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Conformal Vector Fields on Lorentzian Lie Groups of Dimension 4

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Abstract. In this paper, we classify Lorentzian Lie groups of dimension 4 which admit non-Killing left-invariant conformal vector fields.
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1. Introduction

Let $(M, \langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian manifold. A conformal vector field X on M is a vector field satisfying

$$\mathfrak{L}_X\langle\cdot,\cdot\rangle = 2\rho\langle\cdot,\cdot\rangle,$$

where $\mathfrak{L}_X\langle\cdot,\cdot\rangle$ is the Lie derivative and ρ is a smooth function. It connects with the topological structure of the pseudo-Riemannian manifold [7, 8]. An important class of conformal vector fields are *Killing vector fields*, i.e., vector fields such that $\mathfrak{L}_X\langle\cdot,\cdot\rangle = 0$. Killing vector fields provide a close link between the geometry of a manifold M and the algebra of $I(M)$ which is the set of all isometries in M (see [9]).

In this paper, we mainly study Lorentzian Lie groups, i.e. Lie groups with left-invariant Lorentzian metrics. In fact, there are many studies on homogeneous Lorentzian manifolds of dimensions 3 and 4. In dimension 3, similar to the result for Riemannian case in [10], it was proved in [3] that any non-symmetric three-dimensional homogeneous Lorentzian manifold is isometric to a Lorentzian Lie group. The above result also holds for four dimensional Riemannian Lie groups [2]. In [4], Calvaruso and Zaeim proved that for most of the Segre types of the Ricci operator, a four-dimensional locally homogeneous Lorentzian manifold is either Ricci-parallel, or locally isometric to a four-dimensional Lorentzian Lie group. Moreover, they classified four dimensional Einstein Lorentzian Lie groups and described four dimensional Lorentzian Lie group (see [5]).

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Recently, there are some studies on non-Killing conformal vector fields on pseudo-Riemannian Lie groups. Firstly, any left-invariant conformal vector field on a Riemannian Lie group is Killing. Furthermore, it is proved in [1] that any left-invariant conformal vector field on an unimodular pseudo-Riemannian Lie group is a Killing vector field. Also in [1], there is a non-Killing left-invariant conformal vector field on a Lorentzian Lie group of dimension 4.

In this paper, we classify Lorentzian Lie groups of dimension 4 admitting non-Killing left-invariant conformal vector fields, i.e. the following theorem.

Theorem 1.1. *Let G be a four dimensional Lorentzian Lie group with the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ admits a non-Killing left-invariant conformal vector field, then there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g}, \mathfrak{g}]$ is $\{e_1, e_2, e_3\}$, the Lorentzian metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and the non-zero brackets are one of the following cases:

(1) $[e_1, e_4] = \lambda_1 e_1 + \lambda_2 e_2$, $[e_2, e_4] = \lambda_1 e_2 - \lambda_2 e_1$, $[e_1, e_2] = \lambda e_3$, $[e_3, e_4] = 2\lambda_1 e_3$, *where $\lambda\lambda_1\lambda_2 \neq 0$. Here, the non-Killing conformal vector field is the constant multiple of e_4.*

(2) $[e_1, e_4] = \lambda_1 e_1$, $[e_2, e_4] = \lambda_1 e_2$, $[e_3, e_4] = 2\lambda_1 e_3$, $[e_1, e_2] = \lambda e_3$, *where $\lambda\lambda_1 \neq 0$. For this case, the non-Killing conformal vector field is the constant multiple of e_4.*

(3) $[e_1, e_2] = \lambda e_3$, $[e_4, e_3] = -2\lambda_1 e_3$, $[e_4, e_2] = \lambda_2 e_1 - \lambda_1 e_2 - \lambda_3 e_3$, $[e_4, e_1] = -\lambda_1 e_1 - \lambda_2 e_2 + (\lambda_4 - \frac{\lambda_2\lambda_3}{\lambda_1})e_3$, *where $\lambda\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0$. For this case, the non-Killing conformal vector field is the constant multiple of $e_1 - \frac{\lambda_2}{\lambda_1}e_2 + \frac{\lambda_2}{2\lambda_1}e_3 + \frac{\lambda}{\lambda_3}e_4$.*

