

# Category $\mathcal{J}$ Modules for Hamiltonian Vector Fields on a Torus

John Talboom

Communicated by A. Fialowski

**Abstract.** Modules for the Lie algebra of Hamiltonian vector fields on a torus, which admit a compatible action for the commutative algebra of multivariate Laurent polynomials, and have finite dimensional weight spaces, are called category  $\mathcal{J}$ . This paper classifies the indecomposable and the irreducible modules in category  $\mathcal{J}$ .

*Mathematics Subject Classification:* 17B10, 17B66.

*Key Words and Phrases:* Indecomposable representations, irreducible representations, Lie algebra of Hamiltonian vector fields, weight module.

## 1. Introduction

An important class of simple infinite dimensional Lie algebras are the Lie algebras of Cartan type, which arise in the study of vector fields on a manifold. These are categorized into four series, the general, special, Hamiltonian, and contact, denoted  $W_N, S_N, H_N$  and  $K_N$  respectively. Type  $W_N$  is the Lie algebra of derivations of  $R_N = \mathbb{C}[x_1, \dots, x_N]$ , whose elements are vector fields of the form  $X = \sum_{i=1}^N f_i \frac{\partial}{\partial x_i}$  where  $f_i \in R_N$  (some sources allow  $R_N$  to be all formal power series,  $\mathbb{C}[[x_1, \dots, x_N]]$ ). The remaining three series are defined using the Lie derivative  $\mathcal{L}_X \omega$ , of a specific differential form  $\omega$  with respect to vector field  $X$ . Specifically  $S_N = \{X | \mathcal{L}_X(dx_1 \wedge \dots \wedge dx_N) = 0\}$ ,  $H_{2m} = \{X | \mathcal{L}_X(\sum_{i=1}^m dx_i \wedge dx_{m+i}) = 0\}$  and  $K_{2m+1} = \{X | \mathcal{L}_X(\omega) = P\omega\}$ , where  $\omega = dx_{2m+1} + \sum_{i=1}^m x_{m+i} dx_i - x_i dx_{m+i}$  and  $P \in R_{2m+1}$  (see [8] for more details).

Recent works have studied the representation theory of the Lie algebra of derivations  $\text{Der}(A_N)$ , where  $A_N = \mathbb{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$ . Elements of  $\text{Der}(A_N)$  can be identified with the polynomial vector fields on a torus (see Section 2). An interesting class of modules, called  $(A_N, \text{Der}(A_N))$ -modules is considered in [7], which are modules that admit a compatible action by both  $\text{Der}(A_N)$  and the commutative algebra  $A_N$ . In said paper Eswara Rao classifies all irreducible  $(A_N, \text{Der}(A_N))$ -modules with finite dimensional weight spaces. These modules are presented in [2] as category  $\mathcal{J}$ , and the classification is extended to all indecomposable modules by showing that  $\text{Der}(A_N)$  has polynomial action.

A similar result is achieved in [3], again using the strategy of showing polynomial action, where category  $\mathcal{J}$  is considered for  $\mathcal{S}_N$ , the divergence zero vector fields on a torus. An inductive proof is used to show that the general case follows from the case  $N = 2$ , which turns out to be exceptional.

The current paper now considers category  $\mathcal{J}$  for the Lie algebra of Hamiltonian vector fields on an  $N$  dimensional torus, denoted  $\mathcal{H}_N$ , where  $N$  is even. The goal is to classify all indecomposable and all irreducible category  $\mathcal{J}$  modules. Theorem 4.4 is the main result and is summarized below (the action of  $\mathcal{H}_N$  will be given later).

**Theorem.** *Let  $\mathfrak{H}_N^+$  be the vector fields in  $H_N$  with non-negative degree, and  $\mathfrak{h}_N$  the  $2m + 1$  dimensional Heisenberg algebra. Let  $\lambda \in \mathbb{C}^N$  and  $\mathcal{J}_\lambda$  a subcategory of modules in  $\mathcal{J}$  supported on  $\lambda + \mathbb{Z}^N$ , where  $N = 2m \geq 2$ . There is an equivalence of categories between the category of finite dimensional modules for  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$  and  $\mathcal{J}_\lambda$ . This equivalence maps  $V$  to  $A_N \otimes V$  where  $V$  is a finite dimensional module for  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ .*

To obtain this result the general strategy of [3] is applied. Since  $\mathcal{H}_2 \cong \mathcal{S}_2$ , the basis of induction has already been shown (see [3] Proposition 3.7). Irreducible representations for the case  $N = 2$  are studied in [6] by Jiang and Lin and, although not shown here, their paper provides a crucial step in proving the  $N = 2$  case (see [3] Lemma 3.1).

This paper is organized in the following way. Section 2 will present Lie algebra  $\mathcal{H}_N$  and define its category  $\mathcal{J}$  modules. It will be shown here that the action of  $\mathcal{H}_N$  on any module from category  $\mathcal{J}$  is determined completely by the action restricted to a single weight space  $V$ . Section 3 is main part of the proof for the classification theorem. Starting with the previously proven  $N = 2$  case as the basis of induction, it is shown that  $\mathcal{H}_N$  acts by certain  $\text{End}(V)$ -valued polynomials for all  $N$ . In Section 4 the polynomial action of  $\mathcal{H}_N$  is seen to be a representation of Lie algebras  $\mathfrak{H}_N^+$  and  $\mathfrak{h}_N$ , and the main result can be stated. For simple modules of category  $\mathcal{J}$  the action simplifies considerably. Section 5 is devoted to irreducible representations, and exhibits the main result when only simple modules are considered.

## 2. Preliminaries

Let  $A_N = \mathbb{C}[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$  be the algebra of Laurent polynomials over  $\mathbb{C}$ . Elements of  $A_N$  are presented with multi-index notation  $t^r = t_1^{r_1} \dots t_N^{r_N}$  where  $r = (r_1, \dots, r_N) \in \mathbb{Z}^N$ . Let  $\{e_1, \dots, e_N\}$  denote the standard basis for  $\mathbb{Z}^N$ . For  $k \in \mathbb{Z}^N$ ,  $|k| = k_1 + \dots + k_N$ ,  $k! = k_1! \dots k_N!$  and  $\binom{r}{k} = \frac{r!}{k!(r-k)!}$ . Denote the set of non-negative integers by  $\mathbb{Z}_{\geq 0}$ .

