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Extending Generalized Spin Representations

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Abstract. We revisit the construction of higher spin representations by Kleinschmidt and Nicolai for E_{10}, generalize it to arbitrary simply laced types, and provide a coordinate-free approach to the $\frac{3}{2}$-spin and $\frac{5}{2}$-spin representations. Moreover, we discuss the relationship between our findings and the representation theory of Sym_3 pointed out to us by Levy.

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1. Introduction

Generalized spin representations of the maximal compact subalgebra of the split real Kac-Moody algebra of type E_{10} have been introduced in [4], [5] and generalized to arbitrary symmetrizable types in [7]. The purpose of this note is to revisit some of the higher spin representations of type E_{10} studied in [9], notably $\frac{3}{2}$-spin and $\frac{5}{2}$-spin, generalize these to arbitrary simply laced types, and propose a coordinate-free approach which we carry out for $\frac{3}{2}$-spin and $\frac{5}{2}$-spin.

The terminology $\frac{n}{2}$-spin representation originates in the representation theory of $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$: To each irreducible unitary representation of $\rho\colon \mathrm{SU}(2) \to \mathrm{GL}(V)$ there exists $n \in \mathbb{N}$ such that V is isomorphic to the space of complex homogeneous polynomials of degree n in two variables (i.e., $V \cong \mathrm{Sym}^n(\mathbb{C}^2)$) endowed with the natural action of $\mathrm{SU}(2)$. (See, e.g., [3, Proposition II.5.3].)

It is convenient to denote $V(\frac{n}{2}) := \mathrm{Sym}^n(\mathbb{C}^2)$ as one then arrives at the concise Clebsch-Gordan formula

$$V(a) \otimes V(b) \cong V(|a-b|) \oplus V(|a-b|+1) \oplus \cdots \oplus V(a+b),$$

cf. e.g. [3, p. 87]. The module $V(\frac{n}{2})$ is called the $\frac{n}{2}$-spin representation of $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ and those for odd n are exactly the ones on which -1 acts non-trivially, i.e., the ones that do not induce a representation of $\mathrm{SO}(3)$.

The terminology that we use in the present note and that we borrowed from string theory is based on the analogy to the situation of $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$. A generalized spin representation (ρ, W) as in Definition 2.2 is by definition of spin $\frac{1}{2}$. Under certain circumstances such a representation on the vector space W can be extended

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to the tensor product $V \otimes W$ where $V = \mathrm{Sym}^n(\mathfrak{h}^*)$; the resulting representation is called to be of spin $n + \frac{1}{2}$.

Our main result is the following coordinate-free criterion for the existence of such extensions of generalized spin representations:

Theorem. *Let $\mathfrak{g}$ be a simply laced split real Kac-Moody algebra, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let λ be the set consisting of the simple roots $\{\alpha_1, \dots, \alpha_n\}$ of $\mathfrak{g}$ and roots that are sums of two distinct simple roots, let $\mathfrak{k}$ be the maximal compact subalgebra of $\mathfrak{g}$, and let $(\cdot|\cdot)$ denote the induced invariant bilinear form on $\mathfrak{h}^*$. Let $(\cdot|\cdot)$ be normalized in the usual way such that $(\alpha|\alpha) = 2$ for all real roots α which will be denoted by Δ^{re}. Recall that α is real if $\exists\, \omega \in W$ s.t. $\alpha = \omega(\alpha_i)$ for a simple root α_i and W is the Weyl group of $\mathfrak{g}$. A map $X\colon \lambda \to \mathrm{End}\,(V)$ satisfying the following (anti-)commutator relations for all $\alpha, \beta \in \lambda$*

$$\begin{aligned} [X(\alpha), X(\beta)] &= 0 & &\text{if } (\alpha|\beta) = 0 \\ \{X(\alpha), X(\beta)\} &= X(\alpha \pm \beta) & &\text{if } (\alpha|\beta) = \mp 1 \text{ and } \alpha \pm \beta \in \lambda \end{aligned}$$

provides a finite-dimensional representation σ of $\mathfrak{k}$ via the assignment

$$\sigma\,(X_i) := X\,(\alpha_i) \otimes \Gamma\,(\alpha_i)$$

on the Berman generators $X_1, \dots, X_n$ of $\mathfrak{k}$, where the $\Gamma(\alpha_i)$, $1 \le i \le n$ are the anti-symmetric real matrices from (11) *on page 923 induced by the generalized spin representation of $\mathfrak{k}$. Define $X_{\frac{3}{2}}\colon \Delta^{\mathrm{re}} \to \mathrm{End}\,(\mathfrak{h}^*)$ via*

$$\alpha \mapsto X_{\frac{3}{2}}(\alpha) \;:=\; -\alpha\,(\alpha|\cdot) + \frac{1}{2}\mathrm{id}_{\mathfrak{h}^*}.$$

For $\alpha \in \Delta^{\mathrm{re}}$ let $\pi_\alpha := \alpha\,(\alpha|\cdot) \in \mathrm{End}\,(\mathfrak{h}^*)$ and define $X_{\frac{5}{2}}\colon \Delta^{\mathrm{re}} \to \mathrm{End}\,\big(\mathrm{Sym}^2(\mathfrak{h}^*)\big)$ via

$$\alpha \mapsto X_{\frac{5}{2}}(\alpha) \;:=\; \pi_\alpha \otimes \pi_\alpha - (\pi_\alpha \otimes \mathrm{id}_{\mathfrak{h}^*} + \mathrm{id}_{\mathfrak{h}^*} \otimes \pi_\alpha) + \frac{1}{2}\mathrm{id}_{\mathfrak{h}^*} \otimes \mathrm{id}_{\mathfrak{h}^*}.$$

Then $X_{\frac{3}{2}}$ and $X_{\frac{5}{2}}$ fulfill the above equalities for all real roots α, β satisfying $(\alpha|\beta) \in \{0, \pm 1\}$ and thus each provides a representation σ of $\mathfrak{k}$.

The results for this note have been obtained during and shortly after the first author's MSc thesis project [11] in mathematics. It would be interesting to understand how these representations decompose into irreducible components. We refer to [9], [10] for some investigations in this direction using coordinates.

Paul Levy pointed out to us that both assignments $X_{\frac{3}{2}}$ and $X_{\frac{5}{2}}$ are of the form

$$X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$$

where $\rho(s_\alpha)$ denotes the natural reflection action of the fundamental generator s_α induced on $\mathfrak{h}^*$, resp. $\mathrm{Sym}^2(\mathfrak{h}^*)$. For simple roots α, β forming a subdiagram of type A_2 one obtains the equivalence

$$\{X(\alpha), X(\beta)\} = X(\alpha \pm \beta) \;\Leftrightarrow\; \rho(s_\alpha s_\beta s_\alpha) - \rho(s_\alpha s_\beta) - \rho(s_\beta s_\alpha) + \rho(s_\alpha) + \rho(s_\beta) - \mathrm{id} = 0.$$

Among the irreducible representations of Sym_3, the trivial and the geometric representations satisfy the above identity, whereas the sign representation does not. One in fact arrives at a characterization of those representations $\rho\colon W \to \mathrm{GL}(V)$ of the Weyl group W of $\mathfrak{g}$ that can be used for extending generalized spin representations via the assignment $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$: exactly those whose restrictions to any standard subgroup $\mathrm{Sym}_3 \cong \langle s_\alpha, s_\beta \rangle \leq W$ (where α, β are adjacent simple roots of $\mathfrak{g}$) do not contain a sign representation as an irreducible component will do.

Since neither of the given W-modules $\mathfrak{h}^*$ and $\mathrm{Sym}^2(\mathfrak{h}^*)$ contain a Sym_3-sign representation, they both can be used for extending generalized spin representations. The module $\mathrm{Sym}^3(\mathfrak{h}^*)$ on the other hand does contain a sign representation and so the $\frac{7}{2}$-spin representations discussed in [9], [10] still remain elusive.

Moreover, note that a map $X\colon \lambda \to \mathrm{End}(V)$ as in the statement of the Theorem naturally extends to the set of all those positive real roots that can be written as iterated sums of simple roots such that each partial sum itself is a positive real root. It is well-known that in the finite-dimensional situation this set equals the set of all positive (real) roots; in the simply-laced affine case it can be shown that this set also equals the set of all positive real roots (cf. [11]). To the best of our knowledge the question what this set looks like in general is open.

Our note contains several redundancies. First, we reproduce the method to obtain extensions of generalized spin representations of E_{10} and its application to $\frac{3}{2}$ and $\frac{5}{2}$-spin representations proposed by Kleinschmidt and Nicolai in order to make their work [9], [10] accessible to a wider mathematical audience and to point out that their approach actually works for any simply-laced Dynkin diagram. Second, we propose and apply our own coordinate-free method. Third, we interpret our findings in terms of Sym_3-representation theory based on Levy's observations. This organization of our note leads to various existence proofs of $\frac{3}{2}$ and $\frac{5}{2}$-spin representations and to a wealth of starting points for further investigation. We refer to the Outlook & Comments section at the end of this note for a discussion of some of these starting points.

