Principal Subspaces for Double Yangian DY(𝔰𝔩₂)

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Abstract. We consider the realization of level 1 infinite-dimensional modules for the double Yangian DY($\mathfrak{sl}_2$) found by K. Iohara. We use the corresponding vertex operators to generate a family of nonlocal h-vertex algebras W_N, $N \in \mathbb{Z}_{\geqslant 0}$. Finally, we construct combinatorial bases of W_N and establish a connection with the sum side of the Rogers–Ramanujan identity.
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Introduction

In [7], B. L. Feigin and A. V. Stoyanovsky introduced the principal subspaces of the integrable highest weight modules for affine Lie algebras; for more information see also [3, 8, 11, 20]. The quasi-particle bases of these subspaces provide an interpretation of the sum sides of various combinatorial identities; see [2, 7, 8]. For example, the principal subspace of the level 1 integrable highest weight $\widehat{\mathfrak{sl}_2}$-module $L \coloneqq L(\Lambda_0)$, with the highest weight vector $v_{\Lambda_0} \in L$, has a basis

$$\mathcal{B}_{W(\Lambda_0)} = \{e(r_m) \cdots e(r_1)v_{\Lambda_0} : r_1 \leqslant -1,\, r_j - r_{j+1} \geqslant 2,\, j = 1, \ldots, m-1,\, m \geqslant 0\},$$

where $e(z) = \sum_{r \in \mathbb{Z}} e(r)z^{-r-1} \in \mathrm{Hom}(L, L((z)))$ denotes the commutative vertex operator associated with the positive simple root of $\mathfrak{sl}_2$. The difference-two condition $r_j - r_{j+1} \geqslant 2$, which is a consequence of the integrability relations on L,

$$e(z)^2 = \left(\frac{\partial}{\partial z}e(z)\right)e(z) = 0 \quad \text{or equivalently} \quad e(z)_{-1}e(z) = e(z)_{-2}e(z) = 0, \quad (1)$$

see [15], establishes an interpretation of the sum side in the Rogers–Ramanujan identity

$$\sum_{n \geqslant 0} \frac{q^{n^2}}{(1-q) \cdots (1-q^n)} = \prod_{n \geqslant 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \quad (2)$$

In this paper, we are interested in finding a vertex-operator theoretic interpretation of the sum side in (2), in terms of the level 1 infinite-dimensional modules for the double Yangian $\mathrm{DY}(\mathfrak{sl}_2)$, whose realization was given by K. Iohara in [9]. The realization provides the explicit formulas for the action of the double Yangian generators on a certain $\mathbb{C}[[h]]$-module $\mathcal{F}_0$. Moreover, these generators can be arranged into formal series $E(z), F(z), H^{\pm}(z)$ in $\mathrm{DY}(\mathfrak{sl}_2)[[z^{\pm 1}]]$, which can be then viewed as elements of $\mathcal{E}_h(\mathcal{F}_0) := \mathrm{Hom}(\mathcal{F}_0, \mathcal{F}_0((z))[[h]])$. Using the approach similar to [16]–[19] we construct the so-called *nonlocal h-vertex algebras* $W_{N,t} \subset \mathcal{E}_h(\mathcal{F}_0)[t]$, $N \in \mathbb{Z}_{\geqslant 0}$, generated by the (nonlocal) vertex operator $E(z)$ in $\mathcal{E}_h(\mathcal{F}_0)$. The additional variable t is introduced in order to simplify the definition of the vertex operator products on $W_{N,t}$. However, our original goal is to consider nonlocal h-vertex algebras in $\mathcal{E}_h(\mathcal{F}_0)$ which are obtained by evaluating the elements of $W_{N,t}$ at $t = 1$.

In contrast to more general constructions of *quantum vertex algebras* in [6], [16]–[19], we employ the h-derivation $\partial_h/\partial_h z \colon a(z) \mapsto (a(z+h) - a(z))/h$ to introduce the new products of vertex operators, so that we obtain the following analogue of (1) in $W_{0,t}$:

$$E(z)^2 = \left(\frac{\partial_h}{\partial_h z} E(z) \right) E(z+h) = 0 \quad \text{or equivalently}$$
$$E(z)_{-1}E(z) = E(z)_{-2}E(z) = 0,$$

cf. quantum integrability [4, 5]. The relations of the above form enable us to construct bases of $W_{N,t}$ and consequently, to show that the $\mathbb{C}[[h]]$-module $W_N := W_{N,t}\,|_{t=1}$ inherits the nonlocal h-vertex algebra structure from $W_{N,t}$. Also, we immediately obtain the corresponding bases $\mathcal{B}_{W_N}$ of W_N. Finally, the suitably defined character for W_0 coincides with the sum side in (2). The similar approach was adopted in [12, 13], where the difference-two conditions in the quantum affine algebra setting were interpreted via q^2-derivation $b(z) \mapsto (b(zq^2) - b(z))/z(q^2 - 1)$.

In the end, motivated by the presentations of principal subspaces in [3, 21], we use the bases $\mathcal{B}_{W_N}$ to determine the kernels of certain natural nonlocal h-vertex algebra maps

$$\ldots \xrightarrow{f_{N+2}} W_{N+1} \xrightarrow{f_{N+1}} W_N \xrightarrow{f_N} \ldots \xrightarrow{f_3} W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} W_0.$$

More precisely, we prove that the maps f_N for $N \geqslant 3$ are isomorphisms while the maps f_2 and f_1 are epimorphisms whose kernels are generated by $E(z)_{-1}E(z)$ and $E(z)_{-2}E(z)$ respectively.

1. Nonlocal h-vertex algebras

The goal of this section is to prove Theorem 1.11 which states that any quasi-commutative set of vertex operators generates a nonlocal h-vertex algebra.

1.1. Preliminaries. Let V be a $\mathbb{C}[[h]]$-module. The *h-adic topology* on V is a topology generated by the basis $v + h^n V$, $v \in V$, $n \in \mathbb{Z}_{\geqslant 0}$. Recall that V is said to be *separated* if $\cap_{n \geqslant 1} h^n V = 0$ and that V is said to be *torsion-free* if $hv \neq 0$ for

all nonzero $v \in V$; see [10] for more details. Denote by $\mathcal{E}_h(V)$ the $\mathbb{C}[[h]]$-module consisting of the operators

$$a(z) = \sum_{r \in \mathbb{Z}} a_r z^{-r-1} \in (\operatorname{End} V)\,[[z^{\pm 1}]] \tag{1.3}$$

such that for every $b \in V$ we have, with respect to the h-adic topology, $a_r b \to 0$ when $r \to \infty$. More precisely, the element $a(z) \in (\operatorname{End} V)\,[[z^{\pm 1}]]$ belongs to $\mathcal{E}_h(V)$ if for every $b \in V$ and nonnegative integer n there exists an integer m such that $a_r b \in h^n V$ for all $r \geqslant m$. We will denote by $\operatorname{Hom}_h(V, V((z_1, \ldots, z_m)))$ the $\mathbb{C}[[h]]$-module $\operatorname{Hom}(V, V((z_1, \ldots, z_m))[[h]])$. In particular, we have $\mathcal{E}_h(V) = \operatorname{Hom}_h(V, V((z)))$. Denote by $\mathcal{E}(V)$ the $\mathbb{C}[[h]]$-submodule $\operatorname{Hom}(V, V((z))) \subset \mathcal{E}_h(V)$. Observe that if element (1.3) belongs to $\mathcal{E}(V)$, then for any $b \in V$ we have $a_r b = 0$ for sufficiently large integer r. The next definition was motivated by [1, Definition 3.3] and [18, Definition 2.9].

Definition 1.1. A *nonlocal h-vertex algebra* is a triple $(V, Y, \mathbf{1})$, where V is a separated and torsion-free $\mathbb{C}[[h]]$-module with a $\mathbb{C}[[h]]$-module map

$$Y(\cdot, z_0)\colon V \to \operatorname{Hom}(V, V((z_0))) \tag{v1}$$

$$a \mapsto Y(a, z_0) = \sum_{r \in \mathbb{Z}} a_r z_0^{-r-1}$$

and a distinguished element $\mathbf{1}$ in V such that the following conditions hold:

$$Y(\mathbf{1}, z_0)a = a \quad \text{for all } a \in V, \tag{v2}$$

$$Y(a, z_0)\,\mathbf{1} \in V[[z_0]] \quad \text{and} \quad \lim_{z_0 \to 0} Y(a, z_0)\,\mathbf{1} = a \quad \text{for all } a \in V, \tag{v3}$$

$$Y(a, z_0 + z_2)Y(b, z_2)c = Y(Y(a, z_0)b, z_2)c \quad \text{for all } a, b, c \in V. \tag{v4}$$

Note that (v4) is equivalent to

$$(a_r b)_s\, c = \sum_{l \geqslant 0} \binom{l - r - 1}{l} a_{r-l}\,(b_{s+l}c) \quad \text{for all } a, b, c \in V,\ r, s \in \mathbb{Z}. \tag{1.4}$$

Requirement (v1) implies that the infinite sum in (1.4) contains only finitely many nonzero summands, so that the right-hand side in (v4) is well-defined. Even though the notion of (h-adic) associativity is usually expressed somewhat differently and also, more generally, see for example [6, 18], its particular form in Definition 1.1 will be suitable for the setting of this paper.

In parallel with vertex algebra theory, see for example [14, Chapter 3], one can straightforwardly introduce the following notions:

Definition 1.2. An *ideal* of a nonlocal h-vertex algebra V is any $\mathbb{C}[[h]]$-submodule I of V, which satisfies $I = \{v \in V : h^n v \in I \text{ for some } n \geqslant 0\}$, such that $a_r v$ and $v_r a$ belong to I for all $a \in I$, $v \in V$ and $r \in \mathbb{Z}$.

