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Characterization of the L^p-Range of the Poisson Transform on the Octonionic Hyperbolic Plane

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Dedicated to the memory of Professor Ahmed Intissar.

Abstract. Let $B(\mathbb{O}^2) = \{x \in \mathbb{O}^2, |x| < 1\}$ be the bounded realization of the exceptional symmetric space $F_{4(-20)}/Spin(9)$. For a non-zero real number λ, we give a necessary and a sufficient condition on eigenfunctions F of the Laplace-Beltrami operator on $B(\mathbb{O}^2)$ with eigenvalue $-(\lambda^2+\rho^2)$ to have an L^p-Poisson integral representations on the boundary $\partial B(\mathbb{O}^2)$. Namely, F is the Poisson integral of an L^p-function on the boundary if and only if it satisfies the following growth condition of Hardy-type:

$$\sup_{0\le r<1}(1-r^2)^{\frac{-\rho}{2}}\left(\int_{\partial B(\mathbb{O}^2)}|F(r\theta)|^p d\theta\right)^{\frac{1}{p}} < \infty.$$

This extends previous results by the first author et al. for classical hyperbolic spaces.

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1. Introduction

Let $X = G/K$ be a Riemannian symmetric space of the noncompact type. It is well known that a function F is an eigenfunction of all G-invariant differential operators on X if and only if F is Poisson integral

$$P_\lambda f(gK) = \int_K f(k)e^{-(i\lambda+\rho)H(g^{-1}k)}dk,$$

of a hyperfunction f on the Furstenberg boundary K/M, for a generic $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, where H denotes the projection on the abelian part A of the Iwasawa decomposition of G, $\mathfrak{a}_{\mathbb{C}}^*$ is the complex dual of the Lie algebra $\mathfrak{a}$ of A and ρ is the half sum of the positive roots with multiplicities.

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This was conjectured by Helgason who proved it for the rank one case [10] and proved in its full generality by Kashiwara et al. [15].
A natural question is then to look for a characterization of the range of the Poisson transform on classical spaces on the Furstenberg boundary K/M such as the spaces $C^\infty(K/M), L^p(K/M)$ and the space of distributions $D'(K/M)$ (see [1], [6], [7], [12], [17], [18], [16], [22], [19], [20], [27], [25]).
In the case $\lambda = -i\rho$, i.e. the harmonic case, the harmonic functions which are Poisson integrals of L^p-functions $(1 < p \leq \infty)$ or bounded measures are characterized by an H^p-condition. This was proved by Stoll [27] (see also Knapp and Williamson [16] and Michelson [22]).
For λ in $\mathfrak{a}^*_\mathbb{C}$ such that $\mathfrak{Re}(i\lambda)$ lies in the open Weyl chamber, another characterization using weak L^p-spaces is given in Sjögren [25] (see also Lohoué and Rychener [20]).
Later on, Ben Said et al. [1] gave a characterization of the image $P_\lambda(L^p(K/M))$, for $1 < p \leq \infty$, in terms of Hardy type norm for $\lambda \in \mathfrak{a}^*_\mathbb{C}$ such that $\mathfrak{Re}(i\lambda)$ lies in the open positive Weyl chamber and the isotropy subgroup of λ and $\mathfrak{Im}(\lambda)$ in the Weyl group coincide.
All the above studies leave out the case $\lambda \in \mathfrak{a}^* \setminus \{0\}$. Namely the characterization of the image of the Poisson transform on the unitary spherical principal series representation.
Our interest on the problem of characterizing the L^p-range of the Poisson transform, for $\lambda \in \mathfrak{a}^*$ has its root in the work of Strichartz [26]. More precisely, the statement that, for λ in $\mathfrak{a}^*\setminus\{0\}$ the joint eigenfunctions which are Poisson integrals of L^2-functions are characterized by an L^2-weight norm, was conjectured by Strichartz in [26] (for details, see conjecture 4.5 in [26]).
The Strichartz conjecture in the case of the complex hyperbolic space, namely $X = SU(n,1)/S(U(n) \times U(1))$, was settled by the first author et al. [4]. A new proof with an extension to all rank one symmetric spaces was given by Ionescu [12]. In the case of higher rank the Strichartz conjecture 4.5 was recently solved by K. Kaizuka [14] (see also [3]).
In [6] the first author et al. dealt with Poisson transform of L^p-functions. More precisely they proved that Poisson integrals of L^p-functions $(2 \leq p < \infty)$ are characterized by a Hardy type norm in the case $U(n,1,\mathbb{F})/U(n,\mathbb{F}) \times U(1,\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$, i.e. the real, complex or quaternionic hyperbolic spaces, respectively (see also Kumar et al. [18]). The method of the proof uses the techniques of singular integrals on the boundary K/M viewed as a space of homogeneous type in the sense of Coifman and Weiss [8]. Unfortunately, this method depends on the classification of rank one symmetric spaces. That is the classical hyperbolic spaces and the exceptional case.
The aim of this paper is to extend the results in [4] and [6] for classical hyperbolic spaces to the case of the octonionic hyperbolic plane $F_{4(-20)}/Spin(9)$. In order to describe our result let us fix some notations, referring to Section 2 for more details. Let $\mathbb{O}$ be the division algebra of Octonions ($\approx$ the Cayley numbers). Let

$$B(\mathbb{O}^2) = \{x \in \mathbb{O}^2,\ |x| < 1\},$$

be the bounded realization of the symmetric space $F_{4(-20)}/Spin(9)$ and let $\partial B(\mathbb{O}^2)$ denote the unit sphere of $\mathbb{O}^2$ with the normalized area measure $d\omega$ on it. Let

$L^p(\partial B(\mathbb{O}^2))$ denote the space of all $\mathbb{C}$-valued measurable (classes) functions f on $\partial B(\mathbb{O}^2)$ with $\|f\|_p < \infty$. Here

$$\|f\|_p = \left(\int_{\partial B(\mathbb{O}^2)} |f(\omega)|^p d\omega\right)^{\frac{1}{p}}.$$

For $f \in L^p(\partial B(\mathbb{O}^2))$ and λ a complex number, we define the *Poisson transform* of f by

$$P_\lambda f(x) = \int_{\partial B(\mathbb{O}^2)} \left(\frac{1-|x|^2}{|1-[x,\omega]|^2}\right)^{\frac{i\lambda+\rho}{2}} f(\omega)d\omega, \quad \rho = 11.$$

For the precise definition of $[x,\omega]$ see equation (7) in Section 2.
Let $\mathcal{E}_\lambda(B(\mathbb{O}^2))$ be the space of all eigenfunctions of the Laplace-Beltrami operator Δ of $B(\mathbb{O}^2)$ with eigenvalue $-(\lambda^2+\rho^2)$. In order to characterize, for $\lambda \in \mathbb{R}\setminus\{0\}$, those $F \in \mathcal{E}_\lambda(B(\mathbb{O}^2))$ which are Poisson transform by P_λ of some f in $L^p(\partial B(\mathbb{O}^2))$ $(1 < p < \infty)$, we introduce the *Hardy type space* $\mathcal{E}^*_{\lambda,p}(B(\mathbb{O}^2))$ consisting of functions $F \in \mathcal{E}_\lambda(B(\mathbb{O}^2))$ such that

$$\|F\|_{*,p} = \sup_{0\le r<1} (1-r^2)^{\frac{-\rho}{2}} \left(\int_{\partial B(\mathbb{O}^2)} |F(r\theta)|^p d\theta\right)^{\frac{1}{p}} < \infty.$$

Finally, we denote by $B(o,t)$ the *geodesic ball* of radius t centered at 0 and $d\mu(x)$ the G-invariant measure of $B(\mathbb{O}^2)$, given by $d\mu(x) = (1-|x|^2)^{-\rho-1}dm(x)$, $dm(x)$ being the Lebesgue measure. In the sequel of this paper we keep $\rho = 11$ and c will denote a numerical positive constant.
The first main result we prove in this paper can be stated as follows:

Theorem 1.1. *Let $\lambda \in \mathbb{R}\setminus\{0\}$. Then we have*

(i) *A function $F \in \mathcal{E}_\lambda(B(\mathbb{O}^2))$ is the Poisson transform by P_λ of some f in $L^2(\partial B(\mathbb{O}^2))$ if and only if $F \in \mathcal{E}^*_{\lambda,2}(B(\mathbb{O}^2))$. Moreover, there exists a positive constant c such that for every $f \in L^2(\partial B(\mathbb{O}^2))$ the following estimates hold:*

$$|c(\lambda)|\|f\|_2 \le \|P_\lambda f\|_{*,2} \le c(1+|\lambda|+\frac{1}{|\lambda|})\|f\|_2. \tag{1}$$

(ii) *Let $F \in \mathcal{E}^*_{\lambda,2}(B(\mathbb{O}^2))$. Then its L^2-boundary value f is given by the following inversion formula in $L^2(\partial B(\mathbb{O}^2))$:*

$$f(\omega) = |c(\lambda)|^{-2} \lim_{t\to+\infty} \frac{1}{t} \int_{B(o,t)} P_{-\lambda}(x,\omega)F(x)d\mu(x). \tag{2}$$

In (1), $c(\lambda)$ is the Harish-Chandra c-function associated to the Octonionic hyperbolic plane $B(\mathbb{O}^2)$, given by (see [10], p. 64)

$$c(\lambda) = \frac{\Gamma(8)\Gamma(i\lambda)}{\Gamma(\frac{i\lambda+\rho}{2}-3)\Gamma(\frac{i\lambda+\rho}{2})}.$$

The second major result of this paper is

Theorem 1.2. *Let $\lambda \in \mathbb{R} \setminus \{0\}$ and let $p \in]1, +\infty[$. Then we have, a function $F \in \mathcal{E}_\lambda(B(\mathbb{O}^2))$ is the Poisson transform by P_λ of some $f \in L^p(\partial B(\mathbb{O}^2))$ if and only if $F \in \mathcal{E}^*_{\lambda,p}(B(\mathbb{O}^2))$. Moreover, there exists a positive constant $\gamma(\lambda, p)$ such that for every $f \in L^p(\partial B(\mathbb{O}^2))$ the following estimates hold:*

$$|c(\lambda)| \|f\|_p \leq \|P_\lambda f\|_{*,p} \leq \gamma(\lambda, p) \|f\|_p. \tag{3}$$

As mentioned in the introduction, the main difficulty in proving our results lies in proving some uniform pointwise estimates on the generalized spherical functions which unlike the case $\lambda \in \mathbb{C} \setminus \mathbb{R}$ have oscillating terms at infinity.
In this paper we prove the right-hand side of the estimate (1) in Theorem 1.1 by adapting to the Octonionic case the method that we used in the case of the classical hyperbolic spaces, see [6]. More precisely, we will discuss a uniform L^p-boundedness of a family of Calderon-Zygmund operators $(\Psi_r(\lambda))_{r \in [0,1[}$ (see Section 4) on the boundary $\partial B(\mathbb{O}^2)$ considered as a space of homogeneous type in the sense of Coifman and Weiss [8]. To prove the sufficiency condition for $p \neq 2$, we follow the method we used in [2] and [5], to characterize Poisson integrals of L^p-functions on the Shilov boundary of bounded symmetric domains.
Although the techniques we use here may seem to be similar to those in [6], however in working in the exceptional case we encounter a prime difficulty, due to the fact that the algebra of Octonions $\mathbb{O}$ is not associative.

