

The Use of Hopf Algebras in the Lie Theory of Loops Related to Reductive Homogeneous Spaces

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Abstract. In his generalization of reductive homogeneous spaces, Lev Sabinin introduced hyporeductive and pseudoreductive local loops. Sabinin proved that these loops admit a satisfactory Lie theoretical approach. In this paper we derive Sabinin's results in an algebraic context by means of nonassociative Hopf algebras that encode the information about the nonassociative products of these local loops.

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1. Introduction

The knowledge of homogeneous spaces firmly relies on their infinitesimal analysis. Symmetric and reductive homogeneous spaces are archetypal examples. In 1954 Nomizu [57] studied invariant affine connections on reductive homogeneous spaces G/H under this premise. In this context, *reductive* means that the Lie tangent algebra \mathfrak{g} of the connected Lie group G splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where \mathfrak{h} is the Lie algebra of the closed subgroup H , and \mathfrak{m} is a complement satisfying $\text{Ad}_H(\mathfrak{m}) \subseteq \mathfrak{m}$. Nomizu proved that invariant affine connections on these manifolds are in bijection with certain nonassociative products on \mathfrak{m} admitting Ad_H as automorphisms. Later, in [80], Yamaguti introduced the notion of general Lie triple system as an abstract model for the complement \mathfrak{m} . Other names for these triple systems are Lie triple algebras [34] and Lie-Yamaguti algebras [35]. The importance of the local study of homogeneous spaces and their generalizations was beyond doubt, and it was confirmed in the landmark paper [79] where Wolf described the geometry and structure of isotropy irreducible homogeneous spaces through heavy and extensive computations based on the representation theory of Lie algebras.

A new approach to the local study of affine manifolds was proposed in [32] by Kikkawa: around any point e of such a manifold, there exists a locally defined

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product

$$xy := \exp_x(\tau_x^e(\exp_e^{-1}(y))) \quad (1)$$

where τ_x^e denotes the parallel transportation from e to x . Thus, it is possible to classify affine connections according to the algebraic properties of these products. In [60] Sabinin devised a program to model affinely connected spaces in terms of algebraic structures, which also allowed him to consider discrete spaces. In the work of Sabinin, *loops* (nonassociative groups to say) play an important role. For instance, the geodesic sum defined in (1) is a (partial) loop.

The relation between geometry and loops is old and, as far as this paper is concerned, authoritative books and surveys are [3, 4, 5, 8, 9, 55, 59, 66, 67, 70, 76] among others. However a concise, and therefore unwittingly partial, introduction is required to put hyporeductive and pseudoreductive loops in context for clarity.

Links between geometry and loops can be traced to the work of Hilbert [18] where, using incidence configurations, he found coordinate systems for certain geometries and introduced two binary operations $+$ and \cdot on the set of coordinates thus obtaining a division ring. The validity of Desargues's theorem was tied to the associativity of this ring. Influenced by Dehn, in [48] Moufang considered a weaker version of Desargues's theorem. The corresponding projective planes were coordinatized by rings whose nonzero elements form a Moufang loop, i.e. a loop that satisfies any of the Moufang identities¹. Thus, properties of the loop reflect geometrical properties of the plane, a point of view also found in Sabinin's work. The Lie theory for Moufang loops began with Malcev who in [39] outlined the way ahead when noticing that, since the Baker-Campbell-Hausdorff formula depends on two elements, the Lie theory for groups could probably be extended to diassociative loops such as Moufang loops and binary Lie algebras such as Malcev algebras. Unfortunately, it was too early for a general nonassociative Lie theory to show up since techniques from other areas were required.

A *3-web* consists of a set \mathcal{P} (points) and three disjoint sets \mathcal{L}_i ($i = 1, 2, 3$) of subsets of \mathcal{P} such that: lines of the same type (say \mathcal{L}_i) have no points in common, two lines of different type have exactly one point in common, and through any point there is exactly one line of each type. The product of two lines of different types is defined to be the third line passing through their point of intersection. After identifying the sets of lines, a *quasigroup*, i.e. a nonunital loop, is obtained. Web geometry originated in 1926-27 in work by Blaschke and Thomsen, and has been studied in detail by Blaschke, Thomsen, Reidemeister, Bol, Akinis, Goldberg, Shelekhov, Chern, Griffiths and others [59]. The theory of quasigroups was developed by Belousov and his school, while Akinis promoted the study of differentiable 3-webs, as for instance in [1] where he studied the local theory of 3-webs through affine connections and locally analytic quasigroups. At that point it was clear that affine connections, rather than a not already known general Baker-Campbell-Hausdorff formula, would become essential to address the problem of formulating a general nonassociative Lie theory.

In [36] Kuzmin established a Lie correspondence between local analytic loops and Malcev algebras. The study of these algebras was then heavily promoted

¹The left, middle and right Moufang identities $x(y(xz)) = ((xy)x)z$, $(xy)(zx) = (x(yz))x$ $((xy)z)y = x(y(zx))$ are equivalent for loops.

by Kuzmin and Sagle. This correspondence was made global in the work of Kerdman [31] and P. Nagy [52]. In [33] Kikkawa completed the study of locally reductive spaces initiated by Yamaguti, and in [34] he dealt with the geometry and Lie theory of homogeneous Lie loops.

In 1976 Akiwis made a major contribution. In his study of local algebras of 3-webs and associated quasigroups [2] he introduced the notion of W -algebra, now termed *Akiwis algebra* by Hofmann and Strambach [23], the first attempt of a general tangent algebra for analytic loops. No going back was possible. In [40, 41, 42] Mikheev and Sabinin established a Lie correspondence for Bol loops in terms of Akiwis algebras. In [23] Hofmann and Strambach developed a nonassociative Lie theory in terms of Akiwis algebras. Also J. D. Smith in [75] used Akiwis and comtran algebras to study formal n -loops. However, the theory was not entirely satisfactory: the same Akiwis algebra might correspond to several nonisomorphic loops. So, there was a feeling that Akiwis algebras were not general enough.

A second major contribution in the study of local loops occurred in 1987. Mikheev and Sabinin [42], assisted by the language of affine connections, well established in this context at that time, classified local analytic loops in terms of *Sabinin algebras*, a broad generalization of Akiwis algebras. Thus, after thirty years of advances, a nonassociative Lie theory for local analytic loops was finally obtained. Since then, other general algebraic structures for tangent spaces of local differentiable loops have been proposed, notably by Sabinin [61] and Figula [10]. Partial expansions of the Baker-Campbell-Hausdorff formula have also been computed for Bruck loops by G. Nagy [49] (see also [54]) and by Hofmann and Strambach for general loops in terms of Akiwis algebras [24] (also see [25]). For a modern treatment of Sabinin algebras see [78]. But this is only half the story.

Another active approach that ties together geometry and groups in the spirit of Klein's Erlangen programm which has strongly influenced the study of loops is based on the properties of the group of collineations. Baer, in [6], studied projective planes by means of their collineation groups, and a general classification along these lines was obtained by Lenz [37] and Barlotti [7]. Many prominent authors such as Pickert, who stimulated the work on the foundations of topological webs and planes, Salzmann and his students, who worked on collineation groups of compact connected projective planes, or Hofmann, who investigated a general theory of topological loops, contributed to the development of a global approach to loops [19, 20, 21] (see [24] for an historical account and [22] for topological left loops). The group generated by all the left or all the right translations, or by both types of translations, acts transitively on a loop, so the loop is some sort of homogeneous space. Although for analytic loops these groups might not be Lie groups in general, and this is an active line of research, for finite loops this approach lead to important results on varieties of loops such as Lagrange, Sylow or Feit-Thompson theorems among many others. Thus, while the work of Sabinin and his school was mostly local because of his approach to affinely connected manifolds as manifolds with a covering of local loops, the study of global loops has been an essential part of the theory of loops and quasigroups from the very beginning, and it is also related to other topics such as H -spaces. In [56] Niemenmaa and Kepka

described the groups that appear as multiplication groups of loops (see also the work of Figula [12]). An important contribution was the work of P. Nagy and Strambach [53] on loops as invariant sections of groups. The book [55] is a good reference on these techniques. This approach based on sharply transitive sections has been successfully used by Figula and Strambach to classify topological loops of low dimensions with conditions on the group of translations (see [11, 12, 13, 14, 15] and references there in). Algebraic loops have also been studied by G. Nagy [50, 51] and Grishkov and G. Nagy [16] among others.

But let us return to homogeneous reductive spaces and their generalizations from a local point of view. In 1977 Sabinin introduced odular structures to model affinely connected manifolds in an algebraic way [60]. The central idea was that those manifolds are nothing else but manifolds with a covering of local odules (certain loops with some extra operations) satisfying some natural algebraic identities. Thus, it was possible to classify affine connections according to properties of these loops, and vice versa. For instance, given a reductive homogeneous space G/H and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where \mathfrak{g} is the Lie algebra of G , \mathfrak{h} is the Lie algebra of the closed subgroup H and \mathfrak{m} is a complement satisfying $\text{Ad}_H(\mathfrak{m}) \subseteq \mathfrak{m}$, around $\bar{e} := eH \in G/H$ (e denotes the identity element of G) we have a local product

$$\bar{x} \cdot \bar{y} := \overline{\sigma(\bar{x})y}$$

where $\bar{x} := xH$ and $\sigma(\bar{x})$ is the unique element in $\exp(\mathfrak{m})$ such that $\sigma(\bar{x})H = xH$. The reductive property of G/H and $hxH = h\sigma(\bar{x})H = h\sigma(\bar{x})h^{-1}H$ imply $\sigma(h\bar{x}) = h\sigma(\bar{x})h^{-1}$, so

$$(h\bar{x}) \cdot (h\bar{y}) = \overline{\sigma(h\bar{x})hy} = \overline{h\sigma(\bar{x})h^{-1}hy} = \overline{h\sigma(\bar{x})y} = h(\bar{x} \cdot \bar{y}). \quad (2)$$

With the theory of loops and quasigroups in mind, (2) can be easily generalized in many different ways since not only automorphisms of loops have traditionally been of interest.

An *autotopism* of a loop or quasigroup is a triple (f_0, f_1, f_2) of invertible maps satisfying $f_0(xy) = f_1(x)f_2(y)$ for any x and y . Automorphisms are just autotopisms of the form (f, f, f) , while in autotopisms of the form (f, g, f) the component f is called *pseudoautomorphism*. *Hypoautomorphisms* are related to autotopisms (f, f, g) . These triples appear in web and projective geometry because equivalence of 3-webs (resp. projective planes) corresponds to isotopism of quasigroups (resp. ternary rings) rather than to isomorphism. Thus, instead of imposing that the elements of H act as automorphisms of the geodesic sum, Sabinin considered the more natural situation where they appear just as components of autotopisms. This motivated the introduction, among others, of *hyporeductive* and *pseudoreductive* loops as a reasonable generalization of geodesic sums on reductive homogeneous spaces. More precisely, given a loop and the left and right multiplication operators $L_u: x \mapsto ux$ and $R_u: x \mapsto xu$, the map $r(u, v) := R_{uv}^{-1}R_vR_u$ belongs to the stabilizer of the identity element. Instead of imposing that $r(u, v)$ is an automorphism as in (2), in pseudoreductive loops we require autotopisms of the form

$$(P(u, v), r(u, v), P(u, v))$$

for some $P(u, v)$. In the case of hyporeductive loops, this condition is relaxed and we only require autotopisms of the form

$$(H(u, v), H(u, v), H'(u, v))$$

where $H(u, v) := R_{u \circ v} r(u, v)$, $H'(u, v) := L_{u \circ v}^{-1} H(u, v)$ and $u \circ v$ is some element (see sections 4 and 5 for precise definitions). The Lie theory of hyporeductive loops appeared in [62, 63, 64]. A geometric interpretation was given in [26]. See [27, 28] for examples and [29] for the description of the corresponding Akinis algebras. The Lie theory of pseudoreductive loops was developed in [65]. A generalization of symmetric spaces in the spirit of Loos [38] is the notion of transsymmetric space [68, 69]. The deep connection between symmetric spaces and loops is the topic of the work of P. Nagy and Strambach [54].