Definition 1.3. A *homomorphism* of nonlocal h-vertex algebras $(V_1, Y_1, \mathbf{1}^{(1)})$ and $(V_2, Y_2, \mathbf{1}^{(2)})$ is a $\mathbb{C}[[h]]$-module map $f\colon V_1 \to V_2$ such that

$$f(\mathbf{1}^{(1)}) = \mathbf{1}^{(2)} \quad \text{and} \quad f(a_r b) = f(a)_r f(b) \quad \text{for all } a, b \in V_1,\ r \in \mathbb{Z}.$$

Monomorphism, epimorphism and isomorphism of nonlocal h-vertex algebras can be defined as usual. The kernel of any nonlocal h-vertex algebra homomorphism $f \colon V \to W$ is an ideal in V and the quotient $\mathbb{C}[[h]]$-module $V/\ker f$ is a nonlocal h-vertex algebra. Clearly, if f is an epimorphism, nonlocal h-vertex algebras $V/\ker f$ and W are isomorphic.

1.2. Quasi-commutativity and h-derivation. From now on, we assume that V is an arbitrary *topologically free* $\mathbb{C}[[h]]$-module, i.e. a $\mathbb{C}[[h]]$-module which is separated, torsion-free and complete with respect to the h-adic topology. The next definition can be viewed as a special case of *quasi compatibility* [16, Definition 2.10], extended on $\mathcal{E}_h(V)[t]$.

Definition 1.4. A sequence $(\sum_{j=0}^{n_1} a_{1,j}(z)t^j, \ldots, \sum_{j=0}^{n_m} a_{m,j}(z)t^j)$ in $\mathcal{E}_h(V)[t]$ is said to be *quasi-commutative* if for any choice of $(j_1, \ldots, j_m)$ such that $0 \leqslant j_k \leqslant n_k$ we have

$$a_{1,j_1}(z_1)a_{2,j_2}(z_2) \cdots a_{m,j_m}(z_m) \in \mathrm{Hom}_h(V, V((z_1, \ldots, z_m))). \qquad (1.5)$$

A subset $\mathcal{S}$ of $\mathcal{E}_h(V)[t]$ is said to be *quasi-commutative* if every finite sequence in $\mathcal{S}$ is quasi-commutative.

Let $b(z_1, \ldots, z_m)$ be an element of $\mathrm{Hom}_h(V, V((z_1, \ldots, z_m)))$. By applying $b(z_1, \ldots, z_m)$ on an arbitrary $v \in V$ we obtain a series in $V((z_1, \ldots, z_m))[[h]]$. For any nonnegative integer l the coefficient of h^l in this series belongs to $V((z_1, \ldots, z_m))$. Therefore, we can replace the variables $z_1, \ldots, z_m$ in this coefficient by z, thus getting an element of $V((z))$. Consequently, we can replace the variables $z_1, \ldots, z_m$ in $b(z_1, \ldots, z_m)$ by z, thus obtaining the element $c(z) = b(z, \ldots, z) \in \mathcal{E}_h(V)$. We will denote such substitution by

$$c(z) = \lim_{z_1, \ldots, z_m \to z} b(z_1, \ldots, z_m) \quad \text{or, more briefly, by} \quad c(z) = b(z, \ldots, z).$$

In particular, we can apply $\lim_{z_1, \ldots, z_m \to z}$ on (1.5), thus getting an element of $\mathcal{E}_h(V)$.

For any $a(z)$ in $\mathcal{E}_h(V)$ define its h-derivation by

$$\frac{\partial_h}{\partial_h z}a(z) = \frac{a(z+h) - a(z)}{h}. \qquad (1.6)$$

On the right-hand side, as well as in the rest of this paper, we use the usual expansion convention: the negative powers of $(w_1 + \ldots + w_n)$ are expanded in nonnegative powers of $w_2, \ldots, w_n$. Hence, in particular, $(z+h)^n$ for $n \leqslant -1$ is viewed as an element of $\mathbb{C}[z^{-1}][[h]]$. The numerator on the right-hand side of (1.6) is divisible by h, so, since V is h-adically complete, we see that $(\partial_h/\partial_h z)a(z)$ belongs to $\mathcal{E}_h(V)$. In order to simplify our notation we will often denote $(\partial_h^r/\partial_h z^r)a(z)$ by $a^{(r)}(z)$.

Define the operator R_{z_j} on $\mathrm{Hom}_h(V, V((z_1, \ldots, z_m)))$ by

$$R_{z_j} \colon b(z_1, \ldots, z_{j-i}, z_j, z_{j+1}, \ldots, z_m) \mapsto b(z_1, \ldots, z_{j-i}, z_j + h, z_{j+1}, \ldots, z_m).$$

The $\mathbb{C}[[h]]$-module $\mathrm{Hom}_h(V, V((z_1, \ldots, z_m)))$ is invariant under the actions of the $\mathbb{C}[[h]]$-linear operators $\partial_h/\partial_h z_j$ and R_{z_j}. Hence, for an arbitrary quasi-commutative sequence $(a_1(z,t), \ldots, a_m(z,t))$ in $\mathcal{E}_h(V)[t]$, the sequence obtained

by replacing any entry $a_j(z,t)$ by $(\partial_h/\partial_h z)a_j(z,t)$ or $R_z a_j(z,t)$ is again quasi-commutative.

Proposition 1.5.   *For any quasi-commutative pair* $(a(z), b(z))$ *in* $\mathcal{E}_h(V)$ *and for any positive integer* r *we have*

(a) $$a^{(r)}(z) = \frac{1}{h^r} \sum_{l=0}^{r} \binom{r}{l} (-1)^l a(z + (r-l)h) \in \mathcal{E}_h(V),$$

(b) $$(a(z)b(z))^{(r)} = \sum_{l=0}^{r} \binom{r}{l} a^{(l)}(z) b^{(r-l)}(z + lh) \in \mathcal{E}_h(V).$$

1.3. Construction of nonlocal h-vertex algebras.   We now use h-derivation (1.6) to introduce the r-th products among quasi-commutative operators:

Definition 1.6.   Let N be a nonnegative integer and let

$$(a(z,t), b(z,t)) = \left(\sum_{i=0}^{m} a_i(z)t^i, \sum_{j=0}^{n} b_j(z)t^j \right)$$

be a quasi-commutative pair in $\mathcal{E}_h(V)[t]$. For any integer r define the element $a(z,t)_{(-r-1,N)}b(z,t)$ by

$$a(z,t)_{(-r-1,N)}b(z,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} \left(a_i(z)t^i \right)_{(-r-1,N)} \left(b_j(z)t^j \right),$$

where for $r < 0$ we set $a(z,t)_{(-r-1,N)}b(z,t) = 0$ while for $r \geqslant 0$ we define

$$(a_i(z)t^i)_{(-r-1,N)}(b_j(z)t^j) = \frac{1}{r!} \lim_{z_1 \to z} a_i^{(r)}(z_1) b_j(z + (i+r+N)h) t^{i+j+r+N}.$$

Note that the element $a(z,t)_{(-r-1,N)}b(z,t)$, as defined above, belongs to $\mathcal{E}_h(V)[t]$ due to the discussion preceding Proposition 1.5. In order to simplify our notation, we will often denote the expression $a(z,t)_{(r,N)}b(z,t)$ by $a(z,t)_r b(z,t)$. Even though the given product depends on the choice of the nonnegative integer N, this should not cause any confusion because the value of N will be clearly specified in the context. For example, throughout the rest of this section we assume that N is a fixed nonnegative integer.

We should say that different products of (nonlocal) vertex operators which also involve the shift of their argument were already studied earlier; see e.g. [17].

Lemma 1.7.   *Let* $(a_1(z,t), \ldots, a_m(z,t))$ *and* $(a_k(z,t), a_{k+1}(z,t))$ *for some* $k = 1, \ldots, m-1$ *be two quasi-commutative sequences in* $\mathcal{E}_h(V)[t]$. *Then for every integer* r *the sequence*

$$(a_1(z,t), \ldots, a_{k-1}(z,t), a_k(z,t)_{-r-1}a_{k+1}(z,t), a_{k+2}(z,t), \ldots, a_m(z,t))$$

is quasi-commutative.