For  $i \in \{1, \dots, N\}$ , let  $d_i = t_i \frac{\partial}{\partial t_i}$ . The derivations of  $A_N$ ,  $\text{Der}(A_N) = \text{Span}_{\mathbb{C}} \{t^r d_i | i \in \{1, \dots, N\}, r \in \mathbb{Z}^N\}$ , forms a Lie algebra called the *Witt algebra* denoted here by  $\mathcal{W}_N$ , with Lie bracket  $[t^r d_i, t^s d_j] = s_i t^{r+s} d_j - r_j t^{r+s} d_i$ . By setting  $t_j = e^{\sqrt{-1}\theta_j}$  for all  $j \in \{1, \dots, N\}$ ,  $\mathcal{W}_N$  may be identified with the Lie algebra of (complex-valued) polynomial vector fields on an  $N$  dimensional torus, where  $\theta_j$  is the  $j$ th angular coordinate.

The focus of this paper is on the subalgebra of Hamiltonian vector fields  $\mathcal{H}_N$ . Let  $X = \sum_{i=1}^N f_i d_i \in \mathcal{W}_N$  and from now on let  $N = 2m$ . The change of coordinates  $t_j = e^{\sqrt{-1}\theta_j}$ , gives  $\frac{\partial}{\partial \theta_j} = \frac{\partial t_j}{\partial \theta_j} \cdot \frac{\partial}{\partial t_j} = \sqrt{-1} t_j \frac{\partial}{\partial t_j} = \sqrt{-1} d_j$ . Thus  $X$  can be expressed  $X = -\sqrt{-1} \sum_{j=1}^N f_j(t) \frac{\partial}{\partial \theta_j}$ . The defining property,  $\mathcal{L}_X(\sum_{i=1}^m d\theta_i \wedge d\theta_{m+i}) = 0$ , of  $\mathcal{H}_N$  yields that for  $i = 1, \dots, m$ ,  $f_i = \frac{\partial u}{\partial \theta_{m+i}} = \sqrt{-1} t_{m+i} \frac{\partial u}{\partial t_{m+i}}$  and  $f_{m+i} = -\frac{\partial u}{\partial \theta_i} = -\sqrt{-1} t_i \frac{\partial u}{\partial t_i}$  for some  $u \in A_N$ , or  $X = d_i$ . Thus  $\mathcal{H}_N$  is spanned by elements  $X_u = \sum_{i=1}^m t_{m+i} \frac{\partial u}{\partial t_{m+i}} d_i - t_i \frac{\partial u}{\partial t_i} d_{m+i}$ , and  $d_i$ . Obtain homogeneous elements by letting  $u = t^r$ , for  $r \in \mathbb{Z}^N$ , and set  $h(r) = \sum_{i=1}^m r_{m+i} t^r d_i - r_i t^r d_{m+i}$ . So

$$\mathcal{H}_N = \text{Span}_{\mathbb{C}}\{d_i, h(r) \mid i \in \{1, \dots, N\}, r \in \mathbb{Z}^N\}$$

and has commutative Cartan subalgebra  $\text{Span}_{\mathbb{C}}\{d_j \mid j \in \{1, \dots, N\}\}$ . The Lie bracket in terms of homogeneous elements is given by  $[d_i, h(r)] = r_i h(r)$  and

$$[h(r), h(s)] = \sum_{i=1}^m (r_{m+i} s_i - r_i s_{m+i}) h(r + s).$$

Note that  $h(0) = 0$ , and  $[h(r), h(-r)] = 0$ .

Families of modules called *category*  $\mathcal{J}$ , have been defined for  $\mathcal{W}_N$  and  $\mathcal{S}_N$  in [2] and [3] respectively. An analogous category of modules for  $\mathcal{H}_N$  is defined as follows:

**Definition 2.1.** Let  $N \geq 2$  with  $N$  even. An  $\mathcal{H}_N$ -module  $J$  belongs to category  $\mathcal{J}$  if the following properties hold:

- (J1) The action of  $d_i$  on  $J$  is diagonalizable for all  $i \in \{1, \dots, N\}$ .
- (J2) Module  $J$  is a free  $A_N$ -module of finite rank.
- (J3) For any  $X \in \mathcal{H}_N, f \in A_N$  and  $v \in J$ ,  $X(fv) = (Xf)v + f(Xv)$ .

Any submodule of  $J \in \mathcal{J}$  must be invariant under the actions of both  $A_N$  and  $\mathcal{H}_N$ . By (J2) any module in  $\mathcal{J}$  is a finite direct sum of indecomposable modules, hence the goal is to classify indecomposable modules  $J \in \mathcal{J}$ . Property (J1) decomposes  $J$  into weight spaces,  $J = \bigoplus_{\lambda \in \mathbb{C}^N} J_\lambda$  where  $J_\lambda = \{v \in J \mid d_i v = \lambda_i v\}$ . Note that  $h(r)J_\lambda \subset J_{\lambda+r}$  since,

$$d_i(h(r)v) = h(r)d_i v + [d_i, h(r)]v = (\lambda_i + r_i)h(r)v$$

for  $v \in J_\lambda$ . Similarly by (J3)  $t^r J_\lambda \subset J_{\lambda+r}$ . These two relations partition the weights of  $J$  into  $\mathbb{Z}^N$ -cosets of  $\mathbb{C}^N$ , and decompose  $J$  into a direct sum of submodules, each corresponding to a distinct coset. Thus if  $J$  is indecomposable its set of weights is one such coset  $\lambda + \mathbb{Z}^N$  for  $\lambda \in \mathbb{C}^N$  and  $J = \bigoplus_{r \in \mathbb{Z}^N} J_{\lambda+r}$ . Denote by  $\mathcal{J}_\lambda$ , the subcategory of  $\mathcal{J}$  supported on  $\lambda + \mathbb{Z}^N$  for a fixed  $\lambda \in \mathbb{C}^N$  and from now on assume that  $J \in \mathcal{J}_\lambda$ .