(4) $[e_4, e_1] = -\lambda_1 e_1 + \lambda_2 e_3$, $[e_1, e_2] = \lambda e_3$, $[e_4, e_2] = -\lambda_1 e_2 - \frac{\lambda\lambda_2}{\lambda_1}e_3$, $[e_4, e_3] = -2\lambda_1 e_3$, *where $\lambda\lambda_1\lambda_2 \neq 0$. For this case, the non-Killing conformal vector field is the constant multiple of $\frac{\lambda_2}{\lambda_1}e_1 + \frac{\lambda_2^2}{2\lambda_1^2}e_3 + e_4$.*

(5) $[e_1, e_4] = \alpha e_1 + \beta e_2 + \gamma e_3$, $[e_2, e_4] = -\beta e_1 + \alpha e_2 + n e_3$, $[e_3, e_4] = 2\alpha e_3$, *where $\alpha \neq 0$, $\beta, \gamma, n \in \mathbb{R}$. For this case, the non-Killing conformal vector field is the constant multiple of $-2(\alpha\gamma + \beta n)e_1 + 2(\beta\gamma - \alpha n)e_2 + (\gamma^2 + n^2)e_3 + 2(\alpha^2 + \beta^2)e_4$.*

The paper is organized as follows. In Section 2, we recall some facts on non-Killing left-invariant conformal vector field on pseudo-Riemannian Lie groups. In Section 3, we first prove $\dim[\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g} - 1$ for a Lorentzian Lie group G admitting a non-Killing left-invariant conformal vector field, here $\mathfrak{g}$ is the Lie algebra of G. Furthermore, we classify the Lorentzian Lie group G admitting a non-Killing left-invariant conformal vector field of dimension 4, i.e. Theorem 1.1.

2. Preliminaries

Let G be a Lie group with the Lie algebra $\mathfrak{g}$ and let $\langle\cdot,\cdot\rangle$ be a left-invariant pseudo-Riemannian metric on G. Assume that ∇ is the Levi-Civita connection associated with $\langle\cdot,\cdot\rangle$. Then,

$$[X,Y] = \nabla_X Y - \nabla_Y X. \tag{2.1}$$

For a left-invariant metric $\langle\cdot,\cdot\rangle$ on G, we have

$$\langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle = 0, \tag{2.2}$$

for any $X,Y,Z \in \mathfrak{g}$. By (2.1) and (2.2),

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(\langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle),$$

where $X,Y,Z \in \mathfrak{g}$. Assume that $X \in \mathfrak{g}$ is a conformal vector field, i.e.

$$\mathfrak{L}_X\langle\cdot,\cdot\rangle = 2\rho\langle\cdot,\cdot\rangle. \tag{2.3}$$

It follows that, $0 = \mathfrak{L}_X\langle X,X\rangle = 2\rho|X|^2$.
If $\langle\cdot,\cdot\rangle$ is a left-invariant Riemannian metric, then $\rho = 0$ or $X \equiv 0$. That is, X is Killing or X is trivial. For this reason, we focus on a left-invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$.

Lemma 2.1 ([1]). *Let notations be as above. If $X \in \mathfrak{g}$ is a non-Killing conformal vector field, then X is a lightlike vector field, i.e., $\langle X,X\rangle = 0$.*

Lemma 2.2 ([1]). *Let G be an unimodular pseudo-Riemannian Lie group. Then any left-invariant conformal vector field on G is a Killing vector field.*

For non-unimodular pseudo-Riemannian Lie groups, we have the following result.

