On Unitary Representations of
Disconnected Real Reductive Groups

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Abstract. Let G be the real reductive group and let G_0 be the identity component. Let us assume that the unitary dual $\widehat{G_0}$ is known. In this paper (in Sections 6–9) the unitary dual $\widehat{G}$ is constructed. Automorphisms of G_0 generated by elements of G are the main ingredient of the construction. If the automorphism is outer, one has to consider the corresponding intertwining operators S. Operators S and their properties are analyzed in Section 5. Automorphisms of $\mathfrak{g}_0$ are closely related to automorphisms of G_0. They are investigated in Section 3. Automorphisms of $\mathfrak{so}(4,4)$ are analyzed in Section 4.

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1. Introduction

Let G be a real reductive group. The definition of the real reductive group will be introduced in the Section 2. Our definition is more general then the usual one. Namely, we do not require that $\mathrm{Ad}(x)$ is inner automorphism of $\mathfrak{g}$ for all $x \in G$.

The main goal of this paper is to describe the unitary dual of the group G in terms of the unitary dual of the identity component G_0 of G. The natural choice for this construction is the induction from G_0 to G. Since G_0 is a normal subgroup of G, every element of G produces an automorphism of G_0. These automorphisms also play important role in our construction.

In Section 3, we analyze automorphisms of $\mathfrak{g}_0$. Namely, there is isomorphism from Aut G_0 onto the subgroup of Aut $\mathfrak{g}_0$, the group of the automorphisms of $\mathfrak{g}_0$. This subgroup is equal to Aut $\mathfrak{g}_0$, if G_0 is simply connected. For more details, see [3]. Hence, the group Aut $\mathfrak{g}_0$, is close to the group Aut G_0. The most complicated situation (for simple Lie algebras $\mathfrak{g}_0$) appears if $\mathfrak{g}_0$ is equal to $\mathfrak{so}(4,4)$ when Aut $\mathfrak{g}_0$ is equivalent to S_4 and it is analyzed in Section 4. These results are already known but it is hard to find a reference. In particular, we believe that this approach can not be found elsewhere.

Let (π, V) be the representation of G_0 and x representative of some component of G. Then we consider the representation π^x defined by $\pi^x(g) = \pi(x^{-1}gx)$. If π^x is equivalent to π then the corresponding intertwining operator is denoted by S_x. The subgroup of G containing all such elements x is denoted by G_2. Operators S_x naturally lead to coefficients $\beta(x, y)$ which give a better understanding of the structure of G_2. Operators S_x, coefficients $\beta(x, y)$ and some important properties are described in Section 5.

In Section 6 we analyze the induction procedure that goes from G_0 to various subgroups of G. Actually, there is a sequence of subgroups $G_0 \leq G_1 \leq G_2 \leq G$ where G_1 is a subgroup of G such that conjugation by any $x \in G_1$ is equal to conjugation by some $g \in G_0$. In Section 7, we analyze the induction step from G_0 to G_1 and get reducibility in terms of some finite subgroups. This result is mentioned in [10] for discrete series and the request on the Lie group is more restrictive (see also the discussion in the Section 9 of [10]). The next step is to analyze induction that goes from G_1 to G_2. However, it is easier to consider the induction from G_0 to G_2. This time reducibility is described in terms of abstract finite groups and it is derived in Section 8. For this induction step we need intertwining operators S_x and some properties of coefficients $\beta(x, y)$. One could say that Theorem 7.1 is a special case of Theorem 8.1. However, the statement of Theorem 7.1 is more natural. Finally, in Section 9, is shown that $\text{ind}_{G_2}^G \pi$ is irreducible if π is irreducible. It is well known result (see [5]), but we add it in order to complete our picture. Our induction procedure is written in a different way (for example, see [6]), but this terminology is more convenient for calculations. We also add several examples in order to demonstrate our results. These results are also well known, but they describe our theorems and proofs.

The main idea of this paper is to use representations of the group which is slightly bigger (but still finite) then a quotient group G/G_0 in order to describe induced representation $\text{ind}_{G_0}^{G_2} V$ in terms of finite groups. Similar ideas can be found in [9]. Coefficients β (Definition 5.6) and the subgroup F_2 (Section 8) correspond to the group extension mentioned in [9]. Representation η_m of F_0 corresponds to projective representations from [9]. However, this paper is oriented to unitary representations of the disconnect group G and the description is very precise (Theorems 7.1 and 8.1). We also analyze coefficients β and show that $\beta^n = 1$ where $n = |G/G_0|$ (Proposition 5.9). Finally our constructions are very explicit and can be easily performed for small n. Paper [9] is much more general and can not be used to derive (easily) results of this paper. It is oriented to "Mackey theory". In order to understand the flavor of that paper, it is enough to analyze theorems of Section 8 and examples in Section 9.

The results obtained can also be used to analyze multiplicities of irreducible components obtained by parabolic induction (for a different approach, see [7]). There is an example of group G for which multiplicities are 2 (see [7] or [2]). The identity component G_0 has the form $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\pm(I \times I)$ and $G/G_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This example is analyzed in Section 8. Let us denote by ρ the irreducible representation of $P_0 = P \cap G_0$ such that $\text{ind}_{P_0}^{G_0} \rho = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4$. Let us denote by x and y elements of the form $x = \text{diag}(1, -1, 1, 1)$ and $y = \text{diag}(1, 1, 1, -1)$. Let us multiply x by i and y by j in order to obtain $xy = -yx$. Now, $\rho^x \cong \rho$

and $\rho^y \cong \rho$, but $\left(\mathrm{ind}_{P_0}^{P_0 \cup xP_0}\rho\right)^y \not\cong \mathrm{ind}_{P_0}^{P_0 \cup xP_0}\rho$. Hence, $\mathrm{ind}_{P_0}^P \rho = \tau \oplus \tau$. Also, we could use Theorem 7.1 and use that $\mathrm{ind}_{\mathbb{Z}_2}^Q \eta = \zeta \oplus \zeta$ where the quaternion group $Q = Z(G)$, $\mathbb{Z}_2 = Q \cap G_0$ and η is a nontrivial representation. Since, $\pi_1^x \cong \pi_2$, $\pi_1^y \cong \pi_3$ and $\pi_1^{xy} \cong \pi_4$, $\mathrm{ind}_{G_0}^G \pi_l = \sigma$ for $l \in \{1, 2, 3, 4\}$. Now, the induction $\mathrm{ind}_{P_0}^G$ can be described by the following diagram

$$
\begin{array}{ccc}
\rho & \xrightarrow{\mathrm{ind}_{P_0}^{G_0}} & \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4 \\
\Big\downarrow{\mathrm{ind}_{P_0}^P} & & \Big\downarrow{\mathrm{ind}_{G_0}^G} \\
\tau \oplus \tau & \xrightarrow{\mathrm{ind}_P^G} & \sigma \oplus \sigma \oplus \sigma \oplus \sigma.
\end{array}
$$

It follows that $\mathrm{ind}_P^G \tau = \sigma \oplus \sigma$. The terminology is explained in Section 5.

2. Notation

For us, the real reductive group G will be the Lie group G with a compact subgroup K, an involution θ and a nondegenerate, $\mathrm{Ad}(G)$ invariant, θ invariant bilinear form B such that

1. the corresponding Lie algebra $\mathfrak{g}_0$ is a reductive Lie algebra,

2. $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, $\mathfrak{k}_0$ is the Lie algebra of K,

3. $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal under B and B is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$,

4. $G = K \times \exp \mathfrak{p}_0$.