Let  $V = J_\lambda$ . The invertible map  $t^r : V \rightarrow J_{\lambda+r}$  identifies all weight spaces with  $V$  and since  $J$  is a free module for the associative algebra  $A_N$  it follows that any basis for  $V$  is also basis for  $J$  viewed as a free  $A_N$ -module. The finite rank condition of (J2) implies that  $V$  must be finite dimensional. This yields that  $J \cong A_N \otimes V$ . Homogeneous elements of  $J$  will be denoted  $t^s \otimes v$ , for  $s \in \mathbb{Z}^N, v \in V$ .

The action of  $h(r)$  on  $J$  maps  $1 \otimes V \rightarrow t^r \otimes V$ . This induces an endomorphism  $H(r)$  on  $V$  given by  $H(r)v = (t^{-r} \circ h(r))v$ . The action in (J3) can then be expressed as

$$h(r)(t^s \otimes v) = \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i})t^{r+s} \otimes v + t^{r+s} \otimes H(r)v. \tag{1}$$

Thus the action of  $H(r)$  on  $V$  determines the action of  $h(r)$  on  $J$ . The Lie bracket for  $H(r)$  terms is

$$\begin{aligned} [H(r), H(s)] &= [t^{-r}h(r), t^{-s}h(s)] \\ &= t^{-r}(h(r)(t^{-s}))h(s) - t^{-s}(h(s)(t^{-r}))h(r) + t^{-r-s}[h(r), h(s)] \\ &= t^{-r} \left( \sum_{i=1}^m r_{m+i}(-s_i) - r_i(-s_{m+i}) \right) t^{r-s}h(s) \\ &\quad - t^{-s} \left( \sum_{i=1}^m s_{m+i}(-r_i) - s_i(-r_{m+i}) \right) t^{s-r}h(r) \\ &\quad + t^{-r-s} \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i})h(r+s) \\ &= \sum_{i=1}^m (r_i s_{m+i} - r_{m+i} s_i)(H(r) + H(s) - H(r+s)). \end{aligned}$$

### 3. Polynomial action

The concept of a polynomial module was introduced in [1] and then generalized in [4]. Polynomial modules were used in [2] for classifying category  $\mathcal{J}$  modules for  $\mathcal{W}_N$ . This technique was applied in the previous chapter to the  $\mathcal{S}_N$  case and will be used again here. The main idea is to show that  $H(r)$  acts on  $V$  by an  $\text{End}(V)$ -valued polynomial in  $r$  with constant term zero. Specifically,

**Proposition 3.1.** *Let  $N \geq 2$  (with  $N$  even) and let  $J = A_N \otimes V$  be an  $\mathcal{H}_N$ -module in category  $\mathcal{J}$ . Then  $H(r)$  acts on  $V$  by*

$$H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)} - \delta_{r,0} P^{(0)},$$

for all  $r \in \mathbb{Z}^N$ , where  $P^{(r)} \in \text{End}(V)$  and do not depend on  $r$ , and the summation is finite.

The Kronecker delta function is to ensure that  $H(0) = 0$ . This will be proven by induction on  $N$  with the base case  $N = 2$  already shown in [3] since  $\mathcal{H}_2 \cong \mathcal{S}_2$ .

**Lemma 3.2.** *Let  $N = 2$  and  $J = A_2 \otimes V$  be a module for  $\mathcal{H}_2$  in category  $\mathcal{J}$ . Then  $H(r)$  acts on  $V$  by*

$$H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^2} \frac{r^k}{k!} P^{(k)} - \delta_{r,0} P^{(0)},$$

for all  $r \in \mathbb{Z}^2$ , where  $P^{(r)} \in \text{End}(V)$  and do not depend on  $r$ , and the summation is finite.

See [3], Proposition 3.7 for details, where it is shown that the action of  $H(r)$  may be interpolated by an  $\text{End}(V)$ -valued polynomial in  $r$  at all points in the plane except for the origin.

The fact that two (multivariate) polynomials are equal if they agree on a cube of arbitrary size was used repeatedly in [3] as it will be used here. The following Lemma will be required and is presented here without proof (cf. [3] Lemma 3.5).

**Lemma 3.3.** *Let  $S = S_1 \times \cdots \times S_N \in \mathbb{C}^N$ , where each  $S_i$  is a set with  $K + 1$  elements, and let  $F$  and  $G$  be polynomials of degree at most  $K$  in  $N$  variables,  $X_1, \dots, X_N$ , that agree on  $S$ . Then  $F = G$ .*

**Proof of Proposition 3.1.** Proceed by induction on  $N = 2m$  with basis of induction given by Lemma 3.2. Let  $\tilde{r}_i = r - r_i e_i - r_{m+i} e_{m+i}$  for  $i \in \{1, \dots, m\}$ . By induction hypothesis  $H(\tilde{r}_i)$  acts by an  $\text{End}(V)$ -valued polynomial in  $r$  for  $\tilde{r}_i \neq 0$ . Let  $k \in \{1, \dots, m\}$  with  $k \neq i$ . Then  $H(e_i + e_k)$  and  $H(e_i + e_{m+k})$  also act by an  $\text{End}(V)$ -valued polynomials in  $r$  since  $e_i + e_k$  and  $e_i + e_{m+k}$  are constant. Then

$$[H(\tilde{r}_i), H(e_i + e_k)] = -r_{m+k}(H(\tilde{r}_i) + H(e_i + e_k) - H(\tilde{r}_i + e_i + e_k)),$$

and

$$[H(\tilde{r}_i), H(e_i + e_{m+k})] = r_k(H(\tilde{r}_i) + H(e_i + e_{m+k}) - H(\tilde{r}_i + e_i + e_{m+k})).$$

