

## Parity Preservation of $K$ -Types under the Theta Correspondence

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**Abstract.** This note shows a property of degree-parity preservation for  $K$ -types under Howe's theta correspondence. As its application, we deduce the preservation of parity of all  $K$ -types occurring in an arbitrary irreducible  $(\mathfrak{g}, K)$ -module of any Lie group in reductive dual pairs.

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### 1. Degree-parity Preservation

*Howe's duality correspondence* of irreducible admissible representations for reductive dual pairs was introduced by Roger Howe in the 1970s. It is also called the *theta correspondence* as an extension of Weil's representation-theoretic approach to classical  $\theta$ -series. In this short note, a degree-parity preservation property (Theorem 1.5) for  $K$ -types is shown for the local theta correspondence of Lie groups, with an interesting application, Theorem 2.1 (or Theorem 2.6 in its general form), which asserts the preservation of parity of all  $K$ -types occurring in an arbitrary irreducible admissible representation of any Lie group in reductive dual pairs.

For a continuous admissible representation of a real reductive Lie group  $G$ , as we focus on its  $K$ -spectrum, we may replace it by its Harish-Chandra  $(\mathfrak{g}, K)$ -module (consisting of its  $K$ -finite smooth vectors). Here  $K$  is a maximal compact subgroup of  $G$ , and  $\mathfrak{g}$  is the complexified Lie algebra of  $G$ . Throughout this note, we use upper case Latin letters (e.g.,  $G$ ,  $G'$ ,  $K$ ,  $T$ ) to denote Lie groups, and the corresponding lower case Gothic letters (e.g.,  $\mathfrak{g}$ ,  $\mathfrak{g}'$ ,  $\mathfrak{k}$ ,  $\mathfrak{t}$ ) to indicate their complexified Lie algebras.

By a  $K$ -module we mean a continuous representation of  $K$ , and by a  $K$ -type we mean an equivalence class of irreducible  $K$ -modules (which are automatically finite-dimensional and unitary). Let  $\mathcal{R}(K)$  denote the set of all  $K$ -types. By an abuse of notation, for a  $K$ -type  $\sigma \in \mathcal{R}(K)$ , we also understand  $\sigma$  as an irreducible  $K$ -module (up to equivalence).

For a  $(\mathfrak{g}, K)$ -module  $V$ , its  $K$ -spectrum is the  $K$ -module decomposition

$$V \simeq \bigoplus_{\sigma \in \mathcal{R}(K)} W_{\sigma}^{\oplus m(\sigma, V)},$$

where  $W_{\sigma}$  is an underlying space of a  $K$ -type  $\sigma$ , and

$$m(\sigma, V) = \dim \operatorname{Hom}_K(W_{\sigma}, V) = \dim \operatorname{Hom}_K(V, W_{\sigma})$$

is the *multiplicity* of  $\sigma$  in  $V$ . We say that “ $\sigma$  occurs in  $V$ ” if  $m(\sigma, V) \neq 0$ . Denote the set of all  $K$ -types occurring in  $V$  by  $\mathcal{R}(K, V) = \{\sigma \in \mathcal{R}(K) : m(\sigma, V) \neq 0\}$ . A finitely generated  $(\mathfrak{g}, K)$ -module  $V$  is called *admissible* if every  $\sigma \in \mathcal{R}(K)$  has finite multiplicity in  $V$ . It is well-known that every irreducible  $(\mathfrak{g}, K)$ -module is admissible.

A real *reductive dual pair* is a pair  $(G, G')$  of closed reductive subgroups of  $\mathbf{Sp} = Sp_{2N}(\mathbb{R})$  (for some  $N$ ) such that they are mutual centralizers of each other. For a subgroup  $E \subseteq \mathbf{Sp}$ , let  $\widetilde{E}$  denote its preimage in the *metaplectic cover* (the unique non-trivial two-fold central extension)  $\widetilde{\mathbf{Sp}}$  of  $\mathbf{Sp}$ . Indeed we have a short exact sequence:

$$1 \rightarrow \mu_2 \rightarrow \widetilde{\mathbf{Sp}} \rightarrow \mathbf{Sp} \rightarrow 1,$$

where  $\mu_2 = \operatorname{Ker}(\widetilde{\mathbf{Sp}} \rightarrow \mathbf{Sp})$  is a finite group of order 2. Take the Segal-Shale-Weil *oscillator representation* (c.f. [Sha62, Wei64, LV80])  $\omega$  of  $\widetilde{\mathbf{Sp}}$  (associated to the character of  $\mathbb{R}$  that sends  $t$  to  $\exp(2\pi\sqrt{-1}t)$ ). Let  $\mathfrak{sp}$  denote the complexified Lie algebra of  $\mathbf{Sp}$ , and take  $\mathbf{U} = U(N)$  as a maximal compact subgroup of  $\mathbf{Sp}$ . Fock model realizes the  $(\mathfrak{sp}, \widetilde{\mathbf{U}})$ -module of  $\omega$  on the space  $\mathcal{F} = \operatorname{Poly}(\mathbb{C}^N)$  of complex polynomials on  $\mathbb{C}^N$ .