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2. Generalized $\frac{1}{2}$-spin representations

Recall the notion of a Kac-Moody algebra from [8]. Let A be a symmetrizable generalized Cartan matrix and $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A over $\mathbb{R}$ so that for $\mathfrak{h}_\mathbb{C} := \mathfrak{h} \otimes_\mathbb{R} \mathbb{C}$ the triple $(\mathfrak{h}_\mathbb{C}, \Pi, \Pi^\vee)$ is a realization of A over $\mathbb{C}$. Let $\bar{\cdot}\colon \mathbb{C} \to \mathbb{C}$ be complex conjugation and denote by ω_0 the $\bar{\cdot}$-semilinear involution on the complex

Kac-Moody algebra $\mathfrak{g}_{\mathbb{C}}(A)$ determined by

$$\omega_0(e_i) = -f_i \ , \ \omega_0(f_i) = -e_i \ , \ \omega_0(h) = -h \ \forall\, h \in \mathfrak{h}.$$

Call ω_0 the *compact involution* of $\mathfrak{g}_{\mathbb{C}}(A)$ and $\mathfrak{k}_{\mathbb{C}}(A) := \mathrm{Fix}\ \omega_0$ the *maximal compact subalgebra* of $\mathfrak{g}_{\mathbb{C}}(A)$.

Let $\mathfrak{g}(A)$ be the split real form of $\mathfrak{g}_{\mathbb{C}}(A)$, i.e., the real Kac-Moody algebra obtained as the fixed points of complex conjugation $\bar{\cdot}$ acting naturally on the complex vector space underlying $\mathfrak{g}_{\mathbb{C}}(A)$. Let $\omega_{\mathbb{C}}$ and ω denote the Chevalley involutions on these Kac-Moody algebras. Then one has

$$\mathfrak{g}_{\mathbb{C}}(A) \supset \mathrm{Fix}\ \omega_0 \cong \mathrm{Fix}\ \omega \oplus i\omega_{-1},$$

where ω_{-1} denotes the -1 eigenspace of ω on $\mathfrak{g}(A)$. The fixed point subalgebra $\mathfrak{k}(A) = \mathrm{Fix}\ \omega$ is called the *maximal compact subalgebra* of $\mathfrak{g}(A)$.

A Kac-Moody algebra $\mathfrak{g}(A)$ is called *simply laced* if the off-diagonal entries of its generalized Cartan matrix are only 0 or -1.

Theorem 2.1 (Berman). *Let $\mathfrak{g}(A)$ be a simply laced real Kac-Moody algebra and $\mathfrak{k}$ its maximal compact subalgebra. Then $\mathfrak{k}$ is isomorphic to the free Lie algebra over $\mathbb{R}$ of generators $X_1, \ldots, X_n$ modulo the ideal generated by the relations*

$$[X_i, [X_i, X_j]] = -X_j \quad \text{if } a_{ij} = -1, \quad \text{and} \quad [X_i, X_j] = 0 \quad \text{if } a_{ij} = 0$$

via the isomorphism given by $X_i \mapsto e_i - f_i$.

Proof. See [1], also [2] and [7]. ■

Definition 2.2. A representation $\rho\colon \mathfrak{k} \to \mathrm{End}(\mathbb{C}^s)$ is called a *generalized spin representation* if for the generators $X_1, \ldots, X_n$ of $\mathfrak{k}$ one has

$$\rho(X_i)^2 = -\frac{1}{4}\mathrm{id}_s \quad \forall\, i = 1, \ldots, n.$$

Proposition 2.3. *Let $\rho\colon \mathfrak{k} \to \mathrm{End}(\mathbb{C}^s)$ be a generalized spin representation and denote by $[A, B] := AB - BA$ the commutator and by $\{A, B\} := AB + BA$ the anti-commutator. Then for $1 \le i \ne j \le n$ one has*

$$[\rho(X_i), \rho(X_j)] = 0 \text{ if } a_{ij} = 0 \ \Leftrightarrow\ (i, j) \text{ do not form an edge of the Dynkin diagram}$$
$$\{\rho(X_i), \rho(X_j)\} = 0 \text{ if } a_{ij} = -1 \ \Leftrightarrow\ (i, j) \text{ form an edge of the Dynkin diagram.}$$

Proof. If (i, j) do not form an edge of the Dynkin diagram then $a_{ij} = 0$ and so $[X_i, X_j] = 0$ according to Theorem 2.1 which is carried over to $\mathrm{End}(\mathbb{C}^s)$, since ρ is a homomorphism. If (i, j) form an edge, which is to say $a_{ij} = -1$, then by Theorem 2.1 one has $[X_i, [X_i, X_j]] = -X_j$ and setting $A = \rho(X_i)$, $B = \rho(X_j)$ one computes in $\mathrm{End}(\mathbb{C}^s)$:

$$\begin{aligned}&[A,[A,B]] = -B \;\Leftrightarrow\; A^2B - ABA - ABA + BA^2 = -B\\ &\Leftrightarrow\; -\frac{1}{4}B - 2ABA - \frac{1}{4}B = -B \;\Leftrightarrow\; -2ABA = -\frac{1}{2}B \quad \vert \cdot A \text{ from the right}\\ &\Leftrightarrow\; \frac{1}{2}AB = -\frac{1}{2}BA \;\Leftrightarrow\; AB + BA = 0.\end{aligned}$$

Note that multiplication with A preserves equivalence because A is invertible, since $A^{-1} = -4A$. ■

Corollary 2.4. *Given matrices* $A_1, \dots, A_n \in \mathbb{C}^{s\times s}$ *with*
(i) $A_i^2 = -\frac{1}{4}\,\mathrm{id}_s$,
(ii) $[A_i, A_j] = 0$, *if* (i,j) *do not form an edge of the Dynkin diagram,*
(iii) $\{A_i, A_j\} = 0$, *if* (i,j) *form an edge of the Dynkin diagram,*
the extension of the map $X_i \mapsto A_i$ *defines a generalized spin representation* ρ *from* $\mathfrak{k}$ *on* $\mathbb{C}^s$.

Proof. (i) is a necessary condition by the definition of spin representations. Assertion (ii) ensures that the commutation relations between X_i, X_j are respected by ρ if (i,j) do not form an edge, because in this case $[X_i, X_j] = 0$. Finally, (iii) ensures that for $a_{ij} \neq 0$ the relation $[X_i,[X_i,X_j]] = -X_j$ for $i \neq j$ is respected by ρ since according to the proof of Proposition 2.3 the condition $\{A,B\} = 0$ is equivalent to $[A,[A,B]] = -B$ as long as $A^2 = B^2 = -\frac{1}{4}\mathrm{id}_s$. ■

The existence of generalized spin representations has been established in [7].

Theorem 2.5. *For* $1 \le r < n$ *let* $\mathfrak{k}_{\le r} := \langle X_1, \dots, X_r\rangle$ *denote the subalgebra of* $\mathfrak{k}$ *that is generated by the first* r *generators. Furthermore, let* $\rho\colon \mathfrak{k}_{\le r} \to \mathrm{End}(\mathbb{C}^s)$ *be a generalized spin representation as in Definition* 2.2.

If X_{r+1} *centralizes* $\mathfrak{k}_{\le r}$, *that is to say* X_{r+1} *commutes with all generators* $X_1, \dots, X_r$, *then there exists a generalized spin representation* $\rho'\colon \mathfrak{k}_{\le r+1} \to \mathrm{End}(\mathbb{C}^s)$ *with* $\rho'_{|\mathfrak{k}_{\le r}} = \rho$ *given by sending* X_{r+1} *to* $\frac{1}{2}i \cdot \mathrm{id}_s$.

If X_{r+1} *does not centralize* $\mathfrak{k}_{\le r}$, *then* ρ *can be extended to a generalized spin representation* $\rho'\colon \mathfrak{k}_{\le r+1} \to \mathrm{End}(\mathbb{C}^s \oplus \mathbb{C}^s)$. *For this define a sign automorphism* $s_0\colon \mathfrak{k}_{\le r} \to \mathfrak{k}_{\le r}$ *by*

$$s_0(X_i) = \begin{cases} X_i, & \text{if } (i, r+1) \text{ do not form an edge of the Dynkin diagram,}\\ -X_i, & \text{if } (i, r+1) \text{ form an edge of the Dynkin diagram,}\end{cases}$$

and define the extension via

$$\rho'_{|\mathfrak{k}_{\le r}} = \rho \oplus \rho \circ s_0 \quad \text{and} \quad \rho'(X_{r+1}) = \frac{1}{2}\,\mathrm{id}_s \otimes \begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}.$$

Proof. See [7, Theorem 3.9]. ■

Definition 2.6. Given a graph $G = (V, E)$ with vertices V and edges E, a subset $M \subset V$ is called a *coclique* if the subgraph that is induced from G on M does not contain any edges. A coclique M of G is called *maximal* if it is maximal with respect to inclusion.

Corollary 2.7. *Given a simply laced Kac-Moody algebra $\mathfrak{g}(A)$ of rank n and a maximal coclique of size r, then there exists a generalized spin representation $\rho:\ \mathfrak{k} \to \mathrm{End}\,(\mathbb{C}^s)$, where $s = 2^{n-r}$, with compact image.*

Proof. See [7, Corollary 3.10 and Theorem 3.14]. ■

3. Extending a generalized $\frac{1}{2}$-spin representation — following Kleinschmidt and Nicolai

Throughout this section let $\mathfrak{g}$ be a simply laced split real Kac-Moody algebra with maximal compact subalgebra $\mathfrak{k}$. By Corollary 2.7 there exists a generalized $\frac{1}{2}$-spin representation $\rho\colon \mathfrak{k} \to \mathrm{End}(\mathbb{C}^l)$. In this section we make use of Clifford algebras in order to define higher generalized spin representations as carried out by Kleinschmidt and Nicolai [9] for E_{10}.

