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Convolution of Orbital Measures on Complex Grassmannians

Mahmoud Al-Hashami and Boudjemâa Anchouche

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Abstract. In [2] and [1], the regularity of the Radon-Nikodym derivative of the convolutions of orbital measures on a compact symmetric space of rank one was studied. The aim of this paper is to extend the results obtained in [1] to the case of complex Grassmannians. More precisely, let $M = U/K$, where $U = SU(p+q)$ and $K = S(U(p)\times\ U(q))$, be the complex Grassmannian of a p-plane in $\mathbb{C}^{p+q}$, $p \geq q \geq 2$, $a_1, ..., a_r$ be r points in U, and consider the convolution product $\nu_{a_1} * ... * \nu_{a_r}$ of the orbital measures $\nu_{a_1}, ..., \nu_{a_r}$ supported on $Ka_1K, ..., Ka_rK$. By a result of Ragozin [10], if $r \geq \dim M$, then $\nu_{a_1} * ... * \nu_{a_r}$ is absolutely continuous with respect to the Haar measure of U. The aim of this paper is to investigate the C^k-regularity of the Radon-Nikodym derivative of $\nu_{a_1} * ... * \nu_{a_r}$ with respect to the Haar measure of U.
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1. Introduction

The regularity of the convolution of orbital measures on compact symmetric spaces was considered by several authors. In [10] Ragozin proved that the convolution of m orbital measures on a compact symmetric space $M = U/K$, with $\dim M = m$ is absolutely continuous with respect to the Haar measure of U. In [2] and [1], the regularity of the Radon-Nikodym derivative of the convolution of some orbital measures on compact symmetric spaces $M = U/K$ of rank one, with respect to the Haar measure of U, was considered. Following the philosophy which says that if a result is true for Grassmannians (real, complex, queternionic) then it should be true for all compact symmetric spaces ([14], p. 29), we consider in this paper the smoothness of the Radon-Nikodym derivative of a convolution of some orbital measures on complex Grassmannians. Let $M = SU(n)/S(U(p)\times U(q))$, where p and q are integers satisfying $p \geq q \geq 2$, and $n = p+q$, $a_1, \ldots, a_r$ be r-points of $U - N_U(K)$, where $N_U(K)$ is the normalizer of K in U, ν_{a_j} an orbital measure supported on Ka_jK, $j = 1, \cdots r$ (see section 2) and suppose

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that the orbital measure $\nu_{a_1} * ... * \nu_{a_r}$ is absolutely continuous with respect to the Haar measure $\mu_{SU(n)}$ of $SU(n)$, and denote by $f_{a_1,...,a_r}$ its density, or its Radon-Nikodym derivative.

Let k be a positive integer, and let s be a real number such that

$$s > k + \frac{n^2 - 1}{2} \text{ and } C(p,q,s) \geq 2pq, \tag{1}$$

where

$$C(p,q,s) := \frac{1 + 2s + \frac{n(n-1)}{2}}{2p - q}. \tag{2}$$

The aim of this paper is to prove the following

Theorem (Main Theorem). *Let k be a positive integer and s as in* (1) *and $C(p,q,s)$ as in* (2)*. If*

$$r > C(p,q,s),$$

then

$$f_{a_1,...,a_r} \in C^k(SU(n)).$$

The paper is organized as follows. In section 2 we review few basic facts about orbital measures, Fourier transform, and Plancherel theorem. In section 3, we review basic facts about complex Grassmannians and restricted roots. In section 4 we review the properties of spherical functions on complex Grassmannians. In section 5 we give a proof of the main result of the paper.

2. Some Preliminary Results

2.1. Convolution of Orbital Measures.

Let G be a non-compact real semisimple Lie group, $\mathfrak{g}$ its Lie algebra, θ a Cartan involution, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

the corresponding Cartan decomposition. Then

$$\mathfrak{u} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p},$$

is a real Lie subalgebra of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Let U be a Lie group with Lie algebra $\mathfrak{u}$. Then U is compact and semisimple. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. Then

$$G = K \exp(\mathfrak{a}) K \text{ and } U = K \exp\left(\sqrt{-1}\,\mathfrak{a}\right) K,$$

where K is the Lie subgroup of U with the Lie algebra $\mathfrak{k}$.

Let a_1,..., a_r be points in $U - N_U(K)$, and define the following positive functionals

$$I_{a_j}(f) = \int_K \int_K f(k_1 a_j k_2)\, d\mu_K(k_1)\, d\mu_K(k_2), \quad j = 1, \cdots r,$$

where f is an arbitrary continuous function on U and μ_K is the normalized Haar measure of K, i.e., $\mu_K(K) = 1$. By a very well known result of Radon and Riesz, the positive functional I_{a_j} defines a positive Borel measure ν_{a_j}, i.e., there exist a positive Radon measure ν_{a_j} such that for all continuous functions f on U, we have

$$I_{a_j}(f) = \langle \nu_{a_j}, f \rangle = \int_U f(g) d\nu_{a_j}(g).$$

By construction, the measure ν_{a_j} is K-bi-invariant. Hence, without loss of generality, we can assume that

$$a_j = \exp\left(\sqrt{-1} X_j\right),$$

for some X_j in $\mathfrak{a}$.