However, while geometrical methods, either local or global, were relishing the well-deserved landslide victory, another breakthrough was to come. In 1999 [71, 72] Shestakov proved that, in the same way as the product of any Lie algebra can be recovered as the commutator product of a certain associative algebra, the binary and ternary operations of any Akinis algebra can be obtained from the binary product of certain nonassociative algebra. This put Akinis algebras on an algebraic basis. Even more, in 2001 [73, 74] Shestakov and Umirbaev proved that any nonassociative algebra can be viewed as a Sabinin algebra with certain operations derived from its binary product. The space of primitive elements of a nonassociative bialgebra is closed under these operations, so it becomes a Sabinin algebra. Since this space can be identified with the tangent space at the identity element of the corresponding formal loop, a more algebraic approach to a nonassociative Lie theory was now possible. The equivalence, over fields of characteristic zero, between the categories of formal loops, Sabinin algebras and (connected) nonassociative Hopf algebras was obtained in [58], and a bridge between Sabinin's approach and the work of Shestakov and Umirbaev appeared in [43]. See [44] for a survey on this topic.

The use of nonassociative Hopf algebras in the treatment of loops is, in our opinion, indeed valuable. For instance, a general nonassociative version of the Baker-Campbell-Hausdorff formula has been recently found [45, 46]. A Lie theory for commutative automorphic loops has also been obtained with these techniques [17], while no similar approach from differential geometry was available. The freeness of the loop of formal power series, with coefficient 1, generated by free nonassociative variables has been proved with the help of nonassociative Hopf algebras [47]. Thus, the introduction of new algebraic techniques in this field might facilitate some aspects to other researches.

In this paper we continue our project and we transport Sabinin's results on the Lie theory for hyporeductive and pseudoreductive loops to an algebraic setting with the help of nonassociative Hopf algebras. To that end, in sections 4 and 5 we will define hyporeductive and pseudoreductive Hopf algebras, we will describe the space of primitive elements in terms of some triples $(\mathfrak{g}, \mathfrak{s}, \mathfrak{c})$ composed of a Lie algebra \mathfrak{g} , a Lie subalgebra \mathfrak{s} and a complement \mathfrak{c} satisfying certain properties, and finally we will show how to formally integrate these triples to recover hyporeductive and pseudoreductive Hopf algebras. This approach is obviously inspired by the

notion of sharply transitive section [55]. Preliminaries on nonassociative Hopf algebras and Sabinin algebras are included in Section 2, while the general result on the formal integration of triples is proved in Section 3. We hope this paper will contribute to motivate classifications similar to those in [79].

Notation. Given a Lie algebra \mathfrak{g} and a subset $\mathfrak{c} \subseteq \mathfrak{g}$, the Lie subalgebra of \mathfrak{g} generated by \mathfrak{c} will be denoted by $\text{Lie}_{\mathfrak{g}}\langle \mathfrak{c} \rangle$. The subalgebra

$$N_{\mathfrak{g}}(\mathfrak{c}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{c}] \subseteq \mathfrak{c}\}$$

will appear frequently. For a subalgebra \mathfrak{s} of \mathfrak{g} , $\text{core}_{\mathfrak{g}}(\mathfrak{s})$ is, by definition, the largest ideal of \mathfrak{g} contained in \mathfrak{s} . If there is no likelihood of confusion, we will write $N(\mathfrak{s})$ instead of $N_{\mathfrak{g}}(\mathfrak{s})$. The symmetric group of degree n will be denoted by S_n . Since almost all algebras in this paper lack associativity, an order for parentheses in powers is mandatory. We will stick to the convention $c^n := ((cc) \cdots)c$. Finally, all bialgebras in this paper are assumed to be coassociative, cocommutative and unital, and the characteristic of the base field \mathbf{k} is zero.

2. Preliminaries

In this section we will review some basic results and definitions required in the present work. See [44] for an expository paper on nonassociative Hopf algebras.

2.1. Loops. A *loop* (Q, xy, e) is a non-empty set Q endowed with:

- a product xy for which the left and right multiplication operators $L_x: y \mapsto xy$ and $R_x: y \mapsto yx$ are bijective for all $x \in Q$, and
- a distinguished element $e \in Q$, called the *unit element* (or *identity element*), such that $ex = x = xe$ for all $x \in Q$.

Since associativity is not required in the definition, this algebraic structure is usually thought as the no-associative counterpart of groups. The inverse map for groups is replaced in loop theory by the *left* and *right divisions*

$$x \setminus y := L_x^{-1}(y) \quad \text{and} \quad x / y := R_y^{-1}(x).$$

Clearly

$$x \setminus (xy) = y = x(x \setminus y) \quad \text{and} \quad (xy) / y = x = (x / y)y.$$

2.2. Formal loops. Differential manifolds with affine connections have traditionally been a natural source of loops. Given such a manifold M and a point $e \in M$, then the geodesic sum

$$xy := \exp_x(\tau_x^e(\exp_e^{-1}(y))),$$

where τ_x^e stands for the parallel transport from e to x , defines a binary operation $U \times U \rightarrow M$ on a neighborhood U of e . The point e is the unit element of this operation, and the left and right multiplication operators by x are invertible when restricted to small neighborhoods of e . In case $(x, y) \mapsto xy$ is analytic and we only study this product around e , we can assume $M = \mathbb{R}^n$, $e = (0, \dots, 0)$ and $xy = (p_1(x, y), \dots, p_n(x, y))$ where $p_i(x, y)$ are power series ($i = 1, \dots, n$). The

algebra of formal power series on x_1, \dots, x_n can be identified with the dual space of the *symmetric algebra* $\mathbb{R}[V]$ on $V := \mathbb{R}^n$. Under this identifications, xy is nothing else but a linear map $p: \mathbb{R}[V] \otimes \mathbb{R}[V] \rightarrow V$ for which $p(u \otimes 1)$ and $p(1 \otimes u)$ are both equal to the projection of u onto V .

In an algebraic setting the convergence of the series $p_1(x, y), \dots, p_n(x, y)$ is relaxed, and analytic loops are replaced by formal loops. Given a vector space V over a field \mathbf{k} , a *formal loop* (over V) is a linear map $F: \mathbf{k}[V] \otimes \mathbf{k}[V] \rightarrow V$ such that

$$F|_{\mathbf{k}[V] \otimes 1} = \pi_V = F|_{1 \otimes \mathbf{k}[V]}$$

where $\mathbf{k}[V]$ stands for the symmetric \mathbf{k} -algebra on V , and π_V denotes the projection of $\mathbf{k}[V]$ onto V that kills homogeneous elements of degree $\neq 1$.

2.3. Nonassociative connected bialgebras. The category of formal loops over a field \mathbf{k} of characteristic zero is equivalent to the category of connected nonassociative bialgebras [43]. Any formal loop F defines a coalgebra morphism $F' := \exp^*(F)$ by

$$\begin{aligned} F': \mathbf{k}[V] \otimes \mathbf{k}[V] &\rightarrow \mathbf{k}[V] \\ u \otimes v &\mapsto uv := \sum_{n=0}^{\infty} \frac{1}{n!} F(u_{(1)} \otimes v_{(1)}) \circ \dots \circ F(u_{(n)} \otimes v_{(n)}) \end{aligned}$$

where the product \circ in the formula is the usual commutative and associative product on $\mathbf{k}[V]$, that we will not use anymore. The *comultiplication* Δ is the homomorphism of unital commutative and associative algebras

$$\Delta: \mathbf{k}[V] \rightarrow \mathbf{k}[V] \otimes \mathbf{k}[V]$$

determined by $\Delta(a) = a \otimes 1 + 1 \otimes a$ for all $a \in V$. Sweedler's sigma notation for $\Delta(u)$ is $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$ or, with the Einstein convention, just $\Delta(u) = u_{(1)} \otimes u_{(2)}$. The *counit* $\epsilon: \mathbf{k}[V] \rightarrow \mathbf{k}$ is analogously determined by $\epsilon(a) = 0$ for all $a \in V$. Endowed with the new product $uv := F'(u \otimes v)$ and the unit $\eta: \mathbf{k} \rightarrow \mathbf{k}[V]$ ($\xi \mapsto \xi 1$) we obtain a nonassociative connected bialgebra $(\mathbf{k}[V], \Delta, \epsilon, F', \eta)$. The formal loop is recovered as $F = \pi_V F'$.

2.4. Left and right divisions. Non-associative Hopf algebras. Associative connected bialgebras are Hopf algebras, i.e. for any such bialgebra there exists a map, the *antipode*, related to the bialgebra structure by $S(u_{(1)})u_{(2)} = \epsilon(u)1 = u_{(1)}S(u_{(2)})$ for all u . This useful map is no longer available in the nonassociative setting. However, the existence of two maps (the *left* and the *right division*) $\backslash, /: \mathbf{k}[V] \otimes \mathbf{k}[V] \rightarrow \mathbf{k}[V]$ satisfying

$$u_{(1)} \backslash (u_{(2)} v) = \epsilon(u)v = u_{(1)}(u_{(2)} \backslash v) \quad \text{and} \quad (uv_{(1)})/v_{(2)} = \epsilon(v)u = (u/v_{(1)})v_{(2)} \quad (3)$$

can be proved for any nonassociative connected bialgebra [58]. These are the nonassociative counterpart of the antipode S . In the associative case one has $u \backslash v = S(u)v$ and $u/v = uS(v)$. In this paper we will only consider *coassociative* (i.e. $u_{(1)(1)} \otimes u_{(1)(2)} \otimes u_{(2)} = u_{(1)} \otimes u_{(2)(1)} \otimes u_{(2)(2)}$) and *cocommutative* (i.e. $u_{(1)} \otimes u_{(2)} = u_{(2)} \otimes u_{(1)}$) coalgebras. In this setting, a *nonassociative Hopf algebra* is a unital

bialgebra with a left and a right division satisfying (3). The adjective *connected* means that the Hopf algebra is isomorphic to $(\mathbf{k}[V], \Delta, \epsilon, uv, \eta, \backslash, /)$ for some space V (the space of primitive elements).