Rearranging these equations gives

$$r_{m+k}H(\tilde{r}_i + e_i + e_k) = r_{m+k}(H(\tilde{r}_i) + H(e_i + e_k)) + [H(\tilde{r}_i), H(e_i + e_k)],$$

and

$$r_kH(\tilde{r}_i + e_i + e_{m+k}) = r_k(H(\tilde{r}_i) + H(e_i + e_{m+k})) - [H(\tilde{r}_i), H(e_i + e_{m+k})],$$

so that the terms on the right side act by an  $\text{End}(V)$ -valued polynomial in  $r$  for  $\tilde{r}_i \neq 0$ . Thus  $r_{m+k}H(\tilde{r}_i + e_i + e_k)$  and  $r_kH(\tilde{r}_i + e_i + e_{m+k})$  act by polynomial when  $\tilde{r}_i \neq 0$ , and hence for all  $r \in \mathbb{Z}^N$ . It can be shown in a similar fashion that both  $r_{m+k}H(\tilde{r}_i + e_{m+i} + e_k)$  and  $r_kH(\tilde{r}_i + e_{m+i} + e_{m+k})$  act as polynomials for all  $r$ . This holds for all  $i \neq k$ , and in particular it follows that  $r_iH(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_{m+k})$  acts by polynomial for all  $r$ . Now consider the Lie bracket

$$\begin{aligned} & [r_{m+k}H(\tilde{r}_i + e_i + e_k), r_iH(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_{m+k})] \\ &= r_{m+k}r_i(r_{m+i} + 1)(r_k + 1)(H(\tilde{r}_i + e_i + e_k) \\ & \quad + H(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_{m+k}) - H(r + e_i + e_{m+i} + e_k + e_{m+k})). \end{aligned}$$

Isolating the term

$$r_{m+k}r_i(r_{m+i} + 1)(r_k + 1)H(r + e_i + e_{m+i} + e_k + e_{m+k})$$

on the right side shows that it must act by polynomial, call it  $Q_1$ , for all  $r \in \mathbb{Z}^N$  since all other terms do. Similarly form the Lie brackets

$$\begin{aligned} & [r_k H(\tilde{r}_i + e_i + e_{m+k}), r_i H(r_i e_i + r_{m+i} e_{m+i} + e_{m+i} + e_k)], \\ & [r_{m+k} H(\tilde{r}_i + e_{m+i} + e_k), r_{m+i} H(r_i e_i + r_{m+i} e_{m+i} + e_i + e_{m+k})], \\ & [r_k H(\tilde{r}_i + e_{m+i} + e_{m+k}), r_{m+i} H(r_i e_i + r_{m+i} e_{m+i} + e_i + e_k)] \end{aligned}$$

to show that

$$\begin{aligned} & r_k r_i (r_{m+i} + 1)(r_{m+k} + 1) H(r + e_i + e_{m+i} + e_k + e_{m+k}), \\ & r_{m+k} r_{m+i} (r_i + 1)(r_k + 1) H(r + e_i + e_{m+i} + e_k + e_{m+k}), \\ & r_k r_{m+i} (r_i + 1)(r_{m+k} + 1) H(r + e_i + e_{m+i} + e_k + e_{m+k}) \end{aligned}$$

act by polynomials  $Q_2, Q_3$ , and  $Q_4$  respectively, for all  $r \in \mathbb{Z}^N$ . Note that

$$r_k (r_{m+k} + 1) Q_1 = r_{m+k} (r_k + 1) Q_2$$

which implies that both  $r_{m+k}$  and  $r_k + 1$  are factors of  $Q_1$  since they do not divide  $r_k (r_{m+k} + 1)$  (both  $r_k$  and  $r_{m+k} + 1$  are factors of  $Q_2$ ). Comparing  $Q_1$  and  $Q_3$ ,

$$r_{m+i} (r_i + 1) Q_1 = r_i (r_{m+i} + 1) Q_3,$$

and so both  $r_i$  and  $r_{m+i} + 1$  are factors of  $Q_1$  since they do not divide  $r_{m+i} (r_i + 1)$ . Thus  $Q_1 = r_{m+k} r_i (r_{m+i} + 1)(r_k + 1) P_1$  for some polynomial  $P_1$ , and hence  $H(r + e_i + e_{m+i} + e_k + e_{m+k}) = P_1$  for  $r_i, r_{m+k} \neq 0$  and  $r_{m+i}, r_k \neq -1$ . By continuing this process of equating the  $Q_i$  with appropriate linear factors, it can be shown that  $H(r + e_i + e_{m+i} + e_k + e_{m+k})$  acts by a polynomial

$$\begin{aligned} & P_2 \text{ on the region } \{r_i, r_k \neq 0\} \cap \{r_{m+i}, r_{m+k} \neq -1\}, \\ & P_3 \text{ on the region } \{r_{m+i}, r_{m+k} \neq 0\} \cap \{r_i, r_k \neq -1\}, \text{ and} \\ & P_4 \text{ on the region } \{r_{m+i}, r_k \neq 0\} \cap \{r_i, r_{m+k} \neq -1\}. \end{aligned}$$

Since these four regions are obtained by deleting a finite number of hyperplanes from  $\mathbb{Z}^N$ , their intersection contains a cube of arbitrary size. By Lemma 3.3, since  $P_1, P_2, P_3$  and  $P_4$  agree on this cube they must be equal, and so  $H(r + e_i + e_{m+i} + e_k + e_{m+k})$  acts by polynomial on the union of the four regions, which is  $\{(r_i, r_{m+i}) \neq (0, 0), (-1, -1)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (-1, -1)\}$ . Shifting coordinates, this implies that  $H(r)$  acts by polynomial  $P$  on the region

$$\mathcal{R} = \{(r_i, r_{m+i}) \neq (0, 0), (1, 1)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (1, 1)\}.$$

What remains is to extend this region to all  $r \in \mathbb{Z}^N \setminus \{0\}$ .