Let $V \otimes S$ be the tensor product of two $\mathbb{R}$-vector spaces V with basis $\left\{e^1, \dots, e^k\right\}$ and S with basis $\{f_1, \dots, f_l\}$. Then $\{e^i \otimes f_j \mid 1 \le i \le k,\ 1 \le j \le l\}$ is a natural $\mathbb{R}$-basis of $V \otimes S$. Endow V with a nondegenerate symmetric bilinear form q_1 and S with a positive definite bilinear form q_2 such that the basis $\{f_1, \dots, f_l\}$ is orthonormal, i.e.,

$$q_2\,(f_\alpha, f_\beta) = \delta_{\alpha\beta} \qquad \text{for } \alpha, \beta \in \{1, \dots, l\}.$$

Let $\left(G^{ab}\right)_{1 \le a,b \le k}$ denote the Gram matrix of q_1 with respect to the basis $\left\{e^1, \dots, e^k\right\}$, i.e., the matrix whose components are given by

$$G^{ab} = q_1\left(e^a, e^b\right) \qquad \text{for } a, b \in \{1, \dots, k\}.$$

Define $q := q_1 \otimes q_2$ as the bilinear extension of q_1 and q_2 to $V \otimes S$ so that on the chosen basis $\{e^i \otimes f_j \mid 1 \le i \le k,\ 1 \le j \le l\}$ one has for $\alpha, \beta \in \{1, \dots, l\}$, and $a, b \in \{1, \dots, k\}$

$$q\left(e^a \otimes f_\alpha, e^b \otimes f_\beta\right) = q_1\left(e^a, e^b\right) \cdot q_2\,(f_\alpha, f_\beta) = G^{ab}\delta_{\alpha\beta}.$$

The bilinear form q induces a quadratic form

$$Q\colon V \otimes S \to \mathbb{R} : w \mapsto q(w, w).$$

One defines the Clifford algebra $\mathcal{S} = Cl(V \otimes S, Q)$ as the quotient of the tensor algebra $\mathcal{T}\,(V \otimes S)$ modulo the ideal I_Q generated by elements of the form

$$w \otimes w - \frac{1}{2}Q(w) \cdot 1\ ,\ w \in V \otimes S.$$

In $\mathcal{S}$ one therefore has $w^2 = \frac{1}{2}Q(w)$, which can be restated via polarization as

$$wv + vw \;=\; q(v, w). \tag{1}$$

On the level of the basis for $\alpha, \beta \in \{1, \ldots, l\}$, $a, b \in \{1, \ldots, k\}$ this reads as

$$\left(e^a \otimes f_\alpha\right)\left(e^b \otimes f_\beta\right) + \left(e^b \otimes f_\beta\right)\left(e^a \otimes f_\alpha\right) = G^{ab}\delta_{\alpha\beta},$$

which one may repackage in a compact notation by defining[1] for $\alpha \in \{1, \ldots, l\}$, $\mathcal{A} \in \{1, \ldots, k\}$

$$\phi_\alpha^{\mathcal{A}} \quad := \quad e^{\mathcal{A}} \otimes f_\alpha, \tag{2}$$

thus yielding the identity for $\alpha, \beta \in \{1, \ldots, l\}$, $\mathcal{A}, \mathcal{B} \in \{1, \ldots, k\}$

$$\left\{\phi_\alpha^{\mathcal{A}}\,,\,\phi_\beta^{\mathcal{B}}\right\} := \phi_\alpha^{\mathcal{A}}\phi_\beta^{\mathcal{B}} + \phi_\beta^{\mathcal{B}}\phi_\alpha^{\mathcal{A}} = G^{\mathcal{AB}}\delta_{\alpha\beta}. \tag{3}$$

Since $\{e^i \otimes f_j \mid 1 \leq i \leq k,\ 1 \leq j \leq l\}$ is a basis of $V \otimes S$, the set $\left\{\phi_\alpha^{\mathcal{A}} \mid 1 \leq \mathcal{A} \leq k,\right.$ $\left.1 \leq \alpha \leq l\right\}$ is a generating set of the $\mathbb{R}$-algebra $\mathcal{S}$.

Lemma 3.1. *For $X, Y \in \mathbb{R}^{k \times k}$ and $S, T \in \mathbb{R}^{l \times l}$ consider the following elements of $\mathcal{S}$:*

$$\hat{A} := \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} X_{\mathcal{AB}} S^{\alpha\beta} \phi_\alpha^{\mathcal{A}} \phi_\beta^{\mathcal{B}}\,, \qquad \hat{B} := \sum_{\mathcal{C},\mathcal{D}=1}^{k} \sum_{\gamma,\delta=1}^{l} Y_{\mathcal{CD}} T^{\gamma\delta} \phi_\gamma^{\mathcal{C}} \phi_\delta^{\mathcal{D}}\,.$$

Under the hypothesis that for all $\alpha, \beta \in \{1, \ldots, l\}$ and for all $\mathcal{A}, \mathcal{B} \in \{1, \ldots, k\}$

$$X_{\mathcal{AB}} S^{\alpha\beta} = -X_{\mathcal{BA}} S^{\beta\alpha} \quad \textit{and} \quad Y_{\mathcal{AB}} T^{\alpha\beta} = -Y_{\mathcal{BA}} T^{\beta\alpha} \tag{4}$$

the commutator of $\hat{A}$ and $\hat{B}$ is equal to

$$\left[\hat{A}, \hat{B}\right] = \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^{\mathcal{A}} \left([X,Y]_{\mathcal{AB}} \{S,T\}^{\alpha\beta} + \{X,Y\}_{\mathcal{AB}} [S,T]^{\alpha\beta} \right) \phi_\beta^{\mathcal{B}}\,, \tag{5}$$

where (anti-)commutators of X and Y, resp. S and T are taken with respect to the bilinear forms as follows:

$$[X,Y]_{\mathcal{AB}} \quad = \quad \sum_{\mathcal{C},\mathcal{D}=1}^{k} \left(X_{\mathcal{AC}} G^{\mathcal{CD}} Y_{\mathcal{DB}} - Y_{\mathcal{AC}} G^{\mathcal{CD}} X_{\mathcal{DB}} \right), \tag{6}$$

$$\{X,Y\}_{\mathcal{AB}} \quad = \quad \sum_{\mathcal{C},\mathcal{D}=1}^{k} \left(X_{\mathcal{AC}} G^{\mathcal{CD}} Y_{\mathcal{DB}} + Y_{\mathcal{AC}} G^{\mathcal{CD}} X_{\mathcal{DB}} \right), \tag{7}$$

$$[S,T]^{\alpha\beta} \quad = \quad \sum_{\gamma,\delta=1}^{l} \left(S^{\alpha\gamma} \delta_{\gamma\delta} T^{\delta\beta} - T^{\alpha\gamma} \delta_{\gamma\delta} S^{\delta\beta} \right), \tag{8}$$

$$\{S,T\}^{\alpha\beta} \quad = \quad \sum_{\gamma,\delta=1}^{l} \left(S^{\alpha\gamma} \delta_{\gamma\delta} T^{\delta\beta} + T^{\alpha\gamma} \delta_{\gamma\delta} S^{\delta\beta} \right). \tag{9}$$

[1] So far our use of calligraphic letters for indices has no additional meaning. When studying coordinates of spin $\frac{5}{2}$-representations these calligraphic letters will convey an additional structure that fits with the notation introduced here. We refer to Remark 6.3 for details.

Remark 3.2. **(1)** On the level of the tensor product matrices, hypothesis (4) simply requires anti-symmetry: $X^T \otimes S^T = -X \otimes S$.

(2) The lemma can be found as [9, (4.17), p. 13 and footnote 10, p. 14].

