

Isometries of sub-Riemannian Metrics Supported on Martinet Type Distributions

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Abstract. We study 3-dimensional sub-Riemannian manifolds (M, H, g) , where the distribution H , is of Martinet type. Our main concern is to describe the isometries of such manifolds, and to estimate the dimension of the isometry group.

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1. Introduction and statement of the results

Let M be a smooth connected manifold. By a *sub-Riemannian structure* or a *sub-Riemannian metric* on M , we mean a couple (H, g) , where H is a smooth bracket generating distribution of constant rank on M and g is a smooth Riemannian metric on H . The bracket generating property means that for every point q of M , the vector fields defined around q and tangent to H , along with the iterated Lie brackets thereof evaluated at q , span the tangent space T_qM . A triple (M, H, g) , made up of a manifold M and a sub-Riemannian structure (H, g) on it, is called a *sub-Riemannian manifold*.

Fix a sub-Riemannian manifold (M, H, g) . An absolutely continuous curve $\gamma: [a, b] \rightarrow M$ is said to be *horizontal* if $\dot{\gamma}(t) \in H_{\gamma(t)}$ a.e. on $[a, b]$. The number

$$L(\gamma) = \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt,$$

is called the *length* of the curve γ . The following facts are well-known. Under the bracket generating assumption, any two points of M can be joined by a horizontal curve. If $q_1, q_2 \in M$, then the formula

$$d(q_1, q_2) = \inf L(\gamma),$$

where the infimum is taken over all horizontal curves joining q_1 to q_2 , defines a metric on M , and the topology induced by this metric agrees with the manifold

topology of M . Horizontal curves that realize the distance between the points they connect, are called (*length*) *minimizers* or *minimizing geodesics*. By a *geodesic* we mean a horizontal curve, whose each sufficiently short subarc is a minimizer. Suppose that $\{X_1, \dots, X_k\}$ is an orthonormal basis for (H, g) defined on an open set U . The function $\mathcal{H}: T^*U \rightarrow \mathbb{R}$ given by

$$\mathcal{H}(q, p) = \sum_{i=1}^k \langle p, X_i(q) \rangle^2,$$

is called the *geodesic Hamiltonian* (see e.g. [14] for the global definition of \mathcal{H}). Denote by $\vec{\mathcal{H}}$ the Hamiltonian vector field corresponding to the function \mathcal{H} ; by definition, $d\mathcal{H}(V) = \Omega(V, \vec{\mathcal{H}})$, where Ω is the standard symplectic form on T^*M and $V \in T(T^*M)$. The projections onto M of trajectories of the vector field $\vec{\mathcal{H}}$ are called *Hamiltonian* or *normal geodesics*. It can be proved (see [14]) that each sufficiently short subarc of a Hamiltonian geodesic is a minimizer. The *exponential mapping* with pole at a given point q , denoted \exp_q , is defined as follows. Let Ψ^t denote the (local) flow of the field $\vec{\mathcal{H}}$, and for each point $q \in M$, let

$$\mathcal{D}_q = \{\lambda \in T_q^*M : \Psi^t(\lambda) \text{ is defined on } [0, 1]\}.$$

Then $\exp_q: \mathcal{D}_q \rightarrow M$ acts by the formula

$$\exp_q(\lambda) = \pi \circ \Psi^1(\lambda),$$

where $\pi: T^*M \rightarrow M$ is the canonical projection. It is a standard fact that \exp_q is smooth and is not a diffeomorphism at its pole.

The sub-Riemannian geometry constitutes a natural generalization of the Riemannian geometry, so has similar facets that need to be addressed. One of the most important problems studied in the Riemannian case concerns the computation of isometries of a given Riemannian metric. The notion of an isometry admits an obvious extension to the sub-Riemannian case. More precisely, if (M, H, g) is a fixed sub-Riemannian manifold, then an *isometry of (M, H, g)* is understood as a diffeomorphism $f: M \rightarrow M$ such that

- (i) $d_q f(H_q) \subset H_{f(q)}$,
- (ii) $g(d_q f(v), d_q f(w)) = g(v, w)$

for every $q \in M$ and $v, w \in H_q$. Obviously isometries preserve the length of horizontal curves, and the distance d defined above. In particular they map geodesics to geodesics. The group of isometries will be denoted by $\mathcal{I}(M, H, g)$ and the isotropy subgroup of a point $q \in M$ by $\mathcal{I}(M, H, g)_q$. We will also write $\mathcal{I}_0(M, H, g)$ for the component of the identity in $\mathcal{I}(M, H, g)$.