Assume that  $G$  and  $G'$  are embedded in  $\mathbf{Sp}$  in such a way that  $K = \mathbf{U} \cap G$  and  $K' = \mathbf{U} \cap G'$  are maximal compact subgroups of  $G$  and  $G'$  respectively. Hence  $\widetilde{K}$  and  $\widetilde{K}'$  are maximal compact subgroups of  $\widetilde{G}$  and  $\widetilde{G}'$  respectively. Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be the complexified Lie algebras of  $G$  and  $G'$  respectively.

**Lemma 1.1** (Algebraic version of local theta correspondence over  $\mathbb{R}$  [How89b, Theorem 2.1]). *For a real reductive dual pair  $(G, G')$ , if  $\pi$  is an irreducible  $(\mathfrak{g}, \widetilde{K})$ -module, then*

$$\mathcal{F} / \bigcap_{T \in \operatorname{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathcal{F}, \pi)} \operatorname{Ker}(T) \simeq \pi \otimes \Theta(\pi),$$

where  $\Theta(\pi)$  is a finitely generated admissible  $(\mathfrak{g}', \widetilde{K}')$ -module. Moreover, if  $\Theta(\pi)$  is non-zero, then it has a unique irreducible  $(\mathfrak{g}', \widetilde{K}')$ -quotient  $\theta(\pi)$  called the “theta lift” of  $\pi$ .

- The intersection  $\bigcap \operatorname{Ker}(T)$  may be the whole  $\mathcal{F}$ . This case happens  $\Leftrightarrow \operatorname{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathcal{F}, \pi) = 0 \Leftrightarrow \Theta(\pi) = 0$ . In this case, let the theta lift  $\theta(\pi) = 0$ .
- When  $\Theta(\pi) \neq 0$ , the “uniqueness” of  $\theta(\pi)$  means that  $\Theta(\pi)$  has a unique maximal proper sub- $(\mathfrak{g}', \widetilde{K}')$ -module, and it is the kernel of  $\Theta(\pi) \rightarrow \theta(\pi)$ .

Recall that the oscillator representation  $\omega$  is the direct sum of two irreducible unitary representations of  $\widetilde{\mathbf{Sp}}$ , and we have a  $(\mathfrak{sp}, \widetilde{\mathbf{U}})$ -module decomposition  $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ , where  $\mathcal{F}_i$  is the linear span of all homogeneous polynomials of degree  $\equiv i \pmod{2}$  in  $\mathcal{F} = \text{Poly}(\mathbb{C}^N)$ , for  $i \in \{0, 1\}$ . Then as  $(\mathfrak{g}', \widetilde{K}')$ -modules  $\Theta(\pi) = \Theta_0(\pi) \oplus \Theta_1(\pi)$ , with

$$\mathcal{F}_i \Big/ \bigcap_{T \in \text{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathcal{F}_i, \pi)} \text{Ker}(T) \simeq \pi \otimes \Theta_i(\pi), \quad \text{for } i \in \{0, 1\}.$$

**Proposition 1.2.** *Let  $\pi$  be an irreducible  $(\mathfrak{g}, \widetilde{K})$ -module. Then  $\Theta_i(\pi) = 0$  for at least one  $i \in \{0, 1\}$ . In other words,  $\text{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathcal{F}_i, \pi) = 0$  for this  $i$ .*

**Proof.** Otherwise,  $\Theta(\pi) = \Theta_0(\pi) \oplus \Theta_1(\pi)$  with both  $\Theta_i(\pi)$  non-zero finitely generated admissible  $(\mathfrak{g}', \widetilde{K}')$ -modules. By Zorn's Lemma,  $\Theta_i(\pi)$  contains a maximal proper sub- $(\mathfrak{g}', \widetilde{K}')$ -module  $\Pi_i$ . Then both  $\Theta(\pi)/(\Pi_0 \oplus \Theta_1(\pi))$  and  $\Theta(\pi)/(\Theta_0(\pi) \oplus \Pi_1)$  are non-zero irreducible  $(\mathfrak{g}', \widetilde{K}')$ -quotients, in contradiction with the uniqueness in Lemma 1.1. ■

**Corollary 1.3.**  $\mathcal{R}(\widetilde{K}, \mathcal{F}_0) \cap \mathcal{R}(\widetilde{K}, \mathcal{F}_1) = \emptyset$ .

