Lie Algebras Attached to Clifford Modules and Simple Graded Lie Algebras

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Communicated by B. Ørsted

Abstract.   We study possible cases of complex simple graded Lie algebras of depth 2, which are the Tanaka prolongations of pseudo H-type Lie algebras arising through representation of Clifford algebras. We show that the complex simple Lie algebras of type B_n with $|2|$-grading do not contain non-Heisenberg pseudo H-type Lie algebras as their negative nilpotent part, while the complex simple Lie algebras of types A_n, C_n and D_n provide such a possibility. Among exceptional algebras only F_4 and E_6 contain non-Heisenberg pseudo H-type Lie algebras as their negative part of $|2|$-grading. An analogous question addressed to real simple graded Lie algebras is more delicate, and we give results revealing the main differences with the complex situation.

Mathematics Subject Classification: 17B10, 17B22, 17B25, 22E46.
Key Words and Phrases: Simple Lie algebras, root system, Dynkin diagram, graded Lie algebras, parabolic subalgebras, H-type algebra, Clifford algebra, non-degenerate bilinear form.

1.   Introduction

Given a finite dimensional nilpotent graded Lie algebra $\mathfrak{n} = \bigoplus_{p<0} \mathfrak{n}_p$, there is a uniquely associated maximal transitive graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, such that the negative part $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$ coincides with $\mathfrak{n}$. This graded Lie algebra $\mathfrak{g}$, called the *Tanaka prolongation* of $\mathfrak{n}$, may be regarded as the symmetry algebra of $\mathfrak{n}$ and plays a fundamental role in the study of geometric structures associated with differential systems of type $\mathfrak{n}$, see [27]. In particular, if the Tanaka prolongation

is finite dimensional and simple, or more generally, if it satisfies the so-called condition (C) following [22], one can pursuit more detailed studies by constructing Cartan connections, see [22, 23, 28].

A special class of nilpotent Lie algebras, called *H (eisenberg)-type algebras* was introduced by Kaplan in [18]. This class is associated with the Clifford algebra generated by a vector space with a positive definite quadratic form. A generalization related to the Clifford algebras with arbitrary non-degenerate quadratic forms was proposed in [8, 15] and received the name *pseudo H-type Lie algebras.* We are interested in studying the Tanaka prolongation of these Lie algebras and the geometric features of the related Lie groups.

There is a series $\{\mathfrak{n}^{r,s}\}$ of *basic* pseudo H-type algebras defined for integers $r, s \geq 0$, where $\mathfrak{n}^{r,s} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ is a nilpotent graded Lie algebra , $\mathfrak{n}_{-2}$ is endowed with a scalar product $\langle . , . \rangle_{r,s}$ of signature (r, s), and $\mathfrak{n}_{-1}$ is a minimal admissible module of the Clifford algebra $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle . , . \rangle_{r,s})$. The prolongations for the basic pseudo H-type Lie algebras $\mathfrak{n}^{r,s}$, $r + s \leq 8$, were computed by using the Maple software [Tanaka Prolongation] elaborated by Anderson [4] and the tables of structure constants of pseudo H-type Lie algebras [12]. The result, presented in Tables 8 and 9 in Section 7, leads us to a conjecture that the prolongation $\mathrm{Prol}(\mathfrak{n})$ of a pseudo H-type Lie algebra $\mathfrak{n}$ vanishes at order one if it does not contain a simple graded Lie algebra of depth 2. In other words the pseudo H-type Lie algebra has a simple graded Lie algebra of depth two as a symmetry algebra, (not involving metric), otherwise the prolongation vanishes at order one. Inspired by this work, it was shown in [14] that any pseudo H-type algebra $\mathfrak{n}$ with the dimension of the center $\mathfrak{n}_{-2}$ strictly greater than 2 has finite Tanaka prolongation, see also [25]. On the other hand it is known that the prolongation of a subriemannian symbol vanishes at order one [24].

We pose the following question:

Problem 1: Find the pseudo H-type Lie algebras $\mathfrak{n}$, for which the Tanaka prolongation is a simple graded Lie algebra. Equivalently, we can look for a simple graded Lie algebra $\mathfrak{g}$ such that its negative part $\mathfrak{g}_-$ in $|2|$-grading is a pseudo H-type algebra.

The problem of relative Tanaka prolongations of H-type Lie algebras with positive definite quadratic forms was treated in [19]. A similar classification question has been raised in the study of prolongations of (super-)Poincaré algebras, see [1, 2, 3]. These results include some of ours as a particular case, since pseudo H-type algebras are contained in the wider class of so-called extended translation algebras. The fact that the class of pseudo H-type algebras is strictly smaller, allows our results to be more precise than those obtained in [3]. We also call to the attention of the readers the book [29], where the classification question was considered from the point of view of square integrable nilradicals.

The Clifford representations that yield pseudo H-type Lie algebras, so-called admissible Clifford modules, impose some restrictions on the dimensions of the pseudo H-type algebras. These dimensions are related by the Radon-Hurwitz-Eckmann formula. In our present work we study Problem 1 mainly restricting ourselves to the following two problems:

Problem 2: Among all $|2|$-gradings $\mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ of a complex simple Lie algebra $\mathfrak{g}$ find those whose negative nilpotent part $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ has the dimensions of $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ coinciding with those of the complexification of a pseudo H-type Lie algebra (no metric is involved).

Problem 3: Among all candidates found in Problem 2, determine all real pseudo H-type Lie algebras whose complexification coincides with the nilpotent part $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of the gradings of the complex simple Lie algebras.

We find necessary conditions for a $|2|$-grading $\mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ of a complex simple Lie algebra $\mathfrak{g}$ for the dimensions of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ matching the dimensions of a complexified pseudo H-type algebra, by making use of the Radon-Hurwitz-Eckmann formula, as studied in [9, 10, 11, 13]. This way we fix Problem 2. We show that the classical complex Lie algebras of type A_n, C_n and D_n potentially may contain infinitely many pseudo H-type Lie algebras as a negative nilpotent part in their $|2|$-grading by checking carefully the necessary conditions according to each type. At the same time, a classical complex Lie algebra of type B_n with any $|2|$-grading never contains any pseudo H-type Lie algebra as its negative nilpotent part except for the Heisenberg algebra.

We also present some progress in solving Problem 3. We determine all real simple graded Lie algebras whose negative part is a basic pseudo H-type algebra $\mathfrak{n}^{r,s}$ for $r + s \leq 8$. As a typical example we show that the negative part $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of the grading of $\mathfrak{su}(3,3)$ is isomorphic to the pseudo H-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ carrying a scalar product of index $(1,3)$ on the center $\mathfrak{n}_{-2}$. On the other hand we prove that the only H-type Lie algebras with a positive definite bilinear form that appear as the negative part of $|2|$-gradings of $\mathfrak{sl}(n+1, \mathbb{R})$ are the Heisenberg algebras, see Theorem 6.4.

For the exceptional simple Lie algebras we prove a remarkable fact that only F_4 and E_6 are the complexified prolongations of some pseudo H-type Lie algebras which are not the Heisenberg algebra, see Theorem 4.1. To prove that the exceptional simple Lie algebras F_4 and E_6 really appear as the symmetry algebras of a pseudo H-type algebra, we give a computer-aided proof by showing that the prolongation of the basic pseudo H-type algebra $\mathfrak{n}^{r,s}$ is isomorphic to a real form of F_4 for $(r,s) = (7,0), (3,4)$ and to that of E_6 for $(r,s) = (8,0), (7,1), (4,4), (3,5)$, see Table 9. We use the necessary condition obtained from the Radon-Hurwitz-Eckmann formula in order to finish the proof of the theorem.

The structure of the present paper is the following. After reviewing some preliminaries needed for the rest of the paper in Section 2, we proceed presenting the complete answer to Problem 2 in Sections 3 and 4, for the classical complex Lie algebras and for the exceptional ones. In Section 5, we show a concrete example answering affirmatively Problem 3, and in Section 6, we prove that only the Heisenberg algebras appear as the negative part of $|2|$-gradings of $\mathfrak{sl}(n+1, \mathbb{R})$ among all H-type algebras endowed with a positive definite bilinear form. Finally, in Section 7 we list the growth vectors of prolongation $\mathrm{Prol}(\mathfrak{n}^{r,s})$ for $r + s \leq 8$ computed by Maple, and then, identify the Lie algebras $\mathrm{Prol}(\mathfrak{n}^{r,s})$ which are simple.