The simplest sub-Riemannian manifolds are of course those for which $\dim M = 3$, and from now we assume $\dim M = 3$. Among such manifolds, the easiest to deal with are those equipped with either a contact or a Martinet distribution which are defined as follows. Suppose that on a neighborhood U of a fixed point $q_0 \in M$,

H is given as the kernel of a smooth 1-form ω . Clearly, $\omega \wedge d\omega = \Phi\Omega$, where $\Phi: U \rightarrow \mathbb{R}$ is a smooth function, and Ω is a volume form on U . If Φ vanishes nowhere, H is said to be a *contact distribution* on U . Suppose now that Φ vanishes in U , but its differential satisfies $d\Phi_{\{\Phi=0\}} \neq 0$. Then $S_U = \{\Phi = 0\}$ is a smooth hypersurface, which at generic points, is transverse to H . The latter statement is equivalent to $\omega \wedge d\Phi \neq 0$ at generic points of S_U . We say that H is of *Martinet-type*, or simply a *Martinet distribution* on U , if the condition

$$\omega \wedge d\Phi \neq 0$$

holds everywhere on S_U . Finally we say that H is contact (resp. Martinet) on M if it is so in a neighborhood of any point of M . Note that the surfaces S_U are independent of the 1-forms ω used to define it, hence they glue together to form a global hypersurface in M which will be denoted by S , and called the *Martinet surface* of H . A sub-Riemannian manifold (M, H, g) is called a *contact* (resp. *Martinet*) *sub-Riemannian manifold* if H is a contact (resp. Martinet) distribution.

The contact case is well studied. For instance, in the papers [1, 5, 2] the authors study normal forms and isometric types of contact sub-Riemannian structures in dimension 3 (similar studies for Lorentzian metrics can be found in [6, 11]). Contact structures in higher dimensions are studied in [3] and in [9, 10] (the latter papers also deal with indefinite metrics). On the other hand, in the paper [4], the authors investigated geodesics, spheres and wave fronts for the flat sub-Riemannian metrics in the Martinet case. Also [1, 4] present normal forms for sub-Riemannian metrics on Martinet type distributions (see [8] for analogous normal forms in the Lorentzian signature).

The aim of this paper is to estimate the dimension of the group of isometries of Martinet sub-Riemannian manifolds. More precisely, our main result is enclosed in the following theorem:

Theorem 1.1. *Let (M, H, g) be a complete Martinet sub-Riemannian manifold. Then the group $\mathcal{I}(M, H, g)$ is a Lie group with respect to the compact-open topology and $\dim \mathcal{I}(M, H, g) \leq 2$. Moreover, every isometry is uniquely determined by its value at a given point. In particular, the isotropy subgroup $\mathcal{I}(M, H, g)_q$ is trivial for every $q \in M$.*

We also have the following interpretation of the action of the isometry group:

Proposition 1.2. *Suppose that (M, H, g) is a Martinet sub-Riemannian manifold and let S be the Martinet surface. Then a neighbourhood of S has a structure of a locally trivial fibration $E \rightarrow S$ with 1-dimensional fibers. The group $\mathcal{I}(M, H, g)$ acts on E via the fiber maps.*

2. The content of the paper

In Section 3 we construct normal forms for Martinet sub-Riemannian structures. Among many available such forms, we choose those which are the most convenient. In Section 4, we show that the isometry group for a complete Martinet

sub-Riemannian structure is isomorphic to the group of isometries of the structure which arises after removal of the Martinet surface. In Section 5, we estimate the dimension of the isometry group for oriented Martinet sub-Riemannian manifolds. This is done by constructing the invariant frame for isometries and computing its structure functions. In Section 6, we estimate the dimension of the isometry group for general Martinet sub-Riemannian manifolds. Section 7 is devoted to the description of infinitesimal isometries. Section 8 gives some partial results concerning the dependence of the dimension of the isometry group on the coefficients appearing in the normal forms, and finishes with some examples.

3. Normal forms

Let (M, H, g) be a Martinet sub-Riemannian manifold, and let S be the Martinet surface. Every Martinet sub-Riemannian structure can be locally put into one of the possible normal forms given in [4, 8], where the normal form from the latter paper can also be applied in the sub-Riemannian case. We choose normal forms which are most convenient for our purposes.

Theorem 3.1. *Suppose that (M, H, g) is a smooth Martinet sub-Riemannian manifold and let S be the Martinet surface for H . Then around any point $q \in S$ there exist smooth coordinates (x, y, z) in which $x(q) = y(q) = z(q) = 0$ and H admits an orthonormal frame (X, Y) in the following normal form:*

$$X = (1 + yA)\frac{\partial}{\partial x} + y^2(1 + xB_1 + yB_2)\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}, \quad (1)$$

where A, B_1, B_2 are smooth functions defined in a neighborhood of the origin, and $S = \{y = 0\}$.