2.5. Primitive elements and Sabinin algebras. V can be thought as the tangent space at the identity element of the formal loop $(\mathbf{k}[V], F)$. In terms of the comultiplication, V is the nothing else but the vector space of *primitive elements*, $\text{Prim}(\mathbf{k}[V])$, of $\mathbf{k}[V]$, i.e. those elements satisfying

$$\Delta(a) = a \otimes 1 + 1 \otimes a.$$

The space of primitive elements is not closed under the product uv . However, many multilinear maps for which V is stable, such as the commutator $[u, v] := uv - vu$ or the associator $(u, v, w) := (uv)w - u(vw)$, can be obtained out of the binary product uv . The precise algebraic structure of $\text{Prim}(\mathbf{k}[V])$ was determined by Shestakov and Umirbaev [74]. Later, in [58], it was named *Sabinin algebra* since it was first introduced by Sabinin and Mikheev as the right algebraic structure on the tangent space at the identity element to classify local loops. A Sabinin algebra is a rather complex algebraic structure endowed with two infinite families of multilinear operations satisfying certain axioms that generalize those of Lie algebras.

2.6. Shestakov-Umirbaev operations. Let $X := \{x_1, x_2, \dots, y, y_1, y_2, \dots, z\}$ be a set of symbols, and let $\mathbf{k}\{X\}$ be the free unital nonassociative algebra generated by X . Define homomorphisms of unital algebras

$$\Delta: \mathbf{k}\{X\} \rightarrow \mathbf{k}\{X\} \otimes \mathbf{k}\{X\} \quad \text{and} \quad \epsilon: \mathbf{k}\{X\} \rightarrow \mathbf{k}$$

determined by $\Delta(a) = a \otimes 1 + 1 \otimes a$ and $\epsilon(a) = 0$ for any $a \in X$. Consider $u := ((x_1x_2)\cdots)x_m$, $v := ((y_1y_2)\cdots)y_n$, $\underline{u} := x_1 \otimes \cdots \otimes x_m$ and $\underline{v} := y_1 \otimes \cdots \otimes y_n$. The element $\Delta(u) = u_{(1)} \otimes u_{(2)}$ has been defined, and the notation $\Delta(\underline{u}) = \underline{u}_{(1)} \otimes \underline{u}_{(2)}$ has the obvious meaning. We can recursively define intermediate elements $p(\underline{u}; \underline{v}; z) = p(x_1, \dots, x_m; y_1, \dots, y_n; z)$ from the associator $(u, v, w) := (uv)w - u(vw)$ by

$$(u, v, z) = (u_{(1)}v_{(1)})p(\underline{u}_{(2)}; \underline{v}_{(2)}; z)$$

to obtain elements $\langle x_1, \dots, x_m; y, z \rangle$ and $\Phi(x_1, \dots, x_m; y_1, \dots, y_n; y_{n+1})$ ($m, n \geq 1$) by

$$\begin{aligned} \langle 1; y, z \rangle &:= -[y, z], & \langle x_1, \dots, x_m; y, z \rangle &:= -p(\underline{u}; y; z) + p(\underline{u}; z; y) \quad \text{and} \\ \Phi(x_1, \dots, x_n; y_1, \dots, y_m; y_{m+1}) & \\ &:= \frac{1}{n!(m+1)!} \sum_{\sigma \in S_n, \tau \in S_{m+1}} p(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}; y_{\tau(1)}, y_{\tau(2)}, \dots, y_{\tau(m)}; y_{\tau(m+1)}) \end{aligned}$$

where S_k is the symmetric group of degree k . These nonassociative polynomials $\langle x_1, \dots, x_n; y, z \rangle$ and $\Phi(x_1, \dots, x_n; y_1, \dots, y_m; y_{m+1})$ can be evaluated on any nonassociative algebra H , thus they define multilinear maps on H called *Shestakov-Umirbaev operations*. With these new operations H becomes a Sabinin algebra denoted by $\mathcal{YIII}(H)$, and in case H is a nonassociative Hopf algebra then $\text{Prim}(H)$ is a Sabinin subalgebra of $\mathcal{YIII}(H)$. In Section 6 we will use this construction.

2.7. Formal integration. Formal loops are equivalent to nonassociative connected Hopf algebras. Thus, the formal integration of a Sabinin algebra to a formal loop amounts to constructing such a Hopf algebra out of a given Sabinin algebra. The only requirement for this construction is that the Sabinin algebra structure is recovered from the Hopf algebra as the space of primitive elements endowed with the Shestakov-Umirbaev operations. For instance, the formal integration of a Lie algebra corresponds essentially to the construction of its universal enveloping algebra and the proof of the Poincaré-Birkhoff-Witt Theorem.

In the approach to hyporeductive and pseudoreductive local loops, triples consisting of a Lie algebra, a subalgebra and a complement of this subalgebra overshadow Sabinin algebras. In fact, all multilinear operations conforming the Sabinin algebra structure of the tangent space at the identity element of any of these loops can be obtained from just three products of varieties 2, 2 and 3 derived from a certain triple related to the fundamental vector fields of an adequate affine connection; hence, no need for the whole power of Sabinin algebras is required. We adopt this point of view in this paper, and we formally integrate triples rather than Sabinin algebras.

3. Right monoalternative Hopf algebras

3.1. Triples. Let us consider a category whose objects are *triples* $\tau = (\mathfrak{g}, \mathfrak{s}, \mathfrak{c})$ where \mathfrak{g} is a Lie algebra, \mathfrak{s} is a subalgebra of \mathfrak{g} , and \mathfrak{c} is a subspace such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c}$. A morphism $\varphi: \tau \rightarrow \tau'$ is a homomorphism of Lie algebras $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ satisfying $\varphi(\mathfrak{s}) \subseteq \mathfrak{s}'$ and $\varphi(\mathfrak{c}) \subseteq \mathfrak{c}'$. Associated to any triple τ we have two more triples τ_{inn} and τ_{red} defined by $\tau_{\text{inn}} = (\mathfrak{g}_{\text{inn}}, \mathfrak{s}_{\text{inn}}, \mathfrak{c})$, where $\mathfrak{g}_{\text{inn}} := \text{Lie}_{\mathfrak{g}}(\mathfrak{c})$, $\mathfrak{s}_{\text{inn}} := \mathfrak{g}_{\text{inn}} \cap \mathfrak{s}$ and $\tau_{\text{red}} := (\mathfrak{g}_{\text{inn}}/\text{core}(\mathfrak{s}_{\text{inn}}), \mathfrak{s}_{\text{inn}}/\text{core}(\mathfrak{s}_{\text{inn}}), \mathfrak{c})^2$. We will say that the triples τ and τ' are *equivalent* if τ_{red} is isomorphic to τ'_{red} .

3.2. Triples from bialgebras. Any unital bialgebra U determines a triple

$$T(U) := (\mathfrak{g}(U), \mathfrak{s}(U), \mathfrak{c}(U))$$

where

$$\begin{aligned} \mathfrak{g}(U) &:= \{f \in \text{End}_{\mathbf{k}}(U) \mid \Delta(f(u)) = f(u_{(1)}) \otimes u_{(2)} + u_{(1)} \otimes f(u_{(2)}) \ \forall u \in U\}, \\ \mathfrak{s}(U) &:= \{f \in \mathfrak{g}(U) \mid f(1) = 0\} \quad \text{and} \\ \mathfrak{c}(U) &:= \{R_a \mid a \in \text{Prim}(U)\}. \end{aligned}$$

Lemma 3.1. *Let U be a unital connected bialgebra. Then, $T(U)_{\text{inn}} = T(U)_{\text{red}}$.*

Proof. Unital connected bialgebras are spanned by products $((c_1 c_2) \cdots) c_n$ where $c_1, \dots, c_n \in \text{Prim}(U)$ and $n \geq 0$ [74]. We will show by induction on n that any element in $\text{core}(\mathfrak{s}(U)_{\text{inn}})$ kills these generators. The case $n = 0$ is clear by the definition of $\mathfrak{s}(U)$. The general case follows from $f(uc_n) = fR_{c_n}(u) = [f, R_{c_n}](u) + R_{c_n}f(u)$ for any $f \in \text{core}(\mathfrak{s}(U)_{\text{inn}})$ and $u = ((c_1 c_2) \cdots) c_{n-1}$. Thus, $\text{core}(\mathfrak{s}(U)_{\text{inn}}) = \{0\}$. ■

² $\text{core}(\mathfrak{s}_{\text{inn}}) := \text{core}_{\mathfrak{g}_{\text{inn}}}(\mathfrak{s}_{\text{inn}})$.

3.3. Formal integration of triples. A *right monoalternative* bialgebra is a bialgebra satisfying

$$((xy_{(1)}) \cdots)y_{(n)} = x((y_{(1)}y_{(2)}) \cdots y_{(n)})$$

for any x, y and any $n \geq 0$. The following result can be essentially found in [43, 58], although there it is formulated in terms of Sabinin algebras. An approach through these algebras is natural when the families of Hopf algebras under consideration are rooted to varieties of loops.

Theorem 3.2. *Let $\tau = (\mathfrak{g}, \mathfrak{s}, \mathfrak{c})$ be a triple. We have:*

- (1) *There exist a right monoalternative (connected) Hopf algebra U_τ and a morphism $\iota: \tau \rightarrow T(U_\tau)$ whose restriction to \mathfrak{c} gives a linear isomorphism of \mathfrak{c} and $\mathfrak{c}(U_\tau)$.*
- (2) *For any right monoalternative unital bialgebra U' and any morphism $\iota': \tau \rightarrow T(U')$ there exists a unique homomorphism of bialgebras $\psi: U_\tau \rightarrow U'$ such that $\iota'_c(1) = \psi(\iota_c(1))$ for any $c \in \mathfrak{c}$.*
- (3) *τ and $T(U_\tau)$ are equivalent.*
- (4) *Given triples τ and τ' , U_τ is isomorphic to $U_{\tau'}$ if and only if τ is equivalent to τ' .*

Proof. Consider the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . The left ideal $U(\mathfrak{g})\mathfrak{s}$ and the left module $U_\tau := U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{s}$, which is also a coalgebra isomorphic to $\mathfrak{k}[\mathfrak{c}]$. The element $u + U(\mathfrak{g})\mathfrak{s} \in U_\tau$ will be denoted by \bar{u} . Let us fix a basis \mathcal{B} of \mathfrak{c} . By the Poincaré-Birkhoff-Witt theorem, a basis of U_τ consists of elements

$$\sum_{\sigma \in S_n} \overline{c_{\sigma(1)} \cdots c_{\sigma(n)}}$$

where $c_1, \dots, c_n \in \mathcal{B}$ and $n \geq 0$. We use this basis to define a product on U_τ as follows:

$$\bar{u} \left(\sum_{\sigma \in S_n} \overline{c_{\sigma(1)} \cdots c_{\sigma(n)}} \right) := \sum_{\sigma \in S_n} \overline{c_{\sigma(n)} \cdots c_{\sigma(1)} \bar{u}}. \tag{4}$$

Notice that the indexes in the right-hand side of the previous equality can be arranged in any order. Clearly

$$\bar{u} \left(\sum_{\sigma \in S_n} \overline{c_{\sigma(1)} \cdots c_{\sigma(n)}} \right) = \sum_{\sigma \in S_n} ((\bar{u} \overline{c_{\sigma(1)}}) \cdots) \overline{c_{\sigma(n)}}. \tag{5}$$