Consequences.
(i) Let $\Phi_{\lambda,lm}$ be the generalized spherical function associated to the Octonionic hyperbolic plane (see [11]). Namely

$$\Phi_{\lambda,lm}(r) = (8)_l^{-1} (\frac{i\lambda + \rho}{2})_{\frac{m+l}{2}} (\frac{i\lambda + \rho}{2} - 3)_{\frac{l-m}{2}} r^l (1 - r^2)^{\frac{i\lambda+\rho}{2}} \times$$
$${}_2F_1(\frac{i\lambda + \rho + l + m}{2}, \frac{i\lambda + \rho + l - m}{2} - 3, l + 8; r^2),$$

where (l, m) lies in $\widehat{K}_0$ the set of pairs (l, m) of nonnegative integers such that $l \geq m \geq 0$ and $l - m$ is even.
In the above $(a)_k = a(a+1)...(a+k-1)$ is the Pochammer symbol and ${}_2F_1(a, b, c; x)$ is the Gauss hypergeometric function.
As an immediate consequence of Theorem 1.1 we obtain the following uniform pointwise estimate.

Corollary 1.3. *Let λ be a nonzero real number. Then there exists a positive constant c such that, for all $r \in [0, 1[$,*

$$\sup_{(l,m) \in \widehat{K}_0} |\Phi_{\lambda,lm}(r)| \leq c(1 + |\lambda| + \frac{1}{|\lambda|})((1 - r^2)^{\frac{\rho}{2}}.$$

As $\Phi_{\lambda,lm}$ can be written in terms of Jacobi functions, the above estimate might have independent interest on its own from a view point of special functions.
(ii) Recently in their study of the Roe Theorem in rank one symmetric spaces, Kumar et al. [18] proved a characterization of the L^p-range of the Poisson transform by means of Lorentz type norms in the case of the classical hyperbolic spaces

$B(\mathbb{F}^n)$. Below we will show how to apply Theorem 1.2 to extend their result to the exceptional case.
We follow mainly the notations in [18]. Let $X = G/K$ be a hyperbolic space on $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let μ denote the G-invariant measure of X. For F a μ-measurable complex-valued function on X, we set $\|F\|_{p,\infty} = \sup_{s>0} s d_F(s)^{\frac{1}{p}}$, where d_F is the distribution function of F. Finally define $\mathcal{A}_{2,p}(F) = \|\mathcal{A}_p(F)\|_{2,\infty}$, with $\mathcal{A}_p(F)(x) = \left(\int_{K/M} |F(kx)|^p dk\right)^{\frac{1}{p}}$ (where as usual M is the centralizer of A in K, if $G = KAN$ is an Iwasawa decomposition of G). Then the result of Kumar et al. may be state as follows

Theorem 1.4 ([18]). *Let $1 < p < \infty$, $\lambda \in \mathbb{R} \setminus \{0\}$ and X a hyperbolic space over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. If $F \in \mathcal{E}_\lambda(B(\mathbb{F}^n))$, then F is the Poisson transform by P_λ of some $f \in L^p(\partial B(\mathbb{F}^n))$ if and only if it satisfies $\mathcal{A}_{2,p}(F) < \infty$.*

As a consequence of the method of the proof of Theorem 1.2, we extend the above result to the case X is the Octonionic hyperbolic space.

Theorem 1.5. *Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in]1, +\infty[$. Let F be a $\mathbb{C}$-valued function on the Octonionic hyperbolic plane $B(\mathbb{O}^2)$, satisfying $\Delta F = -(\lambda^2 + \rho^2)F$. Then $F = P_\lambda f$, for some $f \in L^p(\partial B(\mathbb{O}^2))$ if and only if $\mathcal{A}_{2,p}(F) < \infty$.*

Proof. Noting that if $\|F\|_{*,p} < \infty$, then $\mathcal{A}_{2,p}(F) \leq c\|F\|_{*,p}$, the necessary condition follows from the right-hand side of the estimate (3) in Theorem 1.2. Next, using the L^2 inversion formula (2), we may follow the same method as in the proof of the sufficiency condition of Theorem 4.3.6 in [18] to get the result. So we omit it. ∎

Now we give the organization of this paper. In Section 2, some preliminaries of harmonic analysis on the Cayley plane are described. In Section 3 we prove our main results. The proof relies on establishing the Key Lemma of this paper giving a uniform L^2-boundedness of the Calderon-Zygmund operators $(\Psi_r(\lambda))$, $r \in [0, 1[$, associated to the Poisson transform P_λ. To prove the Key Lemma, we will adapt on $\partial B(\mathbb{O}^2)$ – in a uniform manner in $r \in [0, 1[$ – the method of proving the $T(1)$-Theorem of David-Journé and Semens [9] and for this we follow the program accomplished by Y.Meyer in his new proof of the $T(1)$-Theorem in $L^2(\mathbb{R}^n)$ which is based on the Cotlar-Stein Lemma. This is the subject of Section 4.
We end this section with a brief discussion on all rank one symmetric spaces. Let $X = G/K$ be a noncompact Riemannian symmetric space of rank one. Then X can be realized as the unit ball in $\mathbb{F}^n$, with $\mathbb{F}$ is either the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, or the quaternionic $\mathbb{H}$, or the Cayley numbers $\mathbb{O}$, in the last case $n = 2$. Moreover,

$$\begin{aligned}
&\text{if } \mathbb{F} = \mathbb{R}, \text{ then } (G, K) = (SO_e(n, 1), SO(n)) \\
&\text{if } \mathbb{F} = \mathbb{C}, \text{ then } (G, K) = (SU(n, 1), S(U(n) \times U(1))) \\
&\text{if } \mathbb{F} = \mathbb{H}, \text{ then } (G, K) = (Sp(n, 1), Sp(n) \times Sp(1)) \\
&\text{if } \mathbb{F} = \mathbb{O}, \text{ then } n = 2 \text{ and } (G, K) = (F_{4(-20)}, Spin(9)).
\end{aligned}$$

Let $\rho_{\mathbb{F}}$ denote the half sum of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}$ being the Lie algebra of G and $\mathfrak{a}$ a Cartan subalgebra of $\mathfrak{g}$ ($\mathfrak{a} = \mathbb{R}H_0$, since $rank(X) = 1$). By abuse of notation we will denote $\rho_{\mathbb{F}}(H_0)$ by $\rho_{\mathbb{F}}$. Then $\rho_{\mathbb{F}} = \frac{n-1}{2}$, n, $2n+1$ or 11 according to $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$.
Denote by A the analytic subgroup of G that corresponds to $\mathfrak{a}$. Then A may be parametrized by $a_t = \exp tH_0$.
For $p \in]1, \infty[$, let $M_p^*(F) = \sup_{t>0} e^{\rho t} \left(\int_K |F(ka_t)|^p dk\right)^{\frac{1}{p}}$, where dk is the normalized Haar measure of K.
Now, taking into account Theorem 1.1, Theorem 1.2 and the results in ([6], Theorem A) and ([18], Theorem 4.3.6), a characterization of the L^p-range of the Poisson transform for the rank one symmetric spaces is now completed. With the help of the above notations these results may be stated in a unified manner as follows:

Theorem 1.6. *Let $\lambda \in \mathfrak{a}^* \setminus \{0\}$ and let $1 < p < \infty$. Let F be a $\mathbb{C}$-valued function on X satisfying $\Delta F = -(\lambda^2 + \rho_{\mathbb{F}}^2)F$. Then we have*

(i) *F has an L^p-Poisson integral representation on K/M if and only if the condition $M_p^*(F) < \infty$ is satisfied. Moreover, there exists a positive constant $\gamma(\lambda, p)$ such that for every $f \in L^p(K/M)$ the following estimates hold:*
$$|c(\lambda)| \|f\|_p \leq M_p^*(P_\lambda f) \leq \gamma(\lambda, p) \|f\|_p.$$
(ii) *If $M_2^*(F) < \infty$. Then F has an L^2-boundary value f given by the following inversion formula in $L^2(K/M)$:*
$$f(kM) = \lim_{t \to +\infty} \frac{1}{t} \int_{B(0,t)} F(x) P_{-\lambda}(x, kM) d\mu(x).$$

In the intergral above $B(0, t)$ denotes the geodesic ball of radius t centered at 0 and the G-invariant measure $d\mu$ on X.
Below we give the last result of this section. For any locally integrable function F with respect to $d\mu$, we set
$$M_2(F)^2 = \sup_{t>0} \frac{1}{t} \int_{B(0,t)} |F(x)|^2 d\mu(x).$$
Writing $x = ka_t.0$ we easily see that if $M_2^*(F)$ is finite then
$$M_2(F) \leq c M_2^*(F), \tag{4}$$
for some c positive constant. Therefore, if $f \in L^2(\partial B(\mathbb{F}^n))$, then
$$M_2(P_\lambda f) \leq c(1 + |\lambda| + \frac{1}{|\lambda|}) \|f\|_2,$$
by the above Theorem. Thus we have proved

Corollary 1.7. *Let $\lambda \in \mathbb{R} \setminus \{0\}$ and let $f \in L^2(\partial B(\mathbb{F}^n))$. Then*
$$\left(\sup_{t>0} \frac{1}{t} \int_{B(0,t)} |P_\lambda f(x)|^2 d\mu(x)\right)^{\frac{1}{2}} \leq c(1 + |\lambda| + \frac{1}{|\lambda|}) \|f\|_2.$$

The above result has already been established by Ionescu using different method (see [12]).