Proof. The idea comes from [1] and the details are found in [7]. For convenience we recall the proof. Fix a point $q \in S$. Since we are considering a local problem, we can assume that $M = \mathbb{R}^3$ and $q = 0$. We start by choosing coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ around zero such that $S = \{\tilde{y} = 0\}$ and $\frac{\partial}{\partial \tilde{x}}|_S, \frac{\partial}{\partial \tilde{y}}|_S$ form an orthonormal frame for $H|_S$. Next, let us take the dual coordinates $p_{\tilde{x}}, p_{\tilde{y}}, p_{\tilde{z}}$ and consider the following subset of the cotangent bundle T^*M :

$$A = \{(\tilde{x}, 0, \tilde{z}, 0, p_{\tilde{y}}, 0) : |\tilde{x}|, |\tilde{z}| < \varepsilon\},$$

where ε is a small positive number. Next define a map $\mu: A \rightarrow \mu(A) \subset M$ by the formula

$$\mu(\tilde{x}, \tilde{z}, p_{\tilde{y}}) = \exp_{(\tilde{x}, 0, \tilde{z})}(0, p_{\tilde{y}}, 0).$$

Clearly there exists a neighborhoods N of the set $\{(\tilde{x}, 0, \tilde{z}, 0, 0, 0) : |\tilde{x}|, |\tilde{z}| < \varepsilon\}$ in A on which μ is a diffeomorphism onto an open set $U = \mu(N) \subset M$. It follows that the map

$$U \xrightarrow{\mu^{-1}} N \xrightarrow{T} \mathbb{R}^3,$$

where $T(a, b, c) = (a, c, b)$ and where we identify N with the suitable open set in \mathbb{R}^3 , defines coordinates (x, y, z) on U , with the property that a point q has

coordinates (x, y, z) if and only if $q = \exp_{(x,0,z)}(0, y, 0)$. Such coordinates will be called *normal*. Note that in terms of these normal coordinates, the Martinet surface is given in U by $S = \{y = 0\}$. Let us also notice that the straight lines

$$t \longrightarrow (x, t, z)$$

are Hamiltonian geodesics which satisfy the transversality condition with respect to the Martinet surface S , and are parametrized by arc-length. Indeed, $\frac{d}{dt}(x, t, z) = (0, 1, 0)$ is a unit horizontal vector for $t = 0$ and thus for every t by elementary properties of Hamiltonian geodesics. In particular, it follows that $Y = \frac{\partial}{\partial y}$ is the unit horizontal vector field. Now it is not hard to see (cf. [1] and [7]) that if X, Y is an orthonormal frame for (H, g) defined on U then X is of the form

$$X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial z}, \quad (2)$$

where a, b are smooth functions. Recalling the fact that $\text{Span}\{X, Y\}$ is a Martinet distribution, we have

- (i) $[X, Y](q) \in H_q$,
- (ii) $\text{Span}\{X(q), Y(q), [X, [X, Y]](q), [Y, [X, Y]](q)\} = T_q M$ for every $q \in S$.

Using these facts, we deduce that a and b in (2) have the form as in (1), and the proof is complete. \blacksquare

4. Isometries in the Martinet case

Suppose that (M, H, g) is a Martinet sub-Riemannian manifold. As usual, by S we denote the Martinet surface of H and observe that $(M \setminus S, H_{M \setminus S}, g)$ is a contact manifold. From the very definition of a Martinet distribution, it follows that the correspondence

$$S \ni q \longrightarrow H_q \cap T_q M$$

is a non-degenerate line field on S . The trajectories of this line field are known to be the so-called *abnormal curves* (see [15] for a definition and properties of abnormal curves).

Suppose that $f: (M, H, g) \longrightarrow (M, H, g)$ is an isometry. The definition of abnormal curves implies that f maps the abnormal curves to abnormal curves, and consequently S to S . It follows that

$$f: (M \setminus S, H_{M \setminus S}, g) \longrightarrow (M \setminus S, H_{M \setminus S}, g),$$

i.e., every isometry of the Martinet structure (H, g) induces an isometry of the contact structure $(H_{M \setminus S}, g)$.

From now on we assume that the distance d induced by the structure (H, g) is *complete*. In such a case, we say that (M, H, g) is a complete sub-Riemannian manifold. Under the assumption of completeness, we can prove the following converse result. Recall that $\mathcal{I}_0(M \setminus S, H_{M \setminus S}, g)$ stands for the component of the identity in $\mathcal{I}(M \setminus S, H_{M \setminus S}, g)$.

Proposition 4.1. *Let (M, H, g) be a complete Martinet sub-Riemannian manifold. Suppose that $f: (M \setminus S, H_{M \setminus S}, g) \rightarrow (M \setminus S, H_{M \setminus S}, g)$ is an element of $\mathcal{I}_0(M \setminus S, H_{M \setminus S}, g)$. Then there exists an isometry $F: (M, H, g) \rightarrow (M, H, g)$ such that $F|_{M \setminus S} = f$.*

Proof. Fix a point $q_0 \in S$ and take an arbitrary sequence $\{q_n\} \in M \setminus S$ such that $q_n \rightarrow q_0$. Since f preserves the distance d , $\{f(q_n)\}$ is a Cauchy sequence. It follows that the limit $\lim f(q_n)$ exists and depends on q_0 only. Denote this limit by $F(q_0)$. Repeating the same procedure to any element of S , and letting $F(q) = f(q)$ for every $q \in M \setminus S$, we obtain the continuous extension F of f , which is defined on the whole of M . Using a continuity argument, it is clear that $d(F(q_1), F(q_2)) = d(q_1, q_2)$ for every $q_1, q_2 \in M$.