It is easy to check that endowed with this product and the coalgebra structure previously mentioned, U_τ is a connected unital bialgebra. Property (5) is in fact independent of the chosen basis \mathcal{B} , i.e. for any other basis \mathcal{B}' of \mathfrak{c} we also have

$$\bar{u} \left(\sum_{\sigma \in S_n} \overline{c'_{\sigma(1)} \cdots c'_{\sigma(n)}} \right) = \sum_{\sigma \in S_n} ((\bar{u} \overline{c'_{\sigma(1)}}) \cdots) \overline{c'_{\sigma(n)}}$$

for any $c'_1, \dots, c'_n \in \mathcal{B}'$. Thus $\bar{u} \bar{c}^n = ((\bar{u} \bar{c}) \cdots) \bar{c}$ (6)

for any $c \in \mathfrak{c}$. Since $\{\bar{c}^n \mid c \in \mathfrak{c}, n \geq 0\}$ linearly spans U_τ , and

$$\bar{c}_{(1)}^n \otimes \cdots \otimes \bar{c}_{(m)}^n = \sum_{i_1 + \cdots + i_m = n} \frac{n!}{i_1! \cdots i_m!} \bar{c}^{i_1} \otimes \cdots \otimes \bar{c}^{i_m},$$

we can conclude that U_τ is right monoalternative. The action of \mathfrak{g} on U_τ defines a homomorphism of Lie algebras $\iota: \mathfrak{g} \rightarrow \mathfrak{gl}(U_\tau)$ with

$$\begin{aligned} \Delta(\iota_x(\bar{u})) &= \Delta(\overline{xu}) = \overline{xu_{(1)}} \otimes \overline{u_{(2)}} + \overline{u_{(1)}} \otimes \overline{xu_{(2)}} \\ &= \iota_x(\overline{u_{(1)}}) \otimes \overline{u_{(2)}} + \overline{u_{(1)}} \otimes \iota_x(\overline{u_{(2)}}). \end{aligned}$$

Hence, ι determines a homomorphism of Lie algebras $\iota: \mathfrak{g} \rightarrow \mathfrak{g}(U_\tau)$. Moreover, for any $d \in \mathfrak{s}$ and $c \in \mathfrak{c}$, $\iota(d)(\bar{1}) = \bar{d} = \bar{0}$ and $\iota_c(\bar{u}) = \bar{c}u = \bar{u}c = R_{\bar{c}}(\bar{u})$. Hence, ι is a morphism from τ to $T(U_\tau)$. The restriction of ι to \mathfrak{c} is a linear isomorphism of \mathfrak{c} , and $\{R_{\bar{c}} \mid c \in \mathfrak{c}\} = \{R_{\bar{c}} \mid \bar{c} \in \text{Prim}(U_\tau)\} = \mathfrak{c}(U_\tau)$.

Now we will prove (2). Let U' be a right monoalternative unital bialgebra and $\iota': \tau \rightarrow T(U')$ a morphism. We can extend ι' to a homomorphism of unital associative algebras $\varphi: U(\mathfrak{g}) \rightarrow \text{End}_{\mathbf{k}}(U')$ to obtain a linear map $\psi: U(\mathfrak{g}) \rightarrow U'$ defined by $\psi(u) := \varphi_u(1)$, where φ_u denotes the image of u under φ . For any $d \in \mathfrak{s}$ and any $u \in U(\mathfrak{g})$ we have $\psi(ud) = \varphi_{ud}(1) = \varphi_u \varphi_d(1) = \varphi_u(\iota'_d(1)) = 0$. Therefore, ψ induces a linear map $\psi: U_\tau \rightarrow U'$ which satisfies

$$\begin{aligned} \psi(\bar{u}\bar{c}) &= \psi(\overline{cu}) = \psi(cu) = \varphi_{cu}(1) = \varphi_c \varphi_u(1) = R_{\varphi_c(1)} \varphi_u(1) = \varphi_u(1) \varphi_c(1) \\ &= \psi(\bar{u})\psi(\bar{c}), \end{aligned}$$

for any $c \in \mathfrak{c}$ and $u \in U(\mathfrak{g})$. Together with the right monoalternativity of U_τ and U' —recall (6)—this proves that ψ is a homomorphism of algebras. With little extra effort we get that ψ is also a homomorphism of unital bialgebras.

For any $c \in \mathfrak{c}$, $\iota'_c = R_{\iota'_c(1)} = R_{\varphi_c(1)} = R_{\psi(c)} = R_{\psi(\bar{c})} = R_{\psi(\iota_c(\bar{1}))}$, and since $\{\iota_c(\bar{1}) \mid c \in \mathfrak{c}\}$ generates the algebra U_τ , ψ is the unique homomorphism $U_\tau \rightarrow U'$ that satisfies this property.

In order to prove (3) we consider the morphism $\iota: \tau \rightarrow T(U_\tau)$. It restricts to a surjective homomorphism $\mathfrak{g}_{\text{inn}} \rightarrow \mathfrak{g}(U_\tau)_{\text{inn}}$ whose kernel is contained in $\text{core}(\mathfrak{s}_{\text{inn}})$. In fact, $\text{core}(\mathfrak{s}_{\text{inn}})$ is also contained in this kernel since the image of $\text{core}(\mathfrak{s}_{\text{inn}})$ is an ideal of $\mathfrak{g}(U_\tau)_{\text{inn}}$ living in $\mathfrak{s}(U_\tau)_{\text{inn}}$, and the only such an ideal is $\{0\}$. This proves the isomorphism $\tau_{\text{red}} \cong T(U_\tau)_{\text{inn}} = T(U_\tau)_{\text{red}}$.

The statement in item (4) will follow from the existence of isomorphisms $U_\tau \cong U_{T(U_\tau)_{\text{inn}}} \cong U_{\tau_{\text{red}}}$ that we proceed to prove. As in item (2), U_τ is a $\mathfrak{g}(U_\tau)_{\text{inn}}$ -module and we have a map $\psi: U(\mathfrak{g}(U_\tau)_{\text{inn}}) \rightarrow U_\tau$ with $\psi(f) := f(\bar{1})$ which, thanks to the monoalternativity, induces a homomorphism of unital bialgebras $\psi: U_{T(U_\tau)_{\text{inn}}} \rightarrow U_\tau$. On the one hand, $U_{T(U_\tau)_{\text{inn}}}$ has a basis

$$\left\{ \sum_{\sigma \in S_n} \overline{R_{c_{\sigma(n)}} \cdots R_{c_{\sigma(1)}}} \mid c_1, \dots, c_n \in \mathfrak{c}, n \geq 0 \right\};$$

on the other hand, $\{\sum_{\sigma \in S_n} \overline{c_{\sigma(1)} \cdots c_{\sigma(n)}} \mid c_1, \dots, c_n \in \mathfrak{c}, n \geq 0\}$ is a basis of U_τ , and ψ bijectively maps the former basis onto the latter; thus ψ is an isomorphism.

This shows the existence of an isomorphism $U_\tau \cong U_{T(U_\tau)_{\text{red}}}$. The isomorphism $U_{T(U_\tau)_{\text{red}}} \cong U_{\tau_{\text{red}}}$ is induced by the isomorphism $T(U_\tau)_{\text{red}} \cong \tau_{\text{red}}$.

Let us assume that τ and τ' are equivalent triples, i.e. $\tau_{\text{red}} \cong \tau'_{\text{red}}$. Thus, $U_{\tau_{\text{red}}} \cong U_{\tau'_{\text{red}}}$ and, by the previous paragraph, U_τ is isomorphic to $U_{\tau'}$. Conversely, if U_τ is isomorphic to $U_{\tau'}$ then $T(U_\tau)_{\text{red}}$ is isomorphic to $T(U_{\tau'})_{\text{red}}$, which leads to an isomorphism $\tau_{\text{red}} \cong \tau'_{\text{red}}$. ■

While in the previous proof we have used the notation \bar{u} for elements in U_τ , once this algebra is obtained there is no need for writing the bar, and we will do not so. Even more, it will become apparent that it is much more convenient to consider an isomorphic copy ρ_c of \mathfrak{c} as being a part of the triple τ , while keeping \mathfrak{c} for the space of primitive elements of U_τ ; thus,

$$\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c) \quad \text{and} \quad \mathfrak{c} = \text{Prim}(U_\tau).$$

The conceptual benefit of this change is that now the morphism $\iota: \tau \rightarrow T(U_\tau)$ maps ρ_c to R_c , i.e. ρ_c is sought as the abstract counterpart of R_c , while \mathfrak{g} behaves as an abstract version of, and probably larger than, the Lie algebra generated by the right multiplication operators by primitive elements in U_τ . The action of \mathfrak{g} on U_τ extends to a homomorphism $U(\mathfrak{g}) \rightarrow \text{End}_{\mathfrak{k}}(U_\tau)$. Again, although not totally correct, it would be helpful to use the same symbol to denote both, the element in $U(\mathfrak{g})$ and its image in $\text{End}_{\mathfrak{k}}(U_\tau)$. Thus, we will freely write $f(u)$ for elements $f \in U(\mathfrak{g})$ and $u \in U_\tau$.

3.4. Natural factorization of $U(\mathfrak{g})$. For any triple $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c)$ we will show that a natural factorization of $U(\mathfrak{g})$, needed in Section 4, exists. The following lemma follows easily from the existence of Poincaré-Birkhoff-Witt bases.

Lemma 3.3. *Let $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c)$ be a triple. Then, $U(\mathfrak{s})$ is the largest subcoalgebra contained in $\{f \in U(\mathfrak{g}) \mid f(1) = \epsilon(f)1\}$.*

Now we define elements ρ_u ($u \in U_\tau$) in $U(\mathfrak{g})$ that play the role of the right multiplication operators R_u ($u \in U_\tau$). Let us fix a basis \mathcal{B} of \mathfrak{c} , and recall the basis

$$\left\{ \sum_{\sigma \in S_n} ((c_{\sigma(1)}c_{\sigma(2)}) \cdots) c_{\sigma(n)} \mid c_1, \dots, c_n \in \mathcal{B}, n \geq 0 \right\}$$

of U_τ . Define $\rho_{\sum_{\sigma \in S_n} ((c_{\sigma(1)}c_{\sigma(2)}) \cdots) c_{\sigma(n)}} := \sum_{\sigma \in S_n} \rho_{c_{\sigma(n)}} \cdots \rho_{c_{\sigma(1)}} \in U(\mathfrak{g})$ (7)

for any $c_1, \dots, c_n \in \mathfrak{c}$ and any $n \geq 0$. We obtain in this way an injective morphism of coalgebras $U_\tau \rightarrow U(\mathfrak{g})$ defined by $u \mapsto \rho_u$. Clearly $\rho_u(x) = xu$ for all $u, x \in U_\tau$.

Proposition 3.4. *Let $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c)$ be a triple. The coalgebra morphism*

$$\pi: U(\mathfrak{g}) \rightarrow U(\mathfrak{g}), \quad f \mapsto S(\rho_{f(1)})f(2)$$

is a projection of $U(\mathfrak{g})$ onto $U(\mathfrak{s})$.