2.   Preliminaries

2.1. Notation

We follow the notational conventions of [7]. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with a Cartan subalgebra $\mathfrak{h}$. Denote by Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$. Suppose we have chosen a set Δ^+ of positive roots, thus $\Delta = \Delta^+ \cup \Delta^-$, and let $\Delta^0 \subset \Delta^+$ be a set of simple roots. We say that $\mathfrak{p}$ is a *parabolic subalgebra* of $\mathfrak{g}$ if it contains the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Let $\Sigma \subset \Delta^0 = \{\alpha_1, \ldots, \alpha_r\}$ be a given subset of simple roots. For a root $\alpha = \sum_i a_i \alpha_i \in \Delta$, we define the *height* with respect to Σ as

$$\mathrm{ht}_\Sigma(\alpha) = \sum_{i:\, \alpha_i \in \Sigma} a_i.$$

If $\alpha_{\max}$ is the highest root of $\mathfrak{g}$, then we denote $k = \mathrm{ht}_\Sigma(\alpha_{\max})$. We obtain a $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$, where

$$\mathfrak{g}_i = \bigoplus_{\alpha:\, \mathrm{ht}_\Sigma(\alpha)=i} \mathfrak{g}_\alpha, \quad i \neq 0, \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha:\, \mathrm{ht}_\Sigma(\alpha)=0} \mathfrak{g}_\alpha.$$

In the present paper we are interested in the case $k = 2$ in order to study the relation to a class of 2-step nilpotent Lie algebras, which we proceed to define.

2.2. Pseudo H-type algebras and Clifford modules

A pseudo H-type algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ is a real graded 2-step nilpotent Lie algebra endowed with a non-degenerate symmetric bilinear form $\langle .\,, .\rangle = \langle .\,, .\rangle_{\mathfrak{n}_{-2}} + \langle .\,, .\rangle_{\mathfrak{n}_{-1}}$, such that $\langle .\,, .\rangle_{\mathfrak{n}_{-2}}$ is non-degenerate and all the maps $J_z \in \mathrm{End}(\mathfrak{n}_{-1})$ defined by

$$\langle J_z x, y\rangle_{\mathfrak{n}_{-1}} = \langle [x, y], z\rangle_{\mathfrak{n}_{-2}}, \quad \text{for all } z \in \mathfrak{n}_{-2}, \quad x, y \in \mathfrak{n}_{-1},$$

satisfy the Clifford condition $J_z^2 = -\langle z, z\rangle_{\mathfrak{n}_{-2}} \mathrm{Id}_{\mathfrak{n}_{-1}}$. The last identity implies that the map $J\colon \mathfrak{n}_{-2} \to \mathrm{End}(\mathfrak{n}_{-1})$ can be extended to a representation of the Clifford algebra $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle .\,, .\rangle_{\mathfrak{n}_{-2}})$.

Given a representation of the Clifford algebra $\tilde{J}\colon \mathrm{Cl}(\mathfrak{n}_{-2}, \langle .\,, .\rangle_{\mathfrak{n}_{-2}}) \to \mathrm{End}(V)$, where V is a real vector space, we say that V is admissible as a Clifford module if it admits a non-degenerate scalar product $\langle .\,, .\rangle_V$ such that

$$\langle \tilde{J}_z x, y\rangle_V = -\langle x, \tilde{J}_z y\rangle_V, \quad \text{for all } z \in \mathfrak{n}_{-2}, \quad x, y \in \mathfrak{n}_{-1}. \tag{1}$$

There is a strong restriction on which Clifford modules can be admissible, as noted in [8]. In particular, not all irreducible Clifford modules can be admissible; thus we define a minimal admissible module as an admissible module of minimal dimension. Note that if $\dim \mathfrak{n}_{-2} > 1$, then $\dim \mathfrak{n}_{-1}$ must be divisible by 4.

If $\langle .\,, .\rangle_{\mathfrak{n}_{-2}}$ as a bilinear form has r positive and s negative eigenvalues, and if $\mathfrak{n}_{-1}$ is an admissible $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle .\,, .\rangle_{\mathfrak{n}_{-2}})$-module of minimal dimension, then we denote by $\mathfrak{n}^{r,s}$ the pseudo H-type algebra $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$.

2.3. Choices of roots for the classical Lie algebras

In this subsection we construct $|2|$-gradings of the complex simple Lie algebras using the conventions from [6]. Following the procedure described in Section 2, the only possibilities for $|2|$-gradings are given in Table 1.

Table 1: Dynkin diagrams and choices of Σ for $|2|$-gradings of classical Lie algebras

Algebra	Σ	Dynkin diagram
$A_n\colon \mathfrak{sl}(n+1,\mathbb{C})$ $n \geq 2$	$\Sigma_{i,j} = \{\alpha_i, \alpha_j\}$ $1 \leq i \leq \left[\frac{n}{2}\right]$ $i < j \leq n+1-i$	
$B_n\colon \mathfrak{so}(2n+1,\mathbb{C})$ $n \geq 2$	$\Sigma_i = \{\alpha_i\}$ $i = 2,\ldots,n$	
$C_n\colon \mathfrak{sp}(2n,\mathbb{C})$ $n \geq 3$	$\Sigma_i = \{\alpha_i\}$ $i = 1,\ldots,n-1$	
$D_n\colon \mathfrak{so}(2n,\mathbb{C})$ $n \geq 4$	$\Sigma_i = \{\alpha_i\}$ $i = 2,\ldots,n-2$	
	$\Sigma_{1,n} = \{\alpha_1, \alpha_n\}$	
	$\Sigma_{n-1,n} = \{\alpha_{n-1}, \alpha_n\}$	

An important remark is that since automorphisms of a Dynkin diagram induce outer automorphisms of the corresponding Lie algebra, some choices of roots yield isomorphic gradings. An example of this fact is the choices of the roots $\{\alpha_1, \alpha_{n-1}\}$ and $\{\alpha_1, \alpha_n\}$ for the Lie algebra $\mathfrak{so}(2n,\mathbb{C})$. In Table 1, only non-isomorphic graded Lie algebras appear.

Note that A_1 does not appear in Table 1, because it has only one root of height one, and thus it cannot produce a $|2|$-grading. Also we have the following exceptional isomorphisms:

- $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$,
- $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$,
- $\mathfrak{so}(2,\mathbb{C}) \cong \mathbb{C}$,
- $\mathfrak{sp}(4,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$,
- $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$,
- $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$,

which explains the choices of n in each case of Table 1.

3. Complexified pseudo H-type algebras in $|2|$-gradings of classical Lie algebras

We employ Table 1 in order to calculate the dimensions of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ of a $|2|$-grading of a classical complex Lie algebra $\mathfrak{g}$, summarizing the computations in the following statement.

