : E \rightarrow B$  admits a section  $s$ . One can then identify the open set  $V = \pi^{-1}(U)$  with  $K \times U$ , where  $K$  is the stabilizer of a base point  $x_0$  in  $U$ . Such a trivialization is given by

$$(k, x) \mapsto h_x k h_x^{-1} s(x),$$

where  $h_x$  is an element of  $\text{SO}_0(n, 1)$  sending  $x_0$  to  $x$  (so that  $h_x K h_x^{-1}$  is the stabilizer of  $x$ ) and depending smoothly on  $x$ . Through this trivialization, the volume  $\text{vol}^\pi$  identifies with the product volume on  $K \times U$  (where  $K \simeq \text{SO}(n)$  is provided with the volume  $\text{vol}^K$ ). On the other side, we have

$$\text{vol}_{(k,x)}^\kappa = \det(\text{Id} - d_x(h_x k h_x^{-1} s(x) \circ f)) \text{vol}_{(k,x)}^\pi$$

by Proposition 5.5.

When  $k$  runs through all elements of  $K$ ,  $h_x k h_x^{-1}$  runs through  $\mathrm{SO}(T_x \mathbb{H}^n)$ . By Lemma 5.6, we thus obtain

$$\begin{aligned} \int_K \det(\mathrm{Id} - d_x(x k h_x^{-1} s(x) \circ f)) \, \mathrm{vol}^K &= \mathbf{V}_n [1 + (-1)^n \det(d_x(s(x) \circ f))] \\ &= \mathbf{V}_n (1 + (-1)^n \mathrm{Jac}_f(x)) \, , \end{aligned}$$

where  $\mathrm{Jac}_f(x)$  is defined by

$$f^* (\mathrm{vol}_{f(x)}^{\mathbb{H}^n}) = \mathrm{Jac}_f(x) \mathrm{vol}_x^{\mathbb{H}^n} \, .$$

Therefore,

$$\int_{K \times U} \mathrm{vol}^k = \int_U \mathbf{V}_n (1 + (-1)^n \mathrm{Jac}_f(x)) \, \mathrm{vol}^{\mathbb{H}^n} \, .$$

We finally obtain

$$\begin{aligned} \int_E \mathrm{vol}^k &= \int_B \mathbf{V}_n (1 + (-1)^n \mathrm{Jac}_f(x)) \, \mathrm{vol}^{\mathbb{H}^n} \\ &= \mathbf{V}_n \int_B \mathrm{vol}^{\mathbb{H}^n} + (-1)^n f^* (\mathrm{vol}^{\mathbb{H}^n}) \, . \end{aligned}$$

This concludes the proof of Theorem 2.11.

## 6. Some open questions

In this last section, we discuss further some questions raised by this work.

**6.1. Are finite covolume discrete groups geometrically finite?** It is a famous result of Kazhdan (see [4, Proposition 3.14]) and Raghunathan [27] that lattices of a Lie group are finitely presented. However, to our knowledge, there is no answer to the following more general question:

**Question 6.1.** Let  $G$  be a Lie group and  $X$  a  $G$ -homogeneous space such that  $G$  preserves a volume form on  $X$ . Let  $\Gamma$  be a discrete subgroup of  $G$  acting properly discontinuously on  $X$  such that  $\Gamma \backslash X$  has finite volume. Is  $\Gamma$  finitely generated?

At first, we expected this to be true for quotients of  $\mathrm{SO}_0(n, 1)$ , but we fell short of proving it here. The main issue is that Guéritaud–Kassel’s theorem *assumes* that  $\Gamma$  is a geometrically finite subgroup of  $\mathrm{SO}_0(n, 1)$  (in particular it is finitely generated).

If we remove this assumption, there are examples of proper actions on  $\mathrm{SO}_0(n, 1)$  given by  $j \times \rho(\Gamma)$ , where  $j$  is discrete and faithful and for which there is a  $(j, \rho)$ -equivariant map  $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$  that satisfies

$$d(f(x), f(y)) < d(x, y)$$

for all  $x \neq y$ , but that is not  $\lambda$ -Lipschitz for some  $\lambda < 1$  (see [13, Paragraph 10.1]).

Conversely, if such a map exists, then  $j \times \rho(\Gamma)$  acts properly discontinuously on the domain  $U$  formed by all the  $g \in \text{SO}_0(n, 1)$  such that  $g \circ f$  has a fixed point. The integral

$$\int_{j(\Gamma) \backslash \mathbb{H}^n} \text{vol}^{\mathbb{H}^n} + (-1)^n f^* \text{vol}^{\mathbb{H}^n}$$

then computes the volume of  $j \times \rho(\Gamma) \backslash U$ . Whether this domain  $U$  is the whole hyperbolic space  $\mathbb{H}^n$  depends on how contracting the map  $f$  is.