Proof. On the one hand, $\text{Im } \pi$ is a subcoalgebra of $U(\mathfrak{g})$ and any $f \in \text{Im } \pi$ satisfies $\pi(f)(1) = S(\rho_{f(1)})f(2)(1) = S(\rho_{f(1)})\rho_{f(2)(1)}(1) = \epsilon(\rho_{f(1)})1 = \epsilon(f)1$; thus, by Lemma 3.3, $\text{Im } \pi \subseteq U(\mathfrak{s})$. On the other hand, for any $f \in U(\mathfrak{s})$, $\pi(f) = S(\rho_{f(1)})f(2) = S(\rho_{\epsilon(f(1))})f(2) = \epsilon(f(1))f(2) = f$. Therefore, π is a projection of $U(\mathfrak{g})$ onto $U(\mathfrak{s})$. ■

The following corollary is an immediate consequence of: 1) the existence of Poincaré-Birkhoff-Witt bases for $U(\mathfrak{g})$, and 2) the way we defined ρ_u in (7).

Corollary 3.5. *Let $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c)$ be a triple. The mapping $f \mapsto \rho_{f(1)}\pi(f(2))$ defines a coalgebra isomorphism $U(\mathfrak{g}) \cong \rho_{U_\tau} \otimes_{\mathbf{k}} U(\mathfrak{s})$.*

4. Hyporeductive Hopf algebras

In this section we will introduce hyporeductive Hopf algebras by analogy with hyporeductive loops. We will describe hyporeductive triples associated to these Hopf algebras, closely following Sabinin’s work, and finally we will formally integrate hyporeductive triples to obtain the corresponding hyporeductive Hopf algebras.

4.1. Hyporeductive loops, triples and Hopf algebras. A loop Q is called (right) *hyporeductive*³ if the following properties hold:

- *Right hypospeciality*: there exists a map $\circ : Q \times Q \rightarrow Q$ such that $u \circ e = e = e \circ u$, where e stands for the identity element of Q , and

$$H(u, v)(xy) = H(u, v)(x)H'(u, v)(y),$$

where by definition $H(u, v)(x) := r(u, v)(x)(u \circ v)$, $r(u, v)(x) := ((xu)v)/(uv)$ and $H'(u, v)(y) := (u \circ v) \setminus H(u, v)(y)$.

- *Right monoalternativity*: $((xy)y \cdots)y = x((yy) \cdots y)$ for all $n \geq 0$, where y appears n times in each side of the equality.

Definition 4.1. A triple $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c)$ is called *hyporeductive* if $\mathfrak{g} = N_{\mathfrak{g}}(\rho_c) + \rho_c$.

The Lie algebra generated by the fundamental vector fields of any analytic hyporeductive loop determines such a triple [66].

Definition 4.2. A nonassociative Hopf algebra U is called *hyporeductive* if the following properties hold:

- *Right hypospeciality*: there exists a coalgebra morphism $\circ : U \otimes U \rightarrow U$ such that $u \circ 1 = \epsilon(u)1 = 1 \circ u$ and

$$H(u, v)(xy) = H(u_{(1)}, v_{(1)})(x)H'(u_{(2)}, v_{(2)})(y),$$

where by definition $H(u, v)(x) := r(u_{(1)}, v_{(2)})(x)(u_{(2)} \circ v_{(2)})$,

$$r(u, v)(x) := ((xu_{(1)})v_{(1)})/(u_{(2)}v_{(2)})$$

and $H'(u, v)(y) := (u_{(1)} \circ v_{(1)}) \setminus H(u_{(2)}, v_{(2)})(y)$

³For analytic loops this definition agrees with the definition in [62, 66].

- *Right monoalternativity*: $((xy_{(1)})y_{(2)}) \cdots y_{(n)} = x((y_{(1)}y_{(2)}) \cdots y_{(n)})$ for all $n \geq 0$.

Observe that for any $a \in \text{Prim}(U)$ the monoalternativity implies $r(a, a) = R_a^2 - R_{a^2} = 0$. Hence,

$$2r(a, b) = r(a, b) - r(b, a) = R_b R_a - R_{ab} - R_a R_b + R_{ba} = -[R_a, R_b] - R_{[a,b]} \quad (8)$$

for all $a, b \in \text{Prim}(U)$.

Lemma 4.3. *Let U be a hyporeductive Hopf algebra. For any $a, b \in \text{Prim}(U)$, the element $a \circ b$ is primitive.*

Proof. Since \circ is a coalgebra morphism and $1 \circ a = 0 = a \circ 1$ for any primitive a , then $\Delta(a \circ b) = (a \circ b) \otimes 1 + (a \circ 1) \otimes (1 \circ b) + (1 \circ b) \otimes (a \circ 1) + 1 \otimes (a \circ b) = (a \circ b) \otimes 1 + 1 \otimes (a \circ b)$. ■

Proposition 4.4. *Let U be a hyporeductive Hopf algebra. Then, $T(U)_{\text{inn}}$ is a hyporeductive triple.*

Proof. On the one hand, given $a, b \in \text{Prim}(U)$, the hypospeciality of U implies $H(a, b)(xy) = H(a, b)(x)y + xH'(a, b)(y)$, and hence $[H(a, b), R_y] = R_{H'(a,b)(y)}$; on the other hand, by (8), $-2H(a, b) = -2r(a, b) - 2R_{a \circ b} = [R_a, R_b] + R_{[a,b]-2a \circ b}$. Therefore,

$$[[R_a, R_b] + R_{[a,b]-2a \circ b}, R_c] = R_{-2H'(a,b)(c)}.$$

This proves: (1) $\mathfrak{g}(U)_{\text{inn}} = \text{span}\langle R_a, [R_a, R_b] \mid a, b \in \text{Prim}(U) \rangle$, and (2) $\mathfrak{g}(U)_{\text{inn}} = N_{\mathfrak{g}(U)_{\text{inn}}}(\mathfrak{c}(U)) + \mathfrak{c}(U)$. ■

4.2. Formal integration of hyporeductive triples. We will prove that for any hyporeductive triple τ , the nonassociative Hopf algebra U_τ is hyporeductive. To this end, let us fix a hyporeductive triple $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_\mathfrak{c})$ and let us denote $N_\mathfrak{g}(\rho_\mathfrak{c})$ just by \mathfrak{n} .

Lemma 4.5. *There exists a coalgebra morphism $\sigma: U(\mathfrak{s}) \rightarrow U(\mathfrak{n})$ such that $\pi\sigma$ is the identity map of $U(\mathfrak{s})$.*

Proof. We include a proof for completeness. Since $\mathfrak{g} = \mathfrak{s} \oplus \rho_\mathfrak{c} = \mathfrak{n} + \rho_\mathfrak{c}$, then we can chose a linear map $\theta: U(\mathfrak{s}) \rightarrow \mathfrak{n}$ such that $\pi\theta(d) = d$ for any $d \in \mathfrak{s}$, and $\theta(1) = 0$. Let us extend this map to a coalgebra morphism $\theta': U(\mathfrak{s}) \rightarrow U(\mathfrak{n})$ by $\theta' = \exp^*(\theta)$, where

$$\exp^*(\theta)(x) := \sum_{n=0}^{\infty} \frac{1}{n!} \theta(x_{(1)}) \cdots \theta(x_{(n)}).$$

The coalgebra morphism $\Psi := \pi\theta': U(\mathfrak{s}) \rightarrow U(\mathfrak{s})$ is an isomorphism since its restriction to \mathfrak{s} is the identity map of \mathfrak{s} (see [77], Theorem 12.2.6); in fact, $\Psi(d) = \pi\theta'(d) = \pi\theta(d) = d$ for all $d \in \mathfrak{s}$. A coalgebra morphism that fulfills the requirement in the statement is $\sigma := \theta'\Psi^{-1}$. ■

For the coalgebra morphism σ in Lemma 4.5 we have

$$\sigma(f) = \rho_{\sigma(f_{(1)})(1)}\pi\sigma(f_{(2)}) = \rho_{\sigma(f_{(1)})(1)}f_{(2)}. \tag{9}$$

Theorem 4.6. *Let τ be a hyporeductive triple. Then, U_τ is a hyporeductive Hopf algebra.*

Proof. We first define the map \circ required for the hypospeciality. To this end, observe that the elements $\rho(u, v) := S(\rho_{u_{(1)}v_{(1)}})\rho_{v_{(2)}}\rho_{u_{(2)}} \ (u, v \in U_\tau)$ span a coalgebra and they satisfy

$$\rho(u, v)(1) = S(\rho_{u_{(1)}v_{(1)}})(u_{(2)}v_{(2)}) = S(\rho_{u_{(1)}v_{(1)}})\rho_{u_{(2)}v_{(2)}}(1) = \epsilon(u)\epsilon(v)1.$$

Thus, by Lemma 3.3, $\rho(u, v) \in U(\mathfrak{s})$. This ensures that the map

$$u \circ v := \sigma(\rho(u, v))(1)$$

is well-defined. Clearly, $u \circ 1 = \epsilon(u)1$, $1 \circ v = \epsilon(v)1$ and

$$\sigma(\rho(u, v)) = \rho_{u_{(1)}\circ v_{(1)}}\rho(u_{(2)}, v_{(2)}) \in U(\mathfrak{n}). \tag{10}$$

The hypospeciality of U_τ will be a consequence of

$$\rho_{u_{(1)}\circ v_{(1)}}\rho(u_{(2)}, v_{(2)})\rho_{U_\tau}S(\rho_{u_{(3)}\circ v_{(3)}}\rho(u_{(4)}, v_{(4)})) \subseteq \rho_{U_\tau}$$

or, more generally, it will follow from $f_{(1)}\rho_{U_\tau}S(f_{(2)}) \subseteq \rho_{U_\tau}$ for all $f \in U(\mathfrak{n})$. The advantage is that we only have to prove it for generators in \mathfrak{n} of $U(\mathfrak{n})$. Clearly $[f, \rho_c] \subseteq \rho_c$ by definition of \mathfrak{n} . In general,

$$[f, \rho_{c^n}] = [f, \rho_c^n] = \sum \rho_c \cdots \rho_c \rho_{f'(c)} \rho_c \cdots \rho_c$$

for some $f'(c) \in \mathfrak{c}$. Hence, the linearization of $\rho_{c^n} = \rho_c^n$ leads to $[f, \rho_{c^n}] \in \rho_{U_\tau}$.

Finally, consider $f := \rho_{u_{(1)}\circ v_{(1)}}\rho(u_{(2)}, v_{(2)})$. Evaluating at 1 we obtain $f_{(1)}\rho_y S(f_{(2)}) = \rho_{f_{(1)}\rho_y S(f_{(2)})(1)}$; thus, $f_{(1)}(S(f_{(2)})(x)y) = x(f_{(1)}(S(f_{(2)})(1)y))$ and

$$f(xy) = f_{(1)}(x)f_{(2)}(S(f_{(3)})(1)y) = f_{(1)}(x)(f_{(2)}(1)\setminus f_{(3)}(y)).$$

Since $\rho_u(x) = R_u(x)$, $\rho(u, v)(x) = r(u, v)(x)$ and $f(x) = H(u, v)(x)$, we are done. ■

5. Pseudoreductive Hopf algebras

In this section we introduce pseudoreductive Hopf algebras by analogy with pseudoreductive loops. We will describe pseudoreductive triples associated to these Hopf algebras, and finally we will formally integrate pseudoreductive triples to obtain pseudoreductive Hopf algebras. Our approach sticks again to the work of Sabinin on analytic loops, and we cannot avoid his clever use of the formula for the differential of the exponential map without resorting to artificial arguments.