Theorem 3.1.   *Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $\mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ be a $|2|$-grading of $\mathfrak{g}$. Then the dimensions of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ are given in Table 2.*

Table 2: Choice of Σ and dimensions for $|2|$-gradings of the classical Lie algebras

Algebra	Σ	$\dim(\mathfrak{g}_{-1})$	$\dim(\mathfrak{g}_{-2})$
$\mathfrak{sl}(n+1, \mathbb{C})$ $n \geq 2$	$\Sigma_{i,j} = \{\alpha_i, \alpha_j\}$ $1 \leq i \leq \left[\frac{n}{2}\right]$ $i < j \leq n+1-i$	$(n+1-(j-i))(j-i)$	$i(n+1-j)$
$\mathfrak{so}(2n+1, \mathbb{C})$ $n \geq 2$	$\Sigma_i = \{\alpha_i\}$ $i = 2, \ldots, n$	$i(2(n-i)+1)$	$i(i-1)/2$
$\mathfrak{sp}(2n, \mathbb{C})$ $n \geq 3$	$\Sigma_i = \{\alpha_i\}$ $i = 1, \ldots, n-1$	$2i(n-i)$	$i(i+1)/2$
$\mathfrak{so}(2n, \mathbb{C})$ $n \geq 4$	$\Sigma_i = \{\alpha_i\}$ $i = 2, \ldots, n-2$	$2i(n-i)$	$i(i-1)/2$
	$\Sigma_{1,n} = \{\alpha_1, \alpha_n\}$	$n(n-1)/2$	$n-1$
	$\Sigma_{n-1,n} = \{\alpha_{n-1}, \alpha_n\}$	$2(n-1)$	$(n-1)(n-2)/2$

Proof. The values given in Table 2 can be directly found after careful analysis of the matrix representations of the corresponding Lie algebras. For the sake of clarity, let us briefly explain how to obtain diagramatically the dimensions given for the case of $\mathfrak{sl}(n+1, \mathbb{C})$. Considering a choice of roots $\Sigma_{i,j}$ according to Table 1, the induced $|2|$-grading is given schematically by (2). Counting dimensions, it is easy to see that

$$\dim \mathfrak{g}_{-1} = (n+1-(j-i))(j-i) \quad \text{and} \quad \dim \mathfrak{g}_{-2} = i(n+1-j).$$

For the remaining three classical Lie algebras, the gradings are more involved, but using their matrix representations the computations can be made similarly.

$$\begin{array}{c} \quad\quad i \quad\quad j \\ \begin{array}{c} i \\ \\ j \end{array} \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array} \right). \quad \blacksquare \end{array} \tag{2}$$

As a consequence of Theorem 3.1, we recover the following theorem known by Boothby [5], see also [30].

Theorem 3.2.   *For any complex simple Lie algebra $\mathfrak{g}$ there exists a unique contact grading, up to isomorphism, that is a $|2|$-grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ for which $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the complexified Heisenberg algebra.*

The list of the contact gradings is given in Table 3 for the classical simple Lie algebras and later in Section 4 for the exceptional Lie algebras.

Table 3: Contact cases in complex simple Lie algebras

Type of $\mathfrak{g}$	dim $\mathfrak{g}_{-1}$	Choice of root
A_n	$2n - 2$	$\{\alpha_1, \alpha_n\}$
B_n	$4n - 6$	$\{\alpha_2\}$
C_n	$2n - 2$	$\{\alpha_1\}$
D_n	$4n - 8$	$\{\alpha_2\}$

Given a pseudo H-type Lie algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$, where $\mathfrak{n}_{-1}$ is an admissible module of the Clifford algebra $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle . , . \rangle_{\mathfrak{n}_{-2}})$, there is a relation between the dimensions of $\mathfrak{n}_{-1}$ and $\mathfrak{n}_{-2}$ given in terms of the Radon-Hurwitz-Eckmann function ρ. Explicitly, if $n = u2^{4\alpha+\beta}$, where u is odd and $\beta = 0, 1, 2$ or 3, then $\rho(n) := 8\alpha + 2^\beta$. For the pseudo H-type Lie algebra $\mathfrak{n}$, we have that

$$\dim \mathfrak{n}_{-2} \leq \rho(\dim \mathfrak{n}_{-1}) - 1. \tag{3}$$

In the case of a positive definite scalar product $\langle . , . \rangle_{\mathfrak{n}_{-2}}$, it is known that inequality (3) is a necessary and sufficient condition for the existence of an H-type Lie algebra of dimension $\dim \mathfrak{n}_{-1} + \dim \mathfrak{n}_{-2}$ with the center $\mathfrak{n}_{-2}$, see [18]. For an indefinite non-degenerate scalar product $\langle . , . \rangle_{\mathfrak{n}_{-2}}$ inequality (3) is only a necessary condition, see [8]. The reason behind this is that not all Clifford modules are admissible, as discussed in Section 2.

From the definition of the Radon-Hurwitz-Eckmann function, it is clear that if u is odd, then we have

$$\rho(u2^r) = \rho(2^r) = \begin{cases} 8s + 1 = 2r + 1, & \text{if } r = 4s, \\ 8s + 2 = 2r, & \text{if } r = 4s + 1, \\ 8s + 4 = 2r, & \text{if } r = 4s + 2, \\ 8s + 8 = 2r + 2, & \text{if } r = 4s + 3. \end{cases}$$

Based on this relation, we want to find all the nilpotent parts $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of $|2|$-gradings of the classical Lie algebras which satisfy inequality (3) in this context, namely $\dim \mathfrak{g}_{-2} \leq \rho(\dim \mathfrak{g}_{-1}) - 1$. They are the first candidates for a positive answer to Problem 2. From now on, we denote $d_1 = \dim \mathfrak{g}_{-1}$ and $d_2 = \dim \mathfrak{g}_{-2}$.

3.1. The case of A_n

Assume $d_2 > 1$. Since the dimension of $\mathfrak{g}_{-1}$ must be divisible by 4 we exclude the following simple algebras from further consideration.

Proposition 3.3. *Let the system of roots $\Sigma_{i,j}$ for $\mathfrak{sl}(n+1, \mathbb{C})$ be chosen. If either*
(1) *both n and $j - i$ are odd, or*
(2) *n is even and $j - i \equiv 2 \mod 4$,*
then the negative part of the $|2|$-grading of $\mathfrak{sl}(n+1, \mathbb{C})$ cannot be of pseudo H-type.

Given $d_2 > 1$, we want to find all possible choices of roots $\Sigma_{i,j}$ such that the associated negative part of the $|2|$-grading is a candidate for a pseudo H-type Lie algebra.

Theorem 3.4. *For any $d_2 > 1$ there is $n \in \mathbb{N}$ and a $|2|$-grading of $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, defined by some $\Sigma_{i,j}$, such that $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ corresponds to the complexification of a pseudo H-type algebra with the center of dimension d_2.*

Proof.   Since $d_2 = i(n + 1 - j)$ according to Table 2, we see that i must divide d_2 and j can be found as $j = n + 1 - \frac{d_2}{i}$. By the symmetry presented in Table 1, if $D(d_2)$ denotes the number of divisors of d_2, then only the first $\left[\frac{D(d_2)+1}{2}\right]$ divisors i of d_2 have to be analyzed. It is enough to choose n in such a way that

$$d_1 = \left(i + \frac{d_2}{i}\right)\left(n + 1 - \left(i + \frac{d_2}{i}\right)\right)$$

is a multiple of the dimension of some minimal admissible module for a Clifford algebra generated by a d_2-dimensional vector space.   ∎

Corollary 3.5.   *For any $d_2 > 1$, the nilpotent part of the grading of $\mathfrak{sl}(n+1, \mathbb{C})$ determined by $\Sigma_{1,n-d_2}$ satisfies*

$$\dim \mathfrak{g}_{-2} = d_2, \quad and \quad \dim \mathfrak{g}_{-1} = (d_2 + 1)(n - d_2),$$

which correspond to the dimensions d_2 and d_1, respectively, of a pseudo H-type algebra $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$, for an appropriate n.
If d_2 is a prime number, then this is a unique possibility for a pseudo H-type algebra to be the nilpotent part of a grading of $\mathfrak{sl}(n + 1, \mathbb{C})$.

We can use Theorem 3.4 with ease for low dimensions. All the cases for $d_2 = 2, \ldots, 8$ are shown in Table 4.

3.2. The case of B_n

This is the simplest case to analyze, since the only chances of matching the dimensions of the negative part of any $|2|$-grading of $\mathfrak{so}(2n + 1, \mathbb{C})$ with a pseudo H-type Lie algebra is the Heisenberg algebra.