Fix again a point $q \in S$ and a sufficiently small neighborhood U admitting normal coordinates x, y, z centered at q . Fix also a neighborhood \mathcal{U} of the identity in $\mathcal{I}_0(M \setminus S, H_{M \setminus S}, g)$, and suppose for the while that $f \in \mathcal{U}$. Now, if \mathcal{U} is small enough, we can assume that our normal coordinates x, y, z centered at q are defined on $f(U)$.

Next let us observe that for any point in S , up to a reparametrization, there exists exactly one Hamiltonian geodesic starting from this point and satisfying the transversality condition with respect to S . Since the isometries preserve the Martinet surface S , and transform unit-speed Hamiltonian geodesics satisfying the transversality condition with respect to S onto unit-speed Hamiltonian geodesics satisfying the transversality condition with respect to S , it follows that f preserves the field $\frac{\partial}{\partial y}$ when restricted to $M \setminus S$. Hence, in the normal coordinates, f has the following form on $U \setminus S$:

$$f(x, y, z) = (f^1(x, z), y, f^3(x, z)). \quad (3)$$

We conclude from (3), that f has a smooth extension to U . Since q was an arbitrary point of S , we conclude that F is smooth on M . Finally, using continuity of the differential $d_q F(v)$ with respect to both variables, it is easy to see that F is an isometry.

To end the proof, notice that if $f \in \mathcal{I}_0(M \setminus S, H_{M \setminus S}, g)$ is an arbitrary element, then $f = f_1 \circ \dots \circ f_k$ for certain $f_1, \dots, f_k \in \mathcal{U}$, and each f_i admits a smooth extension to an isometry of (M, H, g) , $i = 1, \dots, k$. ■

Since the group of isometries of a contact sub-Riemannian manifold is a Lie group with respect to the compact-open topology (cf. [9, 10]), as a corollary we obtain:

Proposition 4.2. *The group of isometries $\mathcal{I}(M, H, g)$ of a Martinet sub-Riemannian manifold (M, H, g) is a Lie group with respect to the compact-open topology. Moreover, for every point $q \in M$, the isotropy subgroup $\mathcal{I}(M, H, g)_q$ of q is compact.*

We will see below that $\mathcal{I}(M, H, g)_q$ is actually trivial.

Remark 4.3. *Suppose that $F \in \mathcal{I}(M, H, g)$ is sufficiently close to the identity, so it maps a domain of normal coordinates into itself. Since F maps abnormal curves with arc-length parameter to abnormal curves with arc-length parameter, it is clear that in the normal coordinates, the restriction of F to S has the form*

$$F(x, 0, z) = (x + \varphi(z), 0, z + \psi(z)). \quad (4)$$

5. Estimate of the dimension of the group of isometries: oriented case

5.1. Reeb fields on contact manifolds. Fix a contact sub-Riemannian manifold (N, \mathcal{D}, h) , $\dim N = 3$. Suppose that \mathcal{D} is endowed with an orientation as a vector bundle. Under such an assumption, it is known that \mathcal{D} admits a representation $\mathcal{D} = \ker \omega$, where ω is a globally defined contact 1-form. Such an ω is defined up to a multiplication by a non-vanishing function, and we will show how to normalize it in a canonical way. To this end, take an oriented orthonormal basis X, Y defined in a neighborhood of some point q . We choose ω so as to have

$$d\omega(X, Y) = 1. \quad (5)$$

Note that (5) holds for any frame $\{\tilde{X}, \tilde{Y}\}$ in the $SO(2)$ orbit of $\{X, Y\}$. We can repeat the same procedure around any point. As a result, we obtain a global 1-form ω which satisfies (5) for any oriented orthonormal basis around any point. The unique 1-form ω obtained in this way will be called the *normalized contact form*.

Suppose that ω is the normalized contact form. Since the kernel of $d\omega$ is 1-dimensional, there exists a unique vector field Z such that

- (i) $d\omega(Z, \cdot) = 0$,
- (ii) $\omega(Z) = 1$.

We call Z the *Reeb field*. Note that it is uniquely defined by both the sub-Riemannian structure, and the orientation. The following will be useful later:

Proposition 5.1 ([9, 10]). *Every isometry f of (N, \mathcal{D}, h) preserves the Reeb field, i.e. $f_*Z = Z$.*

5.2. Invariant coframe. Let (M, H, g) be a complete Martinet sub-Riemannian manifold. Assume that H is endowed with an orientation. Denote by Z the Reeb field on $M \setminus S$ associated with the orientation. Recall that by $\mathcal{I}_0(M, H, g)$ we denote the component of the identity in $\mathcal{I}(M, H, g)$. Applying Proposition 5.1 to $(M \setminus S, H|_{M \setminus S}, g)$, we see that if $f \in \mathcal{I}_0(M, H, g)$, then $f|_{M \setminus S}$ preserves Z .