5.1. Pseudoreductive loops, triples and Hopf algebras. A loop Q is called (right) *pseudoreductive* if the following properties hold:

- *Right pseudospeciality*: there exists a map $\bullet: Q \times Q \rightarrow Q$ such that $e \bullet u = e = u \bullet e$ (e stands for the identity element of Q) and

$$P(u, v)(xy) = r(u, v)(x)P(u, v)(y)$$

where $P(u, v)(x) := r(u, v)(x)(u \bullet v)$.

- *Right monoalternativity*: $((xy)y \cdots)y = x((yy) \cdots y)$ for all $n \geq 0$, where y appears n times in each side of the equality.

Definition 5.1. A nonassociative Hopf algebra U is called (right) *pseudoreductive* if the following properties hold:

- *Right pseudospeciality*: there exists a coalgebra morphism $\bullet: U \otimes U \rightarrow U$ such that $1 \bullet u = \epsilon(u)1 = u \bullet 1$ and

$$P(u, v)(xy) = r(u_{(1)}, v_{(1)})(x)P(u_{(2)}, v_{(2)})(y)$$

where $P(u, v)(x) := r(u_{(1)}, v_{(2)})(x)(u_{(2)} \bullet v_{(2)})$.

- *Right monoalternativity*: $((xy_{(1)})y_{(2)}) \cdots y_{(n)} = x((y_{(1)}y_{(2)}) \cdots y_{(n)})$ for all $n \geq 0$.

Definition 5.2. We will say that a triple $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c)$ is *pseudoreductive* if there exists a map $\zeta: \mathfrak{s} \rightarrow \rho_c$ such that

$$(PRT1) \quad d + \zeta(d) \in N_{\mathfrak{g}}(\rho_c) \text{ for all } d \in \mathfrak{s}, \text{ and}$$

$$(PRT2) \quad \text{ad}_{\rho_c}^{2n}(\zeta(\mathfrak{s})) \subseteq \rho_c \text{ for all } n \geq 0 \text{ and } \rho_c \in \rho_c.$$

Notice that the first condition is equivalent to τ being hyporeductive.

Lemma 5.3. *Let U be a pseudoreductive Hopf algebra. For any $a, b \in \text{Prim}(U)$, $a \bullet b$ is also primitive.*

Proof. See the proof of Lemma 4.3. ■

Proposition 5.4. *Let U be a pseudoreductive Hopf algebra. For any $a, b, c \in \text{Prim}(U)$ we have $[r(a, b) + \frac{1}{2}R_{a \bullet b}, R_c] = R_{r(a,b)(c) + \frac{1}{2}[c, a \bullet b]}$. Hence, $T(U)_{\text{inn}}$ is a hyporeductive triple.*

Proof. Fix $a, b, c \in \text{Prim}(U)$. On the one hand, $P(a, b) = r(a, b) + R_{a \bullet b}$ and $P(a, b)R_y = R_y r(a, b) + R_{P(a,b)(y)}$ so

$$\begin{aligned} [r(a, b) + \frac{1}{2}R_{a \bullet b}, R_c] &= -R_{a \bullet b}R_c + R_{P(a,b)(c)} + \frac{1}{2}[R_{a \bullet b}, R_c] \\ &= -\frac{1}{2}R_{a \bullet b}R_c - \frac{1}{2}R_cR_{a \bullet b} + R_{P(a,b)(c)}; \end{aligned}$$

on the other hand, the monoalternativity implies $R_{a \bullet b}R_c + R_cR_{a \bullet b} = R_{(a \bullet b)c + c(a \bullet b)}$, i.e. $[r(a, b) + \frac{1}{2}R_{a \bullet b}, R_c]$ is the right multiplication operator by the primitive element $r(a, b)(c) + \frac{1}{2}[c, a \bullet b] = [r(a, b) + \frac{1}{2}R_{a \bullet b}, R_c](1)$. ■

The following result highlights the property behind the axiom (PRT2) in the definition of pseudoreductive triples.

Proposition 5.5. *Let U be a pseudoreductive connected Hopf algebra. Then*

$$R_{a\bullet b}R_y + R_yR_{a\bullet b} = R_{(a\bullet b)y+y(a\bullet b)} \tag{11}$$

for all $a, b \in \text{Prim}(U)$ and $y \in U$.

Proof. By Proposition 5.4 we have $[r(a, b) + \frac{1}{2}R_{a\bullet b}, R_c] = R_{r(a,b)(c) + \frac{1}{2}[c, a\bullet b]}$, and by monoalternativity $[r(a, b) + \frac{1}{2}R_{a\bullet b}, R_{c^n}] \in R_U$ for all $a, b, c \in \text{Prim}(U)$. The set $\{c^n \mid n \geq 0, c \in \text{Prim}(U)\}$ spans U (U is connected), thus

$$-\frac{1}{2}(R_{a\bullet b}R_y + R_yR_{a\bullet b}) + R_{P(a,b)(y)} = [r(a, b) + \frac{1}{2}R_{a\bullet b}, R_y] \in R_U$$

for all $y \in U$. This proves $R_{a\bullet b}R_y + R_yR_{a\bullet b} \in R_U$. After evaluating this operator at 1 we get the result. ■

We can use (11) to prove that $T(U)_{\text{inn}}$ satisfies (PRT2) for the initial value $n = 1$. For any $a, b, c \in \text{Prim}(U)$,

$$\begin{aligned} [R_c, [R_c, R_{a\bullet b}]] &= R_c^2R_{a\bullet b} - 2R_cR_{a\bullet b}R_c + R_{a\bullet b}R_c^2 \\ &= 3(R_{c^2}R_{a\bullet b} + R_{a\bullet b}R_{c^2}) - 2(R_{a\bullet b}R_cR_c + R_cR_{a\bullet b}R_c + R_cR_cR_{a\bullet b}). \end{aligned}$$

Both expressions in parentheses are multiplication operators, so $[R_c, [R_c, R_{a\bullet b}]] \in \mathfrak{c}(U)$. Unfortunately it is much more difficult to prove that $T(U)_{\text{inn}}$ is a pseudoreductive triple if we use this approach. Sabinin [66] gave a short proof for analytic loops. It is convenient to fix two parameters s, t so that we have at our disposal the algebra of formal power series $A[[s, t]]$ with coefficients in another algebra A . If A is associative and unital, expressions such as $\exp(ta)$ make sense in $A[[s, t]]$. If A is nonassociative then although

$$\exp_t(ta) := 1 + ta + \frac{1}{2!}(ta)(ta) + \frac{1}{3!}((ta)(ta))(ta) + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}(ta)^n$$

requires our convention about powers, for the most part, it is again a well-defined element in $A[[s, t]]$.

Proposition 5.6. *Let U be a pseudoreductive Hopf algebra. Then, $T(U)_{\text{inn}}$ is a pseudoreductive triple.*

Proof. Identity (11) implies $\sum_{i=0}^n \binom{n}{i} R_{a\bullet b}^i R_y R_{a\bullet b}^{n-i} \in R_U$ for any $a, b \in \text{Prim}(U)$ and $y \in U$. Thus, $\exp(tR_{a\bullet b})\exp(sR_c)\exp(tR_{a\bullet b}) \in R_{U[[s,t]]}$ for all $a, b, c \in \text{Prim}(U)$. Evaluating at 1 we obtain

$$\exp(tR_{a\bullet b})\exp(sR_c)\exp(tR_{a\bullet b}) = R_{(\exp_t(ta\bullet b)\exp_t(sc))\exp_t(ta\bullet b)}. \tag{12}$$

The element $(\exp_l(ta \bullet b) \exp_l(sc)) \exp_l(ta \bullet b)$ is group-like (see [46] for a discussion on group-like elements), so there exists $c(s, t) = c(s, t; a, b) \in \text{Prim}(U)[[s, t]]$ such that $f(s, t) := R_{c(s,t)}$ satisfies

$$\exp(tR_{a \bullet b}) \exp(sR_c) \exp(tR_{a \bullet b}) = \exp(f(s, t)).$$

Taking derivatives with respect to t we have⁴

$$\begin{aligned} \frac{\exp(-\text{ad}_{f(s,t)}) - \text{Id}}{-\text{ad}_{f(s,t)}} \left(\frac{\partial f}{\partial t} \right) &= \exp(-f(s, t)) \frac{\partial \exp(f(s, t))}{\partial t} \\ &= \exp(-f(s, t)) R_{a \bullet b} \exp(f(s, t)) + R_{a \bullet b} \\ &= \exp(-\text{ad}_{f(s,t)})(R_{a \bullet b}) + R_{a \bullet b}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\text{ad}_{f(s,t)}}{\exp(\text{ad}_{f(s,t)}) - \text{Id}} (R_{a \bullet b}) + \frac{-\text{ad}_{f(s,t)}}{\exp(-\text{ad}_{f(s,t)}) - \text{Id}} (R_{a \bullet b}) \\ &= 2 \sum_{2m \geq 0} \frac{\beta_{2m}}{(2m)!} \text{ad}_{f(s,t)}^{2m} (R_{a \bullet b}) \end{aligned}$$

where $0 \neq \beta_{2m}$ is the $2m$ -th Betti number. Evaluating at $t = 0$, since $f(s, 0) = R_{sc}$ and $f \in R_{U[[s,t]]}$, we obtain $\text{ad}_{R_c}^{2m}(R_{a \bullet b}) \in \mathfrak{c}(U)$ for all $m \geq 0$ and $a, b, c \in \text{Prim}(U)$. This proves the pseudospeciality of $T(U)_{\text{inn}}$. ■

Equality (12) naturally appears in the context of Bruck loops and symmetric homogeneous spaces. These structures are related to Lie triple systems, i.e. subspaces of Lie algebras closed under the double commutator. Axiom (PRT2) evokes these structures, and our presentation aimed for it.

5.2. Formal integration of pseudoreductive triples.

Theorem 5.7. *Let τ be a pseudoreductive triple. Then, U_τ is a pseudoreductive Hopf algebra.*

Proof. Let $\tau = (\mathfrak{g}, \mathfrak{s}, \rho_c)$ be a pseudoreductive triple. The pseudoreductive structure of τ induces via $\iota: \tau \rightarrow T(U_\tau)$ a corresponding pseudoreductive structure on the triple $T(U_\tau)_{\text{inn}}$. Thus, there exists a map $\zeta: \mathfrak{s}(U_\tau)_{\text{inn}} \rightarrow \mathfrak{c}(U_\tau)$ such that $d + \zeta(d) \in N_{\mathfrak{g}(U_\tau)_{\text{inn}}}(\mathfrak{c}(U_\tau))$. Given $a, b \in \text{Prim}(U_\tau)$ we define $a \bullet b$ by

$$R_{a \bullet b} := \zeta(2r(a, b)).$$

With this choice we get $[r(a, b) + \frac{1}{2}R_{a \bullet b}, R_c] = R_{r(a,b)(c) + \frac{1}{2}[c, a \bullet b]}$ for any $c \in \text{Prim}(U_\tau)$. The monoalternativity makes this identity valid for all $y \in U_\tau$, i.e.

$$[r(a, b) + \frac{1}{2}R_{a \bullet b}, R_y] = R_{r(a,b)(y) + \frac{1}{2}[y, a \bullet b]}.$$

⁴Consider the free associative algebra $\mathbf{k}\langle x, x' \rangle$ and the derivation $d := x' \frac{\partial}{\partial x}$. With the natural Hopf algebra structure (x, x' are primitive) the map $\gamma(u) = S(u_{(1)})d(u_{(2)})$ satisfies $\gamma(ux) = [\gamma(u), x] + \epsilon(u)x'$. By allowing formal power series, we easily obtain $\exp(-x)d(\exp(x)) = \gamma(\exp(x)) = \frac{\exp(-\text{ad}_x) - \text{Id}}{-\text{ad}_x}(x')$.