Let  $\mathcal{L}$  be the Lie algebra spanned by elements  $H(r)$  for  $r \in \mathbb{Z}^N$ , and consider the automorphisms  $\sigma_i : \mathcal{L} \rightarrow \mathcal{L}$  given by

$$\sigma_i(H(r)) = H(r_1, \dots, -r_i, \dots, -r_{m+i}, \dots, r_N),$$

for  $i \in \{1, \dots, m\}$ . The region  $\mathcal{R}$  may be expanded by considering automorphisms  $\sigma_i$  of  $\mathcal{L}$  since whatever was proven for  $\mathcal{L}$  will also be true of its image under  $\sigma_i$ .

Thus  $\sigma_i(H(r))$  acts by polynomial  $\bar{P}$  on  $\mathcal{R}$  which implies that  $H(r)$  acts by  $\bar{P}$  on  $\{(r_i, r_{m+i}) \neq (0, 0), (-1, -1)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (1, 1)\}$ . Lemma 3.3 may be applied to the intersection of these regions, showing that  $P = \bar{P}$  there, and hence  $H(r)$  acts by  $P$  on their union

$$\mathcal{R}' = \{(r_i, r_{m+i}) \neq (0, 0)\} \cap \{(r_k, r_{m+k}) \neq (0, 0), (1, 1)\}.$$

In the same way, applying  $\sigma_k$  to  $\mathcal{R}$  yields that  $H(r)$  acts by  $P$  on the region  $\mathcal{R}_{i,k} = \{(r_i, r_{m+i}) \neq (0, 0)\} \cap \{(r_k, r_{m+k}) \neq (0, 0)\}$ .

Now apply Lemma 3.3 to the intersection of regions  $\mathcal{R}_{i,k}$  for various  $k$  to show that  $H(r)$  acts by  $P$  on

$$\mathcal{R}_i = \bigcup_{\substack{k=1 \\ k \neq i}}^m \mathcal{R}_{i,k} = \{(r_i, r_{m+i}) \neq (0, 0)\} \cap \{\tilde{r}_i \neq 0\}.$$

Applying the same lemma again to these regions shows that  $H(r)$  acts by  $P$  on

$$\bigcup_{i=1}^m \mathcal{R}_i = \bigcap_{i=1}^m \{\tilde{r}_i \neq 0\}.$$

Fix  $s \in \mathbb{Z}^N$  and let  $\bar{s}_k = s_k e_k + s_{m+k} e_{m+k}$  for  $k \in \{1, \dots, m\}$ . Further assume that  $\bar{s}_k \neq 0$ . Then

$$[H(\bar{s}_k), H(r - \bar{s}_k)] = (s_k r_{m+k} - s_{m+k} r_k)(H(\bar{s}_k) + H(r - \bar{s}_k) - H(r))$$

and so

$$(s_k r_{m+k} - s_{m+k} r_k)H(r) =$$

$$(s_k r_{m+k} - s_{m+k} r_k)(H(\bar{s}_k) + H(r - \bar{s}_k)) - [H(\bar{s}_k), H(r - \bar{s}_k)].$$

From what was proven above, all operators on the right side of this equation act by polynomial on  $\bigcap_{i=1}^m \{\tilde{r}_i \neq \bar{s}_k\}$ , and hence  $(s_k r_{m+k} - s_{m+k} r_k)H(r)$  acts by polynomial  $T$  on this region. Similarly for  $s' \in \mathbb{Z}^N$ ,  $(s'_k r_{m+k} - s'_{m+k} r_k)H(r)$  acts by polynomial  $T'$  on  $\bigcap_{i=1}^m \{\tilde{r}_i \neq \bar{s}'_k\}$  and on the intersection of these regions

$$(s'_k r_{m+k} - s'_{m+k} r_k)T = (s_k r_{m+k} - s_{m+k} r_k)T',$$

implying that  $(s_k r_{m+k} - s_{m+k} r_k)$  is a factor of  $T$ . Thus  $H(r)$  acts by polynomial when  $(s_k r_{m+k} - s_{m+k} r_k) \neq 0$  and hence acts by polynomial on the region

$$\mathcal{T}_{s,k} = \{s_k r_{m+k} \neq s_{m+k} r_k\} \cap \left( \bigcap_{i=1}^m \{\tilde{r}_i \neq \bar{s}_k\} \right).$$

Again use Lemma 3.3 to patch together the regions which  $H(r)$  acts by polynomial.

First 
$$\left( \bigcap_{i=1}^m \{\tilde{r}_i \neq 0\} \right) \cup \mathcal{T}_{s,k} = \{(r_k, r_{m+k}) \neq (0, 0)\} \cap \{r \neq \bar{s}_k\}.$$

Then 
$$\bigcup_{s \in \mathbb{Z}^N} \left( \bigcap_{i=1}^m \{\tilde{r}_i \neq 0\} \right) \cup \mathcal{T}_{s,k} = \{(r_k, r_{m+k}) \neq (0, 0)\},$$

and 
$$\bigcup_{k=1}^m \{(r_k, r_{m+k}) \neq (0, 0)\} = \{r \neq 0\}.$$

Thus  $H(r)$  acts by  $\text{End}(V)$ -valued polynomial in  $r$  for all  $r \neq 0$ . To include  $r = 0$  and to accommodate the fact that  $H(0) = 0$ , a delta function is added, and so  $H(r)$  may be expressed

$$H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)} - \delta_{r,0} P^{(0)},$$

for all  $r \in \mathbb{Z}^N$ , where  $P^{(r)} \in \text{End}(V)$ , and the sum is finite. ■

#### 4. Classification

The result of the previous section is that the action of the  $H(r)$  operators are now in terms of the  $P^{(k)}$ ,  $k \in \mathbb{Z}_{\geq 0}^N$ . Following the ideas of [2] and [3], the Lie brackets between the operators  $P^{(k)}$  are computed below and show that these operators realize a representation of a familiar Lie algebra. This will yield a classification of the modules in category  $\mathcal{J}$ .