2. Preliminary results

We review in this section some known results of harmonic analysis on the octonionic hyperbolic space. We first recall some properties of octonions that will be needed in this paper, referring to [28] and [29] for more details.
We denote by $\mathbb{O}$ the *algebra of octonions*. $\mathbb{O}$ has a basis over $\mathbb{R}$ given by e_0, $e_1, \ldots, e_7$, where e_0 is the unit element, and e_m are anti-commuting elements satisfying $e_m^2 = -1$.
We define the standard involution of $\mathbb{O}$ over $\mathbb{R}$ by $\overline{x} = x_0 - \sum_{j=1}^{7} x_j e_j$, and we have $\overline{xy} = (\overline{y})(\overline{x})$, for every $x, y \in \mathbb{O}$. If $x = \sum_{j=0}^{7} x_j e_j$, the summand $x_0 e_0 = x_0$ is called the *real part* of x and it is noted by $\mathfrak{Re}(x)$. Furthermore, the norm on $\mathbb{O}$ is defined as $|x|^2 = \sum_{j=0}^{7} x_j^2$, and it satisfies the identity $|xy| = |x||y|$. Every nonzero octonion x has a unique inverse, namely $x^{-1} = |x|^{-2}\overline{x}$.
For $x, y \in \mathbb{O}^2$, we put $\Phi(x,y) = \sum_{j=1}^{2} |x_j|^2 |y_j|^2 + 2\Re((x_1 x_2)(\overline{y_1 y_2}))$. The form $\Phi(x,y)$ may be written as

$$\Phi(x,y) = |(\overline{x_1} y_2)(y_2^{-1} y_1) + x_2 \overline{y_2}|^2, \tag{5}$$

for $y_2 \neq 0$. Also, we consider $O_{\mathbb{O}}(2)$ the group of all $\mathbb{R}$-linear transformations of $\mathbb{R}^{16}$ which preserve the form $\Phi(x,y)$. The group $O_{\mathbb{O}}(2)$ is a subgroup of the orthogonal group $O(16)$ (when identifying $\mathbb{O}^2$ with $\mathbb{R}^{16}$).
Let $\partial B(\mathbb{O}^2) = \{\omega \in \mathbb{O}^2 : |\omega| = 1\}$ be the unit sphere in $\mathbb{O}^2$. Then we have

Lemma 2.1 ([29]). *The group $O_{\mathbb{O}}(2)$ acts transitively on the unit sphere $\partial B(\mathbb{O}^2)$.*

2.1. Bounded realization of $F_{4(-20)}/Spin(9)$.

In this subsection we give, after Takahashi[28], treatments of the octonionic hyperbolic space that are suitable for computation and that are helpful to handle the uniform L^p-characterization of the Poisson transform on $F_{4(-20)}/Spin(9)$ (see also [29]).
Let $A(3,\mathbb{O})$ be the exceptional Jordan algebra of 3×3 Hermitian matrices with entries from $\mathbb{O}$ and let $A(3, \mathbb{O} \otimes \mathbb{C})$ be its complexification. The Jordan product being $A \circ B = \frac{1}{2}(AB + BA)$. Let J denote the Jordan subalgebra of $A(3, \mathbb{O} \otimes \mathbb{C})$, consisting of matrices

$$X = \begin{pmatrix} a_1 & u_3 \otimes (-1)^{\frac{1}{2}} & \overline{u_2} \otimes (-1)^{\frac{1}{2}} \\ \overline{u_3} \otimes (-1)^{\frac{1}{2}} & a_2 & u_1 \\ u_2 \otimes (-1)^{\frac{1}{2}} & \overline{u_1} & a_3 \end{pmatrix},$$

with $a_j \in \mathbb{R}$ and $u_j \in \mathbb{O}$, for $j = 1, 2, 3$. We denote by Ω the subset of J consisting of idempotent elements $X \in J$ such that $tr(X) = 1$ and $tr(X \circ E_1) \geq 1$. Here

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next, let $B(\mathbb{O}^2) = \{(x_1, x_2) \in \mathbb{O}^2, |x_1|^2 + |x_2|^2 < 1\}$ be the unit ball of $\mathbb{O}^2$. Then

$$(x_1, x_2) \longmapsto X(x_1, x_2) = \frac{1}{1 - |x|^2} \begin{pmatrix} 1 & \overline{x_2} \otimes (-1)^{\frac{1}{2}} & \overline{x_1} \otimes (-1)^{\frac{1}{2}} \\ x_2 \otimes (-1)^{\frac{1}{2}} & -|x_2|^2 & -x_2\overline{x_1} \\ x_1 \otimes (-1)^{\frac{1}{2}} & -x_1\overline{x_2} & -|x_1|^2 \end{pmatrix}$$

is a bijective map from the unit ball $B(\mathbb{O}^2)$ onto Ω. Let $G = F_{4(-20)}$ be the identity component of $Aut(J)$ the group of all automorphisms of J. Then using the above map we shall define an action of G on $B(\mathbb{O}^2)$. Namely for $x = (x_1, x_2) \in B(\mathbb{O}^2)$, we set

$$g \cdot (x_1, x_2) = (x'_1, x'_2),$$

according to $gX(x_1, x_2) = X(x'_1, x'_2)$. The action of G on $B(\mathbb{O}^2)$ is transitive and as homogeneous space G/K is then identified to $B(\mathbb{O}^2)$, where K is the isotropy subgroup of E_1. Moreover K is a maximal compact subgroup of G which is isomorphic to $Spin(9)$.
Next, we identify the boundary $\partial B(\mathbb{O}^2)$ to the set

$$\{Y(u, v) = \begin{pmatrix} 0 & v & \overline{u} \\ \overline{v} & 0 & 0 \\ u & 0 & 0 \end{pmatrix} : u, v \in \mathbb{O}, |u|^2 + |v|^2 = 1\}.$$

Then using this identification, the action of K on $\partial B(\mathbb{O}^2)$ is defined as follows: For u, v in $\partial B(\mathbb{O}^2)$ put $k.(u, v) = (u', v')$, via $kY(u, v) = Y(u', v')$. It is shown in [28] that this action is transitive. As homogeneous space we have $\partial B(\mathbb{O}^2) = K/M$, where M is the isotropic subgroup of the element $F_2^1 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. Moreover the group M is isomorphic to $Spin(7)$.
Now we describe in brief the Peter-Weyl decomposition of $L^2(\partial B(\mathbb{O}^2))$ under the action of K. Let $\widehat{K}_0$ be the set of pairs (l, m) of non-negative integers such that $l \geq m \geq 0$ and $l - m$ is even. Then under the action of K, the Peter-Weyl decomposition of $L^2(\partial B(\mathbb{O}^2))$ is given by

$$L^2(\partial B(\mathbb{O}^2)) = \bigoplus_{(l,m) \in \widehat{K}_0} V^{lm},$$

where V^{lm} is the K-cyclic space for the zonal spherical function associated to the pair (K, M).(See [28] and [13]).

2.2. The Poisson transform. In this subsection we recall some known results on the Poisson transform P_λ associated to the octonionic hyperbolic plane $B(\mathbb{O}^2)$ that will be needed in the sequel.

Firstly, we introduce some notations. For $x, y \in \mathbb{O}^2$, put

$$\Psi(x,y) = 1 - 2 < x, y >_{\mathbb{R}} + \Phi(x,y), \tag{6}$$

where $< x, y >_{\mathbb{R}}$ is the Euclidean inner product of x and y as vectors in $\mathbb{R}^{16}$. Also, for $x, y \in \mathbb{O}^2$, we set

$$[x,y] = \begin{cases} (\overline{x_1}y_2)(y_2^{-1}y_1) + x_2\overline{y_2} & if\ y_2 \neq 0 \\ \overline{x_1}y_1 & if\ y_2 = 0. \end{cases} \tag{7}$$

Then Ψ may be written as

$$\Psi(x,y) = |1 - [x,y]|^2, \tag{8}$$

and since $|[x,y]| \leq |x||y|$, we easily see that $\Psi(x,y) > 0$, for $x \in \overline{B(\mathbb{O}^2)}$ and $y \in B(\mathbb{O}^2)$. Note that $|[x,y]|$ is the analogue of the form $|\sum_{j=1}^{n} a_j\overline{b_j}|$ in the real, complex or the quaternionic fields.
The Poisson kernel for the octonionic hyperbolic plane is the function $P(x,\omega)$ defined on $B(\mathbb{O}^2) \times \partial B(\mathbb{O}^2)$ by

$$P(x,\omega) = \left(\frac{1 - |x|^2}{\Psi(x,\omega)}\right)^{\rho}.$$

For $f \in L^1(\partial B(\mathbb{O}^2))$ and $\lambda \in \mathbb{C}$, we define the Poisson integral of f by

$$P_\lambda f(x) = \int_{\partial B(\mathbb{O}^2)} P_\lambda(x,\omega) f(\omega) d\omega, \quad \text{where} \quad P_\lambda(x,\omega) = [P(x,\omega)]^{\frac{i\lambda+\rho}{2\rho}}.$$

Below we recall a result on the precise action of the Poisson transform on the K-types V^{lm}.

Proposition 2.2 ([10]). *Let λ be a complex number and let $f \in V^{lm}$. Then*

$$P_\lambda f(x) = \Phi_{\lambda,lm}(|x|) f\left(\frac{x}{|x|}\right),$$

where $\Phi_{\lambda,lm}(|x|)$ is the generalized spherical function given by

$$\Phi_{\lambda,lm}(|x|) = (8)_l^{-1}\left(\frac{i\lambda+\rho}{2}\right)_{\frac{m+l}{2}}\left(\frac{i\lambda+\rho}{2} - 3\right)_{\frac{l-m}{2}} |x|^l (1 - |x|^2)^{\frac{i\lambda+\rho}{2}} \times$$
$$\times\ {}_2F_1\left(\frac{i\lambda+\rho+l+m}{2}, \frac{i\lambda+\rho+l-m}{2} - 3, l+8; |x|^2\right).$$

We end this section by a result on the asymptotic behaviour of the generalized spherical functions, which is due to Ionescu[12].