The convolution of the measures ν_{a_j}, $j = 1, ..., r$, denoted by $\nu_{a_1} * ... * \nu_{a_r}$, is the positive measure corresponding to the positive functional

$$I_{a_1,...,a_r}(f) = \int_U \cdots \int_U f(x_1 \dots x_r) d\nu_{a_1}(x_1) ... d\nu_{a_r}(x_r)$$

$$= \int_K \cdots \int_K f(k_1 a_1 k_2 \cdots k_r a_r k_{r+1}) d\mu_K(k_1) ... d\mu_K(k_{r+1}),$$

i.e.,

$$I_{a_1,...,a_r}(f) = \int_U f(g) d(\nu_{a_1} * ... * \nu_{a_r})(g).$$

In all what follows, to simplify the notations, we will denote $\nu_{a_1} * ... * \nu_{a_r}$ by $\nu_{a_1 \cdots a_r}$. The absolute continuity of the measure $\nu_{a_1 \cdots a_r}$ with respect to the Haar measure of U is characterized in terms of its support as is explained in the following

Proposition 2.1. *Let $a_1, \dots, a_r$ be as above. Then the measure $\nu_{a_1 \cdots a_r}$ is absolutely continuous with respect to the Haar measure of U if and only if $Ka_1 \dots Ka_r K$ has a non-empty interior.*

The proof of the Proposition relies on the following very well known result, for which we include a proof to make the paper self contained. Recall that the support of a measure μ on U, denoted by $\mathrm{Supp}(\mu)$, is the complement of the largest open subset of U on which the measure μ is zero.

Lemma 2.2.

$$\mathrm{Supp}(\nu_{a_1 \cdots a_r}) = Ka_1 K \dots Ka_r K.$$

Proof. Let $x \notin \overline{\mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r})}$. Then there exists a neighborhood V of x such that

$$V \cap (\mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r})) = \emptyset.$$

Let f be a continuous function in U such that its support is in V, i.e.,

$$\mathrm{Supp}(f) = \overline{\{x \in U \mid f(x) \neq 0\}} \subset V.$$

Since

$$\int_U \cdots \int_U f(x_1 \dots x_r)\, d\nu_{a_1}(x_1) ... d\nu_{a_r}(x_r) =$$
$$= \int_{\mathrm{Supp}(\nu_{a_1})} d\nu_{a_1}(x_1) \cdots \int_{\mathrm{Supp}(\nu_{a_r})} f(x_1 \dots x_r)\, d\nu_{a_r}(x_r),$$

and since for $x_j \in \mathrm{Supp}\left(\nu_{a_j}\right)$, for $j = 1, ..., r$, we deduce that

$$x_1 \dots x_r \in Supp(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r}),$$

hence $f(x_1 \dots x_r) = 0$. Hence $x \notin \mathrm{Supp}(\nu_{a_1 \cdots a_r})$. Consequently, we deduce that

$$\mathrm{Supp}(\nu_{a_1 \cdots a_r}) \subset \overline{\mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r})}. \tag{3}$$

Let $\tilde{V}$ be an open subset of U, such that $\tilde{V}$ is of $(\nu_{a_1} * \cdots * \nu_{a_r})$-measure zero, and let f be a nonzero continuous function with compact support $C \subset \tilde{V}$, and values in $[0, 1]$, $f_{|C} = 1$, and $f = 0$ outside of $\tilde{V}$. By assumption

$$\int_U \cdots \int_U f(x_1 \dots x_r)\, d\nu_{a_1}(x_1) ... d\nu_{a_r}(x_r) = 0.$$

So the open set defined by

$$S = \{(x_1, \dots, x_r) \in U^r \mid f(x_1 \dots x_r) > \frac{1}{10}\}$$

is of $(\nu_{a_1} \otimes \cdots \otimes \nu_{a_r})$-measure zero, and therefore

$$\begin{aligned} S \cap \mathrm{Supp}(\nu_{a_1} \otimes \cdots \otimes \nu_{a_r}) &= S \cap (\mathrm{Supp}(\nu_{a_1}) \times ... \times Supp(\nu_{a_r})) \\ &= \emptyset. \end{aligned}$$

Since the map $(x_1, \dots, x_r) \longrightarrow x_1 \dots x_r$ is continuous, we deduce that

$$C \cap \mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r}) = \emptyset.$$

Consequently,

$$\overline{\mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r})} \subset \mathrm{Supp}(\nu_{a_1 \cdots a_r}). \tag{4}$$

Combining (3) and (4), we get

$$\mathrm{Supp}(\nu_{a_1 \cdots a_r}) = \overline{\mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r})}. \tag{5}$$

Using the fact that, in a locally compact group, a product of a closed subgroup by a compact subgroup is closed (see [7], Theorem 4.4) and the fact that in our setting the group U is compact, an induction argument gives

$$\overline{\mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r})} = \mathrm{Supp}(\nu_{a_1}) ... \mathrm{Supp}(\nu_{a_r}). \tag{6}$$

The Lemma follows from (5), (6) and $\mathrm{Supp}\left(\nu_{a_j}\right) = K a_j K$. ■

As a consequence of the Lemma above, we deduce the following

Proof of Proposition 2.1. Let μ_U be the Haar measure of U. Suppose that $\nu_{a_1\cdots a_r}$ is absolutely continuous with respect to μ_U. By Lemma 2.2 we deduce that $Ka_1K\ldots Ka_rK$ is of non-empty interior, for, if $Ka_1K\ldots Ka_rK$ is of empty interior, then it will be of measure zero with respect to μ_U, and therefore $\nu_{a_1\cdots a_r}$ will be singular with respect to μ_U. Conversely, suppose that $Ka_1K\ldots Ka_rK$ is of non-empty interior. Then the map $f : K^{r+1} \longrightarrow U$ given by $f(k_1,...,k_{r+1}) = k_1a_1\cdots a_rk_{r+1}$ is of maximal rank, i.e., $\operatorname{rank} f = \dim U$. Let E be a μ_U-measurable set and suppose that $\mu_U(E) = 0$. By definition of $\nu_{a_1\cdots a_r}$, we have