Lemma 2.3 ([1]). *Let G be a non-unimodular pseudo-Riemannian Lie group. If G admits a non-Killing left-invariant conformal vector field, then*

$$\dim C(\mathfrak{g}) \leq \min(p,q), \quad \dim[\mathfrak{g},\mathfrak{g}] \geq \dim\mathfrak{g} - \min(p,q).$$

Here $\mathfrak{g}$ is the Lie algebra of G, $C(\mathfrak{g})$ is the center of $\mathfrak{g}$ and (p,q) is the signature of the pseudo-Riemannian metric.

Also, there are non-Killing left-invariant conformal vector fields on non-unimodular Lorentzian Lie groups [1, 6].

3. The proof of Theorem 1.1

By Lemma 2.3, if a Lorentzian Lie group G admits a non-Killing left-invariant conformal vector field, then $\dim[\mathfrak{g},\mathfrak{g}] \geq \dim\mathfrak{g} - 1$. Furthermore, we have the following theorem.

Theorem 3.1. *Let G be a Lorentzian Lie group admitting a non-Killing left-invariant conformal vector field. Then* $\dim[\mathfrak{g},\mathfrak{g}] = \dim\mathfrak{g} - 1$.

Proof. Otherwise, $\dim[\mathfrak{g},\mathfrak{g}] = \dim\mathfrak{g}$. It implies $\mathfrak{g}$ is unimodular. Then the theorem follows from Lemma 2.2. ■

In the following, assume that G is a four dimensional Lorentzian Lie group admitting a non-Killing left-invariant conformal vector field. Then the Lie algebra $\mathfrak{g}$ of G is non-unimodular and $\dim[\mathfrak{g},\mathfrak{g}] = 3$. Furthermore, we have the following lemma.

Lemma 3.2. *The 4-dimensional Lorentzian Lie group G admitting a non-Killing left-invariant conformal vector field is solvable.*

Proof. Otherwise, the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is $\mathfrak{s} \times \mathfrak{r}$. Here $\mathfrak{s}$ is the 3-dimensional simple Lie subalgebra and $\mathfrak{r}$ is the 1-dimensional radical of $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}}$ is unimodular. That is, $\mathfrak{g}$ is unimodular. Then the lemma follows from Lemma 2.2. ■

Let G be a four dimensional Lorentzian Lie group, Then there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g},\mathfrak{g}]$ is $\{e_1, e_2, e_3\}$ and Lorentzian metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.4}$$

Assume that G admits a non-Killing conformal vector field $X = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4$. Then X is lightlike. Set $adX(e_j) = \sum_{i=1}^{4} a_{ij} e_i, x_i, a_{ij} \in \mathbb{R}, i, j \in \{1, 2, 3, 4\}$. By

$$-\langle [X, e_i], e_j \rangle - \langle e_i, [X, e_j] \rangle = 2\rho \langle e_i, e_j \rangle,$$

we have:

$$\begin{aligned} &a_{11} = a_{22} = -\rho, \quad a_{31} = a_{14}, \quad a_{32} = a_{24}, \quad a_{21} = -a_{12}, \\ &a_{33} = -2\rho, \quad a_{13} = a_{23} = a_{34} = 0, \quad a_{4i} = 0, \quad i = 1, 2, 3, 4. \end{aligned}$$

Denote a_{12}, a_{14}, a_{24} by ζ, η, θ respectively, with the basis $\{e_1, e_2, e_3, e_4\}$, we have:

$$adX = \begin{pmatrix} -\rho & \zeta & 0 & \eta \\ -\zeta & -\rho & 0 & \theta \\ \eta & \theta & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, we can change the position of η and θ in the above matrix by changing the order of e_1 and e_2 in the basis. Furthermore, we can adjust the basis of $\mathfrak{g}$ such that $\eta\theta = 0$. Otherwise, $\eta\theta \neq 0$. Then under the basis $\{e_1', e_2' = e_2 - \frac{\theta}{\eta}e_1, e_3, e_4\}$, the Lorentzian metric is defined by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and the matrix expression of adX is

$$\begin{pmatrix} -\rho & \zeta & 0 & \eta \\ -\zeta & -\rho & 0 & 0 \\ \eta & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

We still denote $\{e_1', e_2', e_3, e_4\}$ by $\{e_1, e_2, e_3, e_4\}$.