Proof of Lemma 3.1. One computes

$$\begin{aligned}
&[X,Y]_{\mathcal{AB}}\{S,T\}^{\alpha\beta}+\{X,Y\}_{\mathcal{AB}}[S,T]^{\alpha\beta} = \big((XY)_{\mathcal{AB}}-(YX)_{\mathcal{AB}}\big)\big((ST)^{\alpha\beta}+(TS)^{\alpha\beta}\big)\\
&\quad + \big((XY)_{\mathcal{AB}}+(YX)_{\mathcal{AB}}\big)\big((ST)^{\alpha\beta}-(TS)^{\alpha\beta}\big)\\
&= (XY)_{\mathcal{AB}}(ST)^{\alpha\beta}+(XY)_{\mathcal{AB}}(TS)^{\alpha\beta}-(YX)_{\mathcal{AB}}(ST)^{\alpha\beta}-(YX)_{\mathcal{AB}}(TS)^{\alpha\beta}\\
&\quad +(XY)_{\mathcal{AB}}(ST)^{\alpha\beta}-(XY)_{\mathcal{AB}}(TS)^{\alpha\beta}+(YX)_{\mathcal{AB}}(ST)^{\alpha\beta}-(YX)_{\mathcal{AB}}(TS)^{\alpha\beta}\\
&= 2\,(XY)_{\mathcal{AB}}(ST)^{\alpha\beta}-2\,(YX)_{\mathcal{AB}}(TS)^{\alpha\beta}, \qquad (10)
\end{aligned}$$

where in analogy to the (anti-)commutators one abbreviates

$$(XY)_{\mathcal{AB}} = \sum_{\mathcal{C},\mathcal{D}=1}^{k} X_{\mathcal{AC}}G^{\mathcal{CD}}Y_{\mathcal{DB}}, \qquad (YX)_{\mathcal{AB}} = \sum_{\mathcal{C},\mathcal{D}=1}^{k} Y_{\mathcal{AC}}G^{\mathcal{CD}}X_{\mathcal{DB}},$$

$$(ST)^{\alpha\beta} = \sum_{\gamma,\delta=1}^{l} S^{\alpha\gamma}\delta_{\gamma\delta}T^{\delta\beta}, \qquad (TS)^{\alpha\beta} = \sum_{\gamma,\delta=1}^{l} T^{\alpha\gamma}\delta_{\gamma\delta}S^{\delta\beta}.$$

Several applications of equality (3) yield

$$\begin{aligned}
&\big[\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta,\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta\big] = \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta - \phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta\\
&= \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta + \phi^{\mathcal{C}}_\gamma\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{D}}_\delta\phi^{\mathcal{B}}_\beta - G^{\mathcal{DA}}\delta_{\delta\alpha}\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{B}}_\beta\\
&= \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta - \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta\phi^{\mathcal{B}}_\beta + G^{\mathcal{CA}}\delta_{\gamma\alpha}\phi^{\mathcal{D}}_\delta\phi^{\mathcal{B}}_\beta - G^{\mathcal{DA}}\delta_{\delta\alpha}\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{B}}_\beta\\
&= \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta + \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{B}}_\beta\phi^{\mathcal{D}}_\delta - G^{\mathcal{DB}}\delta_{\delta\beta}\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{C}}_\gamma + G^{\mathcal{CA}}\delta_{\gamma\alpha}\phi^{\mathcal{D}}_\delta\phi^{\mathcal{B}}_\beta - G^{\mathcal{DA}}\delta_{\delta\alpha}\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{B}}_\beta\\
&= \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta - \phi^{\mathcal{A}}_\alpha\phi^{\mathcal{B}}_\beta\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{D}}_\delta + G^{\mathcal{CB}}\delta_{\gamma\beta}\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{D}}_\delta - G^{\mathcal{DB}}\delta_{\delta\beta}\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{C}}_\gamma\\
&\quad + G^{\mathcal{CA}}\delta_{\gamma\alpha}\phi^{\mathcal{D}}_\delta\phi^{\mathcal{B}}_\beta - G^{\mathcal{DA}}\delta_{\delta\alpha}\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{B}}_\beta\\
&= G^{\mathcal{CB}}\delta_{\gamma\beta}\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{D}}_\delta - G^{\mathcal{DB}}\delta_{\delta\beta}\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{C}}_\gamma + G^{\mathcal{CA}}\delta_{\gamma\alpha}\phi^{\mathcal{D}}_\delta\phi^{\mathcal{B}}_\beta - G^{\mathcal{DA}}\delta_{\delta\alpha}\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{B}}_\beta\\
&= G^{\mathcal{BC}}\delta_{\beta\gamma}\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{D}}_\delta - G^{\mathcal{DB}}\delta_{\delta\beta}\phi^{\mathcal{A}}_\alpha\phi^{\mathcal{C}}_\gamma + G^{\mathcal{CA}}\delta_{\gamma\alpha}\phi^{\mathcal{D}}_\delta\phi^{\mathcal{B}}_\beta - G^{\mathcal{DA}}\delta_{\delta\alpha}\phi^{\mathcal{C}}_\gamma\phi^{\mathcal{B}}_\beta
\end{aligned}$$

Extending the above commutator linearly then yields

$$\begin{aligned}
\big[\hat{A},\hat{B}\big] = \sum_{\substack{\mathcal{A},\mathcal{B},\mathcal{C},\\ \mathcal{D}=1}}^{k}\ \sum_{\substack{\alpha,\beta,\gamma,\\ \delta=1}}^{l} &\phi^{\mathcal{A}}_\alpha X_{\mathcal{AB}}G^{\mathcal{BC}}Y_{\mathcal{CD}}S^{\alpha\beta}\delta_{\beta\gamma}T^{\gamma\delta}\phi^{\mathcal{D}}_\delta - \phi^{\mathcal{A}}_\alpha Y_{\mathcal{CD}}G^{\mathcal{DB}}X_{\mathcal{AB}}S^{\alpha\beta}T^{\gamma\delta}\delta_{\delta\beta}\phi^{\mathcal{C}}_\gamma\\
&+\phi^{\mathcal{D}}_\delta Y_{\mathcal{CD}}G^{\mathcal{CA}}X_{\mathcal{AB}}S^{\alpha\beta}\delta_{\alpha\gamma}T^{\gamma\delta}\phi^{\mathcal{B}}_\beta - \phi^{\mathcal{C}}_\gamma Y_{\mathcal{CD}}G^{\mathcal{DA}}X_{\mathcal{AB}}T^{\gamma\delta}\delta_{\delta\alpha}S^{\alpha\beta}\phi^{\mathcal{B}}_\beta
\end{aligned}$$

which can then be transformed by renaming indices and using symmetry of the bilinear forms and anti-symmetry of the tensor product matrices (cf. (4)):

$$\begin{aligned}
\big[\hat{A},\hat{B}\big] = \sum_{\substack{\mathcal{A},\mathcal{B},\mathcal{C},\\ \mathcal{D}=1}}^{k}\ \sum_{\substack{\alpha,\beta,\gamma,\\ \delta=1}}^{l} &\phi^{\mathcal{A}}_\alpha\left(X_{\mathcal{AB}}G^{\mathcal{BC}}Y_{\mathcal{CD}}S^{\alpha\beta}\delta_{\beta\gamma}T^{\gamma\delta} - Y_{\mathcal{DC}}G^{\mathcal{CB}}X_{\mathcal{AB}}S^{\alpha\beta}T^{\delta\gamma}\delta_{\gamma\beta}\right)\phi^{\mathcal{D}}_\delta\\
&+\phi^{\mathcal{D}}_\delta\left(Y_{\mathcal{CD}}G^{\mathcal{CA}}X_{\mathcal{AB}}S^{\alpha\beta}\delta_{\alpha\gamma}T^{\gamma\delta} - Y_{\mathcal{DC}}G^{\mathcal{CA}}X_{\mathcal{AB}}T^{\delta\gamma}\delta_{\gamma\alpha}S^{\alpha\beta}\right)\phi^{\mathcal{B}}_\beta
\end{aligned}$$

$$\overset{(4)}{=} \sum_{\substack{\mathcal{A},\mathcal{B},\mathcal{C},\\ \mathcal{D}=1}}^{k} \sum_{\substack{\alpha,\beta,\gamma,\\ \delta=1}}^{l} \phi_\alpha^{\mathcal{A}} \left(X_{\mathcal{AB}} G^{\mathcal{BC}} Y_{\mathcal{CD}} S^{\alpha\beta} \delta_{\beta\gamma} T^{\gamma\delta} + X_{\mathcal{AB}} G^{\mathcal{BC}} Y_{\mathcal{CD}} S^{\alpha\beta} \delta_{\beta\gamma} T^{\gamma\delta} \right) \phi_\delta^{\mathcal{D}}$$
$$+ \phi_\delta^{\mathcal{D}} \left(-Y_{\mathcal{DC}} G^{\mathcal{CA}} X_{\mathcal{AB}} T^{\delta\gamma} \delta_{\gamma\alpha} S^{\alpha\beta} - Y_{\mathcal{DC}} G^{\mathcal{CA}} X_{\mathcal{AB}} T^{\delta\gamma} \delta_{\gamma\alpha} S^{\alpha\beta} \right) \phi_\beta^{\mathcal{B}}$$
$$= 2 \sum_{\mathcal{A},\mathcal{D}=1}^{k} \sum_{\alpha,\delta=1}^{l} \phi_\alpha^{\mathcal{A}} \left(XY\right)_{\mathcal{AD}} \left(ST\right)^{\alpha\delta} \phi_\delta^{\mathcal{D}} - 2 \sum_{B,\mathcal{D}=1}^{k} \sum_{\beta,\delta=1}^{l} \phi_\delta^{\mathcal{D}} \left(YX\right)_{\mathcal{DB}} \left(TS\right)^{\delta\beta} \phi_\beta^{\mathcal{B}}$$
$$= 2 \sum_{\mathcal{A},\mathcal{D}=1}^{k} \sum_{\alpha,\delta=1}^{l} \phi_\alpha^{\mathcal{A}} \left[\left(XY\right)_{\mathcal{AD}} \left(ST\right)^{\alpha\delta} - \left(YX\right)_{\mathcal{AD}} \left(TS\right)^{\alpha\delta} \right] \phi_\delta^{\mathcal{D}}$$

which in view of (10) completes the proof. ∎

Remark 3.3. In fact, we never used in the proof of Lemma 3.1 that the form q_2 was anisotropic. The computations hold in general for arbitrary non-degenerate forms. The definiteness of q_2 only becomes relevant now, when using the preceding lemma in order to construct various representations of $\mathfrak{k}$. The generalized spin representation $\rho\colon \mathfrak{k} \to \mathrm{End}(\mathbb{C}^s)$ from Corollary 2.7 provides anti-symmetric real $2s \times 2s$-matrices

$$\Gamma\left(\alpha_i\right) := 2\rho\left(X_i\right) \tag{11}$$

for all simple roots $\alpha_1, \ldots, \alpha_n$ of $\mathfrak{g}$. Taking these as the matrix S in Lemma 3.1

$$\hat{A} := \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l:=2s} X_{\mathcal{AB}} S^{\alpha\beta} \phi_\alpha^{\mathcal{A}} \phi_\beta^{\mathcal{B}}$$

leaves one with the task of finding suitable symmetric matrices for X.
Note that, since we assumed q_2 to be anisotropic and conducted our computations with respect to an orthonormal basis for that form, the formulae given in (8) and (9) actually coincide with the standard definition of commutators and anti-commutators of matrices. In particular, the results from Proposition 2.3 are applicable.