Theorem 3.6.   *For $d_2 > 1$ there are no $|2|$-gradings of $\mathfrak{so}(2n + 1, \mathbb{C})$ such that the dimensions of the negative part match the dimensions of pseudo H-type Lie algebras.*

Table 4: Examples for Theorem 3.4

d_2	(i, j)	d_1	Restrictions
2	$(1, n-1)$	$3(n-2)$	$n > 2$, $n \equiv 2 \mod 4$
3	$(1, n-2)$	$4(n-3)$	$n > 3$
4	$(1, n-3)$	$5(n-4)$	$n > 4$, $n \equiv 4 \mod 8$
	$(2, n-1)$	$4(n-3)$	$n > 3$, n odd
5	$(1, n-4)$	$6(n-5)$	$n > 5$, $n \equiv 1 \mod 4$
6	$(1, n-5)$	$7(n-6)$	$n > 6$, $n \equiv 6 \mod 8$
	$(2, n-2)$	$5(n-4)$	$n > 4$, $n \equiv 4 \mod 8$
7	$(1, n-6)$	$8(n-7)$	$n > 8$
8	$(1, n-8)$	$9(n-8)$	$n > 9$, $n \equiv 0 \mod 8$
	$(2, n-3)$	$6(n-5)$	$n > 5$, $n \equiv 1 \mod 4$

Proof.   Table 2 states that $d_1 = i(2(n-i)+1)$. The number $2(n-i)+1$ is odd, therefore i must be divisible by 4. We write $i = u2^r$, where $r \geq 2$ and u is odd. Then, for $r \geq 2$, we have

$$d_2 = \frac{i(i-1)}{2} = u2^{r-1}(u2^r - 1) \geq 2^{r-1}(2^r - 1) \geq 2r + 2 \geq \rho(2^r) = \rho(d_1),$$

where we write $d_1 = u(2(n-i)+1)2^r$. Since d_1 and d_2 must satisfy the necessary condition (3), we obtain a contradiction.   ∎

3.3. The case of C_n

In Table 5 one can find the possible candidates for a pseudo H-type Lie algebra which corresponds to the nilpotent part of a $|2|$-grading of $\mathfrak{sp}(2n, \mathbb{C})$ with $d_2 > 1$. Some explanations are needed in order to understand how Table 5 was obtained. The numbering in the second row and in the first column follow because all dimensions d_2 are triangular numbers, and the hypothesis $d_2 > 1$ implies that d_1 must be a multiple of four. The numbers that appear in Table 5 are solutions for n to the dimension equations

$$d_1 = 2i(n-i), \quad d_2 = \frac{i(i+1)}{2}.$$

The numbers in boldface are those values of n for which, in addition, the Radon-Hurwitz-Eckmann condition of equation (3) holds. Once a number appears in boldface in a column, say in dimension $d_1 = d$, the corresponding n also appears in boldface in each d_1 multiple of d. Once a number appears in boldface in a row, all numbers to the left also appear in boldface.

We can say more for $|2|$-gradings determined by certain special choices of roots Σ_i of the algebra $\mathfrak{sp}(2n, \mathbb{C})$.

Table 5: Possible pseudo H-type algebras in the grading of $\mathfrak{sp}(2n,\mathbb{C})$

i	2	3	4	5	6	7	8	9	10
$d_{-1} \setminus d_{-2}$	3	6	10	15	21	28	36	45	55
4	**3**	–	–	–	–	–	–	–	–
8	**4**	–	–	–	–	–	–	–	–
12	**5**	5	–	–	–	–	–	–	–
16	**6**	–	6	–	–	–	–	–	–
20	**7**	–	–	7	–	–	–	–	–
24	**8**	7	7	–	8	–	–	–	–
28	**9**	–	–	–	–	–	–	–	–
32	**10**	–	8	–	–	–	–	–	–
36	**11**	9	–	–	9	–	–	–	–
40	**12**	–	9	9	–	–	–	–	–
⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮
64	**18**	–	**12**	–	–	–	12	–	–
⋮	⋮			⋮	⋮	⋮	⋮	⋮	⋮
640	**162**	–	**84**	**69**	–	–	48	–	42

Theorem 3.7. *For the $|2|$-grading of $\mathfrak{sp}(2n,\mathbb{C})$ determined by the choice of root Σ_i, the following holds:*

(1) *If $n-i$ is an odd number, then only pseudo H-type algebras with $\dim \mathfrak{n}_{-2} = 3$ can match the dimensions of the grading where d_1 is divisible by 4 and this occurs for $i = 2$.*

(2) *If $n-i$ is an even number, then there are three possibilities:*
  - *$n = v2^r + 1$, then $i = u2^r + 1$, $r \geq 1$;*
  - *$n = v2^r$, then $i = u2^r$, $r > 1$;*
  - *$n = 2v$, then $i = 2$,*

where u and v are odd numbers.

Proof. Suppose $n-i$ is odd and $i \geq 2$. Then i can be written as $i = u2^r$, where u is odd. Since $d_1 = 2i(n-i)$ must be divisible by four, we deduce that $r \geq 1$. It follows that $d_1 = v2^{r+1}$, where $v = (n-i)u$ is an odd number. Therefore, using the Radon-Hurwitz-Eckmann function, we obtain

$$\rho(d_1) = \rho(2^{r+1}) \leq 2(r+1) + 2 = 2r + 4.$$

On the other hand, $d_2 = \dfrac{i(i+1)}{2} = u2^{r-1}(u2^r + 1) \geq 2^{r+1}(2^r + 1)$.

It follows that $d_2 > (2r+4) - 1$, for $r \geq 2$, contradicting inequality (3).

If $r = 1$, then $d_2 = u(2u+1)$, but $\rho(d_1) = \rho(4v) = \rho(4) = 4$. Thus $d_2 > 3$ for all $u > 1$, which again contradicts inequality (3).

The remaining case corresponds to $i = 2$, $d_2 = 3$, and d_1 divisible by 4.

Suppose now that $n - i$ is even. Then the proof for the rest of the cases is the following.

- Let $n = v2^s + 1$ and $i = u2^r + 1$, where $1 \leq r < s$ are integers and u and v are odd. We have that $d_1 = 2^{r+1}(u2^r + 1)(v2^{s-r} - u)$, and therefore, $\rho(d_1) = \rho(2^{r+1}) \leq 2(r + 1) + 2$. On the other hand,

$$d_2 = (u2^r + 1)(u2^{r-1} + 1) \geq (2^r + 1)(2^{r-1} + 1) > 2(r + 1) - 1 \geq \rho(d_1) - 1,$$

  which contradicts inequality (3).

- Let $n = v2^s$ and $i = u2^r$ for integers r, s, $1 \leq r < s$ and odd u and v. We have that $d_1 = 2^{2r+1}u(v2^{s-r} - u)$, and therefore $\rho(d_1) = \rho(2^{2r+1}) \leq 2(2r + 1) + 2$. On the other hand,

$$d_2 = u2^{r-1}(u2^r + 1) \geq 2^{r-1}(2^r + 1) > 4r + 3 \geq \rho(d_1) - 1,$$

  for all $r > 2$, which contradicts inequality (3). For $r = 2$ we have that $\rho(d_1) = \rho(32) = 10$, but $d_2 \geq 10 > \rho(d_1) - 1 = 9$, which contradicts inequality (3).

- In the case $r = 1$, we have that $i = 2u$ and $d_1 = 8u(v2^{s-1} - u)$, thus, $\rho(d_1) = 8$ and $d_2 = u(2u + 1) \leq \rho(d_1) - 1 = 7$ only for $u = 1$.

If $s < r$, then all previous inequalities become even stronger. ∎

3.4. The case of D_n

Theorem 3.8. *Under the assumption $d_2 > 1$, the only possible pseudo H-type algebra occurs for $\Sigma_{4,5}$ among $|2|$-gradings of $\mathfrak{so}(2n, \mathbb{C})$ defined by the choice of roots $\Sigma_{n-1,n}$ and $\Sigma_{1,n}$.*

Proof.   In the case $\Sigma_{n-1,n}$, the dimensions are $d_1 = 2(n-1)$ and $d_2 = \frac{(n-1)(n-2)}{2}$, following Table 2. Let $2(n-1) = u2^r$, where u is odd. Let us study four different cases, according to whether $r > 3$ or $r = 1, 2, 3$.

- Assuming that $r > 3$ we have that

$$d_2 \geq 2^{r-2}(2^{r-1} - 1) \geq 2r + 2 \geq \rho(2^r) = \rho(d_1).$$

- If $r = 1$, then $d_1 = 2(n - 1) = 2u$ implies that $n - 1 = u \geq 3$. Thus, $d_2 = \frac{1}{2}u(u - 1) \geq 2 = \rho(2u) = \rho(d_1)$.