$$\begin{aligned}\nu_{a_1\cdots a_r}(E) &= I_{a_1\cdots a_r}(\chi_E)\\ &= \int_K\cdots\int_K \chi_E(k_1a_1\cdots a_rk_{r+1})\,d\mu_K(k_1)\cdots d\mu_K(k_{r+1})\\ &= f_*\mu_{K^{r+1}}(E) = \mu_{K^{r+1}}\left(f^{-1}(E)\right),\end{aligned}$$

where $f_*\mu_{K^{r+1}}$ is the push-forward of the Haar measure on K^{r+1}. Using similar arguments as in the proof of Theorem 2.5 in [10], we deduce that

$$\nu_{a_1\cdots a_r}(E) = 0.$$

We infer that $\nu_{a_1\cdots a_r}$ is absolutely continuous with respect to the Haar measure μ_U. Hence the proposition. ■

The following corollary is implicit in [10]

Corollary 2.3. *Let $a_1,\ldots,a_r$ be as above. If $r \geq U/K$ and U/K is a compact irreducible Hermitian symmetric space, then the measure $\nu_{a_1\cdots a_r}$ is absolutely continuous with respect to the Haar measure of U.*

Proof. Since U/K is a compact irreducible Hermitian symmetric space, by [13], the condition 2.1 in [10] is satisfied. If $r \geq U/K$, then the proof of Theorem 2.5 in [10] implies that $Ka_1K\ldots Ka_rK$ is of nonempty interior. The Corollary follows from Proposition 2.1. ■

2.2. Plancherel Theorem. Let U be a compact Lie group and let π be a unitary irreducible representation of U with a U-invariant inner product $(.,.)$ on E_π. Consider the Hilbert space End (E_π) with respect to the Hilbert-Schmidt inner product

$$(T,S)_{HS} = \operatorname{Tr}\ (S^*\circ T) = \sum_i (Tv_i, Sv_i)$$

where $T,S\in \operatorname{End}\ (E_\pi)$, S^* the adjoint of S and $\{v_i\}$ an orthonormal basis of E_π. Consider the Hilbert space direct sum

$$\operatorname{Op}(\widehat{U}) = \widehat{\bigoplus_{[\pi]\in\widehat{U}}} \operatorname{End}\ (E_\pi),$$

where the inner product is defined by

$$\langle (S_\pi)_{[\pi]\in\widehat{U}}, (T_\pi)_{[\pi]\in\widehat{U}} \rangle_{\mathrm{Op}(\widehat{U})} = \sum_{[\pi]\in\widehat{U}} \langle S_\pi, T_\pi \rangle_{HS}.$$

Plancherel theorem states that there exist an isomorphism

$$\mathcal{F} : L^2(U) \longrightarrow \mathrm{Op}(\widehat{U})$$

defined by

$$\mathcal{F}f = \left((\dim E_\pi)^{\frac{1}{2}} \pi(f) \right)_{[\pi]\in\widehat{U}},$$

such that

$$\langle f_1, f_2 \rangle_{L^2(U)} = \langle \mathcal{F}f_1, \mathcal{F}f_2 \rangle_{\mathrm{Op}(\widehat{U})},$$

for $f_1, f_2 \in L^2(U)$, where

$$\pi(f) = \int_U f(g)\pi(g^{-1})\, d\mu_U(g).$$

Similarly, the Fourier transform of a measure μ on U, denoted by $\mathcal{F}(\mu)$ is defined by

$$\mathcal{F}(\mu) = \left((\dim E_\pi)^{\frac{1}{2}} \pi(\mu) \right)_{[\pi]\in\widehat{U}},$$

where

$$\pi(\mu) = \int_U \pi(g^{-1})\, d\mu(g).$$

Suppose that $\nu_{a_1\cdots a_r}$ is absolutely continuous with respect to the Haar measure of U, and let $f_{a_1,\ldots,a_r}$ be the Radon-Nikodym derivative of $\nu_{a_1\cdots a_r}$ with respect to the Haar measure μ_U of U. Then $\pi(\mu) = \pi(f_{a_1,\ldots,a_r})$, and consequently,

$$\mathcal{F}(\nu_{a_1\cdots a_r}) = \mathcal{F}(f_{a_1,\ldots,a_r}).$$

For more details, see [1] and [11].

Let $H^s(U)$ be the Sobolev space of functions in $L^2(U)$ whose weak derivatives up to order s are in $L^2(U)$, and denote by $||.||_{H^s(U)}$ be the corresponding Sobolev norm. Using Plancherel theorem which says that

$$||f_{a_1,\ldots,a_r}||_{L^2(U)} = ||\mathcal{F}(f_{a_1,\ldots,a_r})||_{\mathrm{Op}(\widehat{U})},$$

and the properties of Fourier Transform, we deduce the following

Proposition 2.4 ([1]). *With the above notation defined, we have*

$$\|f_{a_1,\ldots,a_r}\|^2_{H^s(U)} = \sum_{[\pi]\in\widehat{U}_K} d_\pi (1+\kappa_\pi)^s \prod_{k=1}^{r} |\varphi_\pi(a_k)|^2,$$

where $\widehat{U}_K$ is the set of equivalence classes of irreducible spherical representations of the Gelfand pair (U,K), $d_\pi = \dim E_\pi$, κ_π is the Casimir constant corresponding to the representation π, and φ_π is the spherical function corresponding to the spherical representation π.