To compute the Lie bracket  $[P^{(j)}, P^{(k)}]$  use Proposition 3.1 and consider a region where  $r, s \neq 0$  so that

$$[H(r), H(s)] = \sum_{j \in \mathbb{Z}_{\geq 0}^N} \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^j s^k}{j!k!} [P^{(j)}, P^{(k)}]. \tag{2}$$

The left side of this evaluates to

$$\begin{aligned} & \sum_{i=1}^m (r_i s_{m+i} - r_{m+i} s_i) (H(r) + H(s) - H(r+s)) \\ &= \sum_{i=1}^m (r_i s_{m+i} - r_{m+i} s_i) \left( \sum_{j \in \mathbb{Z}_{\geq 0}^N} \frac{r^j}{j!} P^{(j)} + \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{s^k}{k!} P^{(k)} - \sum_{\ell \in \mathbb{Z}_{\geq 0}^N} \frac{(r+s)^\ell}{\ell!} P^{(\ell)} \right). \end{aligned}$$

Then  $[P^{(j)}, P^{(k)}]$  is obtained by extracting the coefficient of  $\frac{r^j s^k}{j!k!}$  above for various  $j$  and  $k$ . For  $|j|, |k| > 1$

$$[P^{(j)}, P^{(k)}] = \sum_{i=1}^m (j_{m+i} k_i - j_i k_{m+i}) P^{(j+k-e_i-e_{m+i})}.$$

If  $|j| \leq 1$  then  $[P^{(j)}, P^{(k)}] = 0$  for all  $k$  with the exception that  $[P^{(e_i)}, P^{(e_{m+i})}] = P^{(0)}$  (and the usual anticommutativity property holds).

Consider the Lie algebra of derivations of polynomials in  $N$  variables,

$$\text{Der}(\mathbb{C}[x_1, \dots, x_N]) = \text{Span}_{\mathbb{C}} \left\{ x^r \frac{\partial}{\partial x_a} \mid a \in \{1, \dots, N\}, r \in \mathbb{Z}_{\geq 0}^N \right\}.$$

This has as a subalgebra the Hamiltonian Lie algebra given by

$$\mathfrak{H}_N = \text{Span}_{\mathbb{C}} \left\{ X(r) = \sum_{i=1}^m r_{m+i} x^{r-e_{m+i}} \frac{\partial}{\partial x_i} - r_i x^{r-e_i} \frac{\partial}{\partial x_{m+i}} \mid r \in \mathbb{Z}_{\geq 0}^N \right\},$$

with Lie bracket  $[X(r), X(s)] = \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i})X(r + s - e_i - e_{m+i})$ .

(Notation  $\mathfrak{H}_N$  is used in place of  $H_N$  from the introduction so as not to confuse it with operators  $H(r)$ ).

Let  $\mathfrak{H}_N^+ = \text{Span}_{\mathbb{C}} \{X(r) \in \mathfrak{H}_N \mid |r| > 1\}$  and let  $\mathfrak{h}_N$  be the Heisenberg algebra  $\text{Span}_{\mathbb{C}} \{p^i, q^i, \mathbf{c} \mid i \in \{1, \dots, m\}\}$  with  $[p^i, q^i] = \mathbf{c}$  and all other brackets zero. The bracket for the  $P^{(k)}$  operators found above demonstrates the following.

**Proposition 4.1.** *The map  $\rho : \mathfrak{H}_N^+ \oplus \mathfrak{h}_N \rightarrow \text{End}(V)$  given by*

$$\rho(X(r)) = P^{(r)} \text{ for } |r| > 1, \quad \rho(p^i) = P^{(e_i)}, \quad \rho(q^i) = P^{(e_{m+i})}, \quad \rho(\mathbf{c}) = P^{(0)},$$

*is a finite dimensional representation of  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$  on  $V$  for  $N = 2m \geq 2$ .*

**Proposition 4.2.** *Let  $\mathfrak{L}_n = \text{Span}_{\mathbb{C}} \{X(r) \in \mathfrak{H}_N \mid |r| = n + 2\}$ . In the grading  $\mathfrak{H}_N = \bigoplus_{n=-1}^{\infty} \mathfrak{L}_n$ , component  $\mathfrak{L}_0$  is isomorphic to the symplectic algebra  $\mathfrak{sp}_N$  and each  $\mathfrak{L}_n$  is an irreducible  $\mathfrak{sp}_N$ -module.*

**Proof.** Let  $E_{ab}$  be the matrix with a 1 in entry  $(a, b)$  and zeros elsewhere. An isomorphism between  $\mathfrak{L}_0$  and  $\mathfrak{sp}_N$  is obtained by identifying basis elements

$$x_i \frac{\partial}{\partial x_j} - x_{m+j} \frac{\partial}{\partial x_{m+i}}, \quad x_i \frac{\partial}{\partial x_{m+i}} + x_j \frac{\partial}{\partial x_{m+i}}, \quad x_{m+i} \frac{\partial}{\partial x_j} + x_{m+j} \frac{\partial}{\partial x_i},$$

of  $\mathfrak{L}_0$  with  $E_{i,j} - E_{m+j,m+i}, E_{i,m+j} + E_{j,m+i}, E_{m+i,j} + E_{m+j,i}$  respectively from  $\mathfrak{sp}_N$ , where  $i, j \in \{1, \dots, m\}$ . Then each  $\mathfrak{L}_n$  is an  $\mathfrak{sp}_N$ -module via the adjoint action of  $\mathfrak{L}_0$ . The Cartan subalgebra of  $\mathfrak{sp}_N$  is spanned by diagonal elements  $E_{i,i} - E_{m+i,m+i}$ , and raising operators (positive root spaces) are spanned by  $E_{i,m+i}, E_{i,j} - E_{m+j,m+i}, E_{i,m+j} + E_{j,m+i}$  for  $i < j$ .