Lemma 2.3. *Let λ be a nonzero real number. Then there exists a positive constant c such that, for every $(l,m) \in \widehat{K}_0$,*

$$\lim_{t\to+\infty} \frac{1}{t} \int_{B(0,t)} |\Phi_{\lambda,lm}(|x|)|^2 d\mu(x) = c|c(\lambda)|^2. \tag{9}$$

3. Proof of the main results

As explained in the introduction, the main difficulty in proving Theorem 1.1 and Theorem1.2 is to show that the image $P_\lambda(L^p(\partial B(\mathbb{O}^2)$ is continuously embedded in $\mathcal{E}^*_{\lambda,p}(B(\mathbb{O}^2))$. We will overcome this difficulty by discussing the uniform L^p-boundedness of the following family of superficial Poisson-Szegö integrals $(\Psi_r(\lambda))_{r\in[0,1[}$:

$$\Psi_r(\lambda)f(\theta) = \int_{\partial B(\mathbb{O}^2)} \Psi_r(\lambda,\theta,\omega)f(\omega)d\omega, \tag{10}$$

where the Schwartz kernel is given by $\Psi_r(\lambda,\theta,\omega) = (\Psi(r\theta,\omega))^{\frac{-i\lambda-\rho}{2}}$.
To do so, we equip the unit sphere $\partial B(\mathbb{O}^2)$ with the following non-isotropic metric $d(a,b) = |1-[a,b]|^{\frac{1}{2}}$ (see Section 4), so that $(\partial B(\mathbb{O}^2), d)$ becomes a space of homogeneous type in the sense of Coifman and Weiss. Now we state the key lemma of this paper.

Key Lemma 3.1. Let λ be a nonzero real number. Then there exists a positive constant c such that the following estimates hold

$$\sup_{0\le r<1} \|\Psi_r(\lambda)\|_2 \le c(1+|\lambda|+\frac{1}{|\lambda|}), \tag{11}$$

where $\|.\|_2$ stands for the L^2-operatorial norm

$$\sup_{0\le r<1} \int_{d(\omega,e_1)\ge 2d(\theta,e_1)} |\Psi_r(\lambda,\omega,\theta) - \Psi_r(\lambda,\omega,e_1)|d\omega \le c(1+|\lambda|). \tag{12}$$

Notice that from the estimates (11) as well as the Hörmander condition (12) we deduce that the following estimate holds

$$\sup_{0\le r<1} \|\Psi_r(\lambda)\|_p \le \gamma(\lambda,p), \quad for\, p\in]1,\infty[, \tag{13}$$

by the Marcinkiewicz interpolation theorem and duality.

Proof of Theorem 1.1.
(i) The necessary condition. Let $f\in L^2(\partial B(\mathbb{O}^2))$ and write $x = r\theta$ with $r\in[0,1[$ and $\theta\in\partial B(\mathbb{O}^2)$. Then we have

$$P_\lambda f(r\theta) = (1-r^2)^{\frac{i\lambda+\rho}{2}}\Psi_r(\lambda)f(\theta). \tag{14}$$

Therefore

$$\|P_\lambda f\|_{*,2} \le c(1+|\lambda|+\frac{1}{|\lambda|})\|f\|_2,$$

by (11). This proves the right-hand side of the estimate (1) in Theorem 1.1.
To prove the sufficiency condition, let $F\in\mathcal{E}_\lambda(B(\mathbb{O}^2))$ satisfying the growth condition $\|F\|_{*,2}<\infty$. Using the polar coordinates $x = r\theta$, we easily see that if $\|F\|_{*,2}$ is finite, then $M_2(F)\le c\|F\|_{*,2}$, and by Ionescu's result [12], we know that there exists $f\in L^2(\partial B(\mathbb{O}^2))$ such that $F = P_\lambda f$ with $|c(\lambda)|\|f\|_2 \le M_2(P_\lambda f)$. Therefore $|c(\lambda)|\|f\|_2 \le \|F\|_{*,2}$ and the proof of (i) is finished.

(ii) Although the proof of the inversion formula is the same as in the classical hyperbolic spaces [6], but for the sake of completeness we give here the outline of the proof.
Let $F \in \mathcal{E}_\lambda(B(\mathbb{O}^2))$ such that $\|F\|_{*,2} < \infty$. Then $F = P_\lambda f$, for some f in $L^2(\partial B(\mathbb{O}^2))$. Expanding f into its K-type series, $f = \sum_{(l,m)\in\widehat{K}_0} f_{lm}$ and using Proposition 2.2, F may be written as

$$F(r\theta) = \sum_{(l,m)\in\widehat{K}_0} \Phi_{\lambda,lm}(r) f_{lm}(\theta),$$

in $C^\infty([0,1[\times\partial\mathbb{B}(\mathbb{O}^2))$. Next, set

$$g_t(\omega) = |c(\lambda)|^{-2}\frac{1}{t}\int_{B(0,t)} F(x)P_{-\lambda}(x,\omega)dx.$$

Then replacing F by its expansion series and using Proposition 2.2 again, we get

$$g_t(\omega) = \frac{1}{t}\sum_{(l,m)\in\widehat{K}_0}\left(\int_0^{\tanh(t)} |\Phi_{\lambda,lm}(r)|^2(1-r^2)^{-\rho-1}r^{15}dr\right) f_{lm}(\omega).$$

It is easy to see that the functions $(g_t)_{t>0}$ are in $L^2(\partial B(\mathbb{O}^2))$. Furthermore

$$\|g_t - f\|_2^2 = \sum_{(l,m)\in\widehat{K}_0}\left|\frac{|c(\lambda|^{-2}}{t}\int_0^{\tanh(t)} |\Phi_{\lambda,lm}(r)|^2(1-r^2)^{-\rho-1}r^{15}dr - 1\right|^2 \|f_{lm}\|_2^2.$$

Next, using the uniform asymptotic behaviour of the generalized spherical functions $\Phi_{\lambda,lm}(r)$ in Lemma 2.3, we get

$$\lim_{t\to\infty}\|g_t - f\|_2^2 = 0.$$

This finishes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. Let $f \in L^p(\partial B(\mathbb{O}^2)$. Then using the identity (14) as well as the estimate (13), we get

$$\|P_\lambda f\|_{*,p} \le \gamma(\lambda,p)\|f\|_p.$$

This prove the necessary condition in Theorem 1.2. Next, to prove the sufficiency condition of Theorem 1.2 let $(\chi_n)_n$ denote an approximation of the identity in $\mathcal{C}(K)$. That is

$$\chi_n \ge 0,\ \int_K \chi_n(k)dk = 1, \quad \text{and} \quad \lim_{n\mapsto+\infty}\int_{K\setminus U}\chi_n(k)dk = 0,$$

for every neighbourhood U of e in K. Let $F \in \mathcal{E}^*_{\lambda,p}(B(\mathbb{O}^2)$. For each n define the function F_n on $B(\mathbb{O}^2)$ by

$$F_n(x) = \int_K \chi_n(k)F(k^{-1}.x)dk\,.$$

Then $(F_n)_n$ converges pointwise to F as n goes to $+\infty$. Since Δ is $G-$invariant, then F_n lies in $\mathcal{E}_\lambda(B(\mathbb{O}^2))$. For each $r \in [0,1[$ define a function F^r on K, by $F^r(k) = F(rk.e_1)$. Let dk be the Haar measure of K. As

$$\left(\int_{\partial B(\mathbb{O}^2)} |F(r\theta)|^2 d\theta\right)^{\frac{1}{2}} = \left(\int_K |F(rk.e_1)|^2 dk\right)^{\frac{1}{2}},$$

and noting that $F_n^r(k) = (\chi_n * F^r)(k)$, we get

$$\left(\int_{\partial B(\mathbb{O}^2)} |F_n(r\theta)|^2 d\theta\right)^{\frac{1}{2}} \leq \|\chi_n\|_2 \|F^r\|_1 \leq \|\chi_n\|_2 \|F^r\|_p,$$

by the Haussedorf inequality. This shows that

$$(1-r^2)^{-\frac{\varrho}{2}} \left(\int_{\partial B(\mathbb{O}^2)} |F_n(r\theta)|^2 d\theta\right)^{\frac{1}{2}}$$

is uniformly bounded for all $r \in [0,1[$. Thus, the function F_n lies in the space $\mathcal{E}^*_{\lambda,2}(B(\mathbb{O}^2))$. Therefore, for each n there exists a function $f_n \in L^2(\partial B(\mathbb{O}^2))$ such that $F_n = P_\lambda f_n$, by the if part of Theorem 1.1. Moreover, we have

$$f_n(\omega) = |c(\lambda)|^{-2} \lim_{t\to+\infty} \frac{1}{t} \int_{B(0,t)} F_n(x) P_{-\lambda}(x,\omega) d\mu(x),$$

in $L^2(\partial B(\mathbb{O}^2))$, by (2). Next, denote by g_n^t the function

$$g_n^t(\omega) = |c(\lambda)|^{-2} \frac{1}{t} \int_{B(0,t)} F_n(x) P_{-\lambda}(x,\omega) d\mu(x).$$

Below we will show that

$$\sup_{n\in\mathbb{N}, t>0} \|g_n^t\|_p \leq |c(\lambda)|^{-2} \gamma(\lambda,p) \|F\|_{*,p}. \tag{15}$$

To do so, let $\phi \in L^q(\partial B(\mathbb{O}^2)$, with $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$\left|\int_{\partial B(\mathbb{O}^2)} g_n^t(\omega)\overline{\phi(\omega)} d\omega\right| = \frac{|c(\lambda)|^{-2}}{t} \left|\int_{\partial B(\mathbb{O}^2)} \left(\int_{B(0,t)} F_n(x) P_{-\lambda}(x,\omega) dx\right) \overline{\phi(\omega)} d\omega\right|.$$

We may use the polar coordinates $x = r\theta$, $r \in [0,1[$ and $\theta \in \partial B(\mathbb{O}^2)$, to rewrite the right-hand side of the above identity as

$$\frac{|c(\lambda)|^{-2}}{t} \left|\int_{\partial B(\mathbb{O}^2)} \left(\int_0^{\tanh(t)} \left(\int_{\partial B(\mathbb{O}^2)} P_{-\lambda}(r\theta,\omega) F_n(r\theta) d\theta\right) (1-r^2)^{-\rho-1} r^{15} dr\right) \overline{\phi(\omega)} d\omega\right|.$$