3. The Case of $SU(p+q)/S(U(p)\times U(q))$

Let p and q be two integers such that $p\geq q\geq 2$, $n=p+q$, and let

$$I_{p,q}=\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}$$

where I_j is the $j\times j$ identity matrix. Let

$$\begin{array}{rccc}\tau: & SL(n,\mathbb{C}) & \longrightarrow & SL(n,\mathbb{C})\\ & g & \longmapsto & \tau(g)=I_{p,q}(g^*)^{-1}I_{p,q}.\end{array}$$

Using the fact that $I_{p,q}^2=I_{p+q}$, we get

$$\begin{aligned}G=SU(p,q)&=\{g\in SL(n,\mathbb{C})\mid \tau(g)=g\}\\ &=\{g\in SL(n,\mathbb{C})\mid g^*I_{p,q}g=I_{p,q}\}.\end{aligned}$$

The group $SU(p,q)$ is the linear isometry group for the indefinite Hermitian form

$$\sum_{j=1}^{p}|z_j|^2-\sum_{j=1+p}^{p+q}|z_j|^2,$$

and it is a noncompact real form of the complexification $G_{\mathbb{C}}=SL(n,\mathbb{C})$ of G. Let $\mathfrak{su}(p,q)$ be the Lie algebra of $SU(p,q)$, i.e.,

$$\mathfrak{g}=\mathfrak{su}(p,q)=\{A\in M(n,\mathbb{C})\mid A^*I_{p,q}+I_{p,q}A=0,\ \mathrm{Tr}\ (A)=0\},$$

and let

$$\theta(X)=I_{p,q}X^TI_{p,q},\ X \text{ in } \mathfrak{su}(p,q)$$

be a Cartan involution of $\mathfrak{su}(p,q)$. The corresponding Cartan decomposition is given by

$$\mathfrak{g}=\mathfrak{su}(p,q)=\mathfrak{k}\oplus\mathfrak{p},$$

where

$$\mathfrak{k}=\left\{X=\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}\mid A\in\mathfrak{u}(p),\ B\in\mathfrak{u}(q) \text{ and } \mathrm{Tr}\ (A)+\mathrm{Tr}\ (B)=0\right\}$$

and

$$\mathfrak{p}=\left\{X=\begin{pmatrix} 0 & Z\\ \overline{Z}^T & 0\end{pmatrix}\mid Z\in M_{p,q}(\mathbb{C})\right\}.$$

Let K be the compact Lie subgroup of $SU(p,q)$ with Lie algebra $\mathfrak{k}$. Then K is a maximal compact subgroup of $SU(p,q)$ and is given by

$$\begin{aligned}K&=S(U(p)\times U(q))\\ &=\left\{X=\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}\in SL(n,\mathbb{C})\mid A\in U(p),\ B\in U(q)\right\}.\end{aligned}$$

Then it can be seen that

$$SU(p,q)/S(U(p)\times U(q))\cong D_{p,q}^I,$$

where,

$$D^I_{p.q} = \left\{ A \in M\left(p,q,\mathbb{C}\right) \mid I_q - \overline{A}^T A > 0 \right\},$$

$M\left(p,q,\mathbb{C}\right)$ is the set of $p \times q$ matrices with complex entries, and $>$ stands for positive definite. Let

$$\mathfrak{u} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}.$$

Then $\mathfrak{u}$ is a Lie real subalgebra of $\mathfrak{sl}\left(n,\mathbb{C}\right)$. If we denote by U the Lie subgroup of $SL\left(n,\mathbb{C}\right)$ with Lie algebra $\mathfrak{u}$, then

$$U = SU(n).$$

Moreover, the symmetric space

$$\mathrm{Grass}\left(p,q\right) = SU(p+q)/S(U(p) \times U(q))$$

is the Grassmann manifold of q complex planes in $\mathbb{C}^{p+q}$. It can be shown, (see [9]), that the manifold $\mathrm{Grass}\left(p,q\right)$ is a compact Hermitian symmetric space dual of the noncompact Hermitian symmetric space $D^I_{p.q}$. Identifying $M\left(p,q,\mathbb{C}\right)$ with $\mathbb{C}^{pq}$, we get the Borel embedding, which is not needed in this paper,

$$D^I_{p.q} \subset \mathbb{C}^{pq} \subset \mathrm{Grass}\left(p,q\right).$$

Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. Then $\mathfrak{a}$ consists of matrices of the form

$$H\left(t_1,...,t_q\right) = \left(\begin{array}{c:c:c} 0 & 0 & \begin{matrix} & & t_1 \\ & \cdot^{\cdot^{\cdot}} & \\ t_q & & \end{matrix} \\ \hdashline 0 & 0 & 0 \\ \hdashline \begin{matrix} & & t_q \\ & \cdot^{\cdot^{\cdot}} & \\ t_1 & & \end{matrix} & 0 & 0 \end{array}\right).$$

The map $\left(t_1,...,t_q\right) \longrightarrow H\left(t_1,...,t_q\right)$ gives an identification of $\mathfrak{a}$ with $\mathbb{R}^q$. Let α_j be the element of $\mathfrak{a}^*$, the dual of $\mathfrak{a}$, defined by

$$\alpha_j\left(H\left(t_1,...,t_q\right)\right) = t_j.$$

Then the set of restricted roots, denoted by $\Phi\left(\mathfrak{g},\mathfrak{a}\right)$, consists of the following

$$\pm\alpha_i \text{ and } \pm 2\alpha_i,\ i = 1,...,q, \text{ and } \pm\left(\alpha_i \pm \alpha_j\right), 1 \le i < j \le q.$$

with multiplicities

$$m_{\alpha_i} = 2\left(p-q\right), m_{2\alpha_i} = 1, \text{ and } m_{\alpha_i \pm \alpha_j} = 2.$$

Hence, if $p = q$, then $\Phi\left(\mathfrak{g},\mathfrak{a}\right)$ consists of the following roots

$$\pm 2\alpha_i,\ i = 1,...,q, \text{ and } \pm\left(\alpha_i \pm \alpha_j\right), 1 \le i < j \le q.$$