Fix a point $q \in S$ and take its neighbourhood U such that normal coordinates x, y, z are defined on it. Suppose that X, Y is the orthonormal frame for (H, g)

defined on U which is given in the normal form 1. Replacing y with $-y$ if necessary, we can assume that X, Y agree with the given orientation of H . The argument from the proof of Proposition 4.1 implies that $f|_{M \setminus S}$ also preserves the fields X, Y . Indeed, since $f|_{M \setminus S}$ preserves H, Y and the orientation, it must also preserve X . It follows that $f|_{M \setminus S}$ preserves the coframe $\{\theta^1, \theta^2, \theta^3\}$ dual to $\{X, Y, Z\}$. Note that the coframe is defined on $U \setminus S$. Since the same procedure can be carried around any point, it is clear that for every $q \in S$, an invariant coframe is defined on $U_q \setminus S$, where U_q is a sufficiently small neighbourhood of q . In fact, on each U_q there are two invariant coframes: these are exactly the coframes dual to the two frames $\{X, Y, Z\}$ and $\{-X, -Y, Z\}$. These considerations lead to the following result:

Proposition 5.2. *The elements of $\mathcal{I}_0(M, H, g)$ locally coincide with symmetries of the invariant coframe $\{\theta^1, \theta^2, \theta^3\}$.*

We will write the above coframe explicitly. Suppose that we are in the domain U of normal coordinates x, y, z . For convenience we will abbreviate (1) to

$$X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y},$$

remembering however that a and b have the form as in (1). For simplicity we also introduce the following notation:

$$F = \frac{a}{a_y b - a b_y}, \quad G = \frac{b}{a_y b - a b_y}. \tag{6}$$

First, using the very definition, we compute the Reeb field on $U \setminus S$ to be equal to

$$Z = \frac{1}{F_y G - F G_y} \left(F_y \frac{\partial}{\partial x} - (F_x + G_z) \frac{\partial}{\partial y} + G_y \frac{\partial}{\partial z} \right) \tag{7}$$

Now on the set $U \setminus S$ our coframe is given by

$$\begin{cases} \theta^1 = -G_y dx + F_y dz, \\ \theta^2 = b(F_x + G_z) dx + dy - a(F_x + G_z) dz, \\ \theta^3 = G dx - F dz. \end{cases} \tag{8}$$

The structure equations take the form

$$\begin{cases} d\theta^1 = (\ln |a_y b - a b_y|)_y \theta^1 \wedge \theta^2 + T_{13}^1 \theta^1 \wedge \theta^3 + T_{23}^1 \theta^2 \wedge \theta^3, \\ d\theta^2 = \frac{1}{F_y G - F G_y} (F_x + G_z) \theta^1 \wedge \theta^2 + T_{13}^2 \theta^1 \wedge \theta^3 + T_{23}^2 \theta^2 \wedge \theta^3, \\ d\theta^3 = \theta^1 \wedge \theta^2, \end{cases} \tag{9}$$

where $(F_y G - F G_y) T_{13}^1 = -F_{xy} - G_{yz} + (\ln |a_y b - a b_y|)_y (F_x + G_z),$
 $(F_y G - F G_y) T_{23}^1 = F_{yy} G_y - F_y G_{yy},$

$$\begin{aligned} (F_y G - F G_y) T_{23}^2 &= F_{xy} + G_{yz} + (\ln |a_y b - a b_y|)_y (F_x + G_z), \\ (F_y G - F G_y) T_{13}^2 &= -(a_y b - a b_y) (F_x + G_z)^2 + (a_x + b_z) (F_x + G_z) + \\ &\quad + a (F_{xx} + G_{xz}) + b (F_{xz} + G_{zz}). \end{aligned}$$

Looking at (9), and recalling the form of a and b , we see that the coframe (8) is of rank at least 1 which proves the following:

Proposition 5.3. *The group of isometries of an oriented and complete Martinet sub-Riemannian manifold is a Lie group of dimension at most 2.*

Proof. Indeed, $\dim \mathcal{I}(M, H, g) = \dim \mathcal{I}_0(M, H, g) \leq 2$. ■