Since the Poisson kernel is symmetric in θ and ω, then one can use the Fubini Theorem to show that

$$\left|\int_{\partial B(\mathbb{O}^2)} g_n^t(\omega)\overline{\phi(\omega)} d\omega\right| = \frac{|c(\lambda)|^{-2}}{t} \left|\int_0^{\tanh(t)} \left(\int_{\partial B(\mathbb{O}^2)} \overline{P_\lambda \phi}(r\theta) F_n(r\theta) d\theta\right) (1-r^2)^{-\rho-1} r^{15} dr\right|.$$

By the Hölder inequality, the right side of the above equality is less than

$$\frac{|c(\lambda)|^{-2}}{t}\int_0^{\tanh(t)}\left(\int_{\partial B(\mathbb{O}^2)}|P_\lambda\phi(r\theta)|^q d\theta\right)^{\frac{1}{q}}\|F_n^r\|_p(1-r^2)^{-\rho-1}r^{15}dr.$$

By the necessary condition already proved, we have

$$\left(\int_{\partial B(\mathbb{O}^2)}|P_\lambda\phi(r\theta)|^q d\theta\right)^{\frac{1}{q}}\leq\gamma(\lambda,q)(1-r^2)^{-\frac{\rho}{2}}\|\phi\|_q,$$

and using $\|F_n^r\|_p\leq\|F^r\|_p\leq(1-r^2)^{\frac{\rho}{2}}\|F\|_{*,p}$, we deduce that

$$\left|\int_{\partial B(\mathbb{O}^2)}g_n^t(\omega)\overline{\phi(\omega)}d\omega\right|\leq|c(\lambda)|^{-2}\gamma(\lambda,p)\|\phi\|_q\|F\|_{*,p},$$

for every $t>0$ and n. Next, taking the supremum over all ϕ such that $\|\phi\|_q=1$, we get the estimate (15). For q a positive number with $\frac{1}{p}+\frac{1}{q}=1$, let T_n be the linear form on $L^q(\partial B(\mathbb{O}^2))$ defined by:

$$T_n(\phi)=\int_{\partial B(\mathbb{O}^2)}f_n(\omega)\overline{\phi(\omega)}d\omega.$$

Let ϕ be a $\mathbb{C}$-valued continuous function on $\partial B(\mathbb{O}^2)$ we have:

$$T_n(\phi)=\lim_{t\mapsto+\infty}\int_{\partial B(\mathbb{O}^2)}g_n^t(\omega)\overline{\phi(\omega)}d\omega.$$

Using on one hand Hölder's inequality and on the other hand the estimate (15), we see that

$$\left|\int_{\partial B(\mathbb{O}^2)}g_n^t(\omega)\overline{\phi(\omega)}d\omega\right|\leq|c(\lambda)|^{-2}\gamma(\lambda,p)\|F\|_{*,p}\|\phi\|_q.$$

Next, taking the supremum over all continuous functions ϕ with $\|\phi\|_q=1$ in the above inequality, we deduce that the linear functionals T_n are uniformly bounded, with

$$\sup_n\|T_n\|\leq|c(\lambda)|^{-2}\gamma(\lambda,p)\|F\|_{*,p},\tag{16}$$

where $\|.\|$ stands for the operator norm.
Thanks to the Banach-Alaoglu-Bourbaki Theorem, there exists a subsequence of bounded operators $(T_{n_j})_j$ which converges to a bounded linear operator T on $L^q(\partial B(\mathbb{O}^2))$, under the $*$-weak topology, with $\|T\|\leq|c(\lambda)|^{-2}\gamma(\lambda,p)\|F\|_{*,p}$. By the Riesz representation theorem, there exists a unique function $f\in L^p(\partial B(\mathbb{O}^2))$ such that $T(\phi)=\int_{\partial B(\mathbb{O}^2)}f(\omega)\overline{\phi(\omega)}d\omega$ with $\|f\|_p=\|T\|$. Therefore

$$\|f\|_p\leq|c(\lambda)|^{-2}\gamma(\lambda,p)\|F\|_{*,p}.\tag{17}$$

Now, let $\phi_x(\omega)=P_\lambda(x,\omega)$. Then, $T_n(\phi_x)(\omega)=F_n(x)$. Since, on one hand $\lim_{n\to+\infty}F_n(x)=F(x)$ and on the other hand $\lim_{j\to+\infty}T_{n_j}(\phi_x)=T(\phi_x)$, we get $F=P_\lambda f$. The estimate $\|f\|_p\leq|c(\lambda)|^{-2}\gamma(\lambda,p)\|P_\lambda f\|_{*,p}$ follows from (17). This finishes the proof of the main Theorem 1.2. ■

4. Proof of the Key Lemma 3.1.

Recall that
$$[\Psi_r(\lambda)f](\theta) = \int_{\partial B(\mathbb{O}^2)} \Psi_r(\lambda,\theta,\omega)f(\omega)d\omega, \qquad (18)$$
where $\Psi_r(\lambda,\theta,\omega) = \Psi(r\theta,\omega)^{\frac{-i\lambda-\rho}{2}}$. To prove the uniform L^2-uniform boundedness in $r \in [0,1[$ of the family of operators $(\Psi_r(\lambda))_{r\in[0,1[}$, we will adapt on $\partial B(\mathbb{O}^2)$, in a uniform manner in $r \in [0,1[$, the method of proving the $T(1)$-Theorem of David-Journé and Semens [9], for this we follow the program accomplished by Y.Meyer in his new proof of the $T(1)$-Theorem in $L^2(\mathbb{R}^n)$ which is based on the Cotlar-Stein Lemma.
To do this, we endow $\partial B(\mathbb{O}^2)$ with a non-isotropic metric d, so that $(\partial B(\mathbb{O}^2), d)$ becomes a space of homogeneous type in the sense of Coifman and Weiss, see [8]. More precisely, for $a \in \overline{B(\mathbb{O}^2)}, b \in \overline{B(\mathbb{O}^2)}$, we define
$$d(a,b) = [1 - 2 < a,b >_{\mathbb{R}} + \Phi(a,b)]^{\frac{1}{4}}.$$
Since, $\Phi(a,b)$ is $O_{\mathbb{O}}(2)-$invariant and $O_{\mathbb{O}}(2) \subset O(16)$, we have
$$d(h.a, h.b) = d(a,b),$$
for every $h \in O_{\mathbb{O}}(2)$. According to (8) the non-isotropic metric $d(a,b)$ may also be written as
$$d(a,b) = |1 - [a,b]|^{\frac{1}{2}}.$$

Proposition 4.1.

(i) *The triangle inequality* $d(a,c) \leq d(a,b) + d(b,c)$ *holds for all* $a,b,c \in \overline{B(\mathbb{O}^2)}$.

(ii) d *is a metric on* $\partial B(\mathbb{O}^2)$.

(iii) *The volume of* $B(\omega,\delta)$ *with respect to the superficial measure of* $\partial B(\mathbb{O}^2)$ *behaves as* $\delta^{2\rho}$, *where* $B(\omega,\delta) = \{\theta \in \partial B(\mathbb{O}^2) : d(\theta,\omega) < \delta\}$.

Proof. (i) Put $e_2 = (0,1)$. Since d is bi-invariant by $O_{\mathbb{O}}(2)$, we may take $b = re_2$ $(0 \leq r \leq 1)$ and we then have to prove
$$d(a,c) \leq |1 - ra_2|^{\frac{1}{2}} + |1 - rc_2|^{\frac{1}{2}}. \qquad (19)$$
Notice that (19) is obvious if $c_2 = 0$. If $c_2 \neq 0$, then one has to prove that
$$|1 - (\overline{a_1}c_2)(c_2^{-1}c_1) - a_2\overline{c_2}| \leq \left[|1 - ra_2|^{\frac{1}{2}} + |1 - rc_2|^{\frac{1}{2}}\right]^2.$$
The left-hand side of the above inequality is less than $|1 - a_2\overline{c_2}| + |a_1||c_1|$. Next, since $|1 - a_2\overline{c_2}| \leq |1 - ra_2| + |1 - rc_2|$, and
$$|a_1|^2 \leq 1 - |a_2|^2 \leq 1 - r^2|a_2|^2 \leq 2|1 - ra_2|,$$
and a similar estimate holds for $|c_1|$, then (19) holds.

(ii) To prove that d is a metric on $\partial B(\mathbb{O}^2)$, we have only to show that $d(a,b) = 0$ if and only if $a = b$. By the $O_{\mathbb{O}}(2)$-invariance of d it suffices to prove it for $b = e_2$, which is obvious.

For the proof of (iii) we will need the following standard calculus lemma:

Lemma 4.2. *Let f be a $\mathbb{C}$-valued function on $\partial B(\mathbb{O}^2)$ such that $f(\omega_1,\omega_2)=g(\omega_1)$. Then, we have*

$$\int_{\partial B(\mathbb{O}^2)} f(\omega)d\omega = c\int_{\{x\in\mathbb{O}:|x|<1\}} g(x)(1-|x|^2)^3dm(x).$$

Since the metric d is $O_{\mathbb{O}}(2)$-invariant as well as the superficial measure $d\theta$, we have $V(B(\omega,\delta)) = V(B(e_1,\delta))$. Therefore

$$V(B(\omega,\delta)) = \int_{\{\theta\in\mathbb{O}^2,|1-\theta_1|<\delta^2\}} d\theta.$$

Next, use Lemma 4.2 to get:

$$V(B(\omega,\delta)) = c\int_{\{x\in\mathbb{O}:|x|<1;|1-x|<\delta^2\}} (1-|x|^2)^3dm(x).$$

Put $1-x = t(\cos(\alpha)+\sin(\alpha)y)$, where $t>0$, $\alpha\in[0,\pi]$ and $y\in\mathbb{O}$ such that $\mathfrak{Re}(y)=0$ and $|y|=1$. Then the above integral may be rewritten as:

$$V(B(\omega,\delta)) = c\int_{\{(\alpha,t)\in[0,\pi]\times]0,\delta^2[:|1-te^{i\alpha})|<1\}} (2\cos(\alpha)-t)^3t^{10}\sin^6(\alpha)dtd\alpha,$$

from which we deduce easily that $V(B(\omega,\delta))\le c\delta^{22}$. This finishes the proof of Proposition 4.1. ∎

Next, to make Meyer's program work in our case, we follow the same line as in the proof in [4].