- If $r = 2$, then $d_1 = 4u$ and $n - 1 = 2u \geq 3$. Then

$$d_2 = u(2u - 1) \geq 4 = \rho(4u) = \rho(d_1).$$

- If $r = 3$, then $d_1 = 8u$ and $n - 1 = 4u \geq 3$. Thus, for $u \geq 3$, we have that $d_2 = 2u(4u - 1) \geq 8 = \rho(8u) = \rho(d_1)$.

In all the cases above, we obtained a contradiction with the necessary condition (3). It remains to consider the case $r = 3$, $u = 1$; therefore, $d_1 = 8$, which implies $n = 5$. This is our exceptional case.

In the case $\Sigma_{1,n}$, the dimensions are $d_1 = \frac{n(n-1)}{2}$ and $d_2 = n - 1$. Let $d_1 = u2^r$, for $r \geq 2$. Consider two possibilities

- If n is even, then $n = v2^{r+1}$, with v odd. Thus,

$$d_2 = v2^{r+1} - 1 \geq 2r + 2 \geq \rho(2^r) = \rho(d_1).$$

- If $n - 1$ is even, then $n - 1 = v2^{r+1}$, with v odd. Then

$$d_2 = v2^{r+1} \geq 2r + 2 \geq \rho(2^r) = \rho(d_1).$$

So we obtain again a contradiction with the necessary condition (3). ∎

If we choose the roots $\Sigma_i = \{\alpha_i\}$, $i = 2, \ldots, n - 2$, then the table of allowed dimensions for the negative nilpotent part of $|2|$-gradings of $\mathfrak{so}(2n, \mathbb{C})$ with respect to the necessary condition (3) is the same as Table 5, with the shift of 1 to the left in the row of i. This is easy to see comparing the C_n case with D_n in Table 2. Thus, Theorem 3.7 holds true for $\mathfrak{so}(2n, \mathbb{C})$.

Table 6: Dynkin diagrams and choices of Σ for $|2|$-gradings of the exceptional Lie algebras

Algebra	Σ	Dynkin diagram
E_6	$\Sigma_2 = \{\alpha_2\}$ $\Sigma_3 = \{\alpha_3\}$ $\Sigma_{1,6} = \{\alpha_1, \alpha_6\}$	
E_7	$\Sigma_1 = \{\alpha_1\}$ $\Sigma_2 = \{\alpha_2\}$ $\Sigma_6 = \{\alpha_6\}$	
E_8	$\Sigma_1 = \{\alpha_1\}$ $\Sigma_8 = \{\alpha_8\}$	
F_4	$\Sigma_1 = \{\alpha_1\}$ $\Sigma_4 = \{\alpha_4\}$	
G_2	$\Sigma_2 = \{\alpha_2\}$	

4.   Complexified pseudo H-type algebras in $|2|$-gradings of the exceptional Lie algebras

We start by constructing $|2|$-gradings for the exceptional Lie algebras, using the conventions from [6]. The definitions presented in Section 2 show that the only

possibilities for $|2|$-gradings of the five exceptional Lie algebras are given in Table 6. To obtain the dimension of $\mathfrak{g}_{-k}$, $k = 1, 2$, of the $|2|$-grading it suffices to count the number of roots whose root spaces generate $\mathfrak{g}_{-k}$, and this is done with the tables of roots for exceptional Lie algebras in [6]. The results are summarized in Table 7, see also partial results in [26]. Note that each exceptional Lie algebra contains a unique contact grading, as it is stated in Theorem 3.2.

Table 7: Choice of Σ and dimensions for $|2|$-gradings of the exceptional Lie algebras

Algebra	Σ	$\dim(\mathfrak{g}_{-1})$	$\dim(\mathfrak{g}_{-2})$
	Σ_2	20	1
E_6	Σ_3	20	5
	$\Sigma_{1,6}$	**16**	**8**
	Σ_1	32	1
E_7	Σ_2	35	7
	Σ_6	32	10
E_8	Σ_1	64	14
E_8	Σ_8	56	1
F_4	Σ_1	14	1
F_4	Σ_4	**8**	**7**
G_2	Σ_2	4	1

Theorem 4.1.   *Only $(E_6, \Sigma_{1,6})$ and (F_4, Σ_4) are the exceptional simple graded Lie algebras $\mathfrak{g} = \bigoplus \mathfrak{g}_p$ whose negative part $\mathfrak{g}_-$ is the complexification $\mathfrak{n} \otimes \mathbb{C}$ of a pseudo H-type Lie algebra $\mathfrak{n} = \mathfrak{n}_{-2} + \mathfrak{n}_{-1}$ with $\dim \mathfrak{n}_{-2} > 1$.*

Proof.   It follows from Table 7 and from the Radon-Hurwitz-Eckmann inequality (3) that only $(E_6, \Sigma_{1,6})$ and (F_4, Σ_4) are the exceptional simple graded Lie algebras that satisfy the necessary conditions. In fact, among the graded exceptional Lie algebras in Table 7, the contact gradings are excluded from the candidates. Computing the number $\rho(\dim \mathfrak{g}_{-1}) - 1 - \dim \mathfrak{g}_{-2}$ for the other gradings, we note that only $(E_6, \Sigma_{1,6})$ and (F_4, Σ_4) satisfy the Radon-Hurwitz-Eckmann inequality (3), since $\rho(x) = 4, 9, 1, 10, 12, 8$ for $x = 20, 16, 15, 32, 64, 8$.

Now we show by a computer-aided proof that there actually exist pseudo H-type algebras satisfying Theorem 4.1. The prolongation $\text{Prol}(\mathfrak{n}^{8,0})$ of $\mathfrak{n}^{8,0}$ is computed by making use of the Maple program [TanakaProlongation]. It was found that the growth vector of $\text{Prol}(\mathfrak{n}^{8,0})$ is $(8, 16, 30, 16, 8)$ and that the prolongation is semi-simple. Since the minimal admissible module of $\text{Cl}_{8,0}$ is irreducible, the prolongation $\text{Prol}(\mathfrak{n}^{8,0})$ is simple. Due to the fact that a complex simple graded Lie algebra having this growth vector is only $(E_6, \Sigma_{1,6})$, we conclude that the prolongation $\text{Prol}(\mathfrak{n}^{8,0})$ is a real form of $(E_6, \Sigma_{1,6})$.

Computing the prolongations of $\mathfrak{n}^{7,1}$, $\mathfrak{n}^{4,4}$, and $\mathfrak{n}^{3,5}$, one finds that $\text{Prol}(\mathfrak{n}^{7,1})$, $\text{Prol}(\mathfrak{n}^{4,4})$, and $\text{Prol}(\mathfrak{n}^{3,5})$ are also real forms of $(E_6, \Sigma_{1,6})$.

Again by the Maple calculations one can see that the growth vectors of prolongations $\mathrm{Prol}(\mathfrak{n}^{7,0})$ and $\mathrm{Prol}(\mathfrak{n}^{3,4})$ are $(7, 8, 22, 8, 7)$, from which we conclude that $\mathrm{Prol}(\mathfrak{n}^{7,0})$ and $\mathrm{Prol}(\mathfrak{n}^{3,4})$ are real forms of (F_4, Σ_4).   ■

5.   A real example: $\mathfrak{n}^{1,3}$ inside $\mathfrak{su}(3,3)$

Consider the Lie algebra $\mathfrak{su}(3,3)$, which is a real form of the complex simple Lie algebra $\mathfrak{sl}(6, \mathbb{C})$, defined as

$$\mathfrak{su}(3,3) = \{X \in \mathfrak{gl}(6, \mathbb{C}) \colon X + \sigma X^* \sigma = 0\},$$

where X^* denotes the transposed conjugate matrix of X and

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The $|2|$-grading of $\mathfrak{sl}(6, \mathbb{C})$ induced by the choice of roots $\Sigma_{2,4}$ gives a nilpotent Lie algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ with dimensions $\dim \mathfrak{g}_{-1} = 8$ and $\dim \mathfrak{g}_{-2} = 4$. This grading has its natural counterpart $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ in the grading of the real Lie algbra $\mathfrak{su}(3,3)$ satisfying the following statement.