With respect to the following Weyl chamber

$$\mathfrak{a}^+ = \left\{H\left(t_1,...,t_q\right) \in \mathfrak{a} \mid t_1 > t_2 > ... > t_q > 0\right\}$$

the simple roots consist of

$$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_{q-1} - \alpha_q,\ \alpha_q \text{ if } p > q,$$

and

$$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_{q-1} - \alpha_q,\ 2\alpha_q \text{ if } p = q.$$

4. Spherical Functions of the Grassmannian $\mathrm{Grass}(p,q) = SU(p+q)/S(U(p)\times U(q))$

A highest restricted spherical weight λ of the symmetric pair

$$(SU(p+q),\ S(U(p)\times U(q)))$$

corresponding to the spherical irreducible representation π_λ is of the form (see [4])

$$\lambda = \sum_{j=1}^{q} m_j^\lambda \alpha_j,$$

where m_j are even integers satisfying

$$m_1^\lambda \geq m_2^\lambda \geq \cdots \geq m_q^\lambda \geq 0.$$

Let

$$a_l = \exp\left(\sqrt{-1}H\left(t_1^l, ..., t_q^l\right)\right),$$

where $H\left(t_1^l, ..., t_q^l\right) \in \mathfrak{a}^+$, for $l = 1, ..., r$. If we denote the spherical function corresponding to the spherical representation π_λ by φ_λ, then by Berezin-Karpelevich formula (see [3], [8], [4]), we have

$$\varphi_\lambda\left(\exp\left(\sqrt{-1}H\left(t_1^l, ..., t_q^l\right)\right)\right) = \frac{\vartheta\left(t_1^l, ..., t_q^l\right) \det\left[\widetilde{P}_{n_j^\lambda}\left(\cos t_k^l\right)\right]_{1\leq j,k\leq q}}{\prod\limits_{1\leq j<k\leq q}\left(\sigma\left(n_j^\lambda\right) - \sigma\left(n_k^\lambda\right)\right)}, \tag{7}$$

where

$$\vartheta\left(t_1^l, ..., t_q^l\right) = \frac{2^{\frac{q(q-1)}{2}} \prod\limits_{j=1}^{q} j!\,(j+p-q)^{q-j}}{\prod\limits_{1\leq j<k\leq q}\left(\cos t_j^l - \cos t_k^l\right)}, \tag{8}$$

and the integers $n_1^\lambda, ..., n_p^\lambda$ are related to the indices m_j^λ of the weight π_λ above by

$$n_j^\lambda = m_j^\lambda + q - j,\ j = 1, \ldots, q,$$

hence $n_1^\lambda > n_2^\lambda > \cdots > n_q^\lambda \geq 0$, and $\sigma\left(n_j^\lambda\right) = n_j^\lambda\left(n_j^\lambda + p - q + 1\right)$,

$$\begin{aligned}\widetilde{P}_{n_j^\lambda}\left(\cos t_k^l\right) &= \frac{P_{n_j^\lambda}^{(p-q,0)}\left(\cos t_k^l\right)}{P_{n_j^\lambda}^{(p-q,0)}(1)} \\ &= {}_2F_1\left(-n_j^\lambda, n_j^\lambda + p - q + 1, p - q + 1, \sin^2\left(\frac{t_k^l}{2}\right)\right),\end{aligned} \tag{9}$$

where ${}_2F_1(.\,,\,.,.\,,.)$ is the Gauss hyper-geometric function and $P_j^{(a,b)}$ is the Jacobi polynomial. From now on, to simplify the notation, we put $m_j^\lambda = m_j$ and $n_j^\lambda = n_j$. Recall the following result of Darboux which gives the expansion of $P_j^{(a,b)}\left(\cos\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}t\right)$ for j large enough.

Lemma 4.1. (Darboux [5]) *Let a and b be real numbers. Then*

$$P_j^{(a,b)}\left(\cos\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}t\right)=\frac{\cos\left(\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}jt+\frac{1}{2}\left(a+b+1\right)\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}t-\frac{1}{2}a\pi-\frac{1}{4}\pi\right)+O\left(j^{-\frac{1}{2}}\right)}{\sqrt{j\pi}\left(\sin\frac{\langle\beta,\alpha\rangle}{2\langle\alpha,\alpha\rangle}t\right)^{a+\frac{1}{2}}\left(\cos\frac{\langle\beta,\alpha\rangle}{2\langle\alpha,\alpha\rangle}t\right)^{b+\frac{1}{2}}},$$

as $j\to\infty$ uniformly on any subinterval $\delta\le\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}t\le\pi-\delta,\ \delta>0$.

Lemma 4.2. *For j a sufficiently large integer, and t a real number such that $\delta\le t\le\pi-\delta,\ \delta>0$, we have*

$$\left|\widetilde{P}_j(\cos t)\right|\le\frac{\beta\left(t\right)}{j^{p-q+\frac{1}{2}}},$$

where $\widetilde{P}_j(\cos t)$ is defined as in (9).