Step 1: Uniform Calderon-Zygmund type estimates

Proposition 4.3. *There exists a positive constant c such that the following estimates hold*

(i) $$\sup_{0\le r<1} |\Psi_r(\lambda,\theta,\omega)| \le cd(\theta,\omega)^{-2\rho}, \tag{20}$$

(ii) $$\sup_{0\le r<1} |\Psi_r(\lambda,\theta,\omega)-\Psi_r(\lambda,\theta',\omega)| \le c(1+|\lambda|)\frac{d(\theta,\theta')}{d(\theta,\omega)^{2\rho+1}}, \tag{21}$$

for every θ,θ',ω in $\partial B(\mathbb{O}^2)$ such that $d(\theta,\omega)\ge 2d(\theta,\theta')$.

(iii) $$\sup_{0\le r<1}\left|\int_{d(\theta,\omega)\le\delta}\Psi_r(\lambda,\theta,\omega)d\omega\right| \le c(1+\frac{1}{|\lambda|}),\quad \textit{for every } \delta>0. \tag{22}$$

Proof. Notice that $\Psi_r(\lambda,\theta,\omega) = |1-r[\theta,\omega]|^{-i\lambda-\rho}$.

(i) We have

$$|1-[\theta,\omega]| = |1-r[\theta,\omega]-(1-r)[\theta,\omega]| \le |1-r[\theta,\omega]| + (1-r)|[\theta,\omega]|.$$

Since $|[\theta,\omega]|\leq 1$, and for all $r\in[0,1[,\ 1-r\leq|1-r[\theta,\omega]|$, we get

$$|1-r[\theta,\omega]|^{-1}\leq 2|1-[\theta,\omega]|^{-1},\forall r\in[0,1[\tag{23}$$

and (i) follows.
(ii) Now, by the mean calculus lemma we obtain

$$|\Psi_r(\lambda,\theta,\omega)-\Psi_r(\lambda,\theta',\omega)|\leq\frac{|i\lambda+\rho|}{\rho}\left||1-r[\theta,\omega]|^{-\rho}-|1-r[\theta',\omega]|^{-\rho}\right|.$$

Then the proof of (ii) will be based on the identity

$$|1-r[\theta,\omega]|^{-\rho}-|1-r[\theta',\omega]|^{-\rho}=\sum_{j=0}^{\rho-1}\frac{|1-r[\theta',\omega]|-|1-r[\theta,\omega]|}{|1-r[\theta,\omega]|^{j+1}|1-r[\theta',\omega]|^{\rho-j}}.$$

By (23), we have $|1-r[\theta,\omega]|^{-1}\leq 2|1-[\theta,\omega]|^{-1}$, and

$$|1-r[\theta',\omega]|^{-1}\leq 2|1-[\theta',\omega]|^{-1}\leq 8|1-[\theta,\omega]|^{-1}.$$

The last inequality is a consequence of the triangle inequality and the hypotheses $d(\theta,\omega)\geq 2d(\theta,\theta')$. It is clear from above that

$$|\Psi_r(\lambda,\theta,\omega)-\Psi_r(\lambda,\theta',\omega)|\leq c\frac{(\lambda^2+\rho^2)^{\frac{1}{2}}}{\rho}\frac{|[\theta-\theta',\omega]|}{|1-[\theta,\omega]|^{\rho+1}},$$

for some numerical constant c. We claim that

$$|[\theta-\theta',\omega]|\leq d(\theta,\theta')(d(\theta,\theta')+2d(\theta,\omega)), \tag{24}$$

for any θ,θ' *and* ω in $\partial B(\mathbb{O}^2)$.
In the proof of (24) we may take $\theta=e_1=(1,0)$ (by the $O_{\mathbb{O}}(2)$-invariance of the non-isotropic distance) and one have to prove

$$|[e_1-\theta',\omega]|\leq|1-\theta_1'|^{\frac{1}{2}}(|1-\theta_1'|^{\frac{1}{2}}+2|1-\omega_1|^{\frac{1}{2}}). \tag{25}$$

It is obvious for $\omega_2=0$. If $\omega_2\neq 0$, then

$$|[e-\theta',\omega]|=|((1-\overline{\theta_1'})\omega_2)(\omega_2^{-1}\omega_1)-\theta_2'\overline{\omega_2}|\leq|1-\overline{\theta_1'}|+|\theta_2'||\omega_2|.$$

Next, from $|\theta_2'|^2=1-|\theta_1'|^2\leq 2|1-\theta_1'|$ and similar estimate for $|\omega_2|$, we get (25). This finishes the proof of (24). Combining the above results we conclude

$$|\Psi_r(\lambda,\theta,\omega)-\Psi_r(\lambda,\theta',\omega)|\leq c(1+|\lambda|)\frac{d(\theta,\theta')}{d(\theta,\omega)^{2\rho+1}}.$$

This finishes the proof of (ii).
(iii) By the $O_{\mathbb{O}}(2)$-invariance of the metric d and the measure $d\omega$, we have :

$$\int_{d(\omega,\theta)<\delta}\Psi_r(\lambda,\theta,\omega)d\omega=\int_{|1-\omega_1|<\delta^2}|1-r\omega_1|^{-i\lambda-\rho}d\omega. \tag{26}$$

It is clear that (26) is uniformly bounded for every $r \in [0, \dfrac{1}{2}[$ and $\delta > 0$. To show the uniform boundedness for $r \in [\dfrac{1}{2}, 1[$ we use Lemma 4.2 to get

$$\int_{d(\omega,\theta)<\delta} \Psi_r(\lambda,\theta,\omega)d\omega = \int_{\{x\in\mathbb{O}:|x|<1,|1-x|<\delta^2\}} |1-rx|^{-i\lambda-\rho}\left(1-|x|^2\right)^3 dm(x).$$

We put $1 - rx = t(\cos(\alpha) + \sin(\alpha)y)$ where $t > 0$, $\alpha \in [0,\pi]$ and $y \in \mathbb{O}$ such that $\mathfrak{Re}(y) = 0$ and $|y| = 1$. The above integral may be rewritten

$$\int_{d(\omega,\theta)<\delta} \Psi_r(\lambda,\theta,\omega)d\omega = r^{-14}\int_{\Gamma_{r,\delta}} t^{-i\lambda-4}|r^2 - |te^{i\alpha}-1|^2|^3 \sin^6(\alpha)\, dt\, d\alpha,$$

where $\Gamma_{r,\delta}$ is the set of $(t,\alpha) \in \,]0,\infty[\times[0,\pi]$ such that $|te^{i\alpha}-1| < r$ and also $|te^{i\alpha} - (1-r)| < r\delta^2$. Next, replacing $\sin^6(\alpha)$ by

$$\sin^6(\alpha) = \frac{1}{64}\left(20 - 15(e^{2i\alpha}+e^{-2i\alpha}) + 6(e^{4i\alpha}+e^{-4i\alpha}) - (e^{6i\alpha}+e^{-6i\alpha})\right),$$

we are reduced to show the uniform boundedness of integrals of the following type

$$\int_{\Gamma_{r,\delta}} t^{-i\lambda-4}|r^2 - |te^{i\alpha}-1|^2|^3 e^{ik\alpha}\, dt\, d\alpha\,.$$

We use our result in [6] to show that there exists a positive constant c such that

$$\left|\int_{d(\omega,\theta)<\delta} \Psi_r(\lambda,\theta,\omega)d\omega\right| \le c\left(1+\frac{1}{|\lambda|}\right),$$

for every $r \in [\frac{1}{2}, 1[$ and $\delta > 0$. This finishes the proof of the proposition. ∎

Step 2: Uniform action of Szegö-integrals on δ-molecules

For $\eta > 0$ and $0 < \delta \le 1$, we define the weight function

$$\Omega_{\eta,\delta}(\theta,\omega) = a(\delta,\rho)\eta^\delta[\eta + d(\theta,\omega)]^{-\delta-2\rho},$$

for every $\theta, \omega \in \partial B(\mathbb{O}^2)$, where $a(\delta,\rho)$ is a constant such that $\int_{\partial B(\mathbb{O}^2)} \Omega_{\eta,\delta}(\theta,\omega)d\omega \le 1$.

Definition 4.4. A $\mathbb{C}$-valued function m on $\partial B(\mathbb{O}^2)$ is said to be a δ*-molecule centered at* θ_0 *with width* $\eta > 0$ if m satisfies

(i) $|m(\theta)| \le \Omega_{\eta,\delta}(\theta,\theta_0)$,

(ii) $|m(\theta) - m(\theta')| \le (\frac{d(\theta,\theta')}{\eta})^\delta\left(\Omega_{\eta,\delta}(\theta,\theta_0) + \Omega_{\eta,\delta}(\theta',\theta_0)\right)$,

(iii) $\int_{\partial B(\mathbb{O}^2)} m(\theta)d\theta = 0$.

Let $M(\delta,\theta_0,\eta)$ denote the convex set of all δ-molecules centered at θ_0 with width η.

Proposition 4.5. *The operator $\Psi_r(\lambda)$ transforms uniformly in $r \in [0,1[$, δ-molecules into δ'-molecules with $0 < \delta' < \delta \leq 1$. More precisely, for any $0 < \delta' < \delta \leq 1$ there exists a positive constant c such that, for any $m \in M(\delta, \theta_0, \eta)$, the function*

$$m'(\theta) = c(1 + |\lambda| + \frac{1}{|\lambda|})^{-1}(\Psi_r(\lambda)m)(\theta),$$

lies in $M(\delta', \theta_0, \eta)$.

Proof. Firstly, we introduce the following weight function

$$\tilde{\Omega}_{\eta,\delta}(\theta, \omega) = \eta^{\delta} \min\left(\eta^{-\delta-2\rho}, d(\theta,\omega)^{-\delta-2\rho}\right),$$

for $\eta > 0$, $\delta \in]0,1]$ and $\theta, \omega \in \partial B(\mathbb{O}^2)$.
From now on c will denote any positive constant depending only on ρ and on δ. Then it is easy to see that the conditions (i) and (ii) in Definition 4.4 may be replaced by the equivalent conditions

(a) $|m(\theta)| \leq c\tilde{\Omega}_{\eta,\delta}(\theta, \theta_0)$,

(b) $|m(\theta) - m(\theta')| \leq c(\frac{d(\theta,\theta')}{\eta})^{\delta}\tilde{\Omega}_{\eta,\delta}(\theta, \theta_0)$, if $d(\theta, \theta') \leq \eta$.