The definition is taken from [8], but we do not take the last condition: every automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is inner for $\mathfrak{g}$. Let G_0 and K_0 be identity components of G and K respectively. The quotient $G/G_0 \cong K/K_0$ is finite since K is compact.

The representation of G will be denoted by π or (π, V). We say that V is a G-module and sometimes write $g.v$ or simply gv instead of $\pi(g)v$. A representation of $\mathfrak{g}$ on V will be denoted by π again. It naturally extends to a homomorphism $\pi : \mathfrak{U}(\mathfrak{g}) \to End(V)$. We say that V is a $\mathfrak{g}$-module and sometimes write $X.v$ instead of $\pi(X)v$.

Let (π, V) be a representation of a real Lie group G. A $(\mathfrak{g}, K)$-module V is a $\mathfrak{g}$-module and K-module such that the following conditions are satisfied:

1. $k.(X.v) = (Ad(k)X).(k.v)$ for $v \in V_F$ where V_F is the set of K-finite differentiable vectors in V. For more details, see [1].

2. The space $\{k.v \mid k \in K\}$ is finite-dimensional for any $v \in V$ and the action is C^∞.

3. $\dfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\exp tX).v = X.v$ for $v \in V_F$.

The first condition can be written in the form

$$X.(k.v) = k.(\mathrm{Ad}(k^{-1})X.v). \tag{1}$$

To each irreducible unitary representation there corresponds $(\mathfrak{g}, K)$-module and vice verse. Hence, the study of irreducible unitary representations can be done via $(\mathfrak{g}, K)$-modules. More details can be found in [1].

3.   Automorphisms of $\mathfrak{g}_0$

We are interested in automorphisms of the Lie group G. These automorphism form a subgroup of automorphisms of the corresponding Lie algebra $\mathfrak{g}_0$. If G_0 is simply connected then the group of automorphisms of G_0 is equal to the group of automorphisms of $\mathfrak{g}_0$ (see [3]). Hence, we will consider a simpler problem: finding automorphisms of $\mathfrak{g}_0$.

For the rest of this section we assume that $\mathfrak{g}_0$ is simple. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition and we fix $\mathfrak{k}_0$ and $\mathfrak{p}_0$. Complexification of $\mathfrak{k}_0$ will be denoted by $\mathfrak{k}$ and complexification of $\mathfrak{p}_0$ by $\mathfrak{p}$.

If $\mathfrak{g}_0$ is compact then $\mathrm{Aut}_{\mathbb{R}}\mathfrak{g}_0/\mathrm{Int}\mathfrak{g}_0$ is isomorphic to the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$. The same is true for $\mathrm{Aut}_{\mathbb{C}}\mathfrak{g}/\mathrm{Int}\mathfrak{g}$. See, for example [8]. For the general $\mathfrak{g}_0$, this group is slightly more complicated.

Theorem 3.1.   *Outer automorphism φ_0 of simple Lie algebra $\mathfrak{g}_0$ can be obtained as follows:*
(a) *if $\mathfrak{k}_0$ contains $\mathbb{R}$ as a summand then $\varphi_0|_{\mathbb{R}} \equiv -1$,*
(b) *if $\mathfrak{k}_0$ contains isomorphic components then φ_0 mixes these components,*
(c) *if $\varphi_0|_{\mathfrak{k}_0} \equiv 1$ then φ is equal to the Cartan involution.*

Proof.   We can assume that $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are fixed under the action of φ_0. Hence, $\varphi_0|_{\mathfrak{k}_0}$ is the automorphism of $\mathfrak{k}_0$ and $\varphi_0|_{\mathfrak{p}_0}$ is the automorphism of $\mathfrak{p}_0$. The complexification of φ_0 will be denoted by φ.

If $\mathfrak{k}_0$ contains $\mathbb{R}$, then $\mathfrak{k}_0 = \mathfrak{k}_0^1 \oplus \mathbb{R}$ for some semisimple algebra $\mathfrak{k}_0^1$. In that situation, G/K is Hermitian, there is multiplication-by-i map J (acting on $\mathfrak{p}_0$ and $\mathfrak{p}$) with the property $J^2 = -1$ and $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{p}^+$ is the $+i$ eigenspace for J and p^- is the $-i$ eigenspace for J. Also, $\mathbb{R}$ is the center of $\mathfrak{k}_0$, φ_0 maps $\mathbb{R}$ to $\mathbb{R}$ and there is $X_0 \in \mathbb{R}$ such that $[X_0, Y] = JY$ for $Y \in \mathfrak{p}$. (For more details, see [8].) Let us denote the complexification of the center $\mathbb{R}$ by $\mathbb{C}$. Then $\varphi(X_0) = \alpha X_0$ for $\alpha \in \mathbb{C}$. Let us write $\varphi(Y) = Z + T$ for $Y, Z \in \mathfrak{p}^+$ and $T \in \mathfrak{p}^-$. Now, $[X_0, Y] = iY$.

The action of φ produces $[\alpha X_0, Z + T] = i(Z + T)$.

Since the left hand side is equal to $\alpha[X_0, Z + T] = i\alpha Z - i\alpha T$, either $\alpha = 1$ and $T = 0$ ($ad(X)$ maps $\mathfrak{p}^+$ to $\mathfrak{p}^+$ and $\mathfrak{p}^-$ to $\mathfrak{p}^-$) or $\alpha = -1$ and $Z = 0$ ($ad(X)$ maps $\mathfrak{p}^+$ to $\mathfrak{p}^-$ and $\mathfrak{p}^-$ to $\mathfrak{p}^+$). It shows that $\varphi(X_0) = \pm X_0$.

Now, let us consider the situation when $\mathfrak{k}_0$ does not contains $\mathbb{R}$ as a component. If $\mathfrak{k}_0$ contains isomorphic components, then outer isomorphism can mix these components. A nice example is given in Section 4.

It remains to consider the case when $\mathfrak{k}_0$ does not contain neither $\mathbb{R}$ nor isomorphic components. Let us consider an outer automorphisms φ_0 such that $\varphi|_{\mathfrak{k}_0} \equiv 1$. Since G/K is not Hermitian, the representation of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible. Schur's lemma shows that φ is a multiplication by $\lambda \in \mathbb{C}$. Since φ is a complexification

of φ_0, $\lambda \in \mathbb{R}$. Finally, take $X, Y \in \mathfrak{p}_0$ such that $[X, Y] \neq 0$. It shows that $\lambda^2 = 1$. ∎

If $G = SL(2, \mathbb{R})$, then the outer automorphism is obtained as conjugation by $\mathrm{diag}(1, -1)$. Now, $\mathfrak{k}_0 = \mathbb{R}$ and $\varphi_0|_{\mathbb{R}} \equiv -1$. If $G = SL(n, \mathbb{R})$, $n > 2$, the outer automorphism is the Cartan involution.

4. Automorphisms of $\mathfrak{so}(4,4)$

Let us recall that

$$\mathfrak{so}(4, 4) = \{X \in \mathfrak{gl}(8, \mathbb{R}) \mid X^* I_{4,4} + I_{4,4} X = 0\} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \tag{2}$$

where A, B and C are 4-by-4 real matrices and A and C are skew-symmetric. Then $\mathfrak{k}_0 = \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3)$. Let us write $\mathfrak{k}_0 = \prod_{i=1}^{4} \mathfrak{so}(3)_i$. Generators of $\mathfrak{so}(3)_1$ and $\mathfrak{so}(3)_2$ are in A (see (2)) and given by

$$X_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad Y_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad Z_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$X_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad Y_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad Z_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Generators of $\mathfrak{so}(3)_3$ and $\mathfrak{so}(3)_4$ have the same form as generators of $\mathfrak{so}(3)_1$ and $\mathfrak{so}(3)_2$ respectively, but they are in C. Elements of $\mathfrak{p}_0$ are in B

Theorem 4.1. *The group of outer automorphisms of $\mathfrak{so}(4, 4)$ is S_4.*

Proof. The group S_4 permutes components of $\mathfrak{k}_0$. We have to show that these automorphisms of $\mathfrak{k}_0$ can be extended to $\mathfrak{g}_0$. It is enough since $\mathbb{R}$ is not a summand of $\mathfrak{k}_0$ and the conjugation by $\mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1)$ produces the Cartan involution.