Proof.   Although the proof is similar to more general [16, Proposition 2.18], see also [13, Lemma 2.8], we write some details in order to take care of the variations in definition of vertex operator products. The statement of lemma clearly holds for $r < 0$, so suppose that r is a nonnegative integer. Due to Definition 1.6 it is sufficient to consider a sequence of homogeneous elements $(a_1(z)t^{\alpha_1}, \ldots, a_m(z)t^{\alpha_m})$ in $\mathcal{E}_h(V)[t]$ satisfying

$$a_k(z_k)a_{k+1}(z_{k+1}) \in \operatorname{Hom}_h(V, V((z_k, z_{k+1}))) \qquad \text{and} \qquad (1.7)$$

$$a_1(z_1) \cdots a_m(z_m) \in \operatorname{Hom}_h(V, V((z_1, \ldots, z_m))). \qquad (1.8)$$

Due to part (a) of Proposition 1.5 the product $(a_k(z)t^{\alpha_k})_{-r-1}(a_{k+1}(z)t^{\alpha_{k+1}})$ can be written as a $\mathbb{C}[h^{-1}]$-linear combination of $(a_k(z+lh)t^{\alpha_k+r})_{-1}(a_{k+1}(z)t^{\alpha_{k+1}})$, where $l = 0, \ldots, r$. Therefore, it is sufficient to prove that the sequence

$$(a_1(z)t^{\alpha_1}, \ldots, a_{k-1}(z)t^{\alpha_{k-1}}, (a_k(z+lh)t^{\alpha_k+r})_{-1}(a_{k+1}(z)t^{\alpha_{k+1}}),$$
$$a_{k+2}(z)t^{\alpha_{k+2}}, \ldots, a_m(z)t^{\alpha_m})$$

is quasi-commutative for all $l = 0, \ldots, r$. Since

$$(a_k(z+lh)t^{\alpha_k+r})_{-1}(a_{k+1}(z)t^{\alpha_{k+1}})$$
$$= \lim_{z_1 \to z} a_k(z_1 + lh)a_{k+1}(z + (\alpha_k + r + N)h)t^{\alpha_k + \alpha_{k+1} + r + N}$$
$$= a_k(z + lh)a_{k+1}(z + (\alpha_k + r + N)h)t^{\alpha_k + \alpha_{k+1} + r + N},$$

we only have to verify that for any $l = 0, \ldots, r$ the expression

$$a_1(z_1) \cdots a_{k-1}(z_{k-1})a_k(z_k + lh)a_{k+1}(z_k + (\alpha_k + r + N)h)a_{k+2}(z_{k+2}) \cdots a_m(z_m)$$

belongs to $\operatorname{Hom}_h(V, V((z_1, \ldots, z_k, z_{k+2}, \ldots, z_m)))$. This follows from (1.7) and (1.8). ∎

For any quasi-commutative pair $(a(z,t), b(z,t))$ in $\mathcal{E}_h(V)[t]$ define

$$Y(a(z,t), z_0)b(z,t) = \sum_{r \in \mathbb{Z}} a(z,t)_r b(z,t) z_0^{-r-1} \in (\mathcal{E}_h(V)[t])[[z_0]]. \qquad (1.9)$$

Let $\mathbf{1} \in \mathcal{E}_h(V)[t]$ be the identity $V \to V$. The next definition is motivated by the similar notion introduced in [18].

Definition 1.8.   A quasi-commutative set $\mathcal{S} \subset \mathcal{E}_h(V)[t]$ is said to be Y-*closed* if $a(z,t)_r b(z,t)$ belongs to $\mathcal{S}$ for all $a(z,t), b(z,t) \in \mathcal{S}$, $r \in \mathbb{Z}$.

The next corollary is a direct consequence of Definition 1.4 and Lemma 1.7.

Corollary 1.9.   *All maximal quasi-commutative $\mathbb{C}[[h]]$-submodules of $\mathcal{E}_h(V)[t]$ are Y-closed and contain $\mathbf{1}$.*

Lemma 1.10.   *If M is a maximal quasi-commutative $\mathbb{C}[[h]]$-submodule of $\mathcal{E}_h(V)[t]$, then the triple $(M, Y, \mathbf{1})$ is a nonlocal h-vertex algebra with vertex operator map Y given by (1.9).*

Proof.   Since M is a $\mathbb{C}[[h]]$-submodule of $\mathcal{E}_h(V)[t]$, it is clearly separated and torsion-free. By Corollary 1.9, M is Y-closed and contains $\mathbf{1}$. Axiom (v1) clearly holds since the expression in (1.9) possesses only nonnegative powers of the variable z_0. The fact that axioms (v2)–(v4) hold on M can be easily verified by a direct calculation. For example, let us prove (v4).

Let $a(z)t^\alpha$, $b(z)t^\beta$ and $c(z)t^\gamma$ be three homogeneous elements in M. Suppose that $N = 0$. First, we calculate $Y\left(Y(a(z)t^\alpha, z_0)b(z)t^\beta, z_2\right)c(z)t^\gamma$. Using (1.9) we get

$$\sum_{r,s\geqslant 0} \frac{t^{\alpha+\beta+\gamma+r+s}}{r!s!} \left(a^{(r)}(z)b(z+(\alpha+r)h)\right)^{(s)} c(z+(\alpha+\beta+r+s)h)z_0^r z_2^s. \quad (1.10)$$

By part (b) of Proposition 1.5 this equals to

$$\sum_{r,s\geqslant 0}\sum_{l=0}^{s} \frac{t^{\alpha+\beta+\gamma+r+s}}{r!l!(s-l)!} \left(a^{(r+l)}(z)b^{(s-l)}(z+(\alpha+r+l)h)\right) c(z+(\alpha+\beta+r+s)h)z_0^r z_2^s. \quad (1.11)$$

Next, we calculate $Y(a(z)t^\alpha, z_0+z_2)Y(b(z)t^\beta, z_2)c(z)t^\gamma$. Using (1.9) we get

$$\sum_{u,v\geqslant 0} \frac{t^{\alpha+\beta+\gamma+u+v}}{u!v!}a^{(u)}(z)\left(b^{(v)}(z+(\alpha+u)h)c(z+(\alpha+\beta+u+v)h)\right)(z_0+z_2)^u z_2^v. \quad (1.12)$$

By expanding $(z_0+z_2)^u$ we obtain

$$\sum_{u,v\geqslant 0}\sum_{l=0}^{u} \frac{t^{\alpha+\beta+\gamma+u+v}}{(u-l)!l!v!}a^{(u)}(z)\left(b^{(v)}(z+(\alpha+u)h)c(z+(\alpha+\beta+u+v)h)\right)z_0^{u-l} z_2^{v+l}. \quad (1.13)$$

Since the sequences $(a(z)t^\alpha, b(z)t^\beta, c(z)t^\gamma)$, $(a(z)t^\alpha, b(z)t^\beta)$ and $(b(z)t^\beta, c(z)t^\gamma)$ are quasi-commutative, the expressions

$$\lim_{z_3\to z}\left(\lim_{z_2\to z}\left(a^{(r+l)}(z)b^{(s-l)}(z_2+(\alpha+r+l)h)\right) c(z_3+(\alpha+\beta+r+s)h)\right), \quad (1.14)$$

$$\lim_{z_2\to z}\left(a^{(r+l)}(z)\lim_{z_3\to z_2}\left(b^{(s-l)}(z_2+(\alpha+r+l)h)c(z_3+(\alpha+\beta+r+s)h)\right)\right) \quad (1.15)$$

are (well-defined and) equal. Furthermore, both (1.14) and (1.15) are equal to

$$\lim_{z_1,z_2,z_3\to z}\left(a^{(r+l)}(z_1)b^{(s-l)}(z_2+(\alpha+r+l)h)c(z_3+(\alpha+\beta+r+s)h)\right).$$

Thus, by applying the substitutions $r = u-l$ and $s = v+l$ on (1.11), we see that (1.11) and (1.13) coincide, so (v4) follows. Finally, for positive N, the proof of (v4) is immediately obtained by replacing α by $\alpha+N$ and β by $\beta+N$ in (1.10)–(1.15). ∎

Let $\mathcal{S}$ be a quasi-commutative subset of $\mathcal{E}_h(V)[t]$. By Zorn's Lemma, $\mathcal{S}$ is contained in a maximal quasi-commutative $\mathbb{C}[[h]]$-submodule M of $\mathcal{E}_h(V)[t]$. By Lemma 1.10, M is a nonlocal h-vertex algebra. Denote by $\langle\mathcal{S}\rangle$ the intersection of all nonlocal h-vertex algebras which contain $\mathcal{S}\cup\{\mathbf{1}\}$ and whose vertex operator map satisfies (1.9). We have

Theorem 1.11. $(\langle \mathcal{S} \rangle, Y, \mathbf{1})$ *is a nonlocal h-vertex algebra.*

Note that

$$a_{r_1}^{(1)} a_{r_2}^{(2)} \ldots a_{r_m}^{(m)} \mathbf{1} \in \langle \mathcal{S} \rangle \quad \text{for all } r_1, \ldots, r_m \in \mathbb{Z}, \, a^{(1)}, \ldots, a^{(m)} \in \mathcal{S}, \, m \geqslant 0. \quad (1.16)$$

On the other hand, due to (1.4), the $\mathbb{C}[[h]]$-span of all elements of the form (1.16) is a nonlocal h-vertex algebra. Hence, we have

$$\langle \mathcal{S} \rangle = \operatorname*{span}_{\mathbb{C}[[h]]} \left\{ a_{r_1}^{(1)} a_{r_2}^{(2)} \ldots a_{r_m}^{(m)} \mathbf{1} \, : \, r_1, \ldots, r_m \in \mathbb{Z}, \, a^{(1)}, \ldots, a^{(m)} \in \mathcal{S}, \, m \geqslant 0 \right\}.$$
$$(1.17)$$

2. Nonlocal h-vertex algebras associated with double Yangian $\mathrm{DY}(\mathfrak{sl}_2)$

In this section, we use Theorem 1.11 to construct examples of nonlocal h-vertex algebras associated with Iohara's realization of certain $\mathrm{DY}(\mathfrak{sl}_2)$-modules, as well as their combinatorial bases; see Theorem 2.13 and Theorem 2.15.