By Lemma 3.2, we know $\mathfrak{g}$ is solvable. Therefore, $[\mathfrak{g},\mathfrak{g}]$ is a 3-dimensional nilpotent Lie algebra. Firstly, we consider the case when $[\mathfrak{g},\mathfrak{g}]$ is not abelian. That is, $[\mathfrak{g},\mathfrak{g}]$ is a 3-dimensional Heisenberg Lie algebra. Let $\{x, y, z\}$ be a basis of $[\mathfrak{g},\mathfrak{g}]$ satisfying $[x, y] = z$. Then we have

$$adX(z) = adX[x, y] = [adX(x), adX(y)] = kz.$$

That is, the center z of $[\mathfrak{g},\mathfrak{g}]$ is the eigenvector of adX.

Case I: $\eta = 0$

For this case, if $\zeta \neq 0$, we know that there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g},\mathfrak{g}]$ is $\{e_1, e_2, e_3\}$ and the Lorentzian metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by (3.1). Moreover, relative to the basis $\{e_1, e_2, e_3, e_4\}$

$$adX = \begin{pmatrix} -\rho & \zeta & 0 & 0 \\ -\zeta & -\rho & 0 & 0 \\ 0 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that e_3 is the center of $[\mathfrak{g},\mathfrak{g}]$, and $\langle e_3, e_3\rangle = 0$. Set $[e_1, e_2] = \lambda e_3, (\lambda \neq 0)$. Clearly, we have

$$[X, e_1] = -\lambda\beta e_3 + \delta \cdot [e_4, e_1] = -\rho e_1 - \zeta e_2.$$

Notice $\zeta \neq 0$, we have $\delta \neq 0$, so

$$[e_1, e_4] = \frac{\rho e_1 + \zeta e_2 - \lambda\beta e_3}{\delta}.$$

Similarly, we have

$$[e_2, e_4] = \frac{\rho e_2 - \zeta e_1 + \lambda\alpha e_3}{\delta}, \quad [e_3, e_4] = \frac{2\rho e_3}{\delta}.$$

Since $-\langle [X, e_i], X\rangle = 2\rho\langle e_i, X\rangle$, $i = 1, 2, 3, 4$, we obtain

$$\rho\alpha = \zeta\beta, \quad \rho\beta = -\zeta\alpha, \quad 2\rho\gamma = 0.$$

Notice $\rho \neq 0$, we have $\alpha = \beta = \gamma = 0$. So the non-zero brackets are the following

$$[e_1, e_4] = \frac{\rho e_1 + \zeta e_2}{\delta}, \qquad [e_2, e_4] = \frac{\rho e_2 - \zeta e_1}{\delta},$$
$$[e_1, e_2] = \lambda e_3, \qquad [e_3, e_4] = \frac{2\rho e_3}{\delta},$$

and the non-Killing conformal vector field $X = \delta e_4$. Replacing $\frac{\rho}{\delta}$, $\frac{\zeta}{\delta}$ by λ_1 and λ_2, we have the case (1) of Theorem 1.1.

If $\zeta = 0$, relative to the basis $\{e_1, e_2, e_3, e_4\}$,

$$adX = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 \\ 0 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By $-\langle [X, e_i], X\rangle = 2\rho\langle e_i, X\rangle$, $i = 1, 2, 3, 4$, we obtain $\alpha = \beta = \gamma = 0$. Now we assert that e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$.