6. Estimate of the dimension of the group of isometries: general case

We finish the proof of the main theorem 1.1. Let (M, H, g) be an arbitrary sub-Riemannian manifold. If S is the Martinet surface then we consider the contact manifold $(M \setminus S, H|_{M \setminus S}, g)$. Take an orthonormal frame X, Y defined in a neighborhood of a point $q \in M \setminus S$. Similarly, as in subsection 5, we can construct a normalized contact 1-form ω_q (defined on some neighborhood around q) such that $d\omega_q(X, Y) = 1$. This time ω is defined up to a sign, so in fact we have two normalized contact forms around q . We set

$$\sigma_q = \omega_q(q) \in T_q^* M$$

and carry out the same construction around any point in $M \setminus S$. Now (cf. [10]) we define an orientable double cover of $M \setminus S$ by setting

$$\widetilde{M \setminus S} = \{(q, \sigma_q) : q \in M \setminus S\},$$

where σ_q is either of the two normalized covectors at q . The naifold $\widetilde{M \setminus S}$ is endowed with a natural contact sub-Riemannian structure $(\widetilde{H}, \widetilde{g})$ (locally $\widetilde{M \setminus S}$ looks like the disjoint union of two copies of $M \setminus S$, so for instance $\widetilde{H}_{(q, \sigma_q)} = \ker \sigma_q$). Moreover, every isometry of $(M \setminus S, H|_{M \setminus S}, g)$ lifts to an isometry of $(\widetilde{M \setminus S}, \widetilde{H}, \widetilde{g})$. It follows that

$$\dim \mathcal{I}(M \setminus S, H|_{M \setminus S}, g) \leq \dim \mathcal{I}(\widetilde{M \setminus S}, \widetilde{H}, \widetilde{g}), \tag{10}$$

where the latter group has dimension at most 2 due to estimates that have been established in the orientable case.

Finally, let us note that every isometry is determined by its value at a given point. Indeed, if an isometry fixes a point outside the Martinet surface, then it must be the identity because the isometries outside the Martinet surface coincide with symmetries of a certain frame - see [16]. If, on the other hand, an isometry fixes a point lying on the Martinet surface, then by (3) it also fixes the Hamiltonian geodesic passing through this point. Since this geodesic is transverse to the Martinet surface, the argument above applies.

To see that Proposition 1.2 is true, it is enough to notice that the fibers of E are Hamiltonian geodesics transverse to S .

7. Infinitesimal isometries

Fix a sub-Riemannian manifold (M, H, g) . A vector field \mathcal{V} is called an *infinitesimal isometry* if its (local) flow consists of (local) isometries. Denote by $\Gamma(H)$ the set of smooth local sections of H . Similarly as in [11], one proves the following proposition:

Proposition 7.1. \mathcal{V} is an infinitesimal isometry if and only if

- (i) the field $[\mathcal{V}, Z]$ is in $\Gamma(H)$ for every $Z \in \Gamma(H)$;
- (ii) $\mathcal{V}(g(Z_1, Z_2)) = g([\mathcal{V}, Z_1], Z_2) + g(Z_1, [\mathcal{V}, Z_2])$ for every $Z_1, Z_2 \in \Gamma(H)$.

Suppose now that (M, H, g) is a Martinet sub-Riemannian manifold, and let \mathcal{V} be an infinitesimal isometry. Let us also assume that our underlying domain supports normal coordinates x, y, z . Denote by $\varphi_t = (\varphi_t^1, \varphi_t^2, \varphi_t^3)$ the (local) flow of \mathcal{V} ($|t|$ is sufficiently small). Using (3), we can write

$$\varphi_t(x, y, z) = (\varphi_t^1(x, z), y, \varphi_t^3(x, z)). \quad (11)$$

After differentiation we obtain

$$\mathcal{V} = P(x, z) \frac{\partial}{\partial x} + Q(x, z) \frac{\partial}{\partial z}. \quad (12)$$

It turns out that (12) can be strengthened.

Proposition 7.2. If \mathcal{V} is an infinitesimal isometry then, in the normal coordinates,

$$\mathcal{V} = P(z) \frac{\partial}{\partial x} + Q(z) \frac{\partial}{\partial z}. \quad (13)$$

Proof. Let X, Y be an orthonormal frame in the normal form (1). Using (i) from Proposition 7.1, and the formula (12), we obtain $[\mathcal{V}, X] = \lambda X$ for a certain smooth function λ . It is also clear that $[\mathcal{V}, Y] = 0$. Using (ii) from Proposition 7.1 we have $g([\mathcal{V}, X], X) = 0$ as well as $g([\mathcal{V}, X], Y) = 0$. Since $[\mathcal{V}, X]$ is horizontal, it must vanish. To end the proof it suffices to notice that

$$[\mathcal{V}, X]|_{y=0} = \frac{\partial P}{\partial x} \frac{\partial}{\partial x} + \frac{\partial Q}{\partial x} \frac{\partial}{\partial z} = 0. \quad \blacksquare$$

8. Some structural results

In Sections 4 and 5, we obtained an upper estimate for the dimension of the isometry group of Martinet sub-Riemannian structure. We remark that an upper bound for the dimension of the local isometry group of many geometric structures can be established via symbols and the method of Kruglikov [13] which deals with symmetries of filtered structures with nonconstant symbols. In the Martinet case, the symbol degenerates on the Martinet surface, and does not form a Lie algebra

when equipped with the bracket induced by that of vector fields. Hence [13] is not directly applicable since it requires a symbol algebra at every point. The solution to this obstacle is by prolongation to the projectivised horizontal bundle where the symbol over the Martinet surface is the Heisenberg algebra and the Engel algebra elsewhere. The result is the upper bound of 2 which we determined earlier. It should be noted that the method that we use in Section 4 gives more information about isometries.