2.1. Bosonic representations of $\mathrm{DY}(\mathfrak{sl}_2)$. We follow [9] to introduce the double Yangian for $\mathfrak{sl}_2$ and its level 1 infinite-dimensional modules. The *double Yangian* $\mathrm{DY}(\mathfrak{sl}_2)$ is defined as the h-adically completed associative algebra over $\mathbb{C}[[h]]$ generated by the coefficients of the series

$$H^+(z) = 1 + h \sum_{k \geqslant 0} h_k z^{-k-1}, \qquad H^-(z) = 1 - h \sum_{k < 0} h_k z^{-k-1},$$

$$E(z) = \sum_{k \in \mathbb{Z}} e_k z^{-k-1}, \qquad F(z) = \sum_{k \in \mathbb{Z}} f_k z^{-k-1}$$

and the central element c, subject to the defining relations

$$[H^\pm(z), H^\pm(w)] = 0, \tag{2.1}$$

$$(z_\mp - w_\pm + h)(z_\pm - w_\mp - h) H^\pm(z) H^\mp(w)$$
$$= (z_\mp - w_\pm - h)(z_\pm - w_\mp + h) H^\mp(w) H^\pm(z), \tag{2.2}$$

$$H^\pm(z)^{-1} E(w) H^\pm(z) = \frac{z_\pm - w - h}{z_\pm - w + h} E(w), \tag{2.3}$$

$$H^\pm(z) F(w) H^\pm(z)^{-1} = \frac{z_\mp - w - h}{z_\mp - w + h} F(w), \tag{2.4}$$

$$(z - w - h) E(z) E(w) = (z - w + h) E(w) E(z), \tag{2.5}$$

$$(z - w + h) F(z) F(w) = (z - w - h) F(w) F(z), \tag{2.6}$$

$$[E(z), F(w)] = \frac{1}{h} \left(\delta(z_- - w_+) H^+(z_-) - \delta(z_+ - w_-) H^-(w_-) \right), \tag{2.7}$$

where $z_\pm = z \pm hc/4$ and $\delta(z - w) = \sum_{k \in \mathbb{Z}} z^{-k-1} w^k$ is a delta function.

Let $\mathfrak{h} = \mathbb{C}\alpha^\vee$, where $\alpha^\vee = \operatorname{diag}(1, -1)$, be Cartan subalgebra of $\mathfrak{sl}_2$, let $Q = \mathbb{Z}\alpha$ be the root lattice generated by the simple root $\alpha \in \mathfrak{h}^*$, $\alpha \colon \alpha^\vee \mapsto 2$, and

let $P = \mathbb{Z}\lambda$ be the weight lattice generated by the fundamental weight $\lambda = \alpha/2$. Denote by $(\cdot,\cdot)$ the standard bilinear form on $\mathfrak{h}^*$ defined by $(\alpha,\alpha) = 2$. Denote by $\mathfrak{s}$ the Heisenberg algebra generated by the elements a_k, $k \in \mathbb{Z}$, $k \neq 0$, and the central element c subject to the relations

$$[a_k, a_l] = k\delta_{k+l\,0}c \quad \text{for } k,l \in \mathbb{Z},\, k,l \neq 0.$$

Consider the h-adically completed tensor product

$$\mathcal{F}_i := \mathbb{C}[[h]][a_k \,:\, k \in \mathbb{Z}_{<0}] \otimes \mathbb{C}[[h]][Q]e^{\lambda_i}, \quad i = 0, 1,$$

where $\mathbb{C}[[h]][Q]$ is the group algebra of Q over $\mathbb{C}[[h]]$ and $\lambda_i = \delta_{i0}\lambda$. Define the action of the elements c, a_k, ∂_μ, e^μ, where $k \in \mathbb{Z}, k \neq 0$, and $\mu, \nu \in P$, on $\mathcal{F}_i$ by

$$c \cdot (f \otimes e^\nu) = f \otimes e^\nu, \qquad a_k \cdot (f \otimes e^\nu) = \begin{cases} a_k f \otimes e^\nu & \text{for } k < 0, \\ [a_k, f] \otimes e^\nu & \text{for } k > 0, \end{cases} \qquad (2.8)$$

$$\partial_\mu \cdot (f \otimes e^\nu) = (\mu, \nu) f \otimes e^\nu \qquad \text{and} \qquad e^\mu \cdot (f \otimes e^\nu) = f \otimes e^{\mu+\nu}. \qquad (2.9)$$

By action (2.8) $\mathcal{F}_i$ becomes a level 1 module for the Heisenberg algebra $\mathfrak{s}$. Set $c = 1$, so that $z_\pm = z \pm h/4$ and introduce the following notation:

$$E_-^\pm(z) = \exp\left(\pm\sum_{k>0} \frac{a_{-k}}{k}\left(\left(z_\mp + \frac{h}{2}\right)^k + \left(z_\mp - \frac{h}{2}\right)^k\right)\right),$$

$$E_+^\pm(z) = \exp\left(\mp\sum_{k>0} \frac{a_k}{k} z_\pm^{-k}\right).$$

We recall the construction of level 1 $\mathrm{DY}(\mathfrak{sl}_2)$-modules from [9, Theorem 4.7]:

Theorem 2.1. *The following assignments define a $\mathrm{DY}(\mathfrak{sl}_2)$-module structure on $\mathcal{F}_i$:*

$$H^+(z) \mapsto \exp\left(-\sum_{k>0} \frac{a_k}{k}\left(\left(z + \frac{h}{2}\right)^{-k} - \left(z - \frac{h}{2}\right)^{-k}\right)\right)\left(\frac{z - \frac{h}{2}}{z + \frac{h}{2}}\right)^{-\partial_\alpha},$$

$$H^-(z) \mapsto \exp\left(-\sum_{k>0} \frac{a_{-k}}{k}\left((z + h)^k - (z - h)^k\right)\right),$$

$$E(z) \mapsto E_-^+(z)E_+^+(z)e^\alpha\left(z + \frac{h}{4}\right)^{\partial_\alpha},$$

$$F(z) \mapsto E_-^-(z)E_+^-(z)e^{-\alpha}\left(z - \frac{h}{4}\right)^{-\partial_\alpha}.$$

We will also need the operator

$$\Phi_-(z) = \exp\left(\sum_{k>0} \frac{a_{-k}}{k}\left(z + \frac{3}{4}h\right)^k\right) \qquad (2.10)$$

which appears in the construction of the type 1 intertwiners in [9, Theorem 4.8]. In order to simplify our notation, we will denote $E_\pm^+(z)$ by $E_\pm(z)$. In the next proposition, the operators $E_\pm(z)$ and $\Phi_-(z)$ are viewed as the elements of $\mathcal{E}_h(\mathcal{F})$, where $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$.

Proposition 2.2. *The following relations hold on $\mathcal{F}$:*

$$E_+(z_1)E_-(z_2) = \frac{(z_1 - z_2)(z_1 - z_2 + h)}{(z_1 + h/4)^2} E_-(z_2)E_+(z_1), \tag{2.11}$$

$$E_+(z_1)\Phi_-(z_2) = \frac{z_1 - z_2 - h/2}{z_1 + h/4}\Phi_-(z_2)E_+(z_1). \tag{2.12}$$

Furthermore, we have

$$E(z)E(z) = E(z)E(z + h) = 0, \tag{2.13}$$
$$E(z)\Phi_-(z - h/2) = 0. \tag{2.14}$$

Proof. Formulas (2.11) and (2.12) can be verified by a direct calculation. They imply, together with Theorem 2.1 and (2.10), that $E(z_1)E(z_2)$ and $E(z_1)\Phi_-(z_2)$ belong to $\mathrm{Hom}_h(\mathcal{F}, \mathcal{F}((z_1, z_2)))$, so the pairs $(E(z), E(z))$ and $(E(z), \Phi_-(z))$ in $\mathcal{E}_h(\mathcal{F})$ are quasi-commutative. Hence, the expressions

$$\lim_{z_1, z_2 \to z} E(z_1)E(z_2), \quad \lim_{z_1, z_2 \to z} E(z_1)E(z_2 + h) \quad \text{and} \quad \lim_{z_1, z_2 \to z} E(z_1)\Phi_-(z_2 - h/2) \tag{2.15}$$

are well-defined elements of $\mathcal{E}_h(\mathcal{F})$. Finally, we conclude by (2.11) and (2.12) that the operators in (2.15) are equal to zero, thus proving (2.13) and (2.14). ∎

Set $\Psi(z) = \Phi_-(z - h/2)e^\lambda \in \mathcal{E}_h(\mathcal{F})$. Relation (2.12) implies

$$E(z_1)\Psi(z_2) = (z_1 - z_2)\Psi(z_2)E(z_1) \tag{2.16}$$

on $\mathcal{F}$. By applying $\partial_h^n/\partial_h z_1^n$ on (2.16) and using part (b) of Proposition 1.5 we obtain

Proposition 2.3. *For any nonnegative integer n the following relation holds on $\mathcal{F}$:*

$$E^{(n)}(z_1)\Psi(z_2) = \Psi(z_2)\left(nE^{(n-1)}(z_1) + (z_1 - z_2 + nh)E^{(n)}(z_1)\right). \tag{2.17}$$

2.2. Nonlocal h-vertex algebra W_0. Define $\mathcal{S} = \{E(z)\} \subset \mathcal{E}_h(\mathcal{F}_0)$.

Lemma 2.4. *The set $\mathcal{S}$ is quasi-commutative.*

Proof. We will prove by induction over n that the expression $E(z_1) \cdots E(z_n)$ belongs to $\mathrm{Hom}_h(\mathcal{F}_0, \mathcal{F}_0((z_1, \ldots, z_n)))$. Suppose that this holds for some $n \geqslant 1$. Then, we have

$$E(z_1) \cdots E(z_{n+1}) \in \mathrm{Hom}_h(\mathcal{F}_0, \mathcal{F}_0((z_1))((z_2, \ldots, z_{n+1}))). \tag{2.18}$$