**Proof.** Let  $M$  be the centralizer of  $K$  in  $\mathbf{Sp}$ , then  $(K, M)$  is also a reductive dual pair [How89b, Fact 1]. For the dual pair  $(K, M)$  and any  $\widetilde{K}$ -type  $\sigma$ , Proposition 1.2 asserts that  $\text{Hom}_{\widetilde{K}}(\mathcal{F}_i, \sigma) = 0$  for some  $i \in \{0, 1\}$ . Then  $m(\sigma, \mathcal{F}_i) = 0$  and  $\sigma \notin \mathcal{R}(\widetilde{K}, \mathcal{F}_i)$ . ■

For  $\sigma \in \mathcal{R}(\widetilde{K}, \mathcal{F})$ , [How89b] defines the *degree*  $\text{deg}(\sigma)$  with respect to  $(G, G')$  as the minimal degree of polynomials in the  $\sigma$ -isotypic subspace  $\mathcal{F}_\sigma = \sum_{\varphi \in \text{Hom}_{\widetilde{K}}(\sigma, \mathcal{F})} \text{Im}(\varphi)$ .

**Corollary 1.4.** *If  $\sigma \in \mathcal{R}(\widetilde{K}, \mathcal{F}_i)$  for some  $i \in \{0, 1\}$ , then  $\text{deg}(\sigma) \equiv i \pmod{2}$ .*

**Proof.** By Corollary 1.3,  $\sigma \notin \mathcal{R}(\widetilde{K}, \mathcal{F}_{1-i})$ . So  $\mathcal{F}_\sigma \subseteq \mathcal{F}_i$  and  $\text{deg}(\sigma) \equiv i \pmod{2}$ . ■

**Theorem 1.5.** *For a real reductive dual pair  $(G, G')$ , let  $\pi$  be an irreducible  $(\mathfrak{g}, \widetilde{K})$ -module with a non-zero theta lift. If  $\sigma_1, \sigma_2 \in \mathcal{R}(\widetilde{K}, \pi)$ , then  $\sigma_1, \sigma_2 \in \mathcal{R}(\widetilde{K}, \mathcal{F}_i)$  with  $i \equiv \text{deg}(\sigma_1) \equiv \text{deg}(\sigma_2) \pmod{2}$ .*

**Proof.** Let  $V_\pi$  be an underlying space of  $\pi$ . By definition  $\text{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathcal{F}_i, V_\pi) \neq 0$  for some  $i \in \{0, 1\}$ . Take a non-zero  $T \in \text{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathcal{F}_i, V_\pi)$ . As  $\pi$  is an irreducible  $(\mathfrak{g}, \widetilde{K})$ -module, the image  $T(\mathcal{F}_i) = V_\pi$ .

For  $j \in \{0, 1\}$ , let  $W_{\sigma_j}$  be an underlying space of  $\sigma_j$ . As  $m(\sigma_j, V_\pi) = \dim \text{Hom}_{\widetilde{K}}(V_\pi, W_{\sigma_j}) > 0$ , take a non-zero  $\varphi_j \in \text{Hom}_{\widetilde{K}}(V_\pi, W_{\sigma_j})$ . As  $W_{\sigma_j}$  is an irreducible  $\widetilde{K}$ -module, the image  $\varphi_j(V_\pi) = W_{\sigma_j}$ . Now the surjective  $\varphi_j \circ T : \mathcal{F}_i \rightarrow W_{\sigma_j}$  gives a non-zero element of  $\text{Hom}_{\widetilde{K}}(\mathcal{F}_i, W_{\sigma_j})$ . So  $m(\sigma_j, \mathcal{F}_i) > 0$ ,

$\sigma_j \in \mathcal{R}(\tilde{K}, \mathcal{F}_i) \subseteq \mathcal{R}(\tilde{K}, \mathcal{F})$ , and  $\deg(\sigma_j) \equiv i \pmod{2}$  for both  $j \in \{0, 1\}$  by Corollary 1.4. ■

**Remark.** Corollary 1.3 (and the degree-parity preservation of Theorem 1.5 as a consequence) can also be deduced from classical invariant theory, similar to the proof of [Fan17, Lemma 6] based on [How89b, (3.9)(b)] and [How89a].