Let us consider the function $f(s, t)$ in

$$\exp(tR_{a\bullet b}) \exp(sR_c) \exp(tR_{a\bullet b}) = \exp(f(s, t))$$

used in the proof of Proposition 5.6. Beware, now we cannot assume $f(s, t) \in \mathfrak{c}(U_\tau)[[s, t]]$. However, this function is uniquely determined by

$$\frac{\partial f}{\partial t} = 2 \sum_{2m \geq 0} \frac{\beta_{2m}}{(2m)!} \text{ad}_{f(s,t)}^{2m}(R_{a\bullet b})$$

and the initial condition $f(s, 0) = R_{sc} \in \mathfrak{c}(U_\tau)[[s, t]]$, which by (PRT2) ensures $f(s, t) \in \mathfrak{c}(U_\tau)[[s, t]]$ and $\exp(f(s, t)) \in R_{U_\tau[[s,t]]}$. In particular,

$$\frac{\partial \exp(f)}{\partial t}(s, 0) \in R_{U_\tau[[s]]}$$

implies $R_{a\bullet b}R_y + R_yR_{a\bullet b} = R_{(a\bullet b)y+y(a\bullet b)}$ for all $a, b \in \text{Prim}(U_\tau)$ and $y \in U_\tau$. Therefore, if we define $P(a, b) := r(a, b) + R_{a\bullet b}$ then

$$P(a, b)(xy) = r(a, b)(x)y + xP(a, b)(y). \tag{13}$$

Let \mathfrak{p} be the Lie algebra generated by $\{P(a, b) \mid a, b \in \text{Prim}(U_\tau)\}$ and let θ be a linear map $U(\mathfrak{s}(U_\tau)_{\text{inn}}) \rightarrow \mathfrak{p}$ such that $\theta(d) = d + 2\zeta(d)$ for all $d \in \mathfrak{s}(U_\tau)_{\text{inn}}$, i.e. $\theta(r(a, b)) = P(a, b)$, and $\theta(1) = 0$. We extend θ to a coalgebra morphism $\theta' : U(\mathfrak{s}(U_\tau)_{\text{inn}}) \rightarrow U(\mathfrak{p})$ by $\theta' = \exp^*(\theta)$ as in the proof of Lemma 4.5. We also define the isomorphism $\Psi := \pi\theta'$ and elements

$$u \bullet v := \theta'\Psi^{-1}(r(u, v))(1).$$

Any operator $\theta'\Psi(r(u, v)) = R_{u_{(1)}\bullet v_{(1)}}r(u_{(2)}, v_{(2)})$ belongs to $U(\mathfrak{p})$, so (13) implies

$$R_{u_{(1)}\bullet v_{(1)}}r(u_{(2)}, v_{(2)})(xy) = r(u_{(1)}, v_{(1)})(x)R_{u_{(2)}\bullet v_{(2)}}r(u_{(3)}, v_{(3)})(y)$$

as desired. ■

6. Hyporeductive triple algebras

Hyporeductive loops can also be constructed from hyporeductive triple algebras. In this section we discuss these structures from the point of view of Sabinin algebras.

6.1. Hyporeductive triple algebras. The origin of *hyporeductive triple algebras* is linked to the following construction. Let $\tau = (\mathfrak{g}, \mathfrak{s}, \mathfrak{c})$ be a hyporeductive triple (here it would be more convenient to return to the original notation \mathfrak{c} instead of $\rho_{\mathfrak{c}}$), so $\mathfrak{g} = \mathfrak{n} + \mathfrak{c}$ where $\mathfrak{n} = N_{\mathfrak{g}}(\mathfrak{c})$. We can choose a subspace $\mathfrak{h} \subseteq \mathfrak{n}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{c}$. This decomposition induces two binary products $a \cdot b$, $a * b$ and a triple product $[c; a, b]$ on \mathfrak{c} as follows

$$[a, b] = h(a, b) + a \cdot b, \quad h(a, b) = s(a, b) + a * b, \quad [h(a, b), c] = [c; a, b] \tag{14}$$

with $h(a, b) \in \mathfrak{h}$, $s(a, b) \in \mathfrak{s}$ and $a, b, c \in \mathfrak{c}$. The new algebraic structure $(\mathfrak{c}, a \cdot b, a * b, [c; a, b])$ is characterized by twelve axioms (apart from the obvious

skew-commutativity) coming from the Jacobi identity [30]. Conversely, given a hyporeductive triple algebra $(\mathfrak{c}, a \cdot b, a * b, [c; a, b])$ there exists a Lie algebra $E(\mathfrak{c})$ generated by \mathfrak{c} with relations

$$[[a, b] - a \cdot b, c] = [c; a, b]$$

for any $a, b, c \in \mathfrak{c}$. This Lie algebra splits as

$$E(\mathfrak{c}) = \mathfrak{s} \oplus \mathfrak{c} \quad \text{with} \quad \mathfrak{s} := \text{span}\langle [a, b] - a \cdot b \mid a, b \in \mathfrak{c} \rangle$$

being a subalgebra. Thus, $(E(\mathfrak{c}), \mathfrak{s}, \mathfrak{c})$ is a hyporeductive triple [30]. The hyporeductive triple algebra $(\mathfrak{c}, a \cdot b, a * b, [c; a, b])$ is recovered from this triple by (14). Unfortunately, the construction of hyporeductive triple algebras from hyporeductive triples involves the choice of \mathfrak{h} .

The formal integration of a hyporeductive triple algebra $(\mathfrak{c}, a \cdot b, a * b, [c; a, b])$ might be understood as the construction of a hyporeductive (connected) Hopf algebra $U(\mathfrak{c})$ such that $\mathfrak{c} = \text{Prim}(U(\mathfrak{c}))$ and

$$[a, b] = a * b + a \cdot b \quad \text{and} \quad [[R_a, R_b] + R_{a \cdot b}, R_c] = R_{[c; a, b]}. \tag{15}$$

It is not difficult to check that if we begin with the Lie algebra $E(\mathfrak{c})^{\text{opp}}$ (recall that $\mathfrak{g}^{\text{opp}}$ has the opposite product $-[x, y]$ instead of the product $[x, y]$ of \mathfrak{g}) and the triple $(E(\mathfrak{c})^{\text{opp}}, \mathfrak{s}^{\text{opp}}, \mathfrak{c})$ then the corresponding hyporeductive Hopf algebra associated to this triple satisfies the required properties.

6.2. Sabinin algebras associated to hyporeductive triple algebras. As discussed at the beginning of this paper, the algebraic structure of the tangent space of any analytic loop is known as Sabinin algebra. For general loops, two families of multilinear operations $\langle x_1, \dots, x_m; y, z \rangle, \Phi(x_1, \dots, x_m; y_1, \dots, y_n; y_{n+1})$ take part in the description of the corresponding Sabinin tangent algebra. However, for right monoalternative loops only the operations $\langle x_1, \dots, x_n; y, z \rangle, n \geq 0$ are required since $\Phi(x_1, \dots, x_m; y_1, \dots, y_n; y_{n+1}) = 0$ for these loops. The connection of hyporeductive triple algebras with the infinitesimal study of hyporeductive loops suggests that a Sabinin algebra can be obtained out of any such triple algebra. The “underline” notation from Section 2 is quite useful for that.

Theorem 6.1. *Let $(\mathfrak{c}, a \cdot b, a * b, [c; a, b])$ be a hyporeductive triple algebra. The multilinear operations*

$$\begin{aligned} \langle 1; y, z \rangle &:= -y \cdot z - z * y \\ \langle \underline{u}x_{m+1}; y, z \rangle &:= \langle \underline{u}; x_{m+1}, y \cdot z \rangle + \langle \underline{u}_{(1)}; x_{m+1}, \langle \underline{u}_{(2)}; y, z \rangle \rangle + \epsilon(u)[x_{m+1}; y, z] \end{aligned}$$

define a Sabinin algebra structure on \mathfrak{c} .

Proof. Let $U(\mathfrak{c})$ be the hyporeductive Hopf algebra that formally integrates $(\mathfrak{c}, a \cdot b, a * b, [c; a, b])$. We will compute the Shestakov-Umirbaev operations (recall Section 2) $\langle -; -, - \rangle$ of $\mathcal{YIII}(U(\mathfrak{c}))$ to obtain the desired Sabinin algebra $(\mathfrak{c}, \langle -; -, - \rangle)$. By (15) we have

$$\begin{aligned} &\{((ux_{m+1})z)y - ((ux_{m+1})y)z - (ux_{m+1})(y \cdot z)\} - \\ &- \{((uz)y)x_{m+1} - ((uy)z)x_{m+1} + (u(y \cdot z))x_{m+1}\} = u[x_{m+1}; y, z], \end{aligned}$$

for any $x_1, \dots, x_m, x_{m+1}, y, z \in \mathfrak{c}$, $u = ((x_1 x_2) \cdots) x_m$ and $\underline{u} = x_1 \otimes \cdots \otimes x_m$, so

$$\begin{aligned} & (ux_{m+1})_{(1)} \langle \underline{(ux_{m+1})}_{(2)}; y, z \rangle + \\ & + u_{(1)} \langle \underline{u}_{(2)}; y \cdot z, x_{m+1} \rangle - (u_{(1)} \langle u_{(2)}; y, z \rangle) x_{m+1} = u[x_{m+1}; y, z], \end{aligned}$$

which implies

$$\begin{aligned} & u_{(1)} \langle \underline{u}_{(2)}; \langle \underline{u}_{(3)}; y, z \rangle, x_{m+1} \rangle + \\ & + u_{(1)} \langle \underline{u}_{(2)} x_{m+1}; y, z \rangle + u_{(1)} \langle \underline{u}_{(2)}; y \cdot z, x_{m+1} \rangle = u[x_{m+1}; y, z]. \end{aligned}$$

Dividing by $u_{(1)}$ we get

$$\langle \underline{ux_{m+1}}; y, z \rangle = \langle \underline{u}; x_{m+1}, y \cdot z \rangle + \langle \underline{u}_{(1)}; x_{m+1}, \langle \underline{u}_{(2)}; y, z \rangle \rangle + \epsilon(u)[x_{m+1}; y, z],$$

and, by definition of $\langle 1; y, z \rangle$ and the properties of $U(\mathfrak{c})$, $\langle 1; y, z \rangle = -y * z - y \cdot z$. ■