Proof. Suppose that $j\neq 0,\ t>0,\ t\notin\pi\mathbb{Z}$, hence $\sin t\neq 0$. By Lemma 4.1, we have

$$\left|P_j^{(p-q,0)}(\cos t)\right|=\left|\frac{\cos(jt+\frac{1}{2}[p-q+1]t+\frac{1}{2}(p-q)\pi-\frac{1}{4}\pi)+O(j^{-\frac{1}{2}})}{\sqrt{j\pi}(\sin\frac{t}{2})^{p-q+\frac{1}{2}}(\cos\frac{t}{2})^{\frac{1}{2}}}\right|$$

$$\le\frac{c}{\sqrt{j}\left|(\sin\frac{t}{2})^{p-q+\frac{1}{2}}(\cos\frac{t}{2})^{\frac{1}{2}}\right|}\ \le\ \frac{c(t)}{\sqrt{j}},$$

where

$$c(t)=\frac{c}{\left|(\sin\frac{t}{2})^{p-q+\frac{1}{2}}(\cos\frac{t}{2})^{\frac{1}{2}}\right|}.$$

Since

$$P_j^{(p-q,0)}(1)=\frac{(p-q+1)_j}{j!},$$

and

$$\frac{j!}{(p-q+1)_j}=\frac{\Gamma(p-q+1)}{j^{p-q}},$$

where Γ is the Euler Gamma function given by $\Gamma\left(x\right)=\int_0^\infty t^{x-1}\exp\left(-t\right)dt$, we have

$$\left|\widetilde{P}_j(\cos t)\right|\le\frac{\beta(t)}{j^{p-q+\frac{1}{2}}},$$

where

$$\beta(t)=c(t)\Gamma(p-q+1).$$

■

Proposition 4.3. *Let* $\lambda = (m_1, \dots, m_q, 0, ..., 0)$, *with* $m_1 \geq \cdots \geq m_q \geq 0$, *and let* $n_j = m_j + q - j$. *For sufficiently large* n_j *we have*

$$\left|\varphi_{\pi_\lambda}\left(\exp\left(\sqrt{-1}H\left(t_1, ..., t_q\right)\right)\right)\right| \leq \begin{cases} \frac{C(t_1,...,t_q)}{\prod_{j=1}^{q} n_j^{\frac{1}{2}(2p-q)}} & \text{if } \lambda = (m_1, \dots, m_q, 0, ..., 0), \\ & \text{where } m_1 \geq \cdots \geq m_q > 0, \\ \frac{C(t_1,...,t_q)}{\prod_{j=1}^{q-1} n_j^{\frac{1}{2}(2p-q+3)}} & \text{if } \lambda = (m_1, \dots, m_q, 0, ..., 0), \\ & \text{where } m_1 \geq \cdots \geq m_q = 0. \end{cases}$$

Proof. For $j < k$, we have

$$\begin{aligned} \sigma\left(n_j\right) - \sigma\left(n_k\right) &\geq n_j\left(n_j + p - q + 1\right) - n_k\left(n_k + p - q + 1\right) \\ &\geq n_j^2 - n_k^2. \end{aligned} \tag{10}$$

Combining (7), (10), and Hadamard inequality we get,

$$\begin{aligned} \left|\varphi_{\pi_\lambda}\left(\exp\left(\sqrt{-1}H\left(t_1, ..., t_q\right)\right)\right)\right| &\leq \frac{\left|\vartheta\left(t_1, ..., t_q\right)\right|}{\prod_{1\leq j<k\leq q}\left(n_j^2 - n_k^2\right)}\left|\det\left[\widetilde{P}_{n_j}\left(\cos t_k\right)\right]_{1\leq j,k\leq q}\right| \\ &\leq \frac{\left|\vartheta\left(t_1, ..., t_q\right)\right|}{\prod_{1\leq j<k\leq q}\left(n_j^2 - n_k^2\right)}\prod_{j=1}^{q}\sqrt{\sum_{k=1}^{q}\left|\widetilde{P}_{n_j}\left(\cos t_k\right)\right|^2}. \end{aligned}$$

Now we have two cases

Case 1 ($\lambda = (m_1, m_2, \dots, m_q, 0, \dots, 0)$, where $m_q > 0$, hence also $n_q > 0$). *As a consequence of*

$$\begin{aligned} n_j^2 - n_k^2 &= \left(n_j - n_k\right)\left(n_j + n_k\right) && \textit{where } j < k \\ &\geq \left(n_j + n_k\right) && \textit{since } n_j - n_k \geq 1 \\ &\geq \sqrt{n_j n_k}, && \textit{since } \left(n_j + n_k\right) \geq 2\sqrt{n_j n_k}, \end{aligned}$$

$$\prod_{1\leq j<k\leq q} n_j n_k = \prod_{j=1}^{q-1}\prod_{k=j+1}^{q} n_j n_k = \prod_{j=1}^{q-1} n_j^{(q-j)} n_{j+1} \dots n_q = \prod_{j=1}^{q} n_j^{(q-1)},$$

and Lemma 4.2, we get

$$\left|\varphi_{\pi_\lambda}\left(\exp\left(\sqrt{-1}H\left(t_1, ..., t_q\right)\right)\right)\right| \leq \frac{\left|\vartheta\left(t_1, ..., t_q\right)\right|}{\prod_{1\leq j<k\leq q}\left(n_j^2 - n_k^2\right)}\prod_{j=1}^{q}\sqrt{\sum_{k=1}^{q}\left|\widetilde{P}_{n_j}\left(\cos t_k\right)\right|^2}$$

$$\leq \frac{|\vartheta(t_1,...,t_q)|}{\prod\limits_{1\leq j<k\leq q}(n_jn_k)^{\frac{1}{2}}}\prod_{j=1}^{q}\sqrt{\sum_{k=1}^{q}\left|\frac{\beta(t_k)}{n_j^{p-q+\frac{1}{2}}}\right|^2} = \frac{|\vartheta(t_1,...,t_q)|}{\prod_{j=1}^{q}n_j^{\frac{1}{2}(q-1)}}\prod_{j=1}^{q}\left(\frac{\beta(t_1,\ldots,t_q)}{n_j^{p-q+\frac{1}{2}}}\right)$$