Let us set $B(x, y, z) = (1 + xB_1(x, y, z) + y\overline{B_2}(x, y, z))$ in (1) so that the normal form is given by the following expressions:

$$X = (1 + yA)\frac{\partial}{\partial x} + y^2B\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}. \tag{14}$$

The arguments used in the proof of Proposition 7.2 lead to the following observation:

Proposition 8.1. *Suppose that X, Y is an orthonormal frame for (H, g) given in the normal form (1). \mathcal{V} is an infinitesimal isometry if and only if $[\mathcal{V}, X] = 0$ and $[\mathcal{V}, Y] = 0$. In particular, if x, y, z are normal coordinates, then $\mathcal{V} = P(z)\frac{\partial}{\partial x} + Q(z)\frac{\partial}{\partial z}$ is an infinitesimal isometry if and only if*

$$yBP'(z) = A_xP(z) + A_zQ(z) \quad \text{and} \quad BQ'(z) = B_xP(z) + B_zQ(z). \tag{15}$$

Proof. Indeed, these equations express the condition $[\mathcal{V}, X] = 0$. ■

If for example, $B_x \neq 0$, $A_x = 0$ and $A_z = 0$ on some neighbourhood U of the origin, then equations (15) become

$$yBP'(z) = 0 \quad \text{and} \quad B^2\frac{\partial Q}{\partial z} = B_xP(z),$$

or equivalently $P(z) = P(0)$ and $\frac{\partial Q}{\partial z} = \frac{B_x}{B^2}$.

Now if $\frac{\partial}{\partial z}H(x, y, z) = \frac{B_x}{B^2}$ and $H(x, y, 0) = 0$, then we have

$$P(z) = P(0) \quad \text{and} \quad \frac{Q}{B} = P(0)H(x, y, z) + C(x, y),$$

and $B(0, 0, z) = 1$ implies that

$$Q(z) = P(0)H(x, y, z)B(x, y, z) + Q(0)\frac{B(x, y, z)}{B(x, y, 0)}.$$

Obviously this formula requires the condition on B that the right hand side is independent of x and y . For example, the the maximal dimension of 2 is obtained when $B(x, y, z) = (1 + \beta(y))e^{Kx}$, where $\beta(0) = 0$. In this case we have

$$P(z) = P(0) \quad \text{and} \quad Q(z) = P(0)Kz + Q(0).$$

Similarly, by direct integration keeping in mind that $B(0, 0, z) = 1$, we get the following table of naive integrals, contingent on the expressions for P and Q being independent of x and y :

Table 1: Naive integrals

B_x	A_z	A_x	P	Q	max dim
$\neq 0$	$= 0$	$= 0$	$P(0)$	$P(0)H(x, y, z)B(x, y, z) + Q(0)\frac{B(x, y, z)}{B(x, y, 0)}$	2
$= 0$	$= 0$	$= 0$	$P(0)$	$Q(0)\frac{B(y, z)}{B(y, 0)}$	2
$= 0$	$\neq 0$	$= 0$	$P(0)$	0	1
$\neq 0$	$\neq 0$	$\neq 0$	$-Q(z)\frac{A_z(x, 0, z)}{A_x(x, 0, z)}$	$G(x, y, z) + Q(0)\frac{B(x, y, z)}{B(x, y, 0)}$	1
$= 0$	$\neq 0$	$\neq 0$	$-Q(z)\frac{A_z(x, 0, z)}{A_x(x, 0, z)}$	$Q(0)\frac{B(y, z)}{B(y, 0)}$	1
$= 0$	$= 0$	$\neq 0$	0	$Q(0)\frac{B(y, z)}{B(y, 0)}$	1
$\neq 0$	$= 0$	$\neq 0$	0	$Q(0)\frac{B(x, y, z)}{B(x, y, 0)}$	1
$\neq 0$	$\neq 0$	$= 0$	0	0	0

$$\begin{aligned} \frac{\partial}{\partial z}H(x, y, z) &= \frac{B_x}{B^2}, & H(x, y, 0) &= 0, \\ \frac{\partial}{\partial z}G(x, y, z) &= -\frac{B_x A_z(x, 0, z)}{B^2 A_x(x, 0, z)}, & G(x, y, 0) &= 0. \end{aligned}$$

We note that the requirement of x, y -independence means that in most cases $P(0) = 0$ and/or $Q(0) = 0$.