Due to (2.9) and (2.11), we can write $E(z_1) \cdots E(z_{n+1})$ as

$$E(z_1) \cdots E(z_{n+1}) = P(z_1, \ldots, z_{n+1}) E_-(z_1) e^\alpha E(z_2) \cdots E(z_{n+1}) E_+(z_1)(z_1 + h/4)^{\partial_\alpha}$$
(2.19)

for some polynomial $P(z_1, \ldots, z_{n+1})$ in $\mathbb{C}[h, z_1, \ldots, z_{n+1}]$. Since (2.19) is an element of $\mathrm{Hom}_h(\mathcal{F}_0, \mathcal{F}_0((z_2, \ldots, z_{n+1}))((z_1)))$, we conclude by (2.18) that the product $E(z_1) \cdots E(z_{n+1})$ belongs to $\mathrm{Hom}_h(\mathcal{F}_0, \mathcal{F}_0((z_1, \ldots, z_{n+1})))$, as required. ∎

Let N be a nonnegative integer. Recall Lemma 2.4 and Theorem 1.11. Define

$$W_{N,t} := \langle \mathcal{S} \rangle \subset \mathcal{E}_h(\mathcal{F}_0)[t]$$

to be the nonlocal h-vertex algebra with vertex operator map

$$Y^{(N)}(a(z,t), z_0) b(z,t) = \sum_{r \in \mathbb{Z}} a(z,t)_{(r,N)} b(z,t) z_0^{-r-1} \quad \text{for all } a(z,t), b(z,t) \in W_{N,t}.$$
(2.20)

Moreover, associativity (v4), which can be written as

$$\left(a_{(r,N)} b \right)_{(s,N)} c = \sum_{l \geqslant 0} \binom{l-r-1}{l} a_{(r-l,N)} \left(b_{(s+l,N)} c \right), \quad a, b, c \in W_{N,t}, \ r, s \in \mathbb{Z},$$
(2.21)

implies that $W_{N,t}$ is a unital associative algebra with unit $\mathbf{1}$ and the product given by $a \cdot b = a_{(-1,N)} b$ for all $a, b \in W_{N,t}$.

Throughout the rest of this subsection we assume that N is equal to zero and we denote the products $a(z,t)_{(r,0)} b(z,t)$ of the elements $a(z,t), b(z,t) \in W_{0,t}$ by $a(z,t)_r b(z,t)$.

Due to (1.17), nonlocal h-vertex algebra $W_{0,t}$ is equal to the $\mathbb{C}[[h]]$-span of monomials

$$E(z)_{r_m} \ldots E(z)_{r_1} \mathbf{1}, \quad r_1, \ldots, r_m \leqslant -1, \ m \geqslant 0. \quad (2.22)$$

A smaller spanning set can be easily found by using the following consequence of (2.13):

Lemma 2.5. *The following relations hold in $W_{0,t}$:*

$$E(z)_{-1} E(z) = 0 \quad and \quad E(z)_{-2} E(z) = 0. \quad (2.23)$$

Let $\mathcal{B} \subset W_{0,t}$ be the set of monomials

$$E(z)_{r_m} \ldots E(z)_{r_1} \mathbf{1}, \quad r_1 \leqslant -1, r_2, \ldots, r_m \leqslant -3, \ m \geqslant 0. \quad (2.24)$$

Lemma 2.6. *The set $\mathcal{B}$ spans $W_{0,t}$.*

Proof. Suppose that monomial (2.22) satisfies

$$r_m, \ldots, r_{k+1} \leqslant -3 \quad \text{and} \quad r_k \geqslant -2 \qquad \text{for some } 2 \leqslant k \leqslant m. \quad (2.25)$$

We will show that the given monomial can be expressed as a linear combination of monomials (2.22) satisfying

$$r_m, \ldots, r_k \leqslant -3, \quad (2.26)$$

so the lemma will follow by induction.

Suppose that $r_k = -2$. Set $a = b = E(z)$, $c = E(z)_{r_{k-2}} \ldots E(z)_{r_1} \mathbf{1}$, $(r, s) = (-2, r_{k-1})$ and $N = 0$ in (2.21). Due to the second equality in (2.23), the left-hand side in (2.21) equals zero, so we can express the original monomial satisfying (2.25) as a linear combination of the monomials satisfying (2.26).

Suppose that $r_k = -1$. Set $a = b = E(z)$, $c = E(z)_{r_{k-2}} \ldots E(z)_{r_1} \mathbf{1}$, $(r, s) = (-1, r_{k-1})$ and $N = 0$ in (2.21). Due to the first equality in (2.23), the left-hand side in (2.21) equals zero, so we can express the original monomial satisfying (2.25) as a linear combination of the monomials satisfying (2.26) and a monomial satisfying

$$r_m, \ldots, r_{k+1} \leqslant -3, \quad r_k \leqslant -2. \tag{2.27}$$

Now, we may proceed with the monomial satisfying (2.27) as in the case $r_k = -2$. $\blacksquare$

Due to Lemma 2.4, for any element $a(z, t)$ of the nonlocal h-vertex algebra $W_{0,t}$ the sequence $(E(z), \ldots, E(z), a(z, t))$ is quasi-commutative. Therefore, the product $E(z + s_k h)_{-1} \ldots E(z + s_1 h)_{-1} a(z, t)$ is a well-defined element of $\mathcal{E}_h(\mathcal{F}_0)[t]$ for any choice of integers $s_1, \ldots, s_k$. We will usually omit the subscript indices and denote this product by $E(z + s_k h) \cdots E(z + s_1 h) a(z, t)$.

Our next goal is to prove that the set $\mathcal{B}$ is linearly independent. Denote by $\mathcal{L}$ the following subset of $\mathcal{E}_h(\mathcal{F}_0)$:

$$\mathcal{L} = \{ E(z + s_k h) \cdots E(z + s_1 h) \mathbf{1} :$$
$$s_1, \ldots, s_k \in \mathbb{Z}_{\geqslant 0}, \ k \geqslant 0, \ s_{j-1} - s_j \geqslant 2, \ j = 2, \ldots, k \}.$$

For any monomial $L = E(z + s_k h) \cdots E(z + s_1 h) \mathbf{1}$ in $\mathcal{L}$ we say that the integer k is the *length* of L and write $l(L) = k$. Also, we say that the integer s_1 is the *height* of L and write $ht(L) = s_1$. In particular, we have $l(\mathbf{1}) = ht(\mathbf{1}) = 0$.

Lemma 2.7. *The set $\mathcal{L}$ is linearly independent.*

Proof. Let $n > 0$ be a minimal integer such that there exists a linear combination

$$a_1 L_1 + \ldots + a_n L_n = 0, \tag{2.28}$$

where $L_1, \ldots, L_n$ are distinct monomials in $\mathcal{L}$ and $a_1, \ldots, a_n \in \mathbb{C}[[h]]$ are nonzero scalars. Due to Theorem 2.1 and (2.11) the set $\mathcal{L}$ does not contain zero, so $n > 1$. Without loss of generality we can assume that there exists a nonnegative integer s such that $E(z + sh)$ is a factor of L_n and is not a factor of L_{n-1}, i.e.

$$L_j = E(z + s_{m_j}^{(j)} h) \cdots E(z + s_1^{(j)} h) \mathbf{1}, \quad j = 1, \ldots, n, \qquad \text{where}$$
$$s = s_k^{(n)} \text{ for some } k = 1, \ldots, m_n \quad \text{and} \quad s \neq s_k^{(n-1)} \text{ for all } k = 1, \ldots, m_{n-1}.$$

Consider the operator $\Theta_s(z) \colon L \mapsto \Psi(z + sh)^{-1} L \Psi(z + sh)$. Relation (2.16) implies $\Theta_s(z) L_j = b_j L_j$, $j = 1, \ldots, n$, for some $b_j \in \mathbb{C}[[h]]$ such that $b_n = 0$ and $b_{n-1} \neq 0$. Hence, by applying the operator $\Theta_s(z)$ on linear combination (2.28) we obtain contradiction to minimality of n. $\blacksquare$

Lemma 2.7 implies that the set $\mathcal{B}$ does not contain zero and, furthermore, that for any two elements $E_1 = E(z)_{r_k} \ldots E(z)_{r_1} \mathbf{1}$ and $E_2 = E(z)_{s_m} \ldots E(z)_{s_1} \mathbf{1}$ in $\mathcal{B}$ we have

$$E_1 = E_2 \qquad \text{if and only if} \qquad k = m \quad \text{and} \quad r_j = s_j \text{ for } j = 1, \ldots, k.$$

Let $E = E(z)_{r_k} \ldots E(z)_{r_1} \mathbf{1}$ be an arbitrary monomial in $\mathcal{B}$. We say that the integer k is the *length* of the monomial E and write $l(E) = k$. For any nonnegative integer k define

$$\mathcal{B}_{\leqslant k} = \{E \in \mathcal{B} \,:\, l(E) \leqslant k\}. \tag{2.29}$$

We say that the integer $m = -(r_1 + \ldots + r_k + k)$ is the *height* of the monomial E and write $ht(E) = m$. In particular, $l(\mathbf{1}) = ht(\mathbf{1}) = 0$. For any nonnegative integer m define

$$\mathcal{B}^{\leqslant m} = \{E \in \mathcal{B} \,:\, ht(E) \leqslant m\}. \tag{2.30}$$

Example 1. Consider the monomial $E = E(z)_{-3} E(z)_{-4} E(z)_{-6} E(z)_{-1} \mathbf{1} \in \mathcal{B}$. We have

$$l(E) = 4 \quad \text{and} \quad ht(E) = -(-3 - 4 - 6 - 1 + 4) = 10.$$

For any $\mathcal{A} \subseteq \mathcal{E}_h(\mathcal{F}_0)[t]$ denote by $\mathcal{A}|_{t=1}$ the set of elements of $\mathcal{A}$ evaluated at $t = 1$,

$$\mathcal{A}|_{t=1} = \{a(z, 1) \,:\, a(z, t) \in \mathcal{A}\} \subset \mathcal{E}_h(\mathcal{F}_0).$$