To see this, we suppose z is the the center of $[\mathfrak{g}, \mathfrak{g}]$, set $[e_1, e_3] = \lambda z$, $\lambda \in \mathbb{R}$, by Jacobi identity, we have

$$[X, [e_1, e_3]] = [[X, e_1], e_3] + [e_1, [X, e_3]] = -\rho[e_1, e_3] - 2\rho[e_1, e_3] = -3\rho\lambda z.$$

Since z is the eigenvector of adX, and the eigenvalues of the operator adX are $-\rho$ and -2ρ, thus, $[X, [e_1, e_3]] = [X, \lambda z] = k\lambda z$, where $k = -\rho$, or $k = -2\rho$. Thus, $[e_1, e_3] = 0$, and we can also obtain $[e_2, e_3] = 0$. So e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$. In this case, the non-zero brackets are the following:

$$[e_1, e_4] = \frac{\rho e_1}{\delta}, \qquad [e_2, e_4] = \frac{\rho e_2}{\delta},$$
$$[e_3, e_4] = \frac{2\rho e_3}{\delta}, \qquad [e_1, e_2] = \lambda e_3, \lambda \neq 0,$$

the non-Killing conformal vector field $X = \delta e_4$. Replacing $\frac{\rho}{\delta}$ by λ_1, we have the case (2) of Theorem 1.1.

Case II: $\eta \neq 0$

For this case, if $\zeta \neq 0$, we know there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g}, \mathfrak{g}]$ is $\{e_1, e_2, e_3\}$ and Lorentzian metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by (3.1). Moreover, relative to the basis $\{e_1, e_2, e_3, e_4\}$

$$adX = \begin{pmatrix} -\rho & \zeta & 0 & \eta \\ -\zeta & -\rho & 0 & 0 \\ \eta & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

It is easy to see e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$. Set $[e_1, e_2] = \lambda e_3$ where $\lambda \neq 0$. Since $-\langle [X, e_i], X\rangle = 2\rho\langle e_i, X\rangle$, $i = 1, 2, 3, 4$, we obtain

$$\beta = -\frac{\zeta\alpha}{\rho}, \quad \gamma = \frac{\eta\alpha}{2\rho}, \quad \delta = \frac{(\rho^2 + \zeta^2)\alpha}{\rho\eta}. \tag{3.7}$$

By (3.3), (3.4), we can obtain the non-zero brackets relations:

$$[e_4, e_3] = \frac{-2\rho^2\eta e_3}{(\rho^2+\zeta^2)\alpha}, \quad [e_1, e_2] = \lambda e_3, \lambda \neq 0,$$
$$[e_4, e_2] = \frac{\rho\eta(\zeta e_1 - \rho e_2 - \lambda\alpha e_3)}{(\rho^2+\zeta^2)\alpha},$$
$$[e_4, e_1] = \frac{-\rho^2\eta e_1 - \rho\eta\zeta e_2 + (\rho\eta^2 - \lambda\eta\zeta\alpha)e_3}{(\rho^2+\zeta^2)\alpha},$$

and the non-Killing conformal vector field

$$X = \alpha[e_1 - \frac{\zeta}{\rho}e_2 + \frac{\zeta}{2\rho}e_3 + \frac{(\rho^2+\zeta^2)}{\rho\eta}e_4].$$

Replacing $\frac{\rho^2\eta}{(\rho^2+\zeta^2)\alpha}$, $\frac{\rho\eta\zeta}{(\rho^2+\zeta^2)\alpha}$, $\frac{\lambda\rho\eta}{\rho^2+\zeta^2}$, $\frac{\rho\eta^2}{(\rho^2+\zeta^2)\alpha}$ by λ_1, λ_2, λ_3, λ_4, we have the case (3) of Theorem 1.1.

If $\zeta = 0$, we know there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the basis of $[\mathfrak{g}, \mathfrak{g}]$ is $\{e_1, e_2, e_3\}$ and Lorentzian metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is defined by (3.1). Moreover, relative to the basis $\{e_1, e_2, e_3, e_4\}$

$$adX = \begin{pmatrix} -\rho & 0 & 0 & \eta \\ 0 & -\rho & 0 & 0 \\ \eta & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