$$= \frac{\beta(t_1,\ldots,t_q)^q\,|\vartheta(t_1,...,t_q)|}{\prod_{j=1}^{q}n_j^{\frac{1}{2}(q-1)}\prod\limits_{j=1}^{q}n_j^{p-q+\frac{1}{2}}} = \frac{C(t_1,\ldots,t_q)}{\prod_{j=1}^{q}n_j^{\frac{1}{2}(2p-q)}},$$

with

$$\beta(t_1,\ldots,t_q)=\sqrt{\sum_{k=1}^{q}|\beta(t_k)|^2}\ \textit{and}\ C(t_1,\ldots,t_q)=\beta(t_1,\ldots,t_q)^q\,|\vartheta(t_1,\ldots,t_q)|, \quad (11)$$

where $\vartheta(t_1,...,t_q)$ *is defined as in* (8)*.*

Case 2 ($\lambda = (m_1, m_2, \ldots, m_{q-1}, 0, \ldots, 0)$, hence $n_q = 0$)**.** *In this case we have*

$$n_j^2 - n_k^2 = \begin{cases} n_j^2 \ \textit{if}\ k = q, \\ \\ n_j^2 - n_k^2 \geq 2\sqrt{n_jn_k}\ \textit{otherwise.} \end{cases}$$

Since $\widetilde{P}_0(\cos t_k) = 1$*, again, as a consequence of Hadamard inequality, we get*

$$\left|\varphi_{\pi_\lambda}\left(\exp\left(\sqrt{-1}H(t_1,...,t_q)\right)\right)\right| \leq \frac{|\vartheta(t_1,...,t_q)|}{\prod\limits_{1\leq j<k\leq q}\left(n_j^2-n_k^2\right)}\left|\det\left[\widetilde{P}_{n_j}(\cos t_k)\right]_{1\leq j,k\leq q}\right|$$

$$\leq \frac{\sqrt{q}\,|\vartheta(t_1,...,t_q)|}{\prod\limits_{1\leq j<k\leq q}\left(n_j^2-n_k^2\right)}\prod_{j=1}^{q-1}\sqrt{\sum_{k=1}^{q}\left|\widetilde{P}_{n_j}(\cos t_k)\right|^2}$$

$$\leq \frac{\sqrt{q}\,|\vartheta(t_1,...,t_q)|}{\prod\limits_{1\leq j<k\leq q-1}\left(n_j^2-n_k^2\right)\prod\limits_{1\leq j\leq q-1}n_j^2}\prod_{j=1}^{q-1}\sqrt{\sum_{k=1}^{q}\left|\frac{\beta(t_k)}{n_j^{(p-q+\frac{1}{2})}}\right|^2}$$

$$\leq \frac{\sqrt{q}\,|\vartheta(t_1,...,t_q)|}{\prod\limits_{1\leq j<k\leq q-1}\sqrt{n_jn_k}\prod\limits_{1\leq j\leq q-1}n_j^2}\prod_{j=1}^{q-1}\left(\frac{\beta(t_1,\ldots,t_q)}{n_j^{p-q+\frac{1}{2}}}\right)$$

$$\leq \frac{C(t_1,\ldots,t_q)}{\prod\limits_{j=1}^{q-1}n_j^{\frac{q-2}{2}}\prod\limits_{j=1}^{q-1}n_j^2\prod\limits_{j=1}^{q-1}n_j^{p-q+\frac{1}{2}}} \leq \frac{C(t_1,\ldots,t_q)}{\prod\limits_{j=1}^{q-1}n_j^{\frac{1}{2}(2p-q+3)}},$$

with $C(t_1,\ldots,t_q) = \sqrt{q}\beta(t_1,\ldots,t_q)^{q-1}|\vartheta(t_1,...,t_q)|$*, where* $\beta(t_1,\ldots,t_q)$ *is as in* (11) *and* $\vartheta(t_1,...,t_q)$ *is defined as in* (8)*.* ∎

5. Proof of the Main Theorem

Proposition 5.1. *With the above notation defined, we have* $\|f_{a_1,\ldots,a_r}\|^2_{H^s(U)} \leq$

$$C\left(a_1,\ldots,a_r\right)\left(\sum_{n_1>n_2>\cdots>n_q>0}\frac{1}{n_1^{r(2p-q)-2s-\frac{n(n-1)}{2}}\prod_{j=2}^{q}n_j^{r(2p-q)}}\right.$$
$$\left.+\sum_{n_1>n_2>\cdots>n_q=0}\frac{1}{n_1^{r(2p-q+3)-2s-\frac{n(n-1)}{2}}\prod_{j=1}^{q-1}n_j^{r(2p-q+3)}}\right),$$

where $C\left(a_1,\ldots,a_r\right)$ *is a positive constant depending on* $a_1,\ldots,a_r$.

Proof. It is known that $\kappa_\lambda > 0$ (see [6]). By Cauchy Schwarz,

$$\kappa_\lambda = \langle\lambda+2\rho,\lambda\rangle \leq \|\lambda+2\rho\|\,\|\lambda\| \leq \|\lambda\|^2 + 2\,\|\rho\|\,\|\lambda\| \leq C\,\|\lambda\|^2,$$

where C is a positive constant,

$$\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha,$$

and $\Sigma^+ = \{\alpha_i,\, 2\alpha_i\ (i=1,\ldots,q),\, \alpha_i \pm \alpha_j,\, (1\leq i<j\leq q)\}$ is the set of positive restricted roots. Since on a finite dimensional vector space all norms are equivalent, after changing the constant, if necessary, we can assume that

$$\kappa_\lambda \leq C\,m_1^2 \leq C\,n_1^2. \tag{12}$$

Similarly, Weyl dimension formula (see [6]) gives us

$$d_\lambda \leq (1+\|\lambda\|)^{\frac{n(n-1)}{2}} \leq C\,m_1^{\frac{n(n-1)}{2}} \leq C\,n_1^{\frac{n(n-1)}{2}}. \tag{13}$$