Now it is easy to produce an automorphism of $\mathfrak{g}_0$ that sends $\mathfrak{so}(3)_1$ to $\mathfrak{so}(3)_2$, $\mathfrak{so}(3)_2$ to $\mathfrak{so}(3)_1$ and fixes $\mathfrak{so}(3)_3$ and $\mathfrak{so}(3)_4$: $\varphi_{12}(X) = x_{12} X x_{12}^{-1}$ where we defined $x_{12} = \mathrm{diag}(1, 1, -1, 1, 1, 1, 1, 1)$. Similarly, $x_{34} = \mathrm{diag}(1, 1, 1, 1, 1, 1, -1, 1)$ produces an isomorphism φ_{34}. The conjugation by

$$\begin{bmatrix} 0 & I_4 \\ I_4 & 0 \end{bmatrix}$$

produces an automorphism that permutes $\mathfrak{so}(3)_1$ and $\mathfrak{so}(3)_3$ as well as $\mathfrak{so}(3)_2$ and $\mathfrak{so}(3)_4$.

It remains to construct the isomorphism $\varphi_{1234} = \varphi$ which sends X_i, Y_i and Z_i to X_{i+1}, Y_{i+1} and Z_{i+1} respectively. It is enough to write the action of φ on components in B. So, φ sends

$$
\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}
\begin{bmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0 \end{bmatrix}
\begin{bmatrix} 0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{bmatrix}
\begin{bmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0 \end{bmatrix}
\begin{bmatrix} 0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1 \end{bmatrix}
\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}
\begin{bmatrix} 0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1 \end{bmatrix}
\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0 \end{bmatrix}
\begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0 \end{bmatrix}
$$

to

$$
\begin{bmatrix} -1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{bmatrix}
\begin{bmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0 \end{bmatrix}
\begin{bmatrix} 0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{bmatrix}
\begin{bmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0 \end{bmatrix}
\begin{bmatrix} 0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1 \end{bmatrix}
\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0 \end{bmatrix}
\begin{bmatrix} 0&0&-1&0\\0&0&0&-1\\-1&0&0&0\\0&-1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1 \end{bmatrix}
\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}
\begin{bmatrix} 0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0 \end{bmatrix}
\begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix}
$$

respectively.   ∎

5. Operators S

We analyze two subgroups of G: G_1 and G_2. We start with the smaller one.

Definition 5.1. The subgroup G_1 of G is defined by

$$G_1 = \{x \in G \mid x^{-1}gx = h^{-1}gh, \text{ for some } h \in G_0, \ \forall g \in G_0\}.$$

The *center* Z_{G_0} of G_0 will be denoted by Z_0. The *centralizer* $C_G(G_0)$ of G_0 in G_1 will be denoted by Z_1.

It is important to notice that G_1 contains connected components which intersect $C_G(G_0)$. Hence, $Z = C_G(G_0) = C_{G_1}(G_0)$.

Proposition 5.2.   *The subgroup G_1 is a normal subgroup of G.*

Proof.   Let $y \in G_1$, $x \in G$ and $g \in G_0$. We have to show that $x^{-1}yx \in G_1$. Then

$$(x^{-1}yx)^{-1}g(x^{-1}yx) = x^{-1}y^{-1}xgx^{-1}yx =$$
$$x^{-1}h^{-1}xgx^{-1}hx = (x^{-1}hx)^{-1}g(x^{-1}hx).$$

Hence, the conjugation by $x^{-1}yx$ is the same as conjugation by $x^{-1}hx$.   ∎

Let π be the representation of G_0 on the vector space V. Let $x \in G \setminus G_0$ be any representative of the class xG_0. In order to understand the representation $\mathrm{ind}_{G_0}^G \pi$, one considers representations $\pi^x(g) = \pi(x^{-1}gx)$ where $x \in G \setminus G_0$ and $g \in G_0$. Let us assume that π^x is equivalent to π. Then there exists an intertwining operator S_x such that

$$\pi^x(g)S_x = S_x\pi(g). \tag{3}$$

It should be more correct to write $S_x(\pi)$, but we write S_x since π is fixed.

Remark.   Elements $x \in G_2 \setminus G_1$ correspond to outer automorphisms of the group G_0. Theorem 3.1 shows that there are 3 types of such automorphisms. Operators S_x can be easily described for the first type of outer automorphisms. Let $G = SL(2, \mathbb{R})$, V be the $(\mathfrak{g}, K)$-module and $V = \bigoplus_{j \in \mathbb{Z}} V_j$ where V_j are K-types. Let $x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then the conjugation by x is the outer automorphism and $S_x v_j = v_{-j}$ (see (30)). More details will be provided later.

If $x, y \in G$ have the property that π^x and π^y are equivalent to π then π^{xy} is equivalent to π and (one possible) intertwining operator S_{xy} is equal to

$$S_{xy} = S_y S_x. \tag{4}$$

Also, it is easy to see that one possible intertwining operator for x^{-1} is S_x^{-1}.

Definition 5.3.   The subgroup G_2 is the set of elements $x \in G$ for which π^x is equivalent to π.

If the representation π is irreducible, operator S_x is unique up to a scalar value. It follows from (4) that S_x^2 is the intertwining operator for π^{x^2} and in general

$$\pi^{x^k}(g)S_x^k = S_x^k\pi(g), \ \forall k \in \mathbb{N}\,(\mathbb{Z}). \tag{5}$$

Since G/G_0 is finite, $x^{n(x)} = h(x) \in G_0$ for some $n(x) \in \mathbb{N}$ and (5) transforms to

$$\pi((x^{n(x)})^{-1})\pi(g)\pi(x^{n(x)})S_x^{n(x)} = S_x^{n(x)}\pi(g).$$

It shows that $\pi(h(x))S_x^n$ is the intertwining operator for the representation π. Since π is irreducible, Schur's Lemma shows that $\pi(h(x))S_x^{n(x)} = \lambda I$ for some $\lambda \in \mathbb{C}$. Hence, we can take $\dfrac{1}{\sqrt[n(x)]{\lambda}}S_x$ instead of S_x in order to obtain

$$\pi(h(x))S_x^{n(x)} = I. \tag{6}$$

Let $\xi_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then request (6) puts another restriction on S_x: S_x is unique up to a power of $\xi_{n(x)}$. If $h \in G_0$, then

$$S_h = \pi(h^{-1}). \tag{7}$$

Relation (6) is satisfied. Also, (4) is satisfied.

If $x \in G_1$, say $x^{-1}gx = h^{-1}gh$ for $h \in G_0$, then π^x is equivalent to π and $S_x = \pi(h^{-1})$. It is easy to see that (6) is satisfied. Now, let us assume that $x \in G_2/G_0$ and S_x are fixed. For $h, g \in G_0$, S_{xh} and S_{gx} are defined such that (4) and (7) satisfied, i.e.

$$S_{xh} = \pi(h^{-1})S_x \quad \text{and} \quad S_{gx} = S_x\pi(g^{-1}).$$

We have to show that these definitions are good i.e. $S_{xh} = S_{gx}$ if $xh = gx$. However, the first we show that operators S_{xh} and S_{gx} satisfy (6).