Lemma 2.8. *The set* $(\mathcal{B}_{\leqslant 2})|_{t=1}$ *is linearly independent.*

Proof. We first consider the set $\mathcal{B}_{\leqslant 2}$. Note that the linear independence of the set $\mathcal{B}_{\leqslant 1}$ can be easily established using Lemma 2.7. Now, let n be a minimal integer such that there exists a linear combination

$$a_1 E_1 + \ldots + a_n E_n = 0, \quad \text{where } a_1, \ldots, a_n \in \mathbb{C}[[h]] \text{ are nonzero}, \tag{2.31}$$

of distinct monomials

$$E_1, \ldots, E_m \in \mathcal{B}_{\leqslant 1}, \quad E_{m+1}, \ldots, E_n \in \mathcal{B}_2 := \mathcal{B}_{\leqslant 2} \setminus \mathcal{B}_{\leqslant 1}, \qquad 1 \leqslant m \leqslant n - 1.$$

Lemma 2.7 implies that the sets $\mathcal{B}_{\leqslant 2}$ and $(\mathcal{B}_{\leqslant 2})|_{t=1}$ do not contain zero, so $n > 1$. Applying the operator $h^{\partial_\alpha / 2}$ on (2.31) we obtain a new linear combination

$$h\left(h^{-\delta_{E_1, \mathbf{1}}} a_1 E_1 + \ldots + h^{-\delta_{E_m, \mathbf{1}}} a_m E_m\right) + h^2\left(a_{m+1} E_{m+1} + \ldots + a_n E_n\right) = 0,$$

which, together with (2.31), contradicts minimality of n. Therefore, it is sufficient to prove that the set $\mathcal{B}_2$ is linearly independent.

We will prove by induction over k that every set $\mathcal{B}_2^{\leqslant k}$ of monomials $E \in \mathcal{B}_2$ satisfying $ht(E) \leqslant k$ is linearly independent. This is clearly true for $k = 2$ since the set $\mathcal{B}_2^{\leqslant 2}$ contains only one element, $E(z)_{-3} E(z)_{-1} \mathbf{1} \neq 0$. Assume that the set $\mathcal{B}_2^{\leqslant k}$ is linearly independent for some $k \geqslant 2$. The set $\mathcal{B}_2^{k+1} := \mathcal{B}_2^{\leqslant k+1} \setminus \mathcal{B}_2^{\leqslant k}$ consists of the monomials

$$E^r(z) := E(z)_{r-k-3} E(z)_{-r} \mathbf{1}, \quad r = 1, \ldots, k.$$

By Lemma 2.7 the elements

$$L^s(z) := E(z + (k - s)h)E(z + (k + 1)h)t^{k+1}, \quad s = 1, \ldots, k,$$

are linearly independent. Therefore, due to the induction hypothesis, the set

$$\mathcal{B}_2^{\leqslant k} \cup \{L^s(z) \, : \, s = 1, \ldots, k\}$$

is linearly independent; this can be proved by arguing as in the proof of Lemma 2.7.

Using part (a) of Proposition 1.5 we can express every $E^r(z)$ as a $\mathbb{C}[h^{-1}]$-linear combination of the monomials $L^s(z)$ modulo $\mathbb{C}[[h]]$-span of $\mathcal{B}_2^{\leqslant k}$:

$$E^r(z) = h^{-k-1} \sum_{s=1}^{k} \binom{k - r + 2}{s - r + 2}(-1)^{s+r} L^s(z) \mod \operatorname*{span}_{\mathbb{C}[[h]]} \mathcal{B}_2^{\leqslant k}, \qquad r = 1, \ldots, k,$$

where $\binom{a}{b} = 0$ for $b < 0$. Since the matrix $\left(\binom{k-r+2}{s-r+2}(-1)^{s+r}\right)_{r,s=1}^{k}$ is regular, we conclude that the set

$$\mathcal{B}_2^{\leqslant k+1} = \mathcal{B}_2^{\leqslant k} \cup \{E^r(z) \, : \, r = 1, \ldots, k\}$$

is linearly independent, thus completing the induction step.

Finally, one can easily check that the above arguments hold for the elements of the set $(\mathcal{B}_{\leqslant 2})|_{t=1} \subset \mathcal{E}_h(\mathcal{F}_0)$ as well, so the lemma follows. ∎

Recall that $\mathcal{B}_2 = \mathcal{B}_{\leqslant 2} \setminus \mathcal{B}_{\leqslant 1}$ and define $\mathcal{LB}_{\leqslant 2} = \mathcal{LB}_2 \cup \mathcal{B}_{\leqslant 1}$, where

$$\mathcal{LB}_2 = \{L(z)B(z + (ht(L(z)) + 2 + l)h) \, : \, L(z) \in \mathcal{L}, \, B(z) \in \mathcal{B}_2, \, l \geqslant 0\}.$$

By Lemma 2.7 the sets $\mathcal{LB}_{\leqslant 2}$ and $(\mathcal{LB}_{\leqslant 2})|_{t=1}$ do not contain zero.

Lemma 2.9.   *The set $(\mathcal{LB}_{\leqslant 2})|_{t=1}$ is linearly independent.*

Proof.   Due to the first part of the proof of Lemma 2.8, it is sufficient to prove that the set $(\mathcal{LB}_2)|_{t=1}$ is linearly independent. Let $n > 0$ be a minimal integer such that there exists a linear combination

$$\sum_{k=1}^{n} a_k L_k(z) A_k(z) = 0, \tag{2.32}$$

where scalars $a_k \in \mathbb{C}[[h]]$ are nonzero, $L_1(z), \ldots, L_k(z) \in \mathcal{L}$ are distinct and each $A_k(z)$ is a $\mathbb{C}[[h]]$-linear combination of some elements $B(z + (ht(L_k(z)) + 2 + l)h)$, where $l \geqslant 0$ and $B(z) \in (\mathcal{B}_2)|_{t=1}$. Again, due to the first part of the proof of Lemma 2.8, we can assume that all the monomials $L_k(z)$, $k = 1, \ldots, n$, are of the same length. Next, by Lemma 2.8, we can assume that their length is not zero.

If $n > 1$, there exists a minimal nonnegative integer s such that

$$L_i(z)\Psi(z + sh) = 0 \quad \text{and} \quad L_j(z)\Psi(z + sh) \neq 0 \quad \text{for some } i, j = 1, \ldots, n.$$

Recall (2.17) and apply the operator $M \mapsto \Psi(z + sh)^{-1} M \Psi(z + sh)$ on linear combination (2.32). As a result, the summand $a_i L_i(z) A_i(z)$ will get annihilated. Furthermore, we have $\Psi(z + sh)^{-1} a_j L_j(z) A_j(z) \Psi(z + sh) \neq 0$. Some other summands $a_k L_k(z) A_k(z)$, $k \neq i, j$, in (2.32) may also get annihilated. However, since s is minimal, by applying the operator $M \mapsto \Psi(z + sh) M \Psi(z + sh)^{-1}$ on this new linear combination, we obtain

$$\sum_{k=1}^{n} \varepsilon_k a_k L_k(z) A_k(z) = 0, \quad \text{where } \varepsilon_k \in \{0, 1\}, \, \varepsilon_i = 0, \, \varepsilon_j = 1, \qquad (2.33)$$

which contradicts minimality of n.

Hence, in order to prove that the set $(\mathcal{L}\mathcal{B}_2)|_{t=1}$ is linearly independent, it is sufficient to consider only one summand in (2.32), i.e. the expression of the form

$$L(z) C(z) = 0 \quad \text{with} \quad C(z) = \sum_{k=1}^{m} b_k B_k(z + (ht(L(z)) + 2 + l_k)h), \qquad (2.34)$$

where $b_k \in \mathbb{C}[[h]]$ are nonzero scalars, $B_1(z), \ldots, B_m(z)$ are distinct elements of $(\mathcal{B}_2)|_{t=1}$, $L(z) \in \mathcal{L}$ and $l_k \in \mathbb{Z}_{\geqslant 0}$. If $C(z) = 0$, Lemma 2.8 implies $b_k = 0$ for all $k = 1, \ldots, m$, so the proof is over. Otherwise, write $C(z)$ as a $\mathbb{C}((h))$-linear combination of distinct elements $J_1(z), \ldots, J_r(z) \in \mathcal{L}$,

$$C(z) = \sum_{k=1}^{r} c_k J_k(z) \neq 0 \quad \text{with nonzero } c_k \in \mathbb{C}((h)). \qquad (2.35)$$

Choose an integer p such that $d_k := h^p c_k \in \mathbb{C}[[h]]$ for all $k = 1, \ldots, r$. By combining (2.34) and (2.35) we obtain a $\mathbb{C}[[h]]$-linear combination of distinct elements $L(z) J_k(z) \in \mathcal{L}$,

$$\sum_{k=1}^{r} d_k L(z) J_k(z) = 0,$$

thus contradicting Lemma 2.7. ∎

For an arbitrary monomial $E = E(z)_{r_k} \ldots E(z)_{r_1} \mathbf{1}$ in $\mathcal{B}$ define *partial heights*

$$ht_j(E) = ht(E(z)_{r_k} \ldots E(z)_{r_{k-j+1}} \mathbf{1}) \quad \text{for } j = 1, \ldots, k. \qquad (2.36)$$

Notice that $ht_k(E) = ht(E)$. Next, if $l(E) \geqslant 3$, define the *leading term* $\bar{E}$ of E by

$$\bar{E} = E(z) E(z + ht_1(E)h) \cdots E(z + ht_{k-3}(E)h)$$
$$\cdot E(z + ht_{k-2}(E)h)_{r_2} E(z + ht_{k-2}(E)h)_{r_1} \mathbf{1}.$$

For $l(E) \leqslant 2$ we define the *leading term* $\bar{E}$ of E by $\bar{E} = E$.