Every finite dimensional simple  $\mathfrak{sp}_N$ -module is a highest weight module, and so by Weyl's Theorem on complete reducibility it suffices to show that there exists a unique (up to scalar) highest weight vector in each  $\mathfrak{L}_i$ . This requires showing that there is a unique  $v \in \mathfrak{L}_n$  which is annihilated by the adjoint action of all raising operators. Elements  $X(r)$  of  $\mathfrak{L}_n$  are weight vectors as  $\left[ x_i \frac{\partial}{\partial x_i} - x_{m+i} \frac{\partial}{\partial x_{m+i}}, X(r) \right] = (r_i - r_{m+i})X(r)$  for all  $i$ , and any scalar multiple of  $X(ne_1) = -x_1^n \frac{\partial}{\partial x_{m+1}}$  is a highest weight vector of  $\mathfrak{L}_n$  since for any  $i < j$ ,

$$\left[ x_i \frac{\partial}{\partial x_{m+i}}, x_1^n \frac{\partial}{\partial x_{m+1}} \right] = 0, \quad \left[ x_i \frac{\partial}{\partial x_j} - x_{m+j} \frac{\partial}{\partial x_{m+i}}, x_1^n \frac{\partial}{\partial x_{m+1}} \right] = 0,$$

$$\left[ x_i \frac{\partial}{\partial x_{m+j}} + x_j \frac{\partial}{\partial x_{m+i}}, x_1^n \frac{\partial}{\partial x_{m+1}} \right] = 0.$$

It remains to show that there are no other highest weight vectors.

A weight vector  $u \in \mathfrak{L}_n$  of weight  $\lambda$  can be expressed as a finite sum  $u = \sum_{r \in I} C_r X(r)$  where  $X(r)$  has weight  $\lambda$  for each  $r \in I$  and  $C_r \in \mathbb{C}^N$ . The task of

finding highest weight vectors amounts to finding all such  $u$  with  $[X(s), u] = 0$ , for  $s = 2e_i, e_i + e_{m+j}$  and  $e_i + e_j$  with  $i < j$ . By linear independence

$$0 = [X(2e_i), u] = \sum_{r \in I} C_r(-2r_{m+i})X(r + e_i - e_{m+i})$$

implies that  $C_r = 0$  whenever  $r_{m+i} \neq 0$  for some  $i$ . Since  $u$  is a weight vector this means that  $u = X(r)$  for some  $r$  with  $r_{m+1} = \dots = r_{2m} = 0$ . Then

$$0 = [X(e_i + e_{m+j}), u] = r_j X(r + e_i - e_j)$$

implies  $r_j = 0$  for all  $i, j$  with  $i < j$ . This leaves only the case  $r = ne_1$  and thus  $X(ne_1)$  is the unique highest weight vector of  $\mathfrak{L}_n$  up to scalar. ■

Notice that  $[X(e_i + e_{m+i}), X(ne_i)] = nX(ne_i)$  for  $n \geq 0$ , suggesting that in the representation of  $\mathfrak{sp}_N$  on  $\mathfrak{H}_N^+$ , there are infinitely many distinct eigenvalues. However the following Lemma shows that this cannot be the case, meaning that for the representation  $\rho$  of Proposition 4.1, there exists  $k_0$  such that  $\rho(X(ke_i)) = 0$  for all  $k \geq k_0$ . The fact that each  $\mathfrak{L}_k$  is an irreducible  $\mathfrak{sp}_N$ -module implies that all of  $\mathfrak{L}_k$  acts trivially on  $V$  for  $k \geq k_0$ .

**Lemma 4.3.** *Let  $\mathfrak{L}$  be a Lie algebra with nonzero elements  $y, y_1, y_2, \dots$  with the property that  $[y, y_i] = \alpha_i y_i$  for  $i = 1, 2, \dots$ , and  $\alpha_i \in \mathbb{C}$ . Then for a finite dimensional representation  $(U, \rho)$  of  $\mathfrak{L}$ , there are at most  $(\dim U)^2 - \dim U + 1$  distinct eigenvalues for which  $\rho(y_i) \neq 0$ .*

This Lemma and its proof are given in [2]. It is also used in the proof of Lemma 3.2 found in [3]. With this the groundwork is laid for the main result.

**Theorem 4.4.** *Let  $\lambda \in \mathbb{C}^N$  and let  $\mathcal{J}_\lambda$  be a subcategory of modules in  $\mathcal{J}$  supported on  $\lambda + \mathbb{Z}^N$ , where  $N = 2m \geq 2$ . There is an equivalence of categories between the category of finite dimensional modules for  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$  and  $\mathcal{J}_\lambda$ . This equivalence maps  $V$  to  $A_N \otimes V$  where  $V$  is a finite dimensional module for  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ . The action of  $\mathcal{H}_N$  on  $A_N \otimes V$  is given by  $d_a(t^s \otimes v) = (s_a + \lambda_a)t^s \otimes v$ , and for  $r \neq 0$ ,*

$$\begin{aligned} h(r)(t^s \otimes v) &= \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i})t^{r+s} \otimes v + t^{r+s} \otimes \rho(\mathbf{c})v \\ &+ t^{r+s} \otimes \sum_{i=1}^m (r_i \rho(p^i) + r_{m+i} \rho(q^i))v + t^{r+s} \otimes \sum_{\substack{k \in \mathbb{Z}_{\geq 0}^N \\ |k| > 1}} \frac{r^k}{k!} \rho(X(k))v. \end{aligned} \tag{3}$$

**Proof.** Following the definition of category  $\mathcal{J}$ , it was noted that each module in  $\mathcal{J}_\lambda$  may be identified with  $A_N \otimes V$ , where  $V$  is any weight space. Combining (J1) and (J3) gives the action for  $d_i$ , and applying Proposition 3.1 to (1) yields

$$h(r)(t^s \otimes v) = \sum_{i=1}^m (r_{m+i}s_i - r_i s_{m+i})t^{r+s} \otimes v + t^{r+s} \otimes \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)}v,$$

for  $r \neq 0$ , and  $h(0) = 0$  otherwise. Proposition 4.1 shows that  $V$  is a finite dimensional  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ -module, and (3) is obtained from the expression above.