Lemma 5.4.   *Operators S_{xh} and S_{gx} satisfy (6).*

Proof.   Let us prove the lemma for operators S_{xh}.

$$\pi((xh)^{n(x)}(\pi(h^{-1})S_x)^{n(x)} = \pi(xh\ldots x)S_x\pi(h^{-1})\ldots S_x =$$

$$= \pi\left(x^{n(x)}x^{1-n(x)}hx^{n(x)-1}x^{2-n(x)}h\ldots x^{-1}hx\right)S_x\pi(h^{-1})S_x\pi(h^{-1})\ldots S_x =$$

$$= \pi(x^{n(x)})\pi^{x^{n(x)-1}}(h)\pi^{x^{n(x)-2}}(h)\ldots\pi^{x^2}(h)\underbrace{\pi^x(h)S_x}\pi(h^{-1})S_x\pi(h^{-1})\ldots S_x \overset{(3)}{=}$$

$$= \pi(x^{n(x)})\pi^{x^{n(x)-1}}(h)\pi^{x^{n(x)-2}}(h)\ldots\pi^{x^2}(h)\underbrace{S_x\pi(h)}\pi(h^{-1})S_x\pi(h^{-1})\ldots S_x =$$

$$= \pi(x^{n(x)})\pi^{x^{n(x)-1}}(h)\pi^{x^{n(x)-2}}(h)\ldots\pi^{x^2}(h)S_x^2\pi(h^{-1})S_x\pi(h^{-1})\ldots S_x \overset{(5)}{=}\ldots\overset{(5)}{=}$$

$$\overset{(5)}{=}\pi(x^{n(x)})\pi^{x^{n(x)-1}}(h)S_x^{n(x)-1}\pi(h^{-1})S_x \overset{(5)}{=} \pi(x^{n(x)})S_x^{n(x)} \overset{(6)}{=} I.$$

The proof for the operators S_{gx} is similar.   ∎

Proposition 5.5.   *Let $x \in G_2$ and $h, g \in G_0$. If $xh = gx$ then $S_{xh} = S_{gx}$.*

Proof.   Operators S_{xh} and S_{gx} are intertwining operators for $xh = gh$. Hence, $S_{xh}S_{gx}^{-1} = \lambda I$ for some $\lambda \in \mathbb{C}$ and the identity operator I on V. Lemma 5.4 shows that $\lambda = \xi_{n(x)}$. For given $h \in G_0$ and $x \in G_2$, $g(h) = xhx^{-1}$ is a continuous function. Now, let us consider a function

$$f(h) = S_{xh}S_{gx}^{-1} = \pi(h^{-1})S_x\pi(xhx^{-1})S_x^{-1}.$$

Since f is a continuous function and the codomain is a discrete set, f is a constant function. Since $f(e) = I$, $f(h) = I$ for all $h \in G_0$ and $S_{xh} = S_{gx}$.   ∎

It shows that we can work with the class xG_0 instead with a particular representative.

Let us assume that intertwining operators S_x are chosen and fixed for all elements (classes) of G_2. Now, let us consider two particular elements $x, y \in G_2$. The operator $S_y S_x$ is the intertwining operator for xy but it may not satisfy (6). Even if it satisfies (6), it can happen that $S_y S_x = \xi_{n(xy)}^k S_{xy}$ for some k. However, by Schur's lemma, we know that $S_y S_x$ and S_{xy} are proportional. Hence, we define coefficients β.

Definition 5.6. For the given choice of S_x, we define for some $\beta(x, y) \in \mathbb{C}$

$$S_y S_x = \beta(x, y) S_{xy}. \tag{8}$$

Coefficients $\beta(x, y)$ encode information related to reducibility of $\mathrm{ind}_{G_0}^{G_2} \pi$. Hence, let us analyze properties of them. We will need it in Section 8.

Lemma 5.7. *Let* $x, y, z \in G_2/G_0$. *Then* $\beta(x, y)\beta(xy, z) = \beta(x, yz)\beta(y, z)$.

Proof. Relation (8) shows that

$$\beta(x, y)\beta(xy, z)S_{xyz} = \beta(x, y)S_z S_{xy} = S_z S_y S_x =$$
$$= \beta(y, z)S_{yz}S_x = \beta(x, yz)\beta(y, z)S_{xyz}.$$

It remains to compare the first and the last term. ∎

Lemma 5.8. *Let* $x \in G_2/G_0$. *Then*

$$\prod_{y \in G_2/G_0} \beta(x, y) = 1, \qquad \prod_{y \in G_2/G_0} \beta(y, x) = 1.$$

Proof. Exponent $n(x)$ was defined such that $x^{n(x)} = h(x) \in G_0$. Now,

$$S_y \pi(h(x)^{-1}) \overset{(6)}{=} S_y S_x^{n(x)} = \beta(x, y)S_{xy}S_x^{n(x)-1} = \beta(x, y)\beta(x, xy)S_{x^2 y}S_x^{n(x)-2} =$$
$$= \ldots = \beta(x, y)\beta(x, xy) \ldots \beta(x, x^{n(x)-1}y)S_{x^{n(x)}y}. \tag{9}$$

Now, use that
$$S_{x^{n(x)}y} = S_{h(x)y} = S_y \pi(h(x)^{-1}) \tag{10}$$
Relations (9) and (10) show that

$$\beta(x, y)\beta(x, xy) \ldots \beta(x, x^{n(x)-1}) = 1. \tag{11}$$

Since $n(x)$ divides $n = |G_2/G_0|$, G_2/G_0 can be written as a disjoint union of classes $\{x^{n(x)-1}y, \ldots, xy, y\}$. Since the product (11) is equal to 1 for each y in G_2/G_0, the first relation follows. The second formula can be proved similarly. ∎

Proposition 5.9. *Let* $x, y \in G_2$. *Then* $\beta(x, y)^n = 1$.

Proof.   Lemma 5.7 shows that

$$\beta(x,y)\beta(xy,z) = \beta(x,yz)\beta(y,z). \tag{12}$$

Now, we write (12) for all $z \in G_2/G_0$ and multiply them. Lemma 5.8 gives

$$\beta(x,y)^n \cdot 1 = 1 \cdot 1$$

and it proves the proposition. ∎

Remark.   Relation (6) was needed only in the proof of Lemma (5.8). Once, we have Proposition (5.9), we can multiply operators S_x by ξ_n^l for any $l \in \mathbb{Z}$. Proposition (5.9) will be still valid. Sometimes it will lead to a simpler set of coefficients $\beta(x,y)$.

Remark. A reader familiar with cohomology of groups will find some similarities between our construction and the second cohomology group. We say that the group E is the central extension of the group F if there exists a short exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} F \to 1 \tag{13}$$

where A is Abelian and central in E. Central extensions are parametrized by $H^2(F,A)$. A key ingredient in that construction are a set-theoretic section $\sigma : F \to E$ and a function $f : F \times F \to A$ such that

$$\sigma(x)\sigma(y) = i(f(x,y))\sigma(x,y).$$

For more details see, for example, [4]. Let us assume that $G = G_1$. In our construction F from (13) is equal to the quotient G/G_0, $A = Z_0$ and $E = Z_1$ (or $A = G_0$ and $E = Z_1$). One could say that $\beta(x,y) = \zeta(i(f(x,y)))$ where $\zeta : Z_0 \to \mathbb{C}$ is the character of the given representation. However, in our construction of β, intertwining operators S were included. It could be interesting to consider the case when $G = G_2$ and find relationship between possible choices for $\beta : G_2/G_0 \to \mathbb{C}$ (satisfying Lemmas 5.7 and 5.8) and extensions of G_2/G_0.