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References

- [1] A. A. Akivis: *Three-webs of multidimensional surfaces*, Trudy Geometr. Sem. **2** (1969) 7–31.
- [2] —: *The local algebras of a multidimensional three-web*, Sibirsk. Mat. Ž. **17** (1976) 5–11.
- [3] M. A. Akivis, and V. V. Goldberg: *Differential Geometry of Webs*, North-Holland, Amsterdam (2000).
- [4] —: *Local algebras of a differential quasigroup*, Bull. Amer. Math. Soc. (N.S.) **43** (2006), 207–226.
- [5] M. A. Akivis, and A. M. Shelekhov: *Geometry and Algebra of Multidimensional Three-Webs*, Kluwer, Dordrecht (1992).
- [6] R. Baer: *Homogeneity of projective planes*, Amer. J. Math. **64** (1942) 137–152.
- [7] A. Barlotti: *Le possibili configurazioni del sistema delle coppie punto-retta (A, a) per cui un piano grafico risulta (A, a) -transitivo*, Boll. Un. Mat. Ital. **12** (1957) 212–226.
- [8] A. Barlotti, and K. Strambach: *The geometry of binary systems*, Adv. in Math. **49** (1983) 1–105.
- [9] O. Chein, H. O. Pflugfelder, and J. D. H. Smith (eds.), *Quasigroups and Loops: Theory and Applications*, Heldermann Verlag, Berlin (1990).
- [10] A. Figula: *Geodesic loops*, J. Lie Theory **10** (2000) 455–461.
- [11] —: *Multiplication groups of topological loops*, J. Math. Sci. (N.Y.) **193** (2013) 428–432.
- [12] —: *Multiplication groups of topological loops*, J. Math. Sci. (N.Y.) **197** (2014) 735–740.

- [13] —: *Quasi-simple Lie groups as multiplication groups of topological loops*, Adv. Geom. **15** (2015) 315–331.
- [14] —: *Lie groups as multiplication groups of topological loops*, J. Math. Sci. (N.Y.) **218** (2016) 742–747.
- [15] A. Figula, and K. Strambach: *Loops as sections in compact Lie groups*, Abh. Math. Semin. Univ. Hambg. **87** (2017), 61–68.
- [16] A. N. Grishkov, and G. P. Nagy: *Algebraic Bol loops*, Forum Math. **23** (2011), 655–668.
- [17] A. N. Grishkov, and J. M. Pérez-Izquierdo: *Lie’s correspondence for commutative automorphic formal loops*, arXiv:1612.03624v1.
- [18] D. Hilbert: *Grundlagen der Geometrie*, Teubner, Berlin (1899).
- [19] K.-H. Hofmann: *Topologische Loops mit schwachen Assoziativitätsforderungen*, Math. Z. **70** (1958) 125–155.
- [20] —: *Topologische Loops*, Math. Z. **70** (1958) 13–37.
- [21] —: *Connected abelian groups in compact loops*, Trans. Amer. Math. Soc. **104** (1962) 132–143.
- [22] K.-H. Hofmann, and J. R. Martin: *Topological left-loops*, Topology Proc. **39** (2012) 185–194.
- [23] K.-H. Hofmann, and K. Strambach, *Lie’s fundamental theorems for local analytical loops*, Pacific J. Math. **123** (1986) 301–327.
- [24] —: *Topological and analytic loops*, in: O. Chein, H. O. Pflugfelder, and J. D. H. Smith (eds.), *Quasigroups and Loops: Theory and Applications*, Heldermann, Berlin (1990) 205–262.
- [25] —: *Torsion and curvature in smooth loops*, Publ. Math. Debrecen **38** (1991) 189–214.
- [26] A. N. Issa: *Notes on the geometry of smooth hyporeductive loops*, Algebras Groups Geom. **12** (1995) 223–246.
- [27] —: *Examples of hypospecial loops*, Algebras Groups Geom. **13** (1996) 499–513.
- [28] —: *Examples of hypospecial loops. II*, Algebras Groups Geom. **17** (2000), 167–175.
- [29] —: *A note on the Akinis algebra of a smooth hyporeductive loop*, Quasigroups Related Systems **9** (2002) 55–64.
- [30] —: *On enveloping Lie algebras of hyporeductive triple algebras*, Comm. Algebra **38** (2010) 3111–3124.
- [31] F. S. Kerdman: *Analytic Moufang loops in the large*, Algebra i Logika **18** (1979) 523–555.
- [32] M. Kikkawa: *On local loops in affine manifolds*, J. Sci. Hiroshima Univ. Ser. A-I Math. **28** (1964) 199–207.
- [33] —: *On locally reductive spaces and tangent algebras*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci. **5** (1972) 1–13.
- [34] —: *Geometry of homogeneous Lie loops*, Hiroshima Math. J. **5** (1975), 141–179.
- [35] M. K. Kinyon, and A. Weinstein: *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces*, Amer. J. Math. **123** (2001) 525–550.

- [36] E. N. Kuzmine: *La relation entre les algèbres de Malcev et les boucles de Moufang analytiques*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970) A1152–A1155.
- [37] H. Lenz: *Kleiner Desarguesscher Satz und Dualität in projektiven Ebenen*, Jber. Deutsch. Math. Verein. **57** (1954) 20–31.
- [38] O. Loos: *Symmetric Spaces. I: General Theory*, Benjamin, New York et al. (1969).
- [39] A. I. Mal'cev: *Analytic loops*, Mat. Sb. N.S. **36(78)** (1955) 569–576.
- [40] P. O. Mikheev, and L. V. Sabinin: *Analytic Bol loops*, in: *Webs and Quasigroups*, Kalinin. Gos. Univ. Kalinin (1982) 102–109.
- [41] —: *The differential geometry of Bol loops*, Dokl. Akad. Nauk SSSR **281** (1985) 1055–1057.
- [42] —: *Infinitesimal theory of local analytic loops*, Dokl. Akad. Nauk SSSR **297** (1987) 801–804.
- [43] J. Mostovoy, and J. M. Pérez-Izquierdo: *Formal multiplications, bialgebras of distributions and non-associative Lie theory*, Transform. Groups **15** (2010) 625–653.
- [44] J. Mostovoy, J. M. Pérez-Izquierdo, and I. P. Shestakov: *Hopf algebras in non-associative Lie theory*, Bull. Math. Sci. **4** (2014) 129–173.
- [45] —: *Nilpotent Sabinin algebras*, J. Algebra **419** (2014) 95–123.
- [46] —: *A non-associative Baker-Campbell-Hausdorff formula*, Proc. Amer. Math. Soc. **145** (2017) 5109–5122.
- [47] —: *On torsion-free nilpotent loops*, arXiv:1610.06663v1.
- [48] R. Moufang: *Die Schnittpunktsätze des projektiven speziellen Fünfecksnetzes in ihrer Abhängigkeit voneinander*, Math. Ann. **106** (1932), 755–795.
- [49] G. P. Nagy: *The Campbell-Hausdorff series of local analytic Bruck loops*, Abh. Math. Sem. Univ. Hamburg **72** (2002) 79–87.
- [50] —: *On the tangent algebra of algebraic commutative Moufang loops*, Mathematica **45(68)** (2003) 147–160.
- [51] —: *Algebraic commutative Moufang loops*, Forum Math. **15** (2003) 37–62.
- [52] P. T. Nagy: *Moufang loops and Malcev algebras*, Sem. Sophus Lie **3** (1993) 65–68.
- [53] P. T. Nagy, and K. Strambach: *Loops as invariant sections in groups, and their geometry*, Canad. J. Math. **46** (1994) 1027–1056.
- [54] —: *Loops, their cores and symmetric spaces*, Israel J. Math. **105** (1998) 285–322.
- [55] —: *Loops in Group Theory and Lie Theory*, Walter de Gruyter, Berlin (2002).
- [56] M. Niemenmaa, and T. Kepka: *On multiplication groups of loops*, J. Algebra **135** (1990) 112–122.
- [57] K. Nomizu: *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954) 33–65.
- [58] J. M. Pérez-Izquierdo: *Algebras, hyperalgebras, non-associative bialgebras and loops*, Adv. Math. **208** (2007) 834–876.
- [59] H. O. Pflugfelder: *Historical notes on loop theory*, Comment. Math. Univ. Carolinae **41** (2000) 359–370.
- [60] L. V. Sabinin: *Odules as a new approach to geometry with a connection*, Dokl. Akad. Nauk SSSR **233** (1977) 800–803.

- [61] —: *Differential equations of smooth loops*, Trudy Sem. Vektor. Tenzor. Anal. **23** (1988) 133–146.
- [62] —: *Smooth hypo-reductive loops*, in: Variational Methods in Modern Geometry, Univ. Druzhby Narodov, Moscow (1990) 50–69.
- [63] —: *Smooth hyporeductive loops*, Dokl. Akad. Nauk SSSR **314** (1990) 565–568.
- [64] —: *Smooth hyporeductive loops*, in: Webs and Quasigroups, Tver. Gos. Univ., Tver' (1991) 129–137.
- [65] —: *The theory of smooth hyporeductive and pseudoreductive loops*, Algebras Groups Geom. **13** (1996) 1–24.
- [66] —: *Smooth Quasigroups and Loops*, Kluwer, Dordrecht (1999).
- [67] —: *Smooth quasigroups and loops: forty-five years of incredible growth*, Comment. Math. Univ. Carolinae **41** (2000) 377–400.
- [68] L. V. Sabinin, and L. L. Sabinina: *Geometry of transsymmetric spaces*, in: Webs and quasigroups, Kalinin. Gos. Univ., Kalinin (1990) 64–68.
- [69] —: *On geometry of transsymmetric spaces*, in: Webs and quasigroups, Tver. Gos. Univ., Tver' (1991) 117–122.
- [70] V. Shcherbacov: *Elements of Quasigroup Theory and Applications*, CRC Press, Boca Raton (2017).
- [71] I. P. Shestakov: *Linear representability of Akiwis algebras*, Dokl. Akad. Nauk **368** (1999) 21–23.
- [72] —: *Every Akiwis algebra is linear*, Geom. Dedicata **77** (1999) 215–223.
- [73] I. P. Shestakov, and U. U. Umirbaev: *Free Akiwis algebras, primitive elements, and hyperalgebras*, Resenhas **5** (2001) 149–158.
- [74] —: *Free Akiwis algebras, primitive elements, and hyperalgebras*, J. Algebra **250** (2002) 533–548.
- [75] J. D. H. Smith: *Multilinear algebras and Lie's theorem for formal n -loops*, Arch. Math. (Basel) **51** (1988) 169–177.
- [76] —: *Loops and quasigroups: aspects of current work and prospects for the future*, Comment. Math. Univ. Carolinae **41** (2000) 415–427.
- [77] M. E. Sweedler: *Hopf Algebras*, Benjamin, New York (1969).
- [78] G. Weingart: *On the axioms for Sabinin algebras*, Adv. Geom. **16** (2016) 205–229.
- [79] J. A. Wolf: *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math. **120** (1968) 9–148.
- [80] K. Yamaguti: *On the Lie triple system and its generalization*, J. Sci. Hiroshima Univ. Ser. A **21** (1957/1958) 155–160.

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