By Proposition 2.4, the estimates (12), (13) and Proposition 4.3 we get

$$\|f_{a_1,\ldots,a_r}\|^2_{H^s(U)} \leq C\left(a_1,\ldots,a_r\right)\left(\sum_{n_1>n_2>\cdots>n_q>0}\frac{n_1^{2s+\frac{n(n-1)}{2}}}{\prod_{j=1}^{q}n_j^{r(2p-q)}}\right.$$
$$\left.+\sum_{n_1>n_2>\cdots>n_q=0}\frac{n_1^{2s+\frac{n(n-1)}{2}}}{\prod_{j=1}^{q-1}n_j^{r(2p-q+3)}}\right)$$
$$\leq C\left(a_1,\ldots,a_r\right)\left(\sum_{n_1>n_2>\cdots>n_q>0}\frac{1}{n_1^{r(2p-q)-2s-\frac{n(n-1)}{2}}\prod_{j=2}^{q}n_j^{r(2p-q)}}\right.$$
$$\left.+\sum_{n_1>n_2>\cdots>n_q=0}\frac{1}{n_1^{r(2p-q+3)-2s-\frac{n(n-1)}{2}}\prod_{j=1}^{q-1}n_j^{r(2p-q+3)}}\right). \qquad \blacksquare$$

Corollary 5.2. *If* $r > \dfrac{1+2s+\frac{n(n-1)}{2}}{2p-q}$, *then the function* $f_{a_1,\ldots,a_r}$ *is in* $H^s(U)$, *i.e.,* $\|f_{a_1,\ldots,a_r}\|_{H^s(U)} < \infty$.

Proof. Since
$$\sum_{n_1>\cdots>n_q>0} \frac{1}{n_1^{r(2p-q)-2s-\frac{n(n-1)}{2}} \prod_{j=2}^{q} n_j^{r(2p-q)}}$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n_1^{r(2p-q)-2s-\frac{n(n-1)}{2}}} \prod_{i=2}^{q} \left(\sum_{k=1}^{n_1} \frac{1}{k^{r(2p-q)}} \right)$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{n_1^{r(2p-q)-2s-\frac{n(n-1)}{2}}} \prod_{i=2}^{q} \left(\sum_{k=1}^{\infty} \frac{1}{k^{r(2p-q)}} \right),$$
and since $2p-q \geq p \geq 2$, hence $r(2p-q) \geq 2$, we deduce that the series $\sum_{k=1}^{\infty} \frac{1}{k^{r(2p-q)}}$ is convergent. Hence the series
$$\sum_{n_1>n_2>\cdots>n_q>0} \frac{1}{n_1^{r(2p-q)-2s-\frac{n(n-1)}{2}} \prod_{j=2}^{q} n_j^{r(2p-q)}}$$
is convergent for
$$r > \frac{1+2s+\frac{n(n-1)}{2}}{2p-q}.$$
A similar argument implies that the series
$$\sum_{n_1>n_2>\cdots>n_q=0} \frac{1}{n_1^{r(2p-q+3)-2s-\frac{n(n-1)}{2}} \prod_{j=2}^{q-1} n_j^{r(2p-q+3)}}$$
is convergent for
$$r > \frac{1+2s+\frac{n(n-1)}{2}}{2p-q+3}.$$
Hence the Corollary. ■

Proof of the Main Theorem. Note first that
$$\dim\left(SU(n)/S(U(p)\times U(q))\right) = 2pq.$$
Let s be a positive real number for which $C(p,q,s) \geq 2pq$ and let r be a positive integer such that $r > C(p,q,s)$. Then by , $\nu_{a_1\cdots a_r}$ is absolutely continuous with respect to the Haar measure of $SU(n)$, hence
$$d\nu_{a_1\cdots a_r} = f_{a_1,\ldots,a_r} d\mu_{SU(n)}.$$
Since $r > C(p,q,s)$, by Corollary 5.2, $f_{a_1,\ldots,a_r} \in H^s(U)$. Let k be a positive integer and choose the real number s as above and satisfying
$$s > k + \frac{n^2-1}{2}.$$
Then the Main Theorem is a consequence of the Sobolev Embedding Theorem ([1], Theorem 2). ■

Let k be a positive integer, such that

$$k > \frac{\dim SU(n)}{4} = \frac{n^2-1}{4}, \tag{14}$$

and let s be a real number such that

$$s > 2k + \frac{n^2-1}{2} \text{ and } C(p,q,s) \geq 2pq, \tag{15}$$

where $C(p,q,s)$ is as in (2). As a consequence of our Main Theorem and [12] (Theorem 1, part (1)), we get the following

Corollary 5.3. *Let k be a positive integer satisfying* (14) *and s a real number satisfying* (15)*. If $r > C(p,q,s)$, then the Fourier series*

$$\sum_{[\pi]\in\widehat{G}} d_\pi Tr\left(\pi\left(f_{a_1,\ldots,a_r}\right)\pi(g)\right)$$

of the density function $f_{a_1,\ldots,a_r}$ converges absolutely and uniformly to $f_{a_1,\ldots,a_r}$.

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Mahmoud Al-Hashami
Department of Mathematics
and Statistics
College of Science
Sultan Qaboos University
Oman
s100111@student.squ.edu.om

Boudjemâa Anchouche
Department of Mathematics
and Statistics
College of Science
Sultan Qaboos University
Oman
anchouch@squ.edu.om

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