6. Induction

Let H be the subgroup of G and (π, V) the representation of H. The usual approach to the induction procedure is to consider the space

$$\text{ind}_H^G V = \{f : G \to V \mid f(xh) = \pi(h)^{-1}f(x), x \in G, h \in H\}$$

and the action is given by $(\text{ind}_{G_0}^G \pi(g)(f))(x) = f(g^{-1}x)$. We will use a different notation and write

$$W = \text{ind}_H^G V = \left\{ w = \sum_{x \in G/H} xv_x \mid v_x \in V \right\} \tag{14}$$

where sum goes over all classes of G/H and x is some representative. We can write

$$W = \{w(v_x) = \sum_{x \in G/H} xv_x \mid v_x \in V\} \tag{15}$$

or
$$W = \{w = \sum_{x \in G/H} xv_x(w) \mid v_x(w) \in V\}. \tag{16}$$

Relation (15) emphasizes that any collection of $|G/H|$ vectors from V determines an element w in W. Relation (16) emphasizes that any w in W determines the collection of vectors $v_x(w)$. Usually, we will write elements of W in terms of (14). The action of $g \in G$ is given by

$$g. \sum_{x \in G/H} xv_x = \sum_{x \in G/H} xh(x)v_x$$

for some elements $h(x) \in H$. The action can be described more precisely if H is the normal subgroup of G. The identity component, G_0 is the normal subgroup subgroup of G. Also, the subgroup G_1 is the normal subgroup of G (Proposition 5.2).

Let N be the normal subgroup of G and $K(N)$ the maximal compact subgroup of N. Since $G/N \cong K/K(N)$, we can write

$$\sum_{x \in K/K(N)} xv_x \tag{17}$$

instead of (14). The action $\text{ind}_N^G \pi(n)$ for $n \in N$ is given by

$$n.\left(\sum_{x \in G/N} xv_x \right) = \sum_{x \in G/N} x(x^{-1}nx).v_x$$

and the action $\text{ind}_{G_0}^G \pi(y)$ for $y \notin N$ is given by

$$y.\left(\sum_{x \in G/N} xv_x \right) = \sum_{x \in G/N} yx.v_x. \tag{18}$$

The element yx may not be the representative of the class yxG_0. If $z(x,y)$ is the chosen representative of the class yxG_0 in (14) then $yx = z(x,y)n(y,x)$ for $n(y,x) \in G_0$ and the "more precise" form of (18) is

$$y.\left(\sum_{x \in G/N} xv_x \right) = \sum_{x \in G/N} z(x,y)n(y,x).v_x.$$

For $(\mathfrak{g}, K)$-modules, the action of $X \in \mathfrak{g}$ on (17) can be obtained using (1):

$$X.\left(\sum_{x \in K/K(N)} xv_x \right) = \sum_{x \in K/K(N)} x\left(\text{Ad}(k^{-1})X \right).v_x.$$

Sometimes, it is convenient to use operators S in (14). Let us assume that π^x is equivalent to π for all $x \in G/N$. Then we write

$$\text{ind}_N^G V = \left\{ \sum_{x \in G/N} xS_x v_x \mid v \in V \right\}.$$

The action of $y \notin N$ is unchanged but the action of $n \in N$ transforms to

$$n.\left(\sum_{x \in G/N} xS_x v_x \right) = \sum_{x \in G/N} x(x^{-1}nx)S_x v_x \stackrel{(3)}{=} \sum_{x \in G/N} xS_x n v_x.$$

Let us analyze irreducible components of $W = \text{ind}_N^G V$ if π^x in that case. Let us write

$$W = \text{ind}_N^G V = \bigoplus_j W_j.$$

In particular, we are interested in the form of elements $w_j \in W_j$

Lemma 6.1. *Let us assume that N is the normal subgroup of G, π^x is equivalent to π for all $x \in G/N$ and W_j is an irreducible component of $\text{ind}_N^G V$. Then there exists $m \in \mathbb{N}$, and a collection of λ_x^l, $x \in G/N$, $l \in \{1, \ldots, m\}$ such that any element $w_j \in W_j$ has the form*

$$w_j = w_j(v_1, \ldots, v_m) = \sum_{x \in G/N} xS_x(\lambda_x^1 v_1 + \ldots + \lambda_x^m v_m)$$

where $v_i \in V$, S_x are intertwining operators and $\lambda_x^l \in \mathbb{C}$. It should be more correct to write $\lambda_{x,j}^l$, $v_{l,j}$ and $m(j)$ instead of λ_x^l, v_l and m, but we assume that the index j is fixed.

Proof. The subspace W_j contains elements w_j of the form (14). Let us put elements of G/N in a sequence, starting with e. We set $v_1 = v_e$. Now, consider the second element, say x. Let us recall (16) and write $v_x(w_j)$. If $v_x(w_j)$ is not a function of $v_e(w_j)$ for all w_j, then we choose $v_2(w_j)$ such that $S_x v_2(w_j) = v_x(w_j)$. If $v_x(w_j)$ depends on $v_e(w_j)$ for all w_j, then $v_x(w_j) = A_x v_e(w_j)$ for some linear operator A_x since W_j is a linear space. Now, write

$$w_j = ev_1(w_j) + xA_x v_1(w_j) + \sum_{y \in (G/N)\setminus\{eN, xN\}} y v_y(w_j). \tag{19}$$

Now, we act by $n \in N$ on (19) and get

$$n.w_j = e\pi(n)v_1(w_j) + x\pi^x(n)A_x v_1(w_j) + \sum_{y \in (G/N)\setminus\{eN, xN\}} y\pi^y(n)v_y(w_j). \tag{20}$$

If we put $\pi(n)v_1(w_j)$ instead of $v_1(w_j)$ in (19), we get

$$w_j = e\pi(n)v_1(w_j) + xA_x\pi(n)v_1(w_j) + \sum_{y \in (G/N)\setminus\{eN, xN\}} y v_y(w_j). \tag{21}$$

One can compare vectors after x in (20) and (21) and get $\pi^x(n)A_x = A_x\pi(n)$. It shows that $A_x = \lambda_x^1 S_x$. We continue in this way. In order to simplify notation, we switch to terminology of (14) again and write v_z instead of $v_z(w_j)$. Let us assume that we already have independent elements $v_1, \ldots, v_p$ and come to element y. If v_y does not depend on $v_1, \ldots, v_q$ in the same way for all w_j, then we choose

v_{p+1} such that $S_y v_{p+1} = v_y$. If v_y depends on $v_1, \ldots, v_p$ in the same way for all w_j, then put $v_y = A_y^1 v_1 + \ldots + A_y^p v_p$ in (19). In order to simplify notation we will write only element with y. The action of $n \in N$ gives

$$n.w_j = \ldots + y \pi^y(n)(A_y^1 v_1 + \ldots + A_y^p v_p) + \ldots \tag{22}$$

If we put $\pi(n)v_l$ instead of v_l in (19) for $l \in \{1, \ldots, p\}$, we get

$$w_j(\pi(n)v_1, \ldots, \pi(n), \ldots) = \ldots + y(A_y^1 \pi(n)v_1 + \ldots + A_y^p \pi(n)v_p) + \ldots \tag{23}$$

Since v_y depends on $\pi(n)v_1, \ldots, \pi(n)v_p$ and they coincide in $n.w_j$ in (22) and in $w_j(\pi(n)v_1, \ldots, \pi(n)v_p, \ldots)$ in (23), one concludes

$$\pi^y(n)(A_y^1 v_1 + \ldots + A_y^p v_p) = A_y^1 \pi(n)v_1 + \ldots + A_y^p \pi(n)v_p$$

Hence, $\pi^y(n)A_y^l = A_y^l \pi(n)$ for $l \in \{1, \ldots, p\}$. It shows that $A_y^l = \lambda_y^l S_y$ for some $\lambda_y^l \in \mathbb{C}$. ∎

The reducibility of $\mathrm{ind}_{G_0}^G \pi$ is related to equivalence of π and π^x.