Definition 3.4. Now let λ denote the finite set of real roots

$$\lambda := \{\alpha_i \mid 1 \le i \le n\} \cup \left\{ \alpha_i + \alpha_j \in \Delta^{\mathrm{re}} \;\middle|\; \begin{array}{l} (i,j) \text{ form an edge of} \\ \text{the Dynkin diagram} \end{array} \right\}. \tag{12}$$

Note that for $\alpha, \beta \in \lambda$ one has $(\alpha|\beta) \in \{\pm 1, 0\}$.

Proposition 3.5. *A map $X\colon \lambda \to \mathbb{R}^{k\times k}$ that takes values in the set of symmetric matrices which satisfy for all $\alpha, \beta \in \lambda$*

$$[X(\alpha), X(\beta)] = 0, \qquad \text{if } (\alpha|\beta) = 0, \tag{13}$$
$$\{X(\alpha), X(\beta)\} = \frac{1}{2} X\left(\alpha \pm \beta\right), \qquad \text{if } (\alpha|\beta) = \mp 1 \text{ and } \alpha \pm \beta \in \lambda, \tag{14}$$

(*with respect to the commutator and anti-commutator convention from* (6) *and* (7)) *together with the anti-symmetric real matrices* $\Gamma\left(\alpha_1\right), \ldots, \Gamma\left(\alpha_n\right)$ *from* (11) *turns the ansatz*

$$\widehat{\mathrm{J}}(\alpha_i) = \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} X_{\mathcal{AB}}(\alpha_i)\Gamma^{\alpha\beta}(\alpha_i)\phi_\alpha^{\mathcal{A}}\phi_\beta^{\mathcal{B}}$$

into a finite-dimensional representation σ *of* $\mathfrak{k}$ *by defining* σ *on the Berman generators* $X_1, \ldots, X_n$ *of* $\mathfrak{k}$ *as* $\sigma(X_i) := \widehat{\mathrm{J}}(\alpha_i)$*.*

Remark 3.6. The observation that (13) and (14) are the key identities for extending generalized spin representations has been made in [9, (4.23), p. 15; (5.1), p. 18].

Proof of Proposition 3.5. By the homomorphism theorem it suffices to establish that the commutator $\left[\widehat{\mathrm{J}}(\alpha_i), \widehat{\mathrm{J}}(\alpha_j)\right]$ satisfies the relations from Theorem 2.1. By Lemma 3.1 one has

$$\begin{aligned}\left[\widehat{\mathrm{J}}(\alpha_i), \widehat{\mathrm{J}}(\alpha_j)\right] &= \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^{\mathcal{A}} \left[X(\alpha_i), X(\alpha_j)\right]_{\mathcal{AB}} \left\{\Gamma(\alpha_i), \Gamma(\alpha_j)\right\}^{\alpha\beta} \phi_\beta^{\mathcal{B}} \\ &+ \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^{\mathcal{A}} \left\{X(\alpha_i), X(\alpha_j)\right\}_{\mathcal{AB}} \left[\Gamma(\alpha_i), \Gamma(\alpha_j)\right]^{\alpha\beta} \phi_\beta^{\mathcal{B}} .\end{aligned}$$

In case (i, j) is not an edge of the Dynkin diagram this yields

$$\left[\widehat{\mathrm{J}}(\alpha_i), \widehat{\mathrm{J}}(\alpha_j)\right] = 0$$

as desired, because in this case one has $\left[\Gamma(\alpha_i), \Gamma(\alpha_j)\right] = 0$ by Proposition 2.3 and, furthermore, $(\alpha_i|\alpha_j) = 0$, i.e., $\left[X(\alpha_i), X(\alpha_j)\right] = 0$ by hypothesis (13).
In case (i, j) is an edge of the Dynkin diagram one has $\left\{\Gamma(\alpha_i), \Gamma(\alpha_j)\right\} = 0$ by Proposition 2.3 and so

$$\left\{X(\alpha_i), X(\alpha_j)\right\} = \frac{1}{2} X(\alpha_i + \alpha_j)$$

by hypothesis (14). Thus,

$$\left[\widehat{\mathrm{J}}(\alpha_i), \widehat{\mathrm{J}}(\alpha_j)\right] = \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^{\mathcal{A}} \cdot \frac{1}{2} X(\alpha_i + \alpha_j)_{\mathcal{AB}} \left[\Gamma(\alpha_i), \Gamma(\alpha_j)\right]^{\alpha\beta} \phi_\beta^{\mathcal{B}} .$$

Applying the commutator with $\widehat{\mathrm{J}}(\alpha_i)$ again according to Lemma 3.1 yields

$$\begin{aligned}&\left[\widehat{\mathrm{J}}(\alpha_i), \left[\widehat{\mathrm{J}}(\alpha_i), \widehat{\mathrm{J}}(\alpha_j)\right]\right] = \\ &= \frac{1}{2} \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^{\mathcal{A}} \left[X(\alpha_i), X(\alpha_i + \alpha_j)\right]_{\mathcal{AB}} \left\{\Gamma(\alpha_i), \left[\Gamma(\alpha_i), \Gamma(\alpha_j)\right]\right\}^{\alpha\beta} \phi_\beta^{\mathcal{B}} \\ &+ \frac{1}{2} \sum_{\mathcal{A},\mathcal{B}=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^{\mathcal{A}} \left\{X(\alpha_i), X(\alpha_i + \alpha_j)\right\}_{\mathcal{AB}} \left[\Gamma(\alpha_i), \left[\Gamma(\alpha_i), \Gamma(\alpha_j)\right]\right]^{\alpha\beta} \phi_\beta^{\mathcal{B}} .\end{aligned}$$

Since $(\alpha_i|\alpha_i+\alpha_j)=1$, by hypothesis (14) one has

$$\{X(\alpha_i), X(\alpha_i+\alpha_j)\} = \frac{1}{2}X(\alpha_j).$$

Moreover, $[\Gamma(\alpha_i), [\Gamma(\alpha_i), \Gamma(\alpha_j)]] = -4\Gamma(\alpha_j)$ because $\rho(X_i)=\frac{1}{2}\Gamma(\alpha_i)$ is a generalized spin representation of $\mathfrak{k}$ (cf. Proposition 2.3 and its proof). Furthermore,

$$\begin{aligned}\{\Gamma(\alpha_i), [\Gamma(\alpha_i), \Gamma(\alpha_j)]\} &= \Gamma(\alpha_i)\Gamma(\alpha_i)\Gamma(\alpha_j) - \Gamma(\alpha_i)\Gamma(\alpha_j)\Gamma(\alpha_i) \\ &\quad + \Gamma(\alpha_i)\Gamma(\alpha_j)\Gamma(\alpha_i) - \Gamma(\alpha_j)\Gamma(\alpha_i)\Gamma(\alpha_i) = 0,\end{aligned}$$

because $\Gamma(\alpha_i)\Gamma(\alpha_i)$ commutes with $\Gamma(\alpha_j)$ (cf. Corollary 2.4). Altogether,

$$\begin{aligned}\left[\widehat{\mathrm{J}}(\alpha_i), \left[\widehat{\mathrm{J}}(\alpha_i), \widehat{\mathrm{J}}(\alpha_j)\right]\right] &= \frac{1}{2}\sum_{\mathcal{A},\mathcal{B}=1}^{k}\sum_{\alpha,\beta=1}^{l} \phi_\alpha^{\mathcal{A}}\frac{1}{2}X(\alpha_j)_{\mathcal{AB}}\left(-4\Gamma(\alpha_j)\right)^{\alpha\beta}\phi_\beta^{\mathcal{B}} \\ &= -\sum_{\mathcal{A},\mathcal{B}} \phi_\alpha^{\mathcal{A}} X(\alpha_j)_{\mathcal{AB}}\Gamma(\alpha_j)^{\alpha\beta}\phi_\beta^{\mathcal{B}} = -\widehat{\mathrm{J}}(\alpha_j),\end{aligned}$$

again as desired in view of Theorem 2.1. One concludes that the assignment

$$\sigma(X_i) := \widehat{\mathrm{J}}(\alpha_i)$$

defines a finite-dimensional representation of $\mathfrak{k}$. ∎

4. Extending a generalized $\frac{1}{2}$-spin representation – a coordinate-free approach

In this section we discuss a coordinate-free version of Proposition 3.5. We stress that in this section we make use of the usual definition of (anti-)commutators: For $A, B \in End(V)$ let $[A,B] = A\circ B - B\circ A \in End(V)$ and $\{A,B\} = A\circ B + B\circ A \in End(V)$ denote the commutator and the anti-commutator respectively, where $\circ$ denotes composition of (linear) maps.