## 2. Parity Preservation

From Theorem 1.5, we deduce the preservation of parity of all  $K$ -types occurring in an arbitrary irreducible  $(\mathfrak{g}, K)$ -module of a Lie group  $G$  in real reductive dual pairs.

**Theorem 2.1** (Parity preservation). *Let  $G$  and  $K$  be as in the following table, with  $K$  embedded in  $G$  as a maximal compact subgroup in the usual way. If  $\pi$  is an irreducible  $(\mathfrak{g}, K)$ -module and  $\sigma_1, \sigma_2 \in \mathcal{R}(K, \pi)$ , then  $\varepsilon(\sigma_1) = \varepsilon(\sigma_2)$ , where  $\varepsilon : \mathcal{R}(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  is the parity of  $K$ -types defined explicitly in the next subsection.*

$G$	$K$	$G$	$K$	$G$	$K$
$GL_m(\mathbb{R})$	$O(m)$	$O_p(\mathbb{C})$	$O(p)$	$Sp_{2n}(\mathbb{C})$	$Sp(n)$
$GL_m(\mathbb{C})$	$U(m)$	$O(p, q)$	$O(p) \times O(q)$	$Sp_{2n}(\mathbb{R})$	$U(n)$
$GL_m(\mathbb{H})$	$Sp(m)$	$Sp(p, q)$	$Sp(p) \times Sp(q)$	$O^*(2n)$	$U(n)$
		$U(p, q)$	$U(p) \times U(q)$		

### 2.1. Parametrization and parity for $K$ -types.

For  $K = U(n)$ ,  $T = U(1)^n = \text{diag}(U(1), \dots, U(1))$  is a maximal torus, with the standard system of positive roots  $\Delta^+(\mathfrak{t}, \mathfrak{k}) = \{e_i - e_j \mid 1 \leq i < j \leq n\}$ . Write  $\mathfrak{t}_0$  for the real Lie algebra of  $T$ . Write each weight in  $\sqrt{-1}\mathfrak{t}_0^*$  as the  $n$ -tuple of coefficients under the basis  $e_1, \dots, e_n$ . Then a  $U(n)$ -type is parametrized by its highest weight  $(a_1, \dots, a_n)$  with integers  $a_1 \geq a_2 \geq \dots \geq a_n$ .

For  $K = Sp(n)$ ,  $T = Sp(1)^n = \text{diag}(Sp(1), \dots, Sp(1))$  is a maximal torus, with the standard system of positive roots  $\Delta^+(\mathfrak{t}, \mathfrak{k}) = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\}$ . Write each weight in  $\sqrt{-1}\mathfrak{t}_0^*$  as the  $n$ -tuple of coefficients under the basis  $e_1, \dots, e_n$ . Then a  $Sp(n)$ -type is parametrized by its highest weight  $(a_1, \dots, a_n)$  with integers  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ .

**Lemma 2.2** ([Wey39]). *Embed  $O(n)$  in  $U(n)$  as  $O(n) = U(n) \cap GL(n, \mathbb{R})$ . There is a bijection  $\mathcal{R}(O(n)) \xrightarrow{\cong} \{(b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_s, \underbrace{0, \dots, 0}_{n-r-s}) \in \mathcal{R}(U(n)) : 2r + s \leq n, b_r \geq 2\}$ , such that an  $O(n)$ -type  $\sigma$  corresponds to a  $U(n)$ -type  $\lambda$  if and only if the highest weight vectors of  $\lambda$  generate an  $O(n)$ -module equivalent to  $\sigma$ . Therefore, an  $O(n)$ -type  $\sigma$  can be parametrized as follows (with  $[t]$  the greatest*

integer less than or equal to  $t$ ):

$$\begin{cases} (b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_s, \underbrace{0, \dots, 0}_{\lfloor \frac{n}{2} \rfloor - r - s}; +1) & \text{if } r + s \leq \frac{n}{2}, \\ (b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_{n-2r-s}, \underbrace{0, \dots, 0}_{\lfloor \frac{n}{2} \rfloor - n + r + s}; -1) & \text{if } \frac{n}{2} \leq r + s \leq n - r. \end{cases}$$