Example 2. Partial heights of the monomial

$$E = E(z)_{-5} E(z)_{-3} E(z)_{-4} E(z)_{-2} \mathbf{1} \in \mathcal{B}$$

are

$$ht_1(E) = 4, \quad ht_2(E) = 6, \quad ht_3(E) = 9, \quad ht_4(E) = ht(E) = 10,$$

so its leading term equals $\bar{E} = E(z) E(z + 4h) E(z + 6h)_{-4} E(z + 6h)_{-2} \mathbf{1}$.

For any two monomials $E_1, E_2 \in \mathcal{B}$ we have $E_1 = E_2$ if and only if $\bar{E}_1 = \bar{E}_2$. The set of leading terms $\bar{\mathcal{B}} := \left\{ \bar{E} : E \in \mathcal{B} \right\}$ is a subset of $\mathcal{LB}_{\leqslant 2}$, so Lemma 2.9 implies

Corollary 2.10.   *The set* $\bar{\mathcal{B}}\,|_{t=1}$ *is linearly independent.*

For any two elements $E_i = E(z)_{r_m^{(i)}} \ldots E(z)_{r_1^{(i)}} \mathbf{1} \in \mathcal{B}$, $i = 1, 2$, of the same length $m \geqslant 3$ define equivalence relation $\sim$ and partial ordering $\prec$ by

$$E_1 \sim E_2 \quad \text{if} \quad (r_m^{(1)}, \ldots, r_3^{(1)}) = (r_m^{(2)}, \ldots, r_3^{(2)});$$
$$E_1 \prec E_2 \quad \text{if} \quad (r_m^{(1)}, \ldots, r_3^{(1)}) < (r_m^{(2)}, \ldots, r_3^{(2)}) \text{ in lexicographical order.}$$

Example 3.   Observe that the elements of the same length $m \geqslant 3$ are compared with respect to their factors $m, \ldots, 3$, so that, e.g., for $m = 4$ we only consider the factors 4 and 3. For example, we have

$$E(z)_{-3} E(z)_{-4} E(z)_{-6} E(z)_{-2} \mathbf{1} \sim E(z)_{-3} E(z)_{-4} E(z)_{-5} E(z)_{-1} \mathbf{1}$$

because $(-3, -4) = (-3, -4)$ and

$$E(z)_{-4} E(z)_{-3} E(z)_{-6} E(z)_{-2} \mathbf{1} \prec E(z)_{-3} E(z)_{-4} E(z)_{-5} E(z)_{-1} \mathbf{1}$$

because $(-4, -3) < (-3, -4)$.

Theorem 2.11.   *The set* $\mathcal{B}$ *forms a basis for* $W_{0,t}$.

Proof.   Let

$$\sum_{k=1}^{n} a_k B_k = 0, \quad a_k \in \mathbb{C}[[h]],\ a_k \neq 0,\ k = 1, \ldots, n, \qquad (2.37)$$

be a linear combination of distinct elements $B_1, \ldots, B_n \in \mathcal{B}$. Without loss of generality we can assume that all the elements B_k have the same length and that

$$B_n, \ldots, B_{m+1} \prec B_m \sim \ldots \sim B_1 \quad \text{for some } m = 1, \ldots, n.$$

There exist nonzero scalars $b_1, \ldots, b_m \in \mathbb{C}[h^{\pm 1}]$, so that we can express

$$\left(B_1 - b_1 \bar{B}_1 \right)|_{t=1}, \ldots, \left(B_m - b_m \bar{B}_m \right)|_{t=1}, B_{m+1}\,|_{t=1}, \ldots, B_n\,|_{t=1},$$

as a $\mathbb{C}[h^{\pm 1}]$-linear combination of some elements from

$$\mathcal{C} := \left(\mathcal{LB}_{\leqslant 2} \setminus \left\{ \bar{B}_1, \ldots, \bar{B}_m \right\} \right) \big|_{t=1}.$$

By rewriting (2.37)) we obtain

$$\sum_{i=1}^{m} a_i b_i \bar{B}_i\,|_{t=1} = \sum_{i=1}^{m} a_i \left(b_i \bar{B}_i - B_i \right)|_{t=1} - \sum_{j=m+1}^{n} a_j B_j\,|_{t=1}. \qquad (2.38)$$

This implies that the left-hand side in (2.38), which is nonzero by Corollary 2.10, belongs to the $\mathbb{C}((h))$-span of $\mathcal{C}$, thus contradicting Lemma 2.9. Therefore, the set $\mathcal{B}$ is linearly independent. The theorem now follows from Lemma 2.6. ∎

Proof of Theorem 2.11 clearly implies

Corollary 2.12. *The set $\mathcal{B}|_{t=1}$ is linearly independent.*

Consider the evaluation $\mathcal{E}_h(\mathcal{F}_0)[t] \to \mathcal{E}_h(\mathcal{F}_0)$ at $t = 1$,

$$\mathrm{ev}\colon a(z, t) \mapsto a(z, 1) = a(z, t)|_{t=1}. \tag{2.39}$$

By Corollary 2.12, its restriction $\mathrm{ev}_0 := \mathrm{ev}|_{W_{0,t}}$ to $W_{0,t}$ is injective. Hence, we can transfer the nonlocal h-vertex algebra structure from $W_{0,t}$ to $W_0 := W_{0,t}|_{t=1} \subset \mathcal{E}_h(\mathcal{F}_0)$ by defining the vertex operator map $Y_0(\cdot, z_0)\colon W_0 \to \mathrm{Hom}(W_0, W_0((z_0)))$,

$$Y_0(a(z), z_0)b(z) = \sum_{r \in \mathbb{Z}} a(z)_{(r,0)} b(z) z_0^{-r-1}, \quad a(z), b(z) \in W_0, \tag{2.40}$$

by

$$Y_0(a(z), z_0)b(z) = \sum_{r \in \mathbb{Z}} \left((\mathrm{ev}_0^{-1} a(z))_r (\mathrm{ev}_0^{-1} b(z)) \right)|_{t=1} z_0^{-r-1}. \tag{2.41}$$

Note that the r-th products on the right-hand side of (2.41) are products in the nonlocal h-vertex algebra $W_{0,t}$. As in (2.40), we will denote the r-th product of $a(z), b(z) \in W_0$ by $a(z)_{(r,0)}b(z)$. One can easily verify that $(W_0, Y_0, \mathbf{1})$ is a nonlocal h-vertex algebra. Now, we can employ the nonlocal h-vertex algebra structure on W_0 to define the set

$$\mathcal{B}_{W_0} = \left\{ E(z)_{(r_m,0)} \dots E(z)_{(r_1,0)} \mathbf{1} \,:\, r_1 \leqslant -1,\, r_2, \dots, r_m \leqslant -3,\, m \geqslant 0 \right\} \subset W_0, \tag{2.42}$$

which is equal to $\mathcal{B}|_{t=1}$. The next theorem summarizes the main results of this subsection.

Theorem 2.13. $(W_0, Y_0, \mathbf{1})$ *is a nonlocal h-vertex algebra. The set $\mathcal{B}_{W_0}$ forms a basis for W_0.*

2.3. Nonlocal h-vertex algebras W_N for positive N. Throughout this subsection, unless specified otherwise, we assume that N is a fixed positive integer; recall Definition 1.6. In Subsection 2, we introduced the nonlocal h-vertex algebras $W_{N,t} \subset \mathcal{E}_h(\mathcal{F}_0)[t]$ whose vertex operator map $Y^{(N)}$ was given by (2.20).

Using (2.13) one can prove that

$$E(z)_{(-r-1-\delta_{1N},N)} E(z) \neq 0 \text{ for all } r \geqslant 0 \qquad \text{and} \qquad E(z)_{(-1,1)} E(z) = 0. \tag{2.43}$$

Let $\mathcal{B}_{(1)} \subset W_{1,t}$ be the set of all monomials

$$E(z)_{(r_m,1)} \dots E(z)_{(r_1,1)} \mathbf{1}, \quad r_1 \leqslant -1,\, r_2, \dots, r_m \leqslant -2,\, m \geqslant 0.$$

For $N > 1$ let $\mathcal{B}_{(N)} \subset W_{N,t}$ be the set of all monomials

$$E(z)_{(r_m,N)} \dots E(z)_{(r_1,N)} \mathbf{1}, \quad r_1, \dots, r_m \leqslant -1,\, m \geqslant 0.$$

By arguing as in the proof of Lemma 2.6 one can prove

Lemma 2.14. *Nonlocal h-vertex algebra $W_{N,t}$ is equal to the $\mathbb{C}[[h]]$-span of $\mathcal{B}_{(N)}$.*