On the other hand suppose  $V$  is a finite dimensional module for  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$ . Identify the elements of  $\mathfrak{H}_N^+ \oplus \mathfrak{h}_N$  with the  $P^{(k)}$  as in Proposition 4.1 and let  $H(r) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{r^k}{k!} P^{(k)}$ . This sum has only finitely many nonzero terms because in the decomposition  $\mathfrak{H}_N^+ = \bigoplus_{k=0}^{\infty} \mathfrak{L}_k$ , there exists  $k_0$  such that  $\mathfrak{L}_k$  acts trivially on  $V$  for all  $k \geq k_0$ . The Lie bracket for  $H(r)$  terms is obtained from (2), and defining the action of  $h(r)$  on  $A_N \otimes V$  by (1) recovers the commutator relations for  $\mathcal{H}_N$ . Thus  $A_N \otimes V$  is a finite dimensional  $\mathcal{H}_N$ -module. ■

### 5. Irreducible Representations

The action of  $\mathcal{H}_N$  simplifies considerably when only simple modules from category  $\mathcal{J}$  are considered. The central element of the Heisenberg algebra acts trivially in a finite dimensional irreducible representation, and so an  $N$  dimensional abelian algebra  $\mathfrak{a}_N = \text{Span}_{\mathbb{C}} \{C_i | i \in \{1, \dots, N\}\}$  replaces  $\mathfrak{h}_N$ .

The following Lemma will be used to show that the action of  $\mathfrak{H}_N^+$  reduces to that of  $\mathfrak{sp}_N$ . See Lemma 2.4 in [5] for the proof.

**Lemma 5.1** ([5], Lemma 2.4). *Let  $\mathfrak{g}$  be a finite dimensional Lie algebra over  $\mathbb{C}$  with solvable radical  $\text{Rad}(\mathfrak{g})$ . Then  $[\mathfrak{g}, \text{Rad}(\mathfrak{g})]$  acts trivially on any finite dimensional irreducible  $\mathfrak{g}$ -module.*

To apply this lemma let  $\mathfrak{g} = \mathfrak{H}_N^+ / I \oplus \mathfrak{a}_N$ , where  $I$  is the ideal  $I = \bigoplus_{k \geq k_0} \mathfrak{L}_k$  which acts trivially on  $V$  for some  $k_0$ . Then  $\text{Rad}(\mathfrak{g}) = (\bigoplus_{k > 0} \mathfrak{L}_k) / I \oplus \mathfrak{a}_N$  since  $[\mathfrak{L}_n, \mathfrak{L}_k] \subset \mathfrak{L}_{n+k}$ . So  $[\mathfrak{g}, \text{Rad}(\mathfrak{g})] = (\bigoplus_{k > 0} \mathfrak{L}_k) / I$  acts trivially by Lemma 5.1, and hence the ideal  $\bigoplus_{k > 0} \mathfrak{L}_k$  must act trivially on  $V$ .

**Theorem 5.2.** *Let  $\lambda \in \mathbb{C}^N$  and let  $\mathcal{J}_\lambda$  be a subcategory of modules in  $\mathcal{J}$  supported on  $\lambda + \mathbb{Z}^N$ . For  $N = 2m \geq 2$ , there is a one-to-one correspondence between the finite dimensional irreducible modules for  $\mathfrak{sp}_N \oplus \mathfrak{a}_N$  and the irreducible modules in  $\mathcal{J}_\lambda$ . This correspondence maps a finite dimensional irreducible module  $V$  for  $\mathfrak{sp}_N \oplus \mathfrak{a}_N$  to  $A_N \otimes V$ . The action of  $\mathcal{H}_N$  on  $A_N \otimes V$  is given by  $d_a(t^s \otimes u) = (s_a + \lambda_a)t^s \otimes v$  and for all  $r$*

$$\begin{aligned}
 h(r)(t^s \otimes v) &= \sum_{i=1}^m (r_{m+i}(s_i + \mu_{m+i}) - r_i(s_{m+i} + \mu_i))t^{r+s} \otimes v \\
 &+ t^{r+s} \otimes \sum_{i=1}^m (r_{m+i}^2 \varphi(E_{m+i,i}) + r_i r_{m+i} \varphi(E_{i,i} - E_{m+i,m+i}) - r_i^2 \varphi(E_{i,m+i})) v \\
 &+ t^{r+s} \otimes \sum_{\substack{i,j=1 \\ i < j}}^m (r_{m+i} r_{m+j} \varphi(E_{m+j,i} + E_{m+i,j}) + r_i r_{m+j} \varphi(E_{i,j} - E_{m+i,m+j}) \\
 &\qquad\qquad\qquad - r_i r_j \varphi(E_{i,m+j} + E_{j,m+i})) v,
 \end{aligned}$$

where  $\mu_i$  is the action of  $C_i \in \mathfrak{a}_N$ , and  $\varphi$  is an irreducible representation of  $\mathfrak{sp}_N$ .

**Proof.** Since  $\bigoplus_{k>0} \mathfrak{L}_k$  acts trivially on  $V$ , the action of  $\mathfrak{H}_N^+$  reduces to that of  $\mathfrak{L}_0 \cong \mathfrak{sp}_N$ . As mentioned above, the Heisenberg algebra is replaced by abelian algebra  $\mathfrak{a}_N$ , and by Schur's Lemma its elements  $C_i$  act by scalars  $\mu_i$  in an irreducible representation. Thus the action in (3) reduces to that above. Theorem 4.4 gives the correspondence between these sets of modules. ■

**Acknowledgments.** Many thanks to Professor Yuly Billig for his generous help on this paper.

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John Talboom  
 Department of Mathematics  
 Trent University  
 1600 West Bank Drive  
 Peterborough, ON K9L 0G2  
 Canada  
 johntalboom@trentu.ca

Received March 10, 2017  
 and in final form May 8, 2018