Theorem 6.2. *If $G_2 = G_0$ then $\mathrm{ind}_{G_0}^G \pi$ is irreducible.*

Proof. The theorem follows from the Frobenius reciprocity theorem. Since

$$\mathrm{Hom}_G\left(\mathrm{ind}_{G_0}^G \pi, \mathrm{ind}_{G_0}^G \pi\right) = \mathrm{Hom}_{G_0}\left(\pi, \bigoplus_{x \in G/G_0} x\pi^x\right)$$

the left hand side is equal to $\mathbb{C}$ if and only if the right hand side is equal to $\mathbb{C}$ and it is valid if and only if $\pi \not\cong \pi^x$. ∎

It is easy to see directly that $\mathrm{ind}_{G_0}^G \pi$ is reducible if $\pi \cong \pi^x$. Let G_x be the subgroup of G of the form $G_0 \cup xG_0 \cup \ldots \cup x^{n(x)-1}G_0$. In this paper we need the definition and basic properties of coefficients β (we have stopped at Proposition 5.9). Hence, for this comment we assume that $\beta(x^m, x^k) = 1$ for $0 \leq m, k \leq n(x)$. Then $\mathrm{ind}_{G_0}^{G_x} \pi$ decomposes in the form $\mathrm{ind}_{G_0}^{G_x} V = \bigoplus_{j=0}^{n(x)-1} U_j$ where

$$U_j = \{u_j(v) = v + \xi_{n(x)}^j x S_x v + \ldots + \xi_{n(x)}^{jn(x)-1} x^{n(x)-1} S_x^{n(x)-1} v \mid v \in V\}.$$

7. $\mathrm{ind}_{G_0}^{G_1}$

Theorem 7.1. *Let $\zeta_\pi : Z_0 \to \mathbb{C}$ be the character of Z_0 defined by $\pi(z) = \zeta_\pi(z)I_V$ for $z \in Z_0$. The representation $\mathrm{ind}_{G_0}^{G_1} \pi$ decomposes in the same way as the representation $\mathrm{ind}_{Z_0}^{Z_1} \zeta_\pi$.*

Proof. Let us write $\mathbb{C}$ for the vector space of ζ_π. Then

$$\mathrm{ind}_{Z_0}^{Z_1} \mathbb{C} = \bigoplus_j U_j$$

where U_j, by Lemma 6.1, have the form

$$U_j = \left\{ u_j(c_1, \ldots, c_m) = \sum_{x \in Z_1/Z_0} x(\lambda_x^1 c_1 + \ldots + \lambda_x^m c_m) \,|\, c_l \in \mathbb{C} \right\} \qquad (24)$$

for some $\lambda_x^l \in \mathbb{C}$. Now, $\mathrm{ind}_{G_0}^{G_1} V = \bigoplus_j W_j$ where

$$W_j = \left\{ w_j(v_1, \ldots, v_m) = \sum_{x \in Z_1/Z_0} x(\lambda_x^1 v_1 + \ldots + \lambda_x^m v_m) \,|\, v_l \in V \right\}.$$

for the same λ_x^l as in (24). The action of $g \in G_0$ is given by

$$g.w_j(v_1, \ldots, v_m) = g.\left(\sum_{x \in Z_1/Z_0} x(\lambda_x^1 v_1 + \ldots + \lambda_x^m v_m) \right) =$$

$$= \sum_{x \in Z_1/Z_0} x(\lambda_x^1 g.v_1 + \ldots + \lambda_x^m g.v_m) = w_j(g.v_1, \ldots, g.v_m).$$

Hence, $g.w_j(v_1, \ldots, v_m) \in W_j$. The action of $x \in Z \setminus Z_0$ on $w_j(v_1, \ldots, v_m)$ is the same as the action of x on $u_j(c_1, \ldots, c_m)$. ∎

Example 7.2. Let us assume that $Z_1 = Q$, the quaternion group and $Z_0 = \mathbb{Z}_2 = \{\pm 1\}$. It is more convenient to denote elements of Q by 1, x, y and xy instead of i, j and k. Hence $Q = \{\pm 1, \pm x, \pm y, \pm xy\}$. The structure of G_0 is not important. There are two possible cases: $\pi_0|_{\{\pm 1\}}$ is the trivial representation and $\pi_1|_{\{\pm 1\}}$ is a nontrivial representation. The corresponding spaces will be denoted by V_0 and V_1 and restrictions to Z_0 by $\mathbb{C}_0$ and $\mathbb{C}_1$.

Let us consider the first case. We know that

$$\mathrm{ind}_{Z_0}^{Z_1} \mathbb{C}_0 = W_0 \oplus W_1 \oplus W_2 \oplus W_3.$$

It is easy to reconstruct the spaces W_j:

$$W_0 = \{w_0(c) = c + xc + yc + xyc \,|\, c \in \mathbb{C}\}$$
$$W_1 = \{w_1(c) = c + xc - yc - xyc \,|\, c \in \mathbb{C}\}$$
$$W_2 = \{w_2(c) = c - xc + yc - xyc \,|\, c \in \mathbb{C}\}$$
$$W_3 = \{w_3(c) = c - xc - yc + xyc \,|\, c \in \mathbb{C}\}.$$

The action of ± 1 is trivial and the action of other elements is easy to reconstruct. For example,

$$y.w_1(c) = y.(c + xc - yc - xyc) = yc + (-xy)c - (-1)c - (-x)c = -w_1(c).$$

Now, $$\mathrm{ind}_{G_0}^{G_1} V_0 = U_0 \oplus U_1 \oplus U_2 \oplus U_3,$$

where
$$U_0 = \{u_0(v) = v + xv + yv + xyv \,|\, v \in V\}$$
$$U_1 = \{u_1(v) = v + xv - yv - xyv \,|\, v \in V\}$$
$$U_2 = \{u_2(v) = v - xv + yv - xyv \,|\, v \in V\}$$
$$U_3 = \{u_3(v) = v - xv - yv + xyv \,|\, v \in V\}\,.$$

The action of $g \in G_0$ is given by $g.v_j(v) = v_j(g.v)$ and the action of other elements is the same as above. For example,

$$y.u_1(v) = y.(v + xv - yv - xyv) = yv + (-xy)v - (-1)v - (-x)v = -u_1(v).$$

The nontrivial case is more interesting. We know that

$$\text{ind}_{Z_0}^{Z_1}\mathbb{C}_1 = W_0 \oplus W_1$$

and W_0 and W_1 are equivalent. Now,

$$W_0 = \{w_0(c,d) = c + ixc + yd - ixyd \,|\, c,d \in \mathbb{C}\}$$
$$W_1 = \{w_1(c,d) = c - ixc + yd + ixyd \,|\, c,d \in \mathbb{C}\}\,.$$