**Remark.** When  $r + s = \frac{n}{2}$ , these two cases coincide and give the same  $\sigma$ .

Define the parity  $\varepsilon : \mathcal{R}(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  for  $K = U(n)$ ,  $Sp(n)$  or  $O(n)$  as:

$K$	a $K$ -type $\sigma$ is parametrized as	parity $\varepsilon(\sigma) \in \mathbb{Z}/2\mathbb{Z}$
$U(n)$	$(a_1, \dots, a_n)$ with integers $a_1 \geq a_2 \geq \dots \geq a_n$	$\sum_{i=1}^n a_i \pmod{2}$
$Sp(n)$	$(a_1, \dots, a_n)$ with integers $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$	
$O(n)$	$(a_1, \dots, a_{\lfloor \frac{n}{2} \rfloor}; \epsilon)$ with integers $a_1 \geq a_2 \geq \dots \geq a_{\lfloor \frac{n}{2} \rfloor} \geq 0$ , and $\epsilon \in \{\pm 1\}$	$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i + \frac{1-\epsilon}{2} \cdot n \pmod{2}$

The corresponding  $O(n)$ -type and  $U(n)$ -type have the same parity. For  $K = K_1 \times \dots \times K_r$  with  $K_i = U(n_i)$ ,  $Sp(n_i)$  or  $O(n_i)$ , as  $\mathcal{R}(K) = \{\otimes_{i=1}^r \sigma_i : \sigma_i \in \mathcal{R}(K_i)\}$ , define the parity  $\varepsilon : \mathcal{R}(K) \rightarrow \mathbb{Z}/2\mathbb{Z}$  by  $\varepsilon(\otimes_{i=1}^r \sigma_i) = \sum_{i=1}^r \varepsilon(\sigma_i) \in \mathbb{Z}/2\mathbb{Z}$ .

**2.2. Non-vanishing and splitting conditions.**

To prove Theorem 2.1, we recall the non-vanishing and splitting conditions for the local theta correspondence over  $\mathbb{R}$ .

Let  $W$  be a real symplectic vector space. A reductive dual pair  $(G, G')$  in  $Sp(W)$  is called *irreducible* if  $G \cdot G'$  acts irreducibly on  $W$ . Each reductive dual pair  $(G, G')$  in  $Sp(W)$  can be decomposed into a direct sum of irreducible pairs, namely, there is an orthogonal direct sum decomposition  $W = \bigoplus_{i=1}^k W_i$  such that  $G \cdot G'$  acts irreducibly on  $W_i$ , and the restrictions of actions of  $(G, G')$  to  $W_i$  define irreducible reductive dual pairs  $(G_i, G'_i)$  in  $Sp(W_i)$ . Indeed,  $G = G_1 \times \dots \times G_k$  and  $G' = G'_1 \times \dots \times G'_k$ . All irreducible real reductive dual pairs are classified in the following table (c.f. [How79]).

Type I	Sp	Type II	Sp
$(O(p, q), Sp(2n, \mathbb{R}))$	$Sp(2n(p + q), \mathbb{R})$	$(GL_m(\mathbb{R}), GL_n(\mathbb{R}))$	$Sp(2mn, \mathbb{R})$
$(O_p(\mathbb{C}), Sp(2n, \mathbb{C}))$	$Sp(4pn, \mathbb{R})$	$(GL_m(\mathbb{C}), GL_n(\mathbb{C}))$	$Sp(4mn, \mathbb{R})$
$(Sp(p, q), O^*(2n))$	$Sp(4n(p + q), \mathbb{R})$	$(GL_m(\mathbb{H}), GL_n(\mathbb{H}))$	$Sp(8mn, \mathbb{R})$
$(U(p, q), U(r, s))$	$Sp(2(p + q)(r + s), \mathbb{R})$		

An irreducible real reductive dual pair  $(G, G')$  of type I is said to be *in the stable range* with  $G$  the smaller member if the defining module of  $G'$  contains an isotropic subspace of the same dimension as that of the defining module of  $G$ . All irreducible real reductive dual pairs  $(G_1, G_2)$  in the stable range are listed in following table.