Recall the definition of the set λ in Definition 3.4. Then the following holds:

Proposition 4.1. *A map $X\colon \lambda \to \mathrm{End}(V)$ satisfying for all $\alpha, \beta \in \lambda$*

$$\begin{aligned}[X(\alpha), X(\beta)] &= 0 && \text{if } (\alpha|\beta) = 0 && (15)\\ \{X(\alpha), X(\beta)\} &= X(\alpha\pm\beta) && \text{if } (\alpha|\beta) = \mp 1 \text{ and } \alpha\pm\beta\in\lambda && (16)\end{aligned}$$

provides a finite-dimensional representation σ of $\mathfrak{k}$ via the assignment

$$\sigma(X_i) := X(\alpha_i)\otimes\Gamma(\alpha_i) \in \mathrm{End}(V\otimes S)$$

on the Berman generators $X_1, \dots, X_n$ of $\mathfrak{k}$, where the $\Gamma(\alpha_i)$, $1 \le i \le n$ are the anti-symmetric real matrices from (11).

Remark 4.2. **(1)** Defining $X\colon \lambda \to \mathrm{End}(\mathbb{R}) = \mathbb{R}$ as the constant map $X \equiv \frac{1}{2}$ provides the generalized spin representation from [7], cf. Corollary 2.4. On the other hand, in the approach taken by Kleinschmidt and Nicolai described in Proposition 3.5 one needs to define $X\colon \lambda \to \mathbb{R}$ as the constant map $X \equiv \frac{1}{4}$ in order to obtain the generalized spin representation from [7]. This difference in normalization stems from the differences in normalizations of the underlying Clifford algebras when comparing [7, Example 3.2] with (1) on page 920. Similar differences are visible in the formulae for the $\frac{3}{2}$-spin representations given in (18) on page 928 and (19) on page 929 below.

(2) Contrary to Proposition 3.5, the above coordinate-free version does not require the map $X\colon \lambda \to \mathrm{End}(V)$ to take images in the set of self-adjoint/symmetric operators.

(3) Paul Levy pointed out to us the following. Let W be the Weyl group of $\mathfrak{g}$ and let $\rho\colon W \to \mathrm{GL}(V)$ be a representation. The ansatz $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$ leads to

$$\begin{aligned}
&\rho(s_\alpha s_\beta) + \rho(s_\beta s_\alpha) - \rho(s_\alpha) - \rho(s_\beta) + \frac{1}{2}\,\mathrm{id}\\
&= \left(\rho(s_\alpha) - \frac{1}{2}\,\mathrm{id}\right)\left(\rho(s_\beta) - \frac{1}{2}\,\mathrm{id}\right) + \left(\rho(s_\beta) - \frac{1}{2}\,\mathrm{id}\right)\left(\rho(s_\alpha) - \frac{1}{2}\,\mathrm{id}\right)\\
&= \{X(\alpha), X(\beta)\} = X(\alpha+\beta) = \rho(s_{\alpha+\beta}) - \frac{1}{2}\,\mathrm{id} = \rho(s_\alpha s_\beta s_\alpha) - \frac{1}{2}\,\mathrm{id}
\end{aligned}$$

for each pair α, β forming an A_2-subdiagram. One concludes that

$$\{X(\alpha), X(\beta)\} = X(\alpha+\beta)$$

is in fact equivalent to

$$\rho(s_\alpha s_\beta s_\alpha) - \rho(s_\alpha s_\beta) - \rho(s_\beta s_\alpha) + \rho(s_\alpha) + \rho(s_\beta) - \mathrm{id} = 0. \tag{17}$$

Similar computations imply that in fact any case covered by (16) using the ansatz $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\,\mathrm{id}$ is equivalent to (17). Furthermore, one quickly computes that $[\rho(s_\alpha) - \mathrm{id}, \rho(s_\beta) - \mathrm{id}] = 0$ whenever $(\alpha|\beta) = 0$, because this is equivalent to $s_\alpha s_\beta = s_\beta s_\alpha$. We conclude that for the ansatz $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$ it suffices to check (17) for each pair α, β forming an A_2-subdiagram.

Paul Levy also pointed out to us that the identity

$$\rho(s_\alpha s_\beta s_\alpha) - \rho(s_\alpha s_\beta) - \rho(s_\beta s_\alpha) + \rho(s_\alpha) + \rho(s_\beta) - \mathrm{id} = 0$$

holds if and only if the given representation $W \geq \mathrm{Sym}_3 = \langle s_\alpha, s_\beta\rangle \to \mathrm{GL}(V)\colon w \mapsto \rho(w)$ does not contain a sign representation as an irreducible component. Indeed, among the irreducible representations of Sym_3 the trivial and the geometric representations satisfy (17) whereas the sign representation does not.

Proof of Proposition 4.1. By the homomorphism theorem it suffices to establish that the commutator $[\sigma(X_i), \sigma(X_j)]$ satisfies the relations from Theorem 2.1.

In case (i,j) do not form an edge, one computes the following:

$$\begin{aligned}
&[\sigma(X_i), \sigma(X_j)] \\
&= (X(\alpha_i) \otimes \Gamma(\alpha_i)) \circ (X(\alpha_j) \otimes \Gamma(\alpha_j)) - (X(\alpha_j) \otimes \Gamma(\alpha_j)) \circ (X(\alpha_i) \otimes \Gamma(\alpha_i)) \\
&= X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_j) \Gamma(\alpha_i) \\
&= X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) \\
&\qquad\qquad + X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_j) \Gamma(\alpha_i) \\
&= [X(\alpha_i), X(\alpha_j)] \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) + X(\alpha_j) X(\alpha_i) \otimes [\Gamma(\alpha_i), \Gamma(\alpha_j)] = 0,
\end{aligned}$$

because $[X(\alpha_i), X(\alpha_j)] = 0$ by hypothesis (15) and $[\Gamma(\alpha_i), \Gamma(\alpha_j)] = 0$ by Proposition 2.3. In case (i,j) is an edge, Proposition 2.3 and hypothesis (16) yield

$$\{\Gamma(\alpha_i), \Gamma(\alpha_j)\} = 0 \qquad \text{and} \qquad \{X(\alpha_i), X(\alpha_j)\} = X(\alpha_i + \alpha_j).$$

Hence

$$\begin{aligned}
&[\sigma(X_i), \sigma(X_j)] = X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_j) \Gamma(\alpha_i) \\
&= X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) + X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) \\
&\qquad\qquad - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_j) \Gamma(\alpha_i) \\
&= \{X(\alpha_i), X(\alpha_j)\} \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \{\Gamma(\alpha_i), \Gamma(\alpha_j)\} \\
&= X(\alpha_i + \alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j).
\end{aligned}$$

Moreover, since the matrices $\frac{1}{2}\Gamma(\alpha_1)$, ..., $\frac{1}{2}\Gamma(\alpha_n)$ provide a generalized spin representation, by definition one has $\Gamma(\alpha_i)^2 = 4\rho(X_i)^2 = -\mathrm{id}_S$ and by Proposition 2.3 the matrices $\Gamma(\alpha_i)$ and $\Gamma(\alpha_j)$ anti-commute. Therefore one has the following:

$$\begin{aligned}
&[\sigma(X_i), [\sigma(X_i), \sigma(X_j)]] \\
&= (X(\alpha_i) \otimes \Gamma(\alpha_i)) \circ (X(\alpha_i + \alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j)) \\
&\qquad\qquad - (X(\alpha_i + \alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j)) \circ (X(\alpha_i) \otimes \Gamma(\alpha_i)) \\
&= (X(\alpha_i) X(\alpha_i + \alpha_j)) \otimes (\Gamma(\alpha_i) \Gamma(\alpha_i) \Gamma(\alpha_j)) \\
&\qquad\qquad - (X(\alpha_i + \alpha_j) X(\alpha_i)) \otimes (\Gamma(\alpha_i) \Gamma(\alpha_j) \Gamma(\alpha_i)) \\
&= -(X(\alpha_i) X(\alpha_i + \alpha_j)) \otimes \Gamma(\alpha_j) - (X(\alpha_i + \alpha_j) X(\alpha_i)) \otimes \Gamma(\alpha_j) \\
&= -\{X(\alpha_i), X(\alpha_i + \alpha_j)\} \otimes \Gamma(\alpha_j) = -X(\alpha_j) \otimes \Gamma(\alpha_j) = -\sigma(X_j). \qquad \blacksquare
\end{aligned}$$

5. Towards $\frac{3}{2}$-spin representations

Let $V := \mathfrak{h}^*$. If the generalized Cartan matrix A is invertible, then $V = \mathrm{span}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_n\}$; otherwise V is of higher dimension $k := 2n - \mathrm{rk}(A)$. In both cases the invariant bilinear form on $\mathfrak{g}$ induces a nondegenerate bilinear form $(\cdot|\cdot)$ on V. Let v^1, ..., v^k be a basis of V and define

$$G^{ab} := \left(v^a, v^b\right).$$

That is, $(G^{ab})_{1\leq a,b\leq k}$ is the Gram matrix of the bilinear form $(\cdot|\cdot)$ on V with respect to the basis v^1,..., v^k. Moreover, define $(G_{ab})_{1\leq a,b\leq k} := {(G^{ab})_{1\leq a,b\leq k}}^{-1}$, i.e.,

$$\sum_{b=1}^{k} G^{ab} G_{bc} = \delta^a{}_c.$$

Note that the matrix $(G_{ab})_{1\le a,b\le k}$ is symmetric as well since inversion and transposition of matrices commute.