The following estimates will be needed.

Lemma 4.6. *Let $\beta > 0$. Then there exists $c_1 > 0$ such that for $\eta > 0$*

$$\int_{d(\theta,\omega)\geq\eta} d(\theta,\omega)^{-\beta-2\rho} d\omega \leq c_1 \eta^{-\beta}, \tag{27}$$

$$\int_{d(\theta,\omega)\leq\eta} d(\theta,\omega)^{\beta-2\rho} d\omega \leq c_1 \eta^{\beta}. \tag{28}$$

The proof is a simple computation based on Lemma 4.2.
Let $m \in M(\delta, \theta_0, \eta)$, we first prove that $c(1 + |\lambda| + \frac{1}{|\lambda|})^{-1}\Psi_r(\lambda)m$ satisfies the condition (a). To do this we consider first the case where $d(\theta, \theta_0) \leq \eta$ and we decompose $\Psi_r(\lambda)m$ as

$$(\Psi_r(\lambda)m)(\theta) = \int_{d(\theta_0,\omega)\geq 2\eta} \Psi_r(\lambda, \theta, \omega)m(\omega)d\omega + \int_{d(\theta_0,\omega)<2\eta} \Psi_r(\lambda, \theta, \omega)(m(\omega) - m(\theta))d\omega$$
$$+ m(\theta)\int_{d(\theta_0,\omega)<2\eta} \Psi_r(\lambda, \theta, \omega)d\omega = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

Using (20) and noting that $d(\theta, \omega) \geq \eta$ if $d(\theta_0, \omega) \geq 2\eta$, we get $|\mathcal{I}_1| \leq c\eta^{-2\rho}$ (we used the fact that $\int_{\partial B(\mathbb{O}^2)} |m(\omega)|d\omega \leq c$).
On the other hand, using (20), (ii) in Definition 4.4 and (28), we have

$$|\mathcal{I}_2| \leq c\eta^{-\delta-2\rho}\int_{d(\theta_0,\omega)<2\eta} d(\theta,\omega)^{-2\rho+\delta}d\omega \leq c\eta^{-\delta-2\rho}\int_{d(\theta,\omega)<3\eta} d(\theta,\omega)^{-2\rho+\delta}d\omega \leq c\eta^{-2\rho}.$$

For the term $\mathcal{I}_3$, we have

$$|\mathcal{I}_3| \leq \eta^{-2\rho} \left| \int_{d(\theta_0,\omega)<2\eta} \Psi_r(\lambda,\theta,\omega) d\omega \right|,$$

then, following the same line as in the proof of (22), we obtain

$$|\mathcal{I}_3| \leq c\eta^{-2\rho}(|\lambda| + \frac{1}{|\lambda|}).$$

This shows that $c(1+|\lambda|+\frac{1}{|\lambda|})^{-1}\Psi_r(\lambda)m$ satisfies (a) for the case $d(\theta,\theta_0) \leq \eta$. Now, we consider the case $d(\theta,\theta_0) \geq \eta$. Set $A_1 = \{\omega \in \partial B(\mathbb{O}^2), d(\theta,\omega) \leq \frac{d(\theta,\theta_0)}{2}\}$, $A_2 = \{\omega \in \partial B(\mathbb{O}^2),\ d(\theta_0,\omega) < \frac{d(\theta,\theta_0)}{2}\}$, and $A_3 = \partial B(\mathbb{O}^2) \backslash A_1 \cup A_2$. Then write $m(\omega) = \sum_{i=1}^{3} m_i(\omega)$ where $m_i(\omega) = m(\omega)\chi_{A_i}(\omega)$ and χ_{A_i} is the characteristic function of the set A_i. The following lemma follows immediately from the definition of molecules.

Lemma 4.7.

$$|m_1(\omega)| \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho}, \tag{29}$$

$$|m_1(\omega) - m_1(\theta)| \leq c d(\theta,\theta_0)^{-\delta-2\rho} d(\theta,\omega)^{\delta},\ \textit{if } d(\theta,\omega) \leq \frac{\eta}{2}, \tag{30}$$

$$\int_{\partial B(\mathbb{O}^2)} |m_i(\omega)| d\omega \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta},\ \textit{for } i = 1,3, \tag{31}$$

$$\left| \int_{\partial B(\mathbb{O}^2)} m_2(\omega) d\omega \right| \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta}. \tag{32}$$

We write $\Psi_r(\lambda)m$ as $(\Psi_r(\lambda)m)(\theta) = \sum_{i=1}^{3}(\Psi_r(\lambda)m_i)(\theta) = f_1(\theta) + f_2(\theta) + f_3(\theta)$ and decompose f_1 as

$$\begin{aligned} f_1(\theta) = &\int_{d(\theta,\omega)<\frac{\eta}{2}} \Psi_r(\lambda,\theta,\omega)(m_1(\omega) - m_1(\theta)) d\omega + m_1(\theta) \int_{d(\theta,\omega)<\frac{\eta}{2}} \Psi_r(\lambda,\theta,\omega) d\omega \\ &+ \int_{d(\theta,\omega)\geq\frac{\eta}{2}} \Psi_r(\lambda,\theta,\omega) m_1(\omega) d\omega = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

Using the estimates (20), (30) and (27) we obtain $|\mathcal{I}_1| \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho}$. For the term $\mathcal{I}_2$ we have

$$|\mathcal{I}_2| \leq c(1 + \frac{1}{|\lambda|})\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho},$$

by (29) and (22). For $\mathcal{I}_3$, applying the estimates (29) and (20) we obtain

$$|\mathcal{I}_3| \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho} \int_{\frac{\eta}{2}\leq d(\theta,\omega)\leq\frac{d(\theta,\theta_0)}{2}} d(\theta,\omega)^{-2\rho} d\omega.$$

Next, follow the same method as in the proof of (22), to get

$$\int_{\frac{\eta}{2}\leq d(\theta,\omega)\leq\frac{d(\theta,\theta_0)}{2}} d(\theta,\omega)^{-2\rho} d\omega \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho} \log\left(\frac{d(\theta,\theta_0)}{\eta}\right),$$

from which we deduce that $|\mathcal{I}_3| \leq c\eta^{\delta'} d(\theta,\theta_0)^{-\delta'-2\rho}$, for any δ' such that $0<\delta'<\delta\leq 1$. Combining these estimates we see that

$$|f_1(\theta)| \leq c(1+\frac{1}{|\lambda|})\eta^{\delta'} d(\theta,\theta_0)^{-\delta'-2\rho}.$$

For f_2, we have

$$f_2(\theta) = \int_{\partial B(\mathbb{O}^2)} (\Psi_r(\lambda,\theta,\omega) - \Psi_r(\lambda,\theta,\theta_0)) m_2(\omega) d\omega + \Psi_r(\lambda,\theta,\theta_0) \int_{\partial B(\mathbb{O}^2)} m_2(\omega) d\omega.$$

Using the estimates (20) and (32), we get

$$\left| \Psi_r(\lambda,\theta,\theta_0) \int_{\partial B(\mathbb{O}^2)} m_2(\omega) d\omega \right| \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho}.$$

Applying the estimates (21), (i) in Definition 4.4 and (28), we obtain

$$\begin{aligned}
&\left| \int_{\partial B(\mathbb{O}^2)} (\Psi_r(\lambda,\theta,\omega) - \Psi_r(\lambda,\theta,\theta_0)) m_2(\omega) d\omega \right| \\
&\leq c(1+|\lambda|)\eta^{\delta} d(\theta,\theta_0)^{-1-2\rho} \int_{d(\theta_0,\omega)<\frac{d(\theta,\theta_0)}{2}} d(\theta_0,\omega)^{(1-\delta)-2\rho} d\omega \\
&\leq c(1+|\lambda|)\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho}.
\end{aligned}$$

For f_3, noting that on the support of m_3, $d(\theta,\omega) > \frac{d(\theta,\theta_0)}{2}$ and using the estimates (20) and (31), we see that

$$|f_3(\theta)| \leq c\eta^{\delta} d(\theta,\theta_0)^{-\delta-2\rho}.$$

This proves that $c(1+|\lambda|+\frac{1}{|\lambda|})^{-1}\Psi_r(\lambda)m$ satisfies the estimate (a) for $d(\theta,\theta_0)\geq\eta$. It remains to prove that $c(1+|\lambda|+\frac{1}{|\lambda|})^{-1}\Psi_r(\lambda)m$ satisfies (b). As above write

$$\begin{aligned}
(\Psi_r(\lambda)m)(\theta) = &\int_{\partial B(\mathbb{O}^2)} \Psi_r(\lambda,\theta,\omega)(m(\omega)-m(\theta))\phi_1(\omega) d\omega + \\
&+ m(\theta) \int_{\partial B(\mathbb{O}^2)} \Psi_r(\lambda,\theta,\omega)\phi_1(\omega) d\omega + \int_{\partial B(\mathbb{O}^2)} \Psi_r(\lambda,\theta,\omega) m(\omega)\phi_2(\omega) d\omega,
\end{aligned}$$

where $1=\phi_1(\omega)+\phi_2(\omega)$ and $\phi_1(\omega)=\chi_{\{\omega\in\partial B(\mathbb{O}^2):\ d(\theta,\omega)\leq 2d(\theta,\theta')\}}(\omega)$.
Let $f(\theta)$ denote the first term of right-hand side above and let $h(\theta)$ be the sum of the last two terms. Using (20) and (ii) in Definition 4.4, we get

$$|f(\theta)| \leq c\left(\frac{d(\theta,\theta')}{\eta}\right)^{\delta} \tilde{\Omega}_{\eta,\delta}(\theta,\theta_0).$$

The above estimate holds also with θ replaced by θ' for $d(\theta,\theta')\leq\eta$.
For $h(\theta)$, we have

$$\begin{aligned}
h(\theta)-h(\theta') = &(m(\theta)-m(\theta')) \int_{\partial B(\mathbb{O}^2)} \Psi_r(\lambda,\theta',\omega)\phi_1(\omega) d\omega + \\
&+ \int_{\partial B(\mathbb{O}^2)} (\Psi_r(\lambda,\theta,\omega)-\Psi_r(\lambda,\theta',\omega))(m(\omega)-m(\theta))\phi_2(\omega) d\omega = \mathcal{J}_1+\mathcal{J}_2.
\end{aligned}$$