$G_1$	$G_2$	with $G_1$ smaller	with $G_2$ smaller
$O_p(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$n \geq p$	$p \geq 4n$
$O(p, q)$	$Sp_{2n}(\mathbb{R})$	$n \geq p + q$	$p, q \geq 2n$
$Sp(p, q)$	$O^*(2n)$	$n \geq 2(p + q)$	$p, q \geq n$
$U(p, q)$	$U(r, s)$	$r, s \geq p + q$	$p, q \geq r + s$

Write the two elements of  $\text{Ker}(\widetilde{\mathbf{Sp}} \rightarrow \mathbf{Sp}) = \mu_2$  as  $e$  and  $-e$ , such that  $e = (-e)^2$  is the identity element. A  $(\mathfrak{g}, \widetilde{K})$ -module is called *genuine* if  $-e$  acts on it as the scalar multiplication by  $-1$ . Clearly, an irreducible  $(\mathfrak{g}, \widetilde{K})$ -module  $\pi$  with a non-zero theta lift must be genuine, since  $\omega(-e)$  acts on  $\mathcal{F}$  by the scalar  $-1$ . Conversely, two lemmas hold:

**Lemma 2.3** (Non-vanishing of theta liftings in the stable range [PP08]). *If  $(G, G')$  is an irreducible real reductive dual pair of type I in the stable range with  $G$  the smaller member, then each genuine irreducible  $(\mathfrak{g}, \widetilde{K})$ -module has a non-zero theta lift.*

**Lemma 2.4** ([Mœg89, AB95, LPTZ03]). *For  $(G, G') = (GL_m(F), GL_n(F))$  with  $F = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , if  $n \geq m$ , then each genuine irreducible  $(\mathfrak{g}, \widetilde{K})$ -module has a non-zero theta lift.*

Suppose that the covering  $\widetilde{G} \rightarrow G$  splits, namely, there exists an embedding  $G \hookrightarrow \widetilde{G}$  such that the composition  $G \hookrightarrow \widetilde{G} \rightarrow G$  is the identity map on  $G$ . We may identify  $G$  as a subgroup of  $\widetilde{G}$  via this embedding. Then  $\widetilde{G} = G \times \mu_2$  in the sense that the two subgroups  $G$  and  $\mu_2 = \text{Ker}(\widetilde{G} \rightarrow G)$  commute, generate  $\widetilde{G}$ , and  $G \cap \mu_2 = \{e\}$ . Similarly we have  $\widetilde{K} = K \times \mu_2$ . An irreducible  $(\mathfrak{g}, K)$ -module  $\pi$  gives rise to a genuine irreducible  $(\mathfrak{g}, \widetilde{K})$ -module  $\tilde{\pi}$ , with the same underlying space and actions of  $(\mathfrak{g}, K)$ , while  $\tilde{\pi}(-e)$  acts by the scalar  $-1$ . Similarly, a  $K$ -type  $\sigma$  gives rise to a genuine  $\widetilde{K}$ -type  $\tilde{\sigma}$ , with the same underlying space and actions of  $K$ , while  $\tilde{\sigma}(-e)$  acts by the scalar  $-1$ . Consider the actions of  $K$  on the Fock model  $\mathcal{F}$  via the embedding  $K \hookrightarrow \widetilde{K} = K \times \mu_2$ . Clearly,

$$\sigma \in \mathcal{R}(K, \mathcal{F}) \iff \tilde{\sigma} \in \mathcal{R}(\widetilde{K}, \mathcal{F}).$$

In that case we define the *degree*  $\deg(\sigma)$  for  $\sigma \in \mathcal{R}(K, \mathcal{F})$  by  $\deg(\sigma) = \deg(\tilde{\sigma})$ .

**Proposition 2.5.** *Let  $(G, G')$  be an irreducible real reductive dual pair, either in the stable range or of type II, with  $G$  the smaller member. Suppose that  $\widetilde{G} \rightarrow G$  splits over  $G$ . If  $\pi$  is an irreducible  $(\mathfrak{g}, K)$ -module, and  $\sigma_1, \sigma_2 \in \mathcal{R}(K, \pi)$ , then  $\sigma_1, \sigma_2 \in \mathcal{R}(K, \mathcal{F})$  and  $\deg(\sigma_1) \equiv \deg(\sigma_2) \pmod{2}$ .*

**Proof.** As we said,  $\pi$  gives rise to a genuine irreducible  $(\mathfrak{g}, \widetilde{K})$ -module  $\tilde{\pi}$ , while  $\sigma_1$  and  $\sigma_2 \in \mathcal{R}(K, \pi)$  give rise to  $\tilde{\sigma}_1$  and  $\tilde{\sigma}_2 \in \mathcal{R}(\widetilde{K}, \tilde{\pi})$ . By Lemma 2.3, 2.4,  $\theta(\tilde{\pi}) \neq 0$ . By Theorem 1.5,  $\tilde{\sigma}_1, \tilde{\sigma}_2 \in \mathcal{R}(\widetilde{K}, \mathcal{F})$ , and  $\deg(\tilde{\sigma}_1) \equiv \deg(\tilde{\sigma}_2) \pmod{2}$ . Equivalently, we have  $\sigma_1, \sigma_2 \in \mathcal{R}(K, \mathcal{F})$  and  $\deg(\sigma_1) \equiv \deg(\sigma_2) \pmod{2}$ .  $\blacksquare$