Theorem 5.1.   *The graded 2-step nilpotent Lie algebra $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ is isomorphic to the pseudo H-type algebra $\mathfrak{n}^{1,3}$ with $\mathfrak{n}_{-1}$ as the minimal admissible module of $\mathrm{Cl}(\mathfrak{n}_{-2}, \langle .,. \rangle_{\mathfrak{n}_{-2}})$, where $\langle .,. \rangle_{\mathfrak{n}_{-2}}$ is the scalar product of signature $(1,3)$.*

Proof.   Let E_{ij} denote the (6×6)-matrix with 1 at the position (i, j) and 0 elsewhere. The set of matrices

$$\begin{aligned} e_1 &= E_{41} - E_{63} + E_{32} - E_{54}, & e_2 &= i(E_{41} + E_{63} - E_{32} - E_{54}), \\ e_3 &= E_{31} - E_{64} - E_{42} + E_{53}, & e_4 &= i(E_{31} + E_{64} + E_{42} + E_{53}), \\ e_5 &= E_{41} - E_{63} - E_{32} + E_{54}, & e_6 &= i(E_{41} + E_{63} + E_{32} + E_{54}), \\ e_7 &= E_{31} - E_{64} + E_{42} - E_{53}, & e_8 &= i(E_{31} + E_{64} - E_{42} - E_{53}), \end{aligned} \tag{4}$$

is a basis of $\mathfrak{n}_{-1}$, and

$$\begin{aligned} z_1 &= 2(E_{51} - E_{62}), & z_2 &= 2i(E_{51} + E_{62}), \\ z_3 &= 2i(E_{52} - E_{61}), & z_4 &= 2i(E_{52} + E_{61}), \end{aligned} \tag{5}$$

is a basis of $\mathfrak{n}_{-2}$. Consider the scalar product $\langle .,. \rangle_{\mathfrak{n}_{-1}}$ on $\mathfrak{n}_{-1}$ by declaring the basis (4) to be orthonormal, where

$$\langle e_1, e_1 \rangle_{\mathfrak{n}_{-1}} = \langle e_2, e_2 \rangle_{\mathfrak{n}_{-1}} = \langle e_3, e_3 \rangle_{\mathfrak{n}_{-1}} = \langle e_4, e_4 \rangle_{\mathfrak{n}_{-1}} = 1,$$

$$\langle e_5, e_5 \rangle_{\mathfrak{n}_{-1}} = \langle e_6, e_6 \rangle_{\mathfrak{n}_{-1}} = \langle e_7, e_7 \rangle_{\mathfrak{n}_{-1}} = \langle e_8, e_8 \rangle_{\mathfrak{n}_{-1}} = -1.$$

Similarly, consider a scalar product $\langle .\,,.\rangle_{\mathfrak{n}_{-2}}$ on $\mathfrak{n}_{-2}$ by declaring the basis (5) to be orthonormal. Namely

$$-\langle z_1, z_1\rangle_{\mathfrak{n}_{-2}} = -\langle z_2, z_2\rangle_{\mathfrak{n}_{-2}} = -\langle z_3, z_3\rangle_{\mathfrak{n}_{-2}} = \langle z_4, z_4\rangle_{\mathfrak{n}_{-2}} = 1.$$

The following is the commutator table of $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ in terms of the above chosen basis $\{e_1, \ldots, e_8, z_1, \ldots, z_4\}$, using the usual commutators of matrices:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	0	0	$-z_4$	$-z_1$	$-z_2$	0	z_3
e_2	0	0	z_4	0	$-z_2$	z_1	$-z_3$	0
e_3	0	$-z_4$	0	0	0	z_3	z_1	z_2
e_4	z_4	0	0	0	$-z_3$	0	z_2	$-z_1$
e_5	z_1	z_2	0	z_3	0	0	0	$-z_4$
e_6	z_2	$-z_1$	z_3	0	0	0	z_4	0
e_7	0	z_3	$-z_1$	$-z_2$	0	$-z_4$	0	0
e_8	$-z_3$	0	$-z_2$	z_1	z_4	0	0	0

For each $z \in \mathfrak{n}_{-2}$, we define the map $J_z \in \operatorname{End}(\mathfrak{n}_{-1})$ by means of identity (1). In terms of the given basis of $\mathfrak{n}_{-1}$, we can easily compute the following matrix representation of J_{z_i}, $i = 1, 2, 3, 4$.

$$J_{z_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_{z_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_{z_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_{z_4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Simple computations show that

$$J_{z_1}^2 = J_{z_2}^2 = J_{z_3}^2 = -J_{z_4}^2 = \operatorname{Id}_{\mathfrak{n}_{-1}} \quad \text{and} \quad J_{z_i} J_{z_j} + J_{z_j} J_{z_i} = 0 \quad \text{for} \quad i \neq j.$$

This implies that $J_z^2 = -\langle z, z\rangle_{\mathfrak{n}_{-2}} \operatorname{Id}_{\mathfrak{n}_{-1}}$, for all $z \in \mathfrak{n}_{-2}$, and thus, the 2-step nilpotent Lie algebra $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ is the pseudo H-type algebra $\mathfrak{n}^{1,3}$. ∎

6.   Uniqueness of the Heisenberg algebra in $\mathfrak{sl}(n+1, \mathbb{R})$

The aim of this section is to prove that among all $|2|$-gradings $\mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ of $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ the only H-type algebras with a positive definite scalar product that appear as $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ are the Heisenberg algebras. To prove this, we will use an alternative definition of H-type algebras found in [18]. A real graded 2-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ endowed with a positive definite scalar product $\langle .,. \rangle$ is of H-type if and only if the map $\mathrm{ad}_x \colon \ker(\mathrm{ad}_x)^{\perp} \to \mathfrak{n}_{-2}$ is a surjective isometry for any $x \in \mathfrak{n}_{-1}$ with $\langle x, x \rangle = 1$, where $\perp$ stands for the orthogonal complement in $\mathfrak{n}_{-1}$.

Let us assume that the $|2|$-grading is induced by the choice of roots $\Sigma_{i,j}$. With analogous notation as in Section 5, define the following subspaces of $\mathfrak{g}_{-1}$

$$\mathfrak{g}^L_{-1} = \mathrm{span}\{E_{k\ell} \colon i < k \le j, 1 \le \ell \le i\},$$

$$\mathfrak{g}^R_{-1} = \mathrm{span}\{E_{k\ell} \colon j < k \le n+1, i < \ell \le j\}.$$

It is easy to see that $\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$, as vector spaces, and that

$$\dim \mathfrak{g}^L_{-1} = i(j-i), \quad \dim \mathfrak{g}^R_{-1} = (n+1-j)(j-i).$$

It follows immediately that $\dim \mathfrak{g}_{-1} = (n+1-(j-i))(j-i)$. We state a result concerning the products and matrix commutators of the elementary matrices $E_{k\ell}$. The proof follows from simple computations.

Lemma 6.1.   *The product of two elementary matrices is given by*

$$E_{pq}E_{rs} = \begin{cases} E_{ps}, & \textit{if } q = r; \\ 0, & \textit{otherwise.} \end{cases}$$

The commutator between elementary matrices is given by

$$[E_{pq}, E_{rs}] = E_{pq}E_{rs} - E_{rs}E_{pq} = \begin{cases} E_{ps}, & \textit{if } p \ne s, q = r; \\ -E_{rq}, & \textit{if } p = s, q \ne r; \\ E_{pp} - E_{qq}, & \textit{if } p = s, q = r; \\ 0, & \textit{otherwise.} \end{cases} \tag{6}$$

With Lemma 6.1 at hand, we can conclude that the subspaces $\mathfrak{g}^L_{-1}$ and $\mathfrak{g}^R_{-1}$ have the following structure.

Corollary 6.2.   *The subspaces $\mathfrak{g}^L_{-1}$ and $\mathfrak{g}^R_{-1}$, endowed with the usual matrix commutator, form abelian Lie subalgebras of $\mathfrak{g}_{-1}$.*

Since the 2-step nilpotent Lie algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ consists of lower triangular matrices, we have the following result.