We can now proceed as in Subsection 2 and prove that the subsets $\mathcal{B}_{W_N} := \left(\mathcal{B}_{(N)}\right)\big|_{t=1}$ of $\mathcal{E}_h(\mathcal{F}_0)$, where $N \geqslant 1$, are linearly independent. The arguments used therein can be directly generalized to the $N > 0$ case, so we omit the proof. As with the $N = 0$ case, the restriction $\mathrm{ev}_N := \mathrm{ev}\big|_{W_{N,t}}$ of evaluation (2.39) to $W_{N,t}$ for positive N is injective, so we can transfer the nonlocal h-vertex algebra structure from $W_{N,t}$ to $W_N := W_{N,t}\big|_{t=1} \subset \mathcal{E}_h(\mathcal{F}_0)$ as follows. The vertex operator map $Y_N(\cdot, z_0) \colon W_N \to \mathrm{Hom}(W_N, W_N((z_0)))$,

$$Y_N(a(z), z_0)b(z) = \sum_{r \in \mathbb{Z}} a(z)_{(r,N)} b(z) z_0^{-r-1}, \quad a(z), b(z) \in W_N,$$

is defined by

$$Y_N(a(z), z_0)b(z) = \sum_{r \in \mathbb{Z}} \left((\mathrm{ev}_N^{-1} a(z))_{(r,N)} (\mathrm{ev}_N^{-1} b(z)) \right)\big|_{t=1} z_0^{-r-1}.$$

Theorem 2.15. $(W_N, Y_N, \mathbf{1})$ *is a nonlocal h-vertex algebra. The set $\mathcal{B}_{W_N}$ forms a basis for W_N.*

Remark 2.16. Note that the nonlocal h-vertex algebras W_N, $N \geqslant 0$, are not h-adically complete. For example, $\left(\sum_{l=1}^n E(z + lh)h^l\right)_n$ is a Cauchy sequence in W_N, with respect to the h-adic topology, but its h-adic limit, $\sum_{l=1}^\infty E(z + lh)h^l \in \mathcal{E}_h(\mathcal{F}_0)$ does not belong to W_N. Therefore, W_N is not an *h-adic nonlocal vertex algebra*, as defined in [18, Definition 2.9], even though it satisfies all the remaining axioms in the aforementioned definition. However, the triple $(\widetilde{W}_N, \widetilde{Y}_N, \mathbf{1})$, where $\widetilde{Y}_N$ is the $\mathbb{C}((h))$-linear extension of the vertex operator map Y_N to the vector space $\widetilde{W}_N := W_N \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$, is a *nonlocal vertex algebra* over $\mathbb{C}((h))$; see [16, Definition 2.3].

Suppose that $N \geqslant 0$. We will now employ Theorem 2.13 and Theorem 2.15 to determine the connection between the nonlocal h-vertex algebras W_N, which is already implicitly suggested by the form of their bases $\mathcal{B}_{W_N}$.

For any $N \geqslant 1$ there exists a unique $\mathbb{C}[[h]]$-module map $f_N \colon W_N \to W_{N-1}$ such that

$$\mathcal{B}_{W_N} \ni E(z)_{(r_m,N)} \dots E(z)_{(r_1,N)} \mathbf{1} \xmapsto{f_N} E(z)_{(r_m,N-1)} \dots E(z)_{(r_1,N-1)} \mathbf{1} \in W_{N-1}.$$

In particular, $f_N(\mathbf{1}) = \mathbf{1}$. Therefore, we have a sequence of $\mathbb{C}[[h]]$-module maps,

$$\dots \xrightarrow{f_{N+2}} W_{N+1} \xrightarrow{f_{N+1}} W_N \xrightarrow{f_N} \dots \xrightarrow{f_3} W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} W_0.$$

Proposition 2.17. *The $\mathbb{C}[[h]]$-module maps $f_N \colon W_N \to W_{N-1}$ are epimorphisms of nonlocal h-vertex algebras. Furthermore, maps f_N for $N \geqslant 3$ are isomorphisms.*

Proof.   By (2.21), for any nonnegative integer M we have

$$\left(a_{(r,M)}b\right)_{(s,M)}c = \sum_{l \geqslant 0} \binom{l-r-1}{l} a_{(r-l,M)}\left(b_{(s+l,M)}c\right), \quad a,b,c \in W_M,\, r,s \in \mathbb{Z}.$$
(2.44)

Fix negative integer r and choose two arbitrary basis elements

$$a^i = E(z)_{(r^{(i)}_{m_i},N)} \ldots E(z)_{(r^{(i)}_1,N)} \mathbf{1} \in \mathcal{B}_{W_N}, \quad i = 1,2.$$

Using (2.44) for $M = N$ we can express $a^1_{(r,N)}a^2 = (a^1_{(r,N)}a^2)_{(-1,N)}\mathbf{1} \in W_N$ as

$$a^1_{(r,N)}a^2 = \beta_1 b_1 + \ldots \beta_k b_k \quad \text{for some } \beta_1,\ldots,\beta_k \in \mathbb{C},\, b_1,\ldots,b_k \in \mathcal{B}_{W_N}. \quad (2.45)$$

Indeed, this can be done by the technique which was described in the proof of Lemma 2.6. Since formulas in (2.44) for $M = N$ and $M = N - 1$ coincide and

$$f_N(a^i) = E(z)_{(r^{(i)}_{m_i},N-1)} \ldots E(z)_{(r^{(i)}_1,N-1)} \mathbf{1} \in f_N(\mathcal{B}_{W_N}), \quad i = 1,2,$$

we can use (2.44) for $M = N - 1$ analogously, to express $f_N(a^1)_{(r,N-1)}f_N(a^2)$ as

$$f_N(a^1)_{(r,N-1)}f_N(a^2) = \beta_1 f_N(b_1) + \ldots \beta_k f_N(b_k). \quad (2.46)$$

Equalities (2.45) and (2.46) imply that

$$f_N(a^1_{(r,N)}a^2) = f_N(a^1)_{(r,N-1)}f_N(a^2),$$

so we conclude that $f_N \colon W_N \to W_{N-1}$ is a homomorphism of nonlocal h-vertex algebras. Finally, surjectivity of f_N for $N \geqslant 1$ and injectivity of f_N for $N \geqslant 3$ follows from Theorem 2.13 and Theorem 2.15. ∎

For $N = 1,2$ let I_N be an ideal in the nonlocal h-vertex algebra W_N generated by

$$K_N(z) := E(z)_{(-(3-N),N)}E(z) \in W_N.$$

More precisely, I_N is the intersection of all ideals in W_N which contain $K_N(z)$.

Proposition 2.18.   *For $N = 1,2$ the kernel of the epimorphism f_N is equal to I_N. Hence*

$$W_0 \cong W_1/I_1 \quad and \quad W_1 \cong W_2/I_2.$$

Proof.   Let $N = 1,2$. By Lemma 2.5 and (2.43) we have $f_N(K_N(z)) = 0$, so the ideal I_N is contained in the kernel of f_N. On the other hand, by arguing as in the proof of Lemma 2.6 (see also Lemma 2.14) one can prove that there exists a $\mathbb{C}[[h]]$-module epimorphism $g_N \colon W_{N-1} \to W_N/I_N$ such that

$$\mathcal{B}_{W_{N-1}} \ni E(z)_{(r_m,N-1)} \ldots E(z)_{(r_m,N-1)} \mathbf{1} \overset{g_N}{\longmapsto} E(z)_{(r_m,N)} \ldots E(z)_{(r_m,N)} \mathbf{1} + I_N.$$

Since the nonlocal h-vertex algebras W_{N-1} and $W_N/\ker f_N$ are isomorphic, this implies that $\ker f_N \subseteq I_N$ and the proposition follows. ∎

2.4. Character formulas. Suppose that $N \geqslant 0$. Motivated by [12, Section 4.3], we define the *degree* $\deg E$ of the element $E = E(z)_{(r_k, N)} \dots E(z)_{(r_1, N)} \mathbf{1}$ in $\mathcal{B}_{W_N}$ by

$$\deg E = k - \sum_{m=1}^{k} (k - m + 1)(r_m + 1). \tag{2.47}$$

Consider the corresponding direct sum decomposition

$$W_N = \bigoplus_{n \geqslant 0} W_N^{(n)}, \quad \text{where} \quad W_N^{(n)} = \operatorname*{span}_{\mathbb{C}[[h]]} \{E \in \mathcal{B}_{W_N} \,:\, \deg E = n\}.$$

The $\mathbb{C}[[h]]$-submodules $W_N^{(n)}$ are finite-dimensional, so we can define their character by

$$\operatorname{ch}_q W_N = \sum_{n \geqslant 0} \dim W_N^{(n)} q^n.$$

Character formulas for W_N can be easily determined from the bases $\mathcal{B}_{W_N}$ provided by Theorem 2.13 and Theorem 2.15. The character $\operatorname{ch}_q W_0$ is equal to the sum side in the Rogers–Ramanujan identity

$$\sum_{n \geqslant 0} \frac{q^{n^2}}{(1-q) \cdots (1-q^n)} = \prod_{n \geqslant 0} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})}; \tag{2.48}$$

the character $\operatorname{ch}_q W_1$ is equal to the sum side in the Euler identity

$$\sum_{n \geqslant 0} \frac{q^{n(n+1)/2}}{(1-q) \cdots (1-q^n)} = \prod_{n \geqslant 1} (1 + q^n)$$

and finally, the character $\operatorname{ch}_q W_N$ for $N \geqslant 2$ is equal to

$$\operatorname{ch}_q W_N = \prod_{n \geqslant 1} \frac{1}{1 - q^n}.$$

Remark 2.19. Lemma 2.7 and Theorem 2.13 imply that the set $\mathcal{L}$ forms a basis for the nonlocal vertex algebra $\widetilde{W}_0$; recall Remark 2.16. Moreover, the difference-two conditions in the definition of $\mathcal{L}$ provide an interpretation of the sum side in (2.48), as well as the motivation for the definition of degree in (2.47).

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