Proposition 5.1. *Let $\alpha = \sum_{i=1}^{k} \alpha_i v^i \in V = \mathfrak{h}^*$ be a real root*[2]*. Then the map*

$$X\colon \lambda \to \mathbb{R}^{k\times k};\quad \alpha \mapsto (X(\alpha)_{ab})_{1\le a,b\le k}$$

defined via

$$X(\alpha)_{ab} = -\frac{1}{2}\alpha_a\alpha_b + \frac{1}{4}G_{ab} \tag{18}$$

yields a set of matrices that satisfy hypotheses (13) *and* (14) *of Proposition* 3.5. *In particular, this provides a finite-dimensional representation of $\mathfrak{k}$ via $\sigma(X_i) = X(\boldsymbol{\alpha}_i) \otimes \Gamma(\boldsymbol{\alpha}_i)$.*

Remark 5.2. Formula (18) is [9, (4.21), p. 15].

Proof of Proposition 5.1. It suffices to establish the hypotheses of Proposition 3.5. Note first that the matrices $X(\alpha)$ are symmetric by definition. For $\alpha, \beta \in \lambda$ with $(\alpha|\beta) = 0$ one computes

$$\begin{aligned}
[X(\alpha), X(\beta)]_{ad} &\overset{(6)}{=} \sum_{b,c=1}^{k}\left(-\frac{1}{2}\alpha_a\alpha_b + \frac{1}{4}G_{ab}\right)G^{bc}\left(-\frac{1}{2}\beta_c\beta_d + \frac{1}{4}G_{cd}\right)\\
&\qquad - \sum_{b,c=1}^{k}\left(-\frac{1}{2}\beta_a\beta_b + \frac{1}{4}G_{ab}\right)G^{bc}\left(-\frac{1}{2}\alpha_c\alpha_d + \frac{1}{4}G_{cd}\right)\\
&= \sum_{b,c=1}^{k}\Big(\frac{1}{4}\alpha_a\alpha_b G^{bc}\beta_c\beta_d - \frac{1}{4}\beta_a\beta_b G^{bc}\alpha_c\alpha_d - \frac{1}{8}\alpha_a\alpha_b G^{bc}G_{cd} + \frac{1}{8}\beta_a\beta_b G^{bc}G_{cd}\\
&\qquad - \frac{1}{8}G_{ab}G^{bc}\beta_c\beta_d + \frac{1}{8}G_{ab}G^{bc}\alpha_c\alpha_d + \frac{1}{16}G_{ab}G^{bc}G_{cd} - \frac{1}{16}G_{ab}G^{bc}G_{cd}\Big)\\
&= \frac{1}{4}\alpha_a(\alpha|\beta)\beta_d - \frac{1}{4}\beta_a(\alpha|\beta)\alpha_d - \frac{1}{8}\alpha_a\alpha_d + \frac{1}{8}\beta_a\beta_d - \frac{1}{8}\beta_a\beta_d + \frac{1}{8}\alpha_a\alpha_d = 0.
\end{aligned}$$

Moreover, for $(\alpha|\beta) = \mp 1$ one computes

$$\begin{aligned}
\{X(\alpha), X(\beta)\}_{ad} &\overset{(7)}{=} \sum_{b,c=1}^{k}\left(-\frac{1}{2}\alpha_a\alpha_b + \frac{1}{4}G_{ab}\right)G^{bc}\left(-\frac{1}{2}\beta_c\beta_d + \frac{1}{4}G_{cd}\right)\\
&\qquad + \sum_{b,c=1}^{k}\left(-\frac{1}{2}\beta_a\beta_b + \frac{1}{4}G_{ab}\right)G^{bc}\left(-\frac{1}{2}\alpha_c\alpha_d + \frac{1}{4}G_{cd}\right)\\
&= \frac{1}{4}\alpha_a\beta_d(\alpha|\beta) - \frac{1}{8}\alpha_a\alpha_d - \frac{1}{8}\beta_a\beta_d + \frac{1}{16}G_{ab}G^{bc}G_{cd} + \frac{1}{4}\beta_a\alpha_d(\alpha|\beta)\\
&\qquad - \frac{1}{8}\beta_a\beta_d - \frac{1}{8}\alpha_a\alpha_d + \frac{1}{16}G_{ab}G^{bc}G_{cd}
\end{aligned}$$

[2] We warn the reader that in this proposition the α_i do *not* denote simple roots but instead the real coefficients of an arbitrary real root α expanded with respect to some basis of $\mathfrak{h}^*$. In order to avoid ambiguity we will denote simple roots by $\boldsymbol{\alpha}_i$ whenever they appear in the context of this proposition.

$$= \frac{1}{4}\left(-\alpha_a\alpha_d - \beta_a\beta_d \mp (\alpha_a\beta_d + \beta_a\alpha_d) + \frac{1}{2}G_{ad}\right)$$
$$= \frac{1}{2}\left(-\frac{1}{2}(\alpha_a \pm \beta_a)(\alpha_d \pm \beta_d) + \frac{1}{4}G_{ad}\right) = \frac{1}{2}X(\alpha \pm \beta)_{ad}. \quad \blacksquare$$

We conclude this section with the following coordinate-free version of Proposition 5.1.

Proposition 5.3. *For $V = \mathfrak{h}^*$ let $(\cdot|\cdot)$ denote the induced invariant bilinear form on $\mathfrak{h}^*$. Define $X\colon \Delta^{\mathrm{re}} \to \mathrm{End}(\mathfrak{h}^*)$ via*

$$\alpha \mapsto X(\alpha) \;:=\; -\alpha(\alpha|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}. \tag{19}$$

Then X satisfies (15) *and* (16) *for all real roots α, β with $(\alpha|\beta) \in \{0, \pm 1\}$ and thus provides a representation σ of $\mathfrak{k}$.*

Proof. First consider $\alpha, \beta \in \Delta^{\mathrm{re}}$ such that $(\alpha|\beta) = 0$. Then one has

$$\begin{aligned}
[X(\alpha), X(\beta)] &= \left(-\alpha(\alpha|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right)\left(-\beta(\beta|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right) \\
&\qquad - \left(-\beta(\beta|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right)\left(-\alpha(\alpha|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right) \\
&= \alpha(\alpha|\beta)(\beta|\cdot) - \frac{1}{2}\alpha(\alpha|\cdot) - \frac{1}{2}\beta(\beta|\cdot) + \frac{1}{4}id_{\mathfrak{h}^*} - \beta(\beta|\alpha)(\alpha|\cdot) \\
&\qquad + \frac{1}{2}\beta(\beta|\cdot) + \frac{1}{2}\alpha(\alpha|\cdot) - \frac{1}{4}id_{\mathfrak{h}^*} = 0.
\end{aligned}$$

Moreover, for $(\alpha|\beta) = \mp 1$ one has the following:

$$\begin{aligned}
\{X(\alpha), X(\beta)\} &= \left(-\alpha(\alpha|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right)\left(-\beta(\beta|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right) \\
&\qquad + \left(-\beta(\beta|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right)\left(-\alpha(\alpha|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*}\right) \\
&= \alpha(\alpha|\beta)(\beta|\cdot) - \frac{1}{2}\alpha(\alpha|\cdot) - \frac{1}{2}\beta(\beta|\cdot) + \frac{1}{4}id_{\mathfrak{h}^*} \\
&\qquad + \beta(\beta|\alpha)(\alpha|\cdot) - \frac{1}{2}\beta(\beta|\cdot) - \frac{1}{2}\alpha(\alpha|\cdot) + \frac{1}{4}id_{\mathfrak{h}^*} \\
&= \mp\alpha(\beta|\cdot) \mp \beta(\alpha|\cdot) - \alpha(\alpha|\cdot) - \beta(\beta|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*} \\
&= -\left(\pm\alpha(\beta|\cdot) \pm \beta(\alpha|\cdot) + \alpha(\alpha|\cdot) + \beta(\beta|\cdot)\right) + \frac{1}{2}id_{\mathfrak{h}^*} \\
&= -(\alpha \pm \beta)(\alpha \pm \beta|\cdot) + \frac{1}{2}id_{\mathfrak{h}^*} = X(\alpha \pm \beta). \quad \blacksquare
\end{aligned}$$

Remark 5.4. Note that the canonical (non-reduced geometric) Weyl group representation $\rho\colon W \to \mathrm{GL}(\mathfrak{h}^*)$ acts via $\rho(s_\alpha)(x) = x - (\alpha|x)\alpha$ and so one has $X(\alpha) = \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$. Therefore Remark 4.2 applies and the statement of Proposition 5.3 in fact follows from the observation that ρ (restricted to any standard subgroup Sym_3) does not contain the sign representation as an irreducible component.