Lemma 6.3.   *The following equalities hold:*

(1)   *If $E_{k\ell} \in \mathfrak{g}^R_{-1}$, then the dimension of the image of $\mathrm{ad}_{E_{k\ell}}$ is i.*

(2)   *If $E_{k\ell} \in \mathfrak{g}^L_{-1}$, then the dimension of the image of $\mathrm{ad}_{E_{k\ell}}$ is $n+1-j$.*

Proof.   Consider a positive definite scalar product such that the elementary matrices $E_{k\ell}$ form an orthonormal set. Let $E_{k\ell} \in \mathfrak{g}_{-1}$. It is clear from (6), that $\ker(\mathrm{ad}_{E_{k\ell}}) = \mathrm{span}\{E_{pq}\colon p \neq \ell \text{ and } q \neq k\}$. From this equality we can deduce that

$$\ker(\mathrm{ad}_{E_{k\ell}})^\perp = \mathrm{span}\{E_{pq}\colon p = \ell \text{ or } q = k\}.$$

It is easy to see that if $E_{k\ell} \in \mathfrak{g}_{-1}^L$, then

$$\ker(\mathrm{ad}_{E_{k\ell}})^\perp \cap \mathfrak{g}_{-1} = \mathrm{span}\{E_{pk}\colon p = j+1, \ldots, n+1\},$$

and if $E_{k\ell} \in \mathfrak{g}_{-1}^R$, then $\ker(\mathrm{ad}_{E_{k\ell}})^\perp \cap \mathfrak{g}_{-1} = \mathrm{span}\{E_{\ell q}\colon q = 1, \ldots, i\}$. Counting dimensions, the claim follows. ∎

Theorem 6.4.   *The only graded subalgebras of $\mathfrak{sl}(n+1, \mathbb{R})$, which are H-type algebras with a positive definite scalar product are the Heisenberg algebras of dimension $2n-1$.*

Proof.   The map $\mathrm{ad}_{E_{k\ell}}\colon \ker(\mathrm{ad}_{E_{k,\ell}})^\perp \cap \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$, using the notation of Lemma 6.3, is surjective for any $E_{k\ell} \in \mathfrak{g}_{-1}$ only when $i = 1$ and $j = n-1$. ∎

7.   Prolongations of $\mathfrak{n}^{r,s}$

We recall some definitions and basic facts on prolongations of graded Lie algebras following [21]. By a graded Lie algebra we mean a Lie algebra $\mathfrak{g}$ endowed with a $\mathbb{Z}$-grading: $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. We denote the negative part of $\mathfrak{g}$ by $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$. A graded Lie algebra $\mathfrak{g}$ is called *transitive* if $\dim \mathfrak{g}_- < \infty$ and if it satisfies the following condition:

$$\text{if } p \geq 0, \ x \in \mathfrak{g}_p, \text{ and } [x, \mathfrak{g}_-] = 0, \text{then } x = 0.$$

The largest integer μ such that $\mathfrak{g}_{-\mu} \neq 0$ is called the depth of $\mathfrak{g}$. The vector

$$(\dim \mathfrak{g}_{-\mu}, \dim \mathfrak{g}_{-\mu+1}, \ldots, \dim \mathfrak{g}_p, \ldots)$$

is called the growth vector of $\mathfrak{g}$.

Given a transitive graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, then for each $k \geq 0$ there exists a unique maximal transitive graded Lie algebra $\tilde{\mathfrak{g}} = \bigoplus_{p \in \mathbb{Z}} \tilde{\mathfrak{g}}_p$ up to isomorphisms such that $\tilde{\mathfrak{g}}_p = \mathfrak{g}_p$ for $p < k$, which is called the prolongation (or Tanaka prolongation) of the truncated graded Lie algebra $\bigoplus_{p<k} \mathfrak{g}_p$. The prolongation $\tilde{\mathfrak{g}}$ is determined inductively by

$$\tilde{\mathfrak{g}}_{p+1} = \{z \in \mathrm{Hom}(\mathfrak{g}_-, \tilde{\mathfrak{g}})_{p+1}| \ [z(x), y] + [x, z(y)] = z([x, y]), \text{ for all } x, y \in \mathfrak{g}_-\},$$

where $\mathrm{Hom}(*, *)_p$ denotes the set of homogeneous degree p linear maps. The bracket of $\tilde{\mathfrak{g}}$ is defined inductively to satisfy the Jacobi identity and $z(x) = [z, x]$ for $z \in \tilde{\mathfrak{g}}_q$, $x \in \mathfrak{g}_-$. The notion of prolongation of depth $\mu = 1$, had been fundamental in geometry and in the theory of partial differential equations, see for instance [16]. This was generalized to the case of depth greater than 1 in [27] and has played important role in nilpotent geometry and analysis [23].

Note that a transitive graded Lie algebra $\mathfrak{g}$ is the prolongation of a truncated graded algebra $\bigoplus_{p<k} \mathfrak{g}_p$ if and only if the generalized Spencer cohomology group $H^1_r(\mathfrak{g}_-, \mathfrak{g})$ associated with the representation of $\mathfrak{g}_-$ on $\mathfrak{g}$ vanishes for $r \geq k$.

In the case of a simple Lie algebra $\mathfrak{g}$, the Spencer cohomology group $H^p_r(\mathfrak{g}_-, \mathfrak{g})$ may be computed by the method of Kostant [20], and Yamaguchi [30] carried out the computation for H^1, H^2, according to which a simple graded Lie algebra is a prolongation of $\mathfrak{g}_-$ except for a few cases.

If a transitive graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is simple and finite dimensional, then the Killing form B of $\mathfrak{g}$ satisfies $B(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ if $p + q \neq 0$, and therefore, the grading is symmetric; that is B is a non-degenerate pairing between $\mathfrak{g}_p$ and $\mathfrak{g}_{-p}$.

The program [TanakaProlongation] in the Maple Software elaborated by Anderson [4] allows one to calculate the prolongation of a truncated transitive graded Lie algebra $\bigoplus_{p<k} \mathfrak{g}_p$. If one enters into the Maple the data of a truncated graded Lie algebra: a basis and the structure constants, then the [TanakaProlongation] gives back a basis and the structure equations of the prolongation. In particular, we get the dimension of the prolongation at each order.

We used Maple to calculate the prolongation of the basic pseudo H-type Lie algebras $\mathfrak{n}^{r,s} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1} = \mathbb{R}^{r,s} \oplus \mathfrak{n}_{-1}$ for $r + s \leq 8$, where $\mathfrak{n}_{-1}$ is the minimal dimensional admissible module of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s}, \langle .,. \rangle_{r,s})$. The Lie algebra data of $\mathfrak{n}^{r,s}$ are taken from the tables of structure constants found in [12]. We collect the results of computation in Tables 8 and 9, where $\mathfrak{g} = \mathrm{Prol}(\mathfrak{n})$ is the prolongation of a pseudo H-type Lie algebra $\mathfrak{n}$.

Table 8: Prolongation data for $n^{r,s}$ with $\dim \mathfrak{n}_{-2} = 1, 2, 3, 4$.