For an irreducible real reductive dual pair  $(G_1, G_2)$ , The following table gives some sufficient conditions for  $\tilde{G}_i \rightarrow G_i$  to split (c.f. [AB95, AB98, Pau98, Ada07]).

$G_1$	$G_2$	$\tilde{G}_1 \rightarrow G_1$ splits	$\tilde{G}_2 \rightarrow G_2$ splits
$O_p(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	always	
$O(p, q)$	$Sp_{2n}(\mathbb{R})$	if $n$ is even	if $p + q$ is even
$Sp(p, q)$	$O^*(2n)$	always	
$U(p, q)$	$U(r, s)$	if $r + s$ is even	if $p + q$ is even
$GL_m(\mathbb{R})$	$GL_n(\mathbb{R})$	if $n$ is even	if $m$ is even
$GL_m(\mathbb{C})$	$GL_n(\mathbb{C})$	always	
$GL_m(\mathbb{H})$	$GL_n(\mathbb{H})$	always	

Let  $(G, G')$  be an irreducible real reductive dual pair with  $\tilde{G} \rightarrow G$  splitting. The following table lists  $\deg(\sigma)$  explicitly for  $\sigma \in \mathcal{R}(K, \mathcal{F})$  (c.f. [Mœg89, AB95, Pau98, LPTZ03, Pau05]).

$G$	$G'$	$K$	$\sigma \in \mathcal{R}(K, \mathcal{F})$	$\deg(\sigma)$
$O_p(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$O(p)$	$(a_1, \dots, a_{\lfloor \frac{p}{2} \rfloor}; \epsilon)$	$\sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} a_i + \frac{1-\epsilon}{2}(p - 2 \{i : a_i > 0\} )$
$Sp_{2n}(\mathbb{C})$	$O_p(\mathbb{C})$	$Sp(n)$	$(a_1, \dots, a_n)$	$\sum_{i=1}^n a_i$
$O(p, q)$	$Sp_{2n}(\mathbb{R})$	$O(p) \times O(q)$	$(a_1, \dots, a_{\lfloor \frac{p}{2} \rfloor}; \epsilon) \otimes (b_1, \dots, b_{\lfloor \frac{q}{2} \rfloor}; \eta)$	$\sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} a_i + \frac{1-\epsilon}{2}(p - 2 \{i : a_i > 0\} ) + \sum_{j=1}^{\lfloor \frac{q}{2} \rfloor} b_j + \frac{1-\eta}{2}(q - 2 \{j : b_j > 0\} )$
$Sp_{2n}(\mathbb{R})$	$O(p, q)$	$U(n)$	$(a_1, \dots, a_n)$	$\sum_{i=1}^n  a_i - \frac{p-q}{2} $
$Sp(p, q)$	$O^*(2n)$	$Sp(p) \times Sp(q)$	$(a_1, \dots, a_p) \otimes (b_1, \dots, b_q)$	$\sum_{i=1}^p a_i + \sum_{j=1}^q b_j$
$O^*(2n)$	$Sp(p, q)$	$U(n)$	$(a_1, \dots, a_n)$	$\sum_{i=1}^n  a_i - p + q $
$U(p, q)$	$U(r, s)$	$U(p) \times U(q)$	$(a_1, \dots, a_p) \otimes (b_1, \dots, b_q)$	$\sum_{i=1}^p  a_i - \frac{r-s}{2}  + \sum_{j=1}^q  b_j + \frac{r-s}{2} $
$GL_m(\mathbb{R})$	$GL_n(\mathbb{R})$	$O(m)$	$(a_1, \dots, a_{\lfloor \frac{m}{2} \rfloor}; \epsilon)$	$\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} a_i + \frac{1-\epsilon}{2}(m - 2 \{i : a_i > 0\} )$
$GL_m(\mathbb{C})$	$GL_n(\mathbb{C})$	$U(m)$	$(a_1, \dots, a_m)$	$\sum_{i=1}^m  a_i $
$GL_m(\mathbb{H})$	$GL_n(\mathbb{H})$	$Sp(m)$	$(a_1, \dots, a_m)$	$\sum_{i=1}^m a_i$