In the second case the maximal dimension is obtained when $A(x, y, z) = \alpha(y)$ where α is an arbitrary function, and $B(x, y, z) = 1 + C(z)$ ($B_1 = 0$ and $B_2 = C(z)/y$), indeed in this case we have

$$P(z) = P(0) \quad \text{and} \quad Q(z) = Q(0)\frac{1 + C(z)}{1 + C(0)}.$$

Example 8.2. Consider the structure (H, g) defined on \mathbb{R}^3 by an orthonormal basis X, Y given in the normal form (14) with $A = y$ and $B = (1 + y)e^x$. As it is computed above the algebra of infinitesimal isometries is generated by $c_1\frac{\partial}{\partial x}$ and $(c_1z + c_2)\frac{\partial}{\partial z}$, and hence $\dim \mathcal{I}(\mathbb{R}^3, H, g) = 2$.

Example 8.3. Let the structure (H, g) be this time defined by an orthonormal basis X, Y given in the normal form (14) with $A = y$ and $B = 1 + x$. The equations from Proposition 8.1 reduce to $(1 + x)Q'(z) = P(z)$ which implies that $P(z) = 0$ and $Q(z) = c$. Thus, infinitesimal isometries are exactly the fields $c\frac{\partial}{\partial z}$, $c \in \mathbb{R}$, and $\dim \mathcal{I}(\mathbb{R}^3, H, g) = 1$.

Example 8.4. Looking at the above remarks, it is not hard to find conditions for rigidity. For example, one can make sure that every sub-Riemannian structure (H, g) defined by an orthonormal basis in the normal form (14) such that

$$A_x^2(0) + B_x^2(0) \neq 0, \quad A_{xx}(0)A_z(0) - A_x(0)A_{xz}(0) \neq 0 \quad (16)$$

has no nontrivial isometries, meaning that $\dim \mathcal{I}(\mathbb{R}^3, H, g) = 0$ in this case.

References

- [1] Agrachev, A., H. E. A. Chakir, and J. P. Gauthier, *Sub-Riemannian metrics on \mathbb{R}^3* , *Canad. Math. Soc. Conf. Proceedings* **25**, 1998.
- [2] Agrachev, A., and D. Barilari, *Sub-Riemannian structures on 3D Lie groups*, *J. Dyn. Control Systems* **18** (2012) 21–44.
- [3] Agrachev, A., *Exponential mappings for contact sub-Riemannian structures*, *J. Dyn. Control Systems* **2** (1996) 321–358.
- [4] Agrachev, A., B. Bonnard, M. Chyba, and I. Kupka, *Sub-Riemannian sphere in Martinet flat case*, *ESAIM Control Optim. Calc. Var.* **2** (1997) 377–448.
- [5] El-Alaoui, El-H., J.-P. Gauthier, and I. Kupka, *Small sub-Riemannian Balls in R^3* , *J. Dyn. Control Systems* **2** (1996) 359–421.
- [6] Grochowski, M., *Normal forms of germs of contact sub-Lorentzian structures on \mathbb{R}^3 . Differentiability of the sub-Lorentzian distance function*, *J. Dyn. Control Systems* **9** (2003) 531–547.
- [7] Grochowski, M., *Normal forms and reachable sets for analytic Martinet sub-Lorentzian structures of Hamiltonian type*, *J. Dyn. Control Syst.* **17** (2011) 49–75.
- [8] Grochowski, M., *The structure of reachable sets for affine control systems induced by generalized Martinet sub-Lorentzian metrics*, *ESAIM Control Optim. Calc. Var.* **18** (2012) 1150–1177.
- [9] Grochowski, M., and W. Kryński, *Invariants of contact sub-pseudo-Riemannian structures and Einstein-Weyl geometry*, *Radon Series on Computational and Applied Mathematics* **18** (2017) 434–453.
- [10] Grochowski, M., and W. Kryński, *On contact sub-pseudo-Riemannian isometries*, *ESAIM Control Optim. Calc. Var.* **23** (2017) 1751–1765.
- [11] Grochowski, M., and B. Warhurst, *Invariants and infinitesimal transformations for contact sub-Lorentzian structures on 3-dimensional manifolds*, *SIGMA* **11** (2015), No. 31, 23 pages.
- [12] Kobayashi, S., *Transformation Groups in Differential Geometry*, Springer, New York et al. (1972).

- [13] Kruglikov, B., *Symmetries of filtered structures via filtered Lie equations*, J. Geometry Physics **85** (2014).
- [14] Liu, W., and H. Sussmann, *Shortest Paths for Sub-Riemannian Metrics on Rank-Two Distributions*, Mem. Amer. Math. Soc. **564**, American Mathematical Society, Providence (1995).
- [15] Montgomery, R., *A Tour of Subriemannian Geometries, their Geodesics and Applications*, American Mathematical Society, Providence (2006).
- [16] Olver, P. J., *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge (1995).

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