$\dim \mathfrak{n}_{-2}$	$\mathfrak{n}$	Growth vector of $\mathfrak{g}$	$\mathfrak{g}$	$\mathfrak{g} \otimes \mathbb{C}$
1	$\mathfrak{n}^{1,0}$	$(1, 2, 4, 6, 9, \dots)$	$\mathrm{ct}(3, \mathbb{R})$	$\mathrm{ct}(3, \mathbb{C})$
	$\mathfrak{n}^{0,1}$	$(1, 2, 4, 6, 9, \dots)$	$\mathrm{ct}(3, \mathbb{R})$	$\mathrm{ct}(3, \mathbb{C})$
2	$\mathfrak{n}^{2,0}$	$(2, 4, 8, 12, 18, \dots)$	$\mathrm{ct}(3, \mathbb{C})_{\mathbb{R}}$	$\mathrm{ct}(3, \mathbb{C}) \oplus \mathrm{ct}(3, \mathbb{C})$
	$\mathfrak{n}^{1,1}$	$(2, 4, 8, 12, 18, \dots)$	$\mathrm{ct}(3, \mathbb{R}) \oplus \mathrm{ct}(3, \mathbb{R})$	$\mathrm{ct}(3, \mathbb{C}) \oplus \mathrm{ct}(3, \mathbb{C})$
	$\mathfrak{n}^{0,2}$	$(2, 4, 8, 12, 18, \dots)$	$\mathrm{ct}(3, \mathbb{C})_{\mathbb{R}}$	$\mathrm{ct}(3, \mathbb{C}) \oplus \mathrm{ct}(3, \mathbb{C})$
3	$\mathfrak{n}^{3,0}$	$(3, 4, 7, 4, 3)$	$\mathfrak{sp}(2, 1)$	$\mathfrak{sp}(6, \mathbb{C})$
	$\mathfrak{n}^{2,1}$	$(3, 8, 14, 8, 3)$	$\mathfrak{sp}(8, \mathbb{R})$	$\mathfrak{sp}(8, \mathbb{C})$
	$\mathfrak{n}^{1,2}$	$(3, 4, 7, 4, 3)$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{sp}(6, \mathbb{C})$
	$\mathfrak{n}^{0,3}$	$(3, 8, 14, 8, 3)$	$\mathfrak{sp}(2, 2)$	$\mathfrak{sp}(8, \mathbb{C})$
4	$\mathfrak{n}^{4,0}$	$(4, 8, 11, 8, 4)$	$\mathfrak{sl}(3, \mathbb{H})$	$\mathfrak{sl}(6, \mathbb{C})$
	$\mathfrak{n}^{3,1}$	$(4, 8, 11, 8, 4)$	$\mathfrak{su}(4, 2)$	$\mathfrak{sl}(6, \mathbb{C})$
	$\mathfrak{n}^{2,2}$	$(4, 8, 11, 8, 4)$	$\mathfrak{sl}(6, \mathbb{R})$	$\mathfrak{sl}(6, \mathbb{C})$
	$\mathfrak{n}^{1,3}$	$(4, 8, 11, 8, 4)$	$\mathfrak{su}(3, 3)$	$\mathfrak{sl}(6, \mathbb{C})$
	$\mathfrak{n}^{0,4}$	$(4, 8, 11, 8, 4)$	$\mathfrak{sl}(3, \mathbb{H})$	$\mathfrak{sl}(6, \mathbb{C})$

Table 9: Growth vectors for the prolongation of some pseudo H-type algebras with $\dim \mathfrak{n}_{-2} = 5, 6, 7, 8$.

$\dim \mathfrak{n}_{-2}$	$\mathfrak{n}$	Growth vector of $\mathfrak{g}$	$\mathfrak{g}$	$\mathfrak{g} \otimes \mathbb{C}$
5	$\mathfrak{n}^{5,0}$	$(5, 8, 12)$		
	$\mathfrak{n}^{4,1}$	$(5, 16, 17)$		
	$\mathfrak{n}^{3,2}$	$(5, 8, 12)$		
	$\mathfrak{n}^{2,3}$	$(5, 8, 12)$		
	$\mathfrak{n}^{1,4}$	$(5, 8, 12)$		
	$\mathfrak{n}^{0,5}$	$(5, 16, 17)$		
6	$\mathfrak{n}^{6,0}$	$(6, 8, 16)$		
	$\mathfrak{n}^{5,1}$	$(6, 16, 18)$		
	$\mathfrak{n}^{4,2}$	$(6, 16, 18)$		
	$\mathfrak{n}^{3,3}$	$(6, 8, 16)$		
	$\mathfrak{n}^{2,4}$	$(6, 8, 16)$		
	$\mathfrak{n}^{1,5}$	$(6, 16, 18)$		
	$\mathfrak{n}^{0,6}$	$(6, 16, 18)$		
7	$\mathfrak{n}^{7,0}$	$(7, 8, 22, 8, 7)$	F_{II}	F_4
	$\mathfrak{n}^{3,4}$	$(7, 8, 22, 8, 7)$	F_{I}	F_4
8	$\mathfrak{n}^{8,0}$	$(8, 16, 30, 16, 8)$	E_{IV}	E_6
	$\mathfrak{n}^{7,1}$	$(8, 16, 30, 16, 8)$	E_{III}	E_6
	$\mathfrak{n}^{4,4}$	$(8, 16, 30, 16, 8)$	E_{I}	E_6
	$\mathfrak{n}^{3,5}$	$(8, 16, 30, 16, 8)$	E_{II}	E_6

We finish our work stating several remarks.

1. The form of the growth vector of $\mathfrak{g}$ proposes a conjecture whether $\mathfrak{g}$ is simple or not, and then this can be verified rigorously. If it is simple, then one can identify the complex simple Lie algebra such that it is isomorphic to the complexification of $\mathfrak{g}$. However, it is not easy to identify the real form of $\mathfrak{g}$ related to the pseudo H-type Lie algebra $\mathfrak{n}$. This requires a careful case by case analysis.

2. If $\dim \mathfrak{n}_{-2}$ is equal to 1 or 2, then the pseudo H-type Lie algebra is related to the contact algebra. We denote by $\mathrm{ct}(2n+1, \mathbb{F})$ the contact algebra over the field $\mathbb{F}$ of degree n. The contact algebra is a simple infinite Lie algebra. It is known that the contact algebra is the prolongation of its negative part which is the Heisenberg algebra [27]. We refer to [21] for more detailed structure of the contact algebra, its gradation and growth vector to compare the Maple computation to any higher order.

3. If $\dim \mathfrak{n}_{-2}$ is equal to 3 or 4, the growth vector of $\mathfrak{g} = \mathrm{Prol}(\mathfrak{n}^{r,s})$ is symmetric for all (r, s) with $r + s = 3$ or 4 and $\mathfrak{g}$ is a simple graded Lie algebra. The complexification $\mathfrak{g} \otimes \mathbb{C}$ is determined as in Table 8, or more precisely, $(\mathfrak{sp}(6, \mathbb{C}), \Sigma_2)$, $(\mathfrak{sp}(8, \mathbb{C}), \Sigma_2)$, $(\mathfrak{sl}(6, \mathbb{C}), \Sigma_{1,2})$, respectively, as graded Lie algebras. The real forms corresponding to $\mathfrak{g}$ were found by a careful study. An explicit isomorphism between $\mathfrak{n}^{1,3}$ and $\mathfrak{su}(3, 3)$ has been described in Section 5. Similar calculations can be performed in other cases.

4. If $\dim \mathfrak{n}_{-2}$ is equal to 5 or 6, the computation shows that the prolongation $\mathfrak{g} = \mathrm{Prol}(\mathfrak{n}^{r,s})$ all vanishes at order 1 for $r + s = 5, 6$ and therefore is not simple. This seems to be a particular case of our general conjecture: *for any pseudo H-type algebra $\mathfrak{n}$, the prolongation $\mathfrak{g} = \mathrm{Prol}(\mathfrak{n})$ vanishes at order 1 if it does not contain a simple graded Lie algebra of depth 2.* As a result of computations, combined with the analysis of Table 2, shows that the negative part of $\mathfrak{so}(2n, \mathbb{C})$ cannot be the complexification of a non-Heisenberg pseudo H-type algebra for $n \leq 6$.

5. In the case $\dim \mathfrak{n}_{-2}$ is equal to 7 or 8, we list in Table 9 the pseudo H-type Lie algebras $\mathfrak{n}^{r,s}$ with $r + s = 7, 8$ for which the growth vector of $\mathfrak{g} = \mathrm{Prol}(\mathfrak{n}^{r,s})$ is symmetric, which means that $\mathfrak{g}$ is simple. The complexifications $\mathfrak{g} \otimes \mathbb{C}$ are determined to be (F_4, Σ_4) and $(E_6, \Sigma_{1,6})$, respectively, as it was explained in the proof of Theorem 4.1. The real simple Lie algebras $\mathfrak{g}$, identified through group theoretic observation, are listed in Table 9. For the notation of real forms in Table 9 see [17, p. 534].

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