The estimate (b) on m and (22), yields

$$|\mathcal{J}_1| \le c(|\lambda| + \frac{1}{|\lambda|}) \left(\frac{d(\theta, \theta')}{\eta} \right)^{\delta} \tilde{\Omega}_{\eta,\delta}(\theta, \theta_0).$$

In order to estimate $\mathcal{J}_2$, we first observe that the following estimate holds

$$|\Psi_r(\lambda, \theta, \omega) - \Psi_r(\lambda, \theta', \omega)| \le c(1 + |\lambda|) \frac{d(\theta, \theta')}{d(\theta, \omega)^{1+2\rho}},$$

for all ω in the support of ϕ_2.
Next, using (ii) in Definition 4.4 and the estimate (27), we get

$$|\mathcal{J}_2| \le c d(\theta, \theta')^{\delta} \begin{cases} \eta^{-\delta-2\rho} & \text{if } d(\theta, \theta_0) \le 2\eta \\ d(\theta, \theta_0)^{-\delta-2\rho} & \text{if } d(\theta, \theta_0) \ge 2\eta \end{cases}.$$

This shows that

$$|h(\theta) - h(\theta')| \le c(|\lambda| + \frac{1}{|\lambda|}) d(\theta, \theta')^{\delta} \tilde{\Omega}_{\eta,\delta}(\theta, \theta_0).$$

Combining the above estimates and noting that $\tilde{\Omega}_{\eta,\delta}(\theta, \theta_0) \le \tilde{\Omega}_{\eta,\delta'}(\theta, \theta_0)$ for all $0 < \delta' < \delta \le 1$, we get the desired estimates.
Condition (iii) in Definition 4.4 is easy to check. This completes the proof of Proposition 4.5. ■

Step 3: Molecular resolution of $L^2(\partial B(\mathbb{O}^2))$.

We built an adapted δ-molecular resolution by means of the Poisson kernel $P(x, \omega)$. Namely, for $j = 0, 1, \ldots$, we set for $\eta_j = \frac{2(1-2^{-j})}{2^{-2j}+2(1-2^{-j})}$

$$\Delta_j(\theta, \omega) = P(\eta_{j+1}\theta, \omega) - P(\eta_j \theta, \omega).$$

Proposition 4.8. *Let* $\delta \in]0, 1]$. *Then we have*

(i) *The functions* $\theta \to \Delta_j(\theta, \theta_0)$ *are* δ*-molecules centered at* θ_0 *with width* 2^{-j}.

(ii) *Let* $\mathcal{H}$ *be the set of all square integrable functions* f *on* $\partial B(\mathbb{O}^2)$ *such that* $\int_{\partial B(\mathbb{O}^2)} f(\theta) d\theta = 0$. *Then for every* $f \in \mathcal{H}$ *we have*

$$f = \sum_{j \ge 0} \Delta_j f,$$

where Δ_j *is the integral operator on* $\partial B(\mathbb{O}^2)$ *given by*

$$\Delta_j f(\theta) = \int_{\partial B(\mathbb{O}^2)} \Delta_j(\theta, \omega) f(\omega) d\omega.$$

Proof. (i) Let $P_\eta(\theta, \omega) = P(\frac{2(1-\eta)}{\eta^2+2(1-\eta)}\theta, \omega)$, $\eta \in]0, \frac{1}{2}]$.

Lemma 4.9. *There exists* c_2 *such that for* $\delta > 0,\ 0 < \eta \leq \frac{1}{2}$, *and* $d(\theta, \theta') \leq \eta$

$$P_\eta(\theta, \omega) \leq c_2 \Omega_{\eta,\delta}(\theta, \omega), \quad \textit{and} \tag{33}$$

$$|P_\eta(\theta, \omega) - P_\eta(\theta', \omega)| \leq c_2 d(\theta, \theta') \begin{cases} d(\theta, \omega)^{-2\rho-1} & \textit{if } d(\theta, \omega) \geq 2\eta \\ \eta^{-2\rho-1} & \textit{if } d(\theta, \omega) \leq 2\eta \end{cases}. \tag{34}$$

The proof is a simple computation based on the explicit expression of the Poisson kernel.
Let $\eta = 2^{-j}$. Then (i) is a direct consequence of the above estimates.

The assertion (ii) is obvious since the Poisson kernel $P(r\theta, \omega)$ is an approximation of identity in $\partial B(\mathbb{O}^2)$ as $r \to 1$. This finishes the proof of Proposition 4.8. ∎

Now, to get the estimate (11) in the Key Lemma 3.1, we may write the operator $\Psi_r(\lambda)$ as

$$\Psi_r(\lambda) = \sum_{j=0}^{+\infty} \Psi_{r,j}(\lambda),$$

with $\Psi_{r,j}(\lambda) = \Psi_r(\lambda) \circ \Delta_j$.
Next, using the uniform action of $\Psi_r(\lambda)$ on molecules as well as the bounded mean property (iii) in Proposition 4.3, we may apply uniformly in $r \in [0, 1[$ the Cotlar-Stein Lemma to obtain $\|\Psi_r(\lambda)\|_2 \leq c(1 + |\lambda| + \frac{1}{|\lambda|})$.
Finally, combining the estimate (ii) of Proposition 4.3 as well as (iii) of Proposition 4.1, we get the Hörmander condition (12) on the Schwartz kernel and the proof of the Key Lemma 3.1 of this paper is complete.

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References

[1] S. Ben Said, T. Oshima, N. Shimeno: *Fatou's theorem and Hardy-type spaces for eigenfunctions of the invariant differential operators on symmetric spaces*, Int. Math. Res. Note **16** (2003) 915–931.

[2] A. Boussejra: *L^p-Poisson integrals representations of solutions of the Hua system on Hermitian symmetric spaces of tube type*, J. Funct. Anal. **235** (2006) 413–429.

[3] A. Boussejra: *The Poisson transform on Symmetric Spaces*, Seminar On Mathematical Sciences **39**, Keio University, Yokohama (2016).

[4] A. Boussejra, A. Intissar: *Caractérisation des intégrales de Poisson-Szegö de $L^2(\partial\mathbb{B}^n)$ dans la boule de Bergman B^n $(n \geq 2)$*, C. R. Acad. Sci. Paris, Série I, **315** (1992) 1353–1357.

[5] A. Boussejra, K. Koufany: *Characterization of Poisson integrals for non-tube bounded symmetric domains*, J. Math. Pures Appl. **87** (2007) 438–451.

[6] A. Boussejra, H. Sami: *Characterization of the L^p-range of the Poisson transform in Hyparbolic Spaces $B(\mathbb{F}^n)$*, J. Lie Theory **12** (2002) 1–14.

[7] W. O. Bray: *Aspects of harmonic analysis on real hyperbolic space*, in: Fourier Analysis, Lecture Notes in Pure and Appl. Math. **157**, Dekker, New York (1994) 77–102.

[8] R. Coifman, G. Weiss: *Extension of Hardy spaces and their use inanalysis*, Bull. Amer. Math. Soc. **83** (1977) 569–644.

[9] G. David, L. Journe, S. Semmes: *Opérateur de Calderon-Zygmund fonctions para-accrative et interpolation*, Rev. Mat. Iberoamericana **1** (1985) 1–56.

[10] S. Helgason: *A duality for symmetric spaces with applications to group representations*, Advances in Math. **5** (1970) 1–154.

[11] S. Helgason: *Eigenspaces of the Laplacian integral representations and irreducibility*, J. Funct. Anal. **17** (1974) 328–353.

[12] A. D. Ionescu: *On the Poisson transform on symmetric spaces of rank one*, J. Funct. Anal. **174** (2000) 513–523.

[13] K. D. Johnson: *Composition series and intertwining operators for the spherical principal series. II*, Trans. Amer. Math. Soc. **215** (1976) 269–283.

[14] K. Kaizuka: *A characterization of the L^2-range of the Poisson transform related to Strichartz conjecture on symmetric spaces of noncompact type*, Advances in Math. **303** (2016) 464–501.

[15] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, M. Tanaka: *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. **107** (1978) 1–39.

[16] A. W. Knapp, R. E. Williamson, *Poisson integrals and semisimple groups*, J. Analyse Math. **24** (1971) 53–76.

[17] P. Kumar: *Fourier restriction theorem and characterization on weak L^2-eigenfunctions of the Laplace-Beltrami operator*, J. Funct. Anal. **266** (2014) 5584–5597.

[18] P. Kumar, S. K. Ray, R. P. Sarkar: *Characterization of almost L^p-eigenfunctions of the Laplace-Beltrami operator*, Trans. Amer. Math. Soc. **366** (2014) 3191–3225.

[19] J. Lewis: *Eigenfunctions on symmetric spaces with distribution valued boundary form*, J. Funct. Anal. **29** (1978) 331–357.

[20] N. Lohoué, T. Rychener: *Some function spaces on symmetric spaces related to convolution operators*, J. Funct. Anal. **55** (1984) 200–219.

[21] Y. Meyer: *Les nouveaux opérateurs de Calderon-Zygmund*, Astérisque **131** (1985) 237–254.

[22] H. Lee Michelson: *Fatou theorems for eigenfunctions of the invariant differential operators on symmetric spaces*, Trans. Amer. Math. Soc. **177** (1973) 257–274.

[23] W. Rudin: *Function Theory in the Unit Complex Ball of $\mathbb{C}^n$*, Springer, Berlin et al. (1980).

[24] T. Oshima, J. Sekiguchi: *Eigenspaces of invariant differential operators on an affine symmetric space*, Invent. Math. **57** (1980) 1–81.

[25] P. Sjögren: *Characterizations of Poisson integrals on symmetric spaces*, Math. Scand. **49** (1981) 229–249.

[26] R. Strichartz: *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. **87** (1989) 51–148.

[27] M. Stoll: *Hardy-type spaces of harmonic functions on symmetric spaces of noncompact type*, J. Reine Angew. Math. **271** (1974) 63–76.

[28] R. Takahashi: *Quelques résultats sur l'analyse harmonique dans l'éspace symétrique non compact de rang 1 du type exceptionel*, Lecture Notes in Mathematics **739**, Springer, Berlin et al. (1979) 511–567.

[29] Va. V. Volchov, Vi. Volchov: *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group*, Springer, Berlin et al. (2009).

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