**2.3. Proof of Theorem 2.1.**

By Proposition 2.5, it suffices to find a suitable  $G'$  such that  $(G, G')$  is an irreducible real reductive dual pair satisfying three conditions:

- (1) It is either in the stable range or of type II, with  $G$  the smaller member.
- (2) The covering  $\tilde{G} \rightarrow G$  splits.
- (3) For  $(G, G')$ , the degree  $\deg(\sigma) \equiv \varepsilon(\sigma) \pmod{2}$  for any  $\sigma \in \mathcal{R}(K, \mathcal{F})$ .

We can take  $G'$  according to the following table, which lists explicit sufficient conditions to ensure (1), (2) and (3).

$G$	$G'$	condition (1)	condition (2)	condition (3)
$O_p(\mathbb{C})$	$Sp_{2n}(\mathbb{C})$	$n \geq p$		
$Sp_{2n}(\mathbb{C})$	$O_p(\mathbb{C})$	$p \geq 4n$		
$O(p, q)$	$Sp_{2n}(\mathbb{R})$	$n \geq p + q$	$2 \mid n$	
$Sp_{2n}(\mathbb{R})$	$O(p, q)$	$p, q \geq 2n$	$2 \mid p + q$	$4 \mid p - q$
$Sp(p, q)$	$O^*(2n)$	$n \geq 2(p + q)$		
$O^*(2n)$	$Sp(p, q)$	$p, q \geq n$		$2 \mid p - q$
$U(p, q)$	$U(r, s)$	$r, s \geq p + q$	$2 \mid r + s$	$4 \mid r - s$
$GL_m(\mathbb{R})$	$GL_n(\mathbb{R})$	$n \geq m$	$2 \mid n$	
$GL_m(\mathbb{C})$	$GL_n(\mathbb{C})$	$n \geq m$		
$GL_m(\mathbb{H})$	$GL_n(\mathbb{H})$	$n \geq m$		

**2.4. Generalization.**

Theorem 2.1 can be generalized to all members of (maybe reducible) real reductive dual pairs.

**Theorem 2.6.** *Let  $G$  be a member of a real reductive dual pair, with a maximal compact subgroup  $K$ . If  $\pi$  is an irreducible  $(\mathfrak{g}, K)$ -module, and  $\sigma, \sigma' \in \mathcal{R}(K, \pi)$ , then  $\varepsilon(\sigma) = \varepsilon(\sigma')$ .*

**Proof.** Any real reductive dual pair is a direct sum of irreducible ones, so  $G = G_1 \times G_2 \times \dots \times G_r$  with each  $G_i$  a member of an irreducible real reductive dual pair. Then  $K = K_1 \times \dots \times K_r$  with  $K_i$  a maximal compact subgroup of  $G_i$ . By [GK13],  $\pi = \bigotimes_{i=1}^r \pi_i$ , where  $\pi_i$  is an irreducible  $(\mathfrak{g}_i, K_i)$ -module. Moreover,  $\sigma = \bigotimes_{i=1}^r \sigma_i$  and  $\sigma' = \bigotimes_{i=1}^r \sigma'_i$ , with  $\sigma_i$  and  $\sigma'_i \in \mathcal{R}(K_i, \pi_i)$ . Theorem 2.1 holds for  $(G_i, K_i)$  (up to isomorphisms). So  $\varepsilon(\sigma_i) = \varepsilon(\sigma'_i)$  for all  $i$ . Therefore,  $\varepsilon(\sigma) = \sum_{i=1}^r \varepsilon(\sigma_i) = \sum_{i=1}^r \varepsilon(\sigma'_i) = \varepsilon(\sigma')$ . ■

In the end, please note that the phenomenon of parity preservation of  $K$ -types in an irreducible admissible representation is well-known to experts for many kinds of Lie groups for many years. Many cases can be proved in a more elementary way, without the help of Howe’s theory of theta correspondence. If  $G$  is connected with  $\text{rank}(G) = \text{rank}(K)$  and of type A, C or D (for example  $G = U(p, q)$  or  $Sp(2n, \mathbb{R})$ ), this follows easily from the fact that the difference between any two (highest) weights in an irreducible  $(\mathfrak{g}, K)$ -module is a sum of roots, which are “even” in our sense of parity. However, in other cases, especially when  $G$  is disconnected (so that the definition of “parity” is less natural), our approach makes the phenomenon much clearer.

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