It is easy to see that the eigenvalues of the operator adX are $-\rho, -2\rho$. The corresponding eigenvectors are $k_1e_2 + k_2(e_1 + \frac{\eta}{\rho}e_3)$, ke_3, here $k_1, k_2, k \in \mathbb{R}$. Since

$$-\langle [X, e_i], X\rangle = 2\rho\langle e_i, X\rangle, \quad i = 1, 2, 3, 4.$$

We obtain

$$\alpha = \frac{\eta\delta}{\rho}, \quad \beta = 0, \quad \gamma = \frac{\eta^2\delta}{2\rho^2}. \tag{3.9}$$

Notice the center of $[\mathfrak{g}, \mathfrak{g}]$ is the eigenvector adX. By the same argument in case (2), we know that e_3 is the center of $[\mathfrak{g}, \mathfrak{g}]$. Set $[e_1, e_2] = \lambda e_3$ where $\lambda \neq 0$. By (3.5), (3.6), we can obtain the non-zero brackets:

$$[e_4, e_1] = \frac{-\rho e_1 + \eta e_3}{\delta}, \qquad [e_1, e_2] = \lambda e_3, \lambda \neq 0,$$
$$[e_4, e_2] = \frac{-\rho^2 e_2 - \lambda\eta\delta e_3}{\rho\delta}, \quad [e_4, e_3] = \frac{-2\rho e_3}{\delta},$$

and the non-Killing conformal vector field

$$X = \delta(\frac{\eta e_1}{\rho} + \frac{\eta^2 e_3}{2\rho^2} + e_4).$$

Replacing $\frac{\varrho}{\delta}$, $\frac{\eta}{\delta}$ by λ_1 and λ_2, we have the case (4) of Theorem 1.1.

Finally, we study the case when $[\mathfrak{g}, \mathfrak{g}]$ is abelian. First, there is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the Lorentzian metric associated with the basis $\{e_1, e_2, e_3, e_4\}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and the basis of $[\mathfrak{g}, \mathfrak{g}]$ is e_1, e_2, e_3. Let non-zero Lie brackets be the following:

$$[e_1, e_4] = \alpha e_1 + \beta e_2 + \gamma e_3, \quad [e_2, e_4] = q e_1 + m e_2 + n e_3, \quad [e_3, e_4] = w e_1 + v e_2 + y e_3.$$

Notice $\mathfrak{g}$ admits a non-Killing conformal vector field X. So $\mathfrak{g}$ is non-unimodular, i.e. $\alpha + m + y \neq 0$. Suppose $X = \sum_{i=1}^{4} x_i e_i$ is a left-invariant conformal vector field. Then we have

$$0 = \mathfrak{L}_X \langle \cdot, \cdot \rangle - 2\rho \langle \cdot, \cdot \rangle = \begin{pmatrix} 2\alpha x_4 - 2\rho & \beta x_4 + q x_4 & w x_4 & A \\ betax_4 + q x_4 & 2m x_4 - 2\rho & v x_4 & B \\ w x_4 & v x_4 & 0 & -y x_4 + 2\rho \\ A & B & -y x_4 + 2\rho & C \end{pmatrix}$$

where $A = -\gamma x_4 - \alpha x_1 - q x_2 - w x_3$, $B = -n x_4 - \beta x_1 - m x_2 - v x_3$, and $C = 2(\gamma x_1 + n x_2 + y x_3)$. If $y x_4 = 0$, then $\rho = 0$, i.e. X is a Killing vector field. Since X is non-Killing, we have $y x_4 \neq 0$. This implies that $w = v = 0$, $\rho = \alpha x_4$, $\beta = -q$, $\alpha = m = \frac{1}{2} y \neq 0$. Moreover,

$$X = x_4[-\frac{\alpha\gamma + \beta n}{\alpha^2 + \beta^2} e_1 + \frac{\beta\gamma - \alpha n}{\alpha^2 + \beta^2} e_2 + \frac{\gamma^2 + n^2}{2(\alpha^2 + \beta^2)} e_3 + e_4],$$

which is the case (5) of Theorem 1.1. In summary, we have Theorem 1.1.

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