The action of elements of G on W_j is given by

$$(-1).w_0(c,d) = w_0(-c,-d), \qquad (-1).w_1(c,d) = w_1(-c,-d)$$
$$x.w_0(c,d) = w_0(-ic,id), \qquad x.w_1(c,d) = w_1(ic,-id)$$
$$y.w_0(c,d) = w_0(-d,c), \qquad y.w_1(c,d) = w_1(-d,c).$$

Then $\text{ind}_{G_0}^{G_1}V_0 = U_0 \oplus U_1$ where

$$U_0 = \{u_0(v,t) = v + ixv + yt - ixyt \,|\, v,t \in V\} \tag{25}$$
$$U_1 = \{u_1(v,t) = v - ixv + yt + ixyt \,|\, v,t \in V\} \tag{26}$$

and U_0 and U_1 are equivalent and the intertwining operator sends $u_0(v,t)$ to $u_1(v,t)$. The action of elements of G on W_j, $(j = 0,1)$ is given by

$$g.u_j(v,t) = u_j(g.v, g.t)$$
$$x.u_j(v,t) = u_j(-i(-1)^k v, i(-1)^k t)$$
$$y.u_j(v,t) = u_j(-t, v).$$

8. $\text{ind}_{G_0}^{G_2}$

We prefer to induce from G_0 in order to avoid some technical difficulties. This step is similar to previous one. However, intertwining operators S_x have to be used.

Let $n = |G_2/G_0|$, $\xi_n = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$ and $\mathbb{Z}_n = \{\xi_n^j \,|\, j \in \{0,\ldots,n-1\}\}$.

Proposition 5.9 shows that $\beta(x,y) = \xi_n^j$ for $k \in \{0,\ldots,n-1\}$. Let B be the subgroup of $\mathbb{Z}_n$ generated by all $\beta(x,y)$. Then B has the form

$$B = \mathbb{Z}_m = \left\{\xi_m^j = \left(\cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}\right) \,|\, j \in \{0,\ldots,m-1\}\right\}$$

for some m which divides n. It indicates that we should consider the group F_2 for which the underlying set has the form $(G_2/G_0) \times B$ and the group operation is given by

$$(x, \alpha)(y, \beta) = (xy, \beta(x, y)\alpha\beta) \tag{27}$$

It is easy to check that the group operation is well defined. For example, associativity for F follows from associativity for G (or G_2). Associativity can be also proved formally using Proposition 5.7. Let $F_0 = F_2 \cap G_0 = \{e\} \times B = \{(e, \xi^k) \mid k \in \{0, \ldots, m-1\}\}$.

We define the representation η_m of F_0 on $\mathbb{C}$ by $\eta_m((e, \xi^j)).z = \xi^j z$.

Theorem 8.1. *The representation $\mathrm{ind}_{G_0}^{G_2}\pi$ decomposes in the same way as the representation $\mathrm{ind}_{F_0}^{F_2}\eta_m$.*

Proof. The proof is similar to the proof of Theorem 7.1. However, we have to emphasize that this time, we have an abstract group F_2 instead of subgroup Z_1. Actually, we could use an abstract group F_1 in Theorem 7.1 but the subgroup Z_1 is more natural choice. Let

$$\mathrm{ind}_{F_0}^{F_2}\mathbb{C} = \bigoplus_j U_j$$

where U_j have the form

$$U_j = \left\{ u_j(c_1, \ldots, c_m) = \sum_{x \in F_2/F_0} x(\lambda_x^1 c_1 + \ldots + \lambda_x^m c_m) \mid c_j \in \mathbb{C} \right\}. \tag{28}$$

Now, $\mathrm{ind}_{G_0}^{G_2}V = \bigoplus_j W_j$ where

$$W_j = \left\{ w_j(v_1, \ldots, v_m) = \sum_{x \in F_2/F_0} xS_x(\lambda_x^1 v_1 + \ldots + \lambda_x^m v_m) \mid v_l \in V \right\}$$

for the same λ_x^l as in (28). The action of $g \in G_0$ is given by

$$g.w_j(v_1, \ldots, v_m) = g.\left(\sum_{x \in F_2/F_0} xS_x(\lambda_x^1 v_1 + \ldots + \lambda_x^m v_m) \right) \overset{(3)}{=}$$

$$\overset{(3)}{=} \sum_{x \in F_2/F_0} xS_x(\lambda_x^1 g.v_1 + \ldots + \lambda_x^m g.v_m) = w_j(g.v_1, \ldots, g.v_m).$$

Hence, $g.w_j(v_1, \ldots, v_m) \in W_j$. The action of $y \in Z \setminus Z_0$ on $w_j(v_1, \ldots, v_m)$ is given by

$$y.w_j(v_1, \ldots, v_m) = y.\left(\sum_{y \in F_2/F_0} xS_x(\lambda_x^1 v_1 + \ldots + \lambda_x^m v_m) \right) =$$

$$= \sum_{x \in F_2/F_0} yxS_xS_yS_y^{-1}(\lambda_x^1 v_1 + \ldots + \lambda_x^m v_m) = \sum_{x \in F_2/F_0} yxS_xS_y(\lambda_x^1 S_y^{-1}v_1, \ldots, \lambda_x^m S_y^{-1}v_m).$$

It shows that the action of $y \in Z \setminus Z_0$ on $w_j(v_1, \ldots, v_m)$ is the same as the action of y on $u_j(c_1, \ldots, c_m)$ (up to adding S_y^{-1}). Coefficients β appear in the action of y on $w \in \operatorname{ind}_{G_0}^{G_2} \pi$ and in the action of y on $u \in \operatorname{ind}_{F_0}^{F_2} \eta_m$. ∎

Example 8.2. We will consider an example which is very similar to the previous one. Let $G_0 = (SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\{I \times I, (-I) \times (-I)\}$. This time, the action will be written in terms of $(\mathfrak{g}, K)$-modules.

For the beginning, let us consider only one one $SL(2, \mathbb{R})$. We use the terminology and results from [11]. The complexified Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is spanned by

$$H = -i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X = \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),$$

$$Y = \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

It acts on the vector space V spanned by vectors v_j for all $j \in \mathbb{Z}$. The action of $\mathfrak{sl}(2, \mathbb{C})$ on V is given by

$$H.v_j = jv_j, \quad X.v_j = \frac{1}{2} \left(\lambda + (j+1) \right) v_{j+2}, \quad Y.v_j = \frac{1}{2} \left(\lambda - (j-1) \right) v_{j-2}.$$

for some $\lambda \in \mathbb{C}$. Our V is irreducible if and only if $\lambda + j + 1 \neq 0$, $\forall j$. Now, we introduce the element x of the form

$$x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{29}$$

and it is easy to check that

$$Ad(x)H = -H, \quad Ad(x)X = Y, \quad Ad(x)Y = X.$$

Also, it is easy to check that π^x is isomorphic to π and the intertwining operator S_x is given by

$$S_x v_j = v_{-j}. \tag{30}$$

The K-type V_k is spanned by vector v_k. It shows that $S_x V_j = V_{-j}$. Hence $\operatorname{ind}_{G_0}^{G}$ should act on $V_j \oplus V_{-j}$. It is easy to show that

$$\operatorname{ind}_{G_0}^{G} (V_j \oplus V_{-j}) = Z_j^0 \oplus Z_j^1 \quad \text{for} \quad j > 0, \quad \text{and} \quad \operatorname{ind}_{G_0}^{G} V_0 = Z_0^0 \oplus Z_0^1,$$

where

$$Z_j^0 = \operatorname{span}(v_j + x S_x v_j, v_{-j} + x S_x v_{-j}) = \operatorname{span}(v_j + x v_{-j}, v_{-j} + x v_j)$$
$$Z_j^1 = \operatorname{span}(v_j - x S_x v_j, v_{-j} - x S_x v_{-j}) = \operatorname{span}(v_j - x v_{-j}, v_{-j} - x v_j)$$
$$Z_0^0 = \operatorname{span}(v_0 + x S_x v_0) = \operatorname{span}(v_0 + x v_0)$$
$$Z_0^1 = \operatorname{span}(v_0 - x S_x v_0) = \operatorname{span}(v_0 - x v_0).$$