6. Towards $\frac{5}{2}$-spin representations

Definition 6.1. Let $(T_{ab})_{a,b} \in \mathbb{R}^{k\times k}$ and $(U_{cd})_{c,d} \in \mathbb{R}^{k\times l}$. Then $\left(T_{(ab)}\right)_{a,b} \in \mathbb{R}^{k\times k}$ denotes the matrix with components

$$T_{(ab)} := \frac{1}{2}T_{ab} + \frac{1}{2}T_{ba}.$$

Moreover, for $1 \le a, b, c \le k$ and $1 \le d \le l$ define

$$T_{a(b}U_{c)d} := \frac{1}{2}T_{ab}U_{cd} + \frac{1}{2}T_{ac}U_{bd}.$$

This notation is called the *symmetrizer bracket.*

Lemma 6.2. *As in Section* 5 *let* v^1, ..., v^k *be a basis of* $\mathfrak{h}^*$, *let* $(G^{ab})_{a,b}$ *be the Gram matrix of the invariant form with respect to this basis, let* $(G_{ab})_{a,b}$ *be its inverse, let* $\alpha = \sum_{i=1}^k \alpha_i v^i, \beta = \sum_{i=1}^k \beta_i v^i \in \mathfrak{h}^*$, *and let*

$$\alpha^i := \sum_{j=1}^k G^{ij}\alpha_j, \qquad \beta^i := \sum_{j=1}^k G^{ij}\beta_j.$$

Then the following identities hold:

$$\sum_{g,h=1}^k \alpha^g\alpha^h G_{g(c}G_{d)h} = \alpha_c\alpha_d = \alpha_{(c}\alpha_{d)} \tag{20}$$

$$\sum_{g,h=1}^k \alpha^g\alpha^h \beta_{(g}G_{h)(c}\beta_{d)} = (\alpha|\beta)\,\alpha_{(c}\beta_{d)} \tag{21}$$

$$\sum_{e,f,g,h=1}^k \alpha_{(a}G_{b)(e}\alpha_{f)}G^{eg}G^{fh}\beta_{(g}G_{h)(c}\beta_{d)} = \frac{1}{2}\alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)} + \frac{1}{2}(\alpha|\beta)\,\alpha_{(a}G_{b)(c}\beta_{d)} \tag{22}$$

$$\sum_{e,f,g,h=1}^k G_{a(e}G_{f)b}G^{eg}G^{fh}\beta_{(g}G_{h)(c}\beta_{d)} = \beta_{(a}G_{b)(c}\beta_{d)} \tag{23}$$

$$\sum_{e,f,g,h=1}^k G_{a(e}G_{f)b}G^{eg}G^{fh}G_{g(c}G_{d)h} = G_{a(c}G_{d)b}. \tag{24}$$

Proof. Observe first that

$$\sum_{g=1}^k \alpha^g G_{gc} = \sum_{g,i=1}^k G^{gi}\alpha_i G_{gc} = \sum_{g,i=1}^k G_{cg}G^{gi}\alpha_i = \alpha_c$$

and

$$\sum_{g=1}^k \alpha^g\beta_g = \sum_{g,i=1}^k G^{gi}\alpha_i\beta_g = \sum_{g,i=1}^k \alpha_i G^{ig}\beta_g = (\alpha|\beta).$$

Equality (20) can then be established as follows:

$$\begin{aligned}\sum_{g,h=1}^{k}\alpha^g\alpha^h G_{g(c}G_{d)h} &= \frac{1}{2}\sum_{g,h,i,j=1}^{k} G^{gi}\alpha_i G^{hj}\alpha_j\left(G_{gc}G_{dh}+G_{gd}G_{ch}\right)\\ &= \frac{1}{2}\sum_{g,h,i,j=1}^{k} G_{cg}G^{gi}\alpha_i G_{dh}G^{hj}\alpha_j + G_{dg}G^{gi}\alpha_i G_{ch}G^{hj}\alpha_j\\ &= \frac{1}{2}\left(\alpha_c\alpha_d+\alpha_d\alpha_c\right)=\alpha_{(c}\alpha_{d)}=\alpha_c\alpha_d.\end{aligned}$$

A similar computation yields equality (21):

$$\begin{aligned}\sum_{g,h=1}^{k}\alpha^g\alpha^h\beta_{(g}G_{h)(c}\beta_{d)} &= \frac{1}{4}\sum_{g,h=1}^{k}\alpha^g\alpha^h\left(\beta_gG_{hc}\beta_d+\beta_hG_{gc}\beta_d+\beta_gG_{hd}\beta_c+\beta_hG_{gd}\beta_c\right)\\ &= \frac{1}{4}\left((\alpha|\beta)\alpha_c\beta_d+(\alpha|\beta)\alpha_c\beta_d+(\alpha|\beta)\,\alpha_d\beta_c+(\alpha|\beta)\,\alpha_d\beta_c\right)\\ &= \frac{1}{2}\,(\alpha|\beta)\left(\alpha_c\beta_d+\alpha_d\beta_c\right)=(\alpha|\beta)\,\alpha_{(c}\beta_{d)}.\end{aligned}$$

For equality (22) one computes the following:

$$\begin{aligned}&16\sum_{e,f,g,h=1}^{k}\alpha_{(a}G_{b)(e}\alpha_{f)}G^{eg}G^{fh}\beta_{(g}G_{h)(c}\beta_{d)}\\ &=\sum_{e,f,g,h=1}^{k}\left(\alpha_aG_{be}\alpha_f+\alpha_bG_{ae}\alpha_f+\alpha_aG_{bf}\alpha_e+\alpha_bG_{af}\alpha_e\right)G^{eg}G^{fh}\\ &\qquad\left(\beta_gG_{hc}\beta_d+\beta_hG_{gc}\beta_d+\beta_gG_{hd}\beta_c+\beta_hG_{gd}\beta_c\right)\\ &=\sum_{g,h=1}^{k}\left(\alpha_a\delta_{bg}\alpha^h+\alpha_b\delta_{ah}\alpha^g+\alpha_a\delta_{bh}\alpha^g+\alpha_b\delta_{ag}\alpha^h\right)\\ &\qquad\left(\beta_gG_{hc}\beta_d+\beta_hG_{gc}\beta_d+\beta_gG_{hd}\beta_c+\beta_hG_{gd}\beta_c\right)\\ &=\alpha_a\beta_b\alpha_c\beta_d+(\alpha|\beta)\,\alpha_aG_{bc}\beta_d+\alpha_a\beta_b\beta_c\alpha_d+(\alpha|\beta)\,\alpha_aG_{bd}\beta_c\\ &\qquad+(\alpha|\beta)\,\alpha_bG_{ac}\beta_d+\beta_a\alpha_b\alpha_c\beta_d+(\alpha|\beta)\,\alpha_bG_{ad}\beta_c+\beta_a\alpha_b\beta_c\alpha_d\\ &\qquad+(\alpha|\beta)\,\alpha_aG_{bc}\beta_d+\alpha_a\beta_b\alpha_c\beta_d+(\alpha|\beta)\,\alpha_aG_{bd}\beta_c+\alpha_a\beta_b\beta_c\alpha_d\\ &\qquad+\beta_a\alpha_b\alpha_c\beta_d+(\alpha|\beta)\,\alpha_bG_{ac}\beta_d+\beta_a\alpha_b\beta_c\alpha_d+(\alpha|\beta)\,\alpha_bG_{ad}\beta_c\\ &=2\left(\alpha_a\beta_b\alpha_c\beta_d+\alpha_a\beta_b\beta_c\alpha_d+\beta_a\alpha_b\alpha_c\beta_d+\beta_a\alpha_b\beta_c\alpha_d\right)\\ &\qquad+2\,(\alpha|\beta)\left(\alpha_aG_{bc}\beta_d+\alpha_aG_{bd}\beta_c+\alpha_bG_{ac}\beta_d+\alpha_bG_{ad}\beta_c\right)\ .\\ &=8\alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)}+8\,(\alpha|\beta)\,\alpha_{(a}G_{b)(c}\beta_{d)}\end{aligned}$$

Equalities (23) and (24) can be shown in the same manner. ∎

Throughout this section let $\mathfrak{g}$ be a simply laced split real Kac-Moody algebra with maximal compact subalgebra $\mathfrak{k}$. By Corollary 2.7 there exists a generalized $\frac{1}{2}$-spin representation $\rho\colon\mathfrak{k}\to\mathrm{End}(\mathbb{C}^l)$. In analogy to Sections 3 and 5 we make use of Clifford algebras in order to define higher spin representations.

Define $V := \mathrm{Sym}^2(\mathfrak{h}^*)$. Then, given a basis v^1, ..., v^k of $\mathfrak{h}^*$, the vector space V admits the natural basis $\{\frac{1}{2}(v^{i_1} \otimes v^{i_2} + v^{i_2} \otimes v^{i_1}) \mid 1 \le i_1 \le i_2 \le k\}$. These basis elements will be denoted as $v^i v^j := \frac{1}{2}(v^i