Interestingly, when  $n = 2$ , it can happen that  $f$  is close enough to being an orientation reversing isometry asymptotically, so that  $\text{vol}^{\mathbb{H}^2} + f^* \text{vol}^{\mathbb{H}^2}$  is integrable and thus  $j \times \rho(\Gamma) \backslash U$  has finite volume. However, this may imply that  $U \subsetneq \text{SO}_0(2, 1)$ . More generally, we ask the following question:

**Question 6.2.** Let  $\Gamma$  be a discrete group and  $j \times \rho$  a faithful representation of  $\Gamma$  into  $\text{SO}_0(n, 1) \times \text{SO}_0(n, 1)$  such that  $j \times \rho(\Gamma)$  acts properly discontinuously on  $\text{SO}_0(n, 1)$  with finite covolume. Up to switching the factors, assume  $j$  is discrete and faithful. Is  $j(\Gamma)$  a lattice in  $\text{SO}_0(n, 1)$ ?

**6.2. What are the values of the volume of Guéritaud–Kassel’s quotients?.**

Let us fix a torsion-free lattice  $\Gamma$  in  $\text{SO}_0(n, 1)$ . It is natural to ask what values can be taken by the volume of quotients of  $\text{SO}_0(n, 1)$  by some action of  $\Gamma$ . Given Theorem 1, this boils down to understanding the possible values for the volume of a representation  $\rho : \Gamma \rightarrow \text{SO}_0(n, 1)$  which is strictly dominated by a discrete and faithful representation  $j : \Gamma \rightarrow \text{SO}_0(n, 1)$ .

The case  $n = 2$  is well-understood. If  $\Gamma$  is not cocompact, Theorem 3 shows that quotients of  $\text{SO}_0(2, 1)$  can have any volume in the open interval  $(0, 4\pi \mathbf{Vol}(j))$ , where  $j$  is a Fuchsian representation of  $\Gamma$ . On the contrary, if  $\Gamma$  is cocompact, then the volume of a representation  $\rho$  is an integral multiple of  $2\pi$ . If, moreover,  $\rho$  is strictly dominated by  $j$ , then one has  $|\mathbf{Vol}(\rho)| < \mathbf{Vol}(j)$  (note that the weak inequality is true without any hypothesis on  $\rho$  by the Milnor–Wood inequality). It follows that

$$\frac{1}{4\pi^2} \mathbf{Vol}(j \times \rho(\Gamma) \backslash \text{SO}_0(2, 1)) \in \{1, \dots, 4g - 5\},$$

where  $g$  is the genus of the surface  $j(\Gamma) \backslash \mathbb{H}^2$ .

Conversely, Salein [28] proved the existence of pairs  $(j, \rho)$  with  $j$  Fuchsian and  $\rho$  strictly dominated by  $j$  of any non-extremal Euler class. Therefore,  $\frac{1}{4\pi^2} \mathbf{Vol}(j \times \rho(\Gamma) \backslash \text{SO}_0(2, 1))$  can take all the integral values between 1 and  $4g - 5$ .

The picture is much more blurry in higher dimension. When  $n$  is even, it is known that twice the volume of a representation of  $\Gamma$  is an integral multiple of the volume of the sphere of dimension  $n$ . (When  $\Gamma$  is cocompact, it follows from the Chern–Gauss–Bonnet theorem. It was generalized to any lattice by Bucher–Burger–Iozzi [8].) However, it is not known which of this values are actually realized, let alone by representations strictly dominated by a Fuchsian one. In

odd dimension, the only thing that is known is that there are only finitely many possible values, since the space of cusp deteriorating representations has finitely many connected components.

Until recently, the only known examples of finite volume Guéritaud–Kassel quotients when  $n \geq 3$  were constructed by taking the representation  $\rho$  to be a continuous deformation of the identity (see [12] and [20]). In particular, such a representation had volume 0.

Motivated by Question 6.4 below, Lakeland and Leininger recently gave many examples of pairs of lattices  $\Gamma$  and  $\Gamma'$  in  $\mathrm{SO}_0(3, 1)$  such that there exists a contracting map of positive degree from  $\Gamma \backslash \mathbb{H}^3$  to  $\Gamma' \backslash \mathbb{H}^3$ , leading to new exotic examples of Guéritaud–Kassel quotients of  $\mathrm{SO}_0(3, 1)$  for which  $\mathbf{Vol}(\rho)$  is non-zero. Their work raises many questions, such as the following:

**Question 6.3.** Is there a positive constant  $C$  such that any Guéritaud–Kassel quotient  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(3, 1)$  satisfies

$$\mathbf{Vol}(j \times \rho(\Gamma) \backslash \mathrm{SO}_0(3, 1)) \geq C ?$$

Is there a positive constant  $C$  such that any Guéritaud–Kassel quotient  $j \times \rho(\Gamma) \backslash \mathrm{SO}_0(3, 1)$  satisfies

$$\mathbf{Vol}(j \times \rho(\Gamma) \backslash \mathrm{SO}_0(3, 1)) \geq C \mathbf{Vol}(j) ?$$

Finally, in dimension greater than 4, the following question has no answer, to our knowledge:

**Question 6.4.** Let  $\Gamma$  be a lattice in  $\mathrm{SO}_0(n, 1)$ ,  $n \geq 4$  and  $j : \Gamma \rightarrow \mathrm{SO}_0(n, 1)$  the inclusion. Does there exist a representation  $\rho$  strictly dominated by  $j$  whose volume is not zero?

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