Now, $(\mathfrak{g}, K)$-module $\operatorname{ind}_{G_0}^{G} V$ is reducible and

$$\operatorname{ind}_{G_0}^{G} V = U_0 + U_1.$$

If K-types of V are odd, then

$$U_0|_K = \bigoplus_{j \text{ odd}} Z_j^0 \qquad \text{and} \qquad U_1|_K = \bigoplus_{j \text{ odd}} Z_j^1.$$

If K-types of G_0 are even, then

$$U_0|_K = Z_0^0 \oplus \bigoplus_{j \text{ even, } j>0} U_j^0 \qquad \text{and} \qquad U_1|_K = Z_0^1 \oplus \bigoplus_{j \text{ even, } j>0} U_j^1.$$

For example,

$$x.(w_j + xw_{-j}) = w_{-j} + xw_j, \qquad x.(w_{-j} + xw_j) = w_j + xw_{-j},$$
$$x.(w_j - xw_{-j}) = -(w_{-j} - xw_j), \qquad x.(w_{-j} - xw_j) = -(w_j + xw_{-j}),$$
$$X.(w_{-2} + xw_2) = \frac{1}{2}(\lambda - 1)(w_0 + xw_0),$$
$$X.(w_{-2} - xw_2) = \frac{1}{2}(\lambda - 1)(w_0 - xw_0),$$
$$Y.(w_2 + xw_{-2}) = \frac{1}{2}(\lambda - 1)(w_0 + xw_0),$$
$$X.(w_j + xw_{-j}) = \frac{1}{2}(\lambda + j + 1)(w_{j+2} + xw_{-j-2}).$$

Now, let us consider our group G_0 acting on the space

$$\text{span}\{v_j^1 \otimes v_l^2 \mid j, l \in \mathbb{Z}\}.$$

The group G has 4 connected components. We add elements similar to (29) (by abuse of notation we use x again): $x = i \cdot \text{diag}(1, -1, 1, 1)$, $y = j \cdot \text{diag}(1, 1, 1, -1)$ and $z = k \cdot \text{diag}(1, -1, 1, -1)$. Multiplication by i, j and k does not mean multiplication by all entries. We simply want to obtain $xy = -yx$. Thanks to (30), it is easy to find intertwining operators:

$$S_x v_j^1 \otimes v_l^2 = v_{-j}^1 \otimes v_l^2, \quad S_y v_j^1 \otimes v_l^2 = v_j^1 \otimes v_{-l}^2, \quad S_z v_j^1 \otimes v_l^2 = v_{-j}^1 \otimes v_{-l}^2.$$

Actually, we should multiply operators S by i in order to satisfy (6). However, Remark after Proposition 5.9 shows that we can omit multiplication by i. The next step is to find coefficients $\beta(x, y)$. Since intertwining operators S_x commute, $\beta(x, y) = 1$, $\beta(y, z) = 1$, $\beta(z, x) = 1$, $\beta(y, x) = -1$, $\beta(z, y) = -1$ and $\beta(x, z) = -1$. It shows that $B = \{\pm 1\}$ and $F_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times B$ as a set and the group operation (27) shows that $F_2 \cong Q$, where Q is the group of quaternions. Also, $F_0 = \mathbb{Z}_2$. We are in situation of the previous Example. Using (25) and (26), one can easily get

$$U_0 = \left\{ u_0(v, t) = v + ixv + yt - ixyt \mid v = v^1 \otimes v^2, t = t^1 \otimes t^2 \in V \right\}$$
$$U_1 = \left\{ u_1(v, t) = v - ixv + yt + ixyt \mid v = v^1 \otimes v^2, t = t^1 \otimes t^2 \in V \right\}.$$

9. $\mathrm{ind}_{G_2}^{G}$

The following result is known. For example, see [5]. However, we put the theorem in order to complete the induction procedure. Also, our proof follows ideas developed in this paper.

Theorem 9.1. *The representation* $\mathrm{ind}_{G_2}^{G}\rho$ *is irreducible for the irreducible representation* (ρ, W) *of* G_2.

Proof. Let (π, V) be the irreducible representation of G_0 and let (ρ, W) be one irreducible component of $(\mathrm{ind}_{G_0}^{G_2}\pi, \mathrm{ind}_{G_0}^{G_2}V)$. Now, let us consider $\mathrm{ind}_{G_2}^{G}W$ and let U be one irreducible component. We want to show that $U = \mathrm{ind}_{G_2}^{G}W$. Let us mention that G_2 do not have to be a normal subgroup of G. By (14) elements of U can be written in the form

$$u = \sum_{y \in G/G_2} y w_y$$

where $w_y \in W$. If there is no relations among w_y's, then $U = \mathrm{ind}_{G_2}^{G}W$. Hence, we have to assume that at least one w, say w_p is a function of other w's. Since U is a linear space, we can assume that

$$w_p = \sum_{y \in (G/G_2)\backslash\{pG_2\}} A_y w_y \tag{31}$$

for some linear operators A_y and write

$$u = \sum_{y \in (G/G_2)\backslash\{pG_2\}} y w_y + p \sum_{y \in (G/G_2)\backslash\{pG_2\}} A_y w_y. \tag{32}$$

Lemma 6.1 shows that (32) can be written in the form

$$u = \sum_{y \in (G/G_2)\backslash\{pG_2\}} y \sum_{x \in G_2/G_0} x S_x \left(\lambda_{y,x}^1 v_{y,x}^1 + \ldots + \lambda_{y,x}^{m(y,x)} v_{y,x}^{m(y,x)} \right) +$$
$$+ p \sum_{x \in G_2/G_0} x S_x \sum_j A_x^j v_x^j \tag{33}$$

where $v_x^j \in \{v_{y,x}^l\}$ and $A_x^j \in GL(V)$. Now, let us choose one $v_{q,x}^l$ to be an arbitrary element of V, denote it by v and set all other $v_{y,x}^l = 0$. Then (33) transforms to

$$u = u(v) = q \sum_{x \in G_2/G_0} x S_x \lambda_x v + p \sum_{x \in G_2/G_0} x S_x A_x v. \tag{34}$$

Elements $u(v)$ form a G_0-invariant linear subspace of U. The action of $g \in G_0$ on (34) produces

$$\pi(g)u(v) = q \sum_{x \in G_2/G_0} x S_x \lambda_x \pi^q(g) v + p \sum_{x \in G_2/G_0} x S_x \pi^p(g) A_x v. \tag{35}$$

We can write $\pi^q(g)v$ instead of v in (34) and get

$$u(\pi^q(g)) = q \sum_{x \in G_2/G_0} x S_x \lambda_x \pi^q(g)v + p \sum_{x \in G_2/G_0} x S_x A_x \pi^q(g)v. \qquad (36)$$

One can compare (35) and (36) and obtain

$$\pi^p(g)A_x v = A_x \pi^q(g)v, \quad \text{or} \quad \pi^{q^{-1}p}(g)A_x v = A_x \pi(g)v.$$

It gives a contradiction since $q^{-1}p \notin G_2$. Hence, the assumption (31) is not valid. It shows that $U = \operatorname{ind}_{G_2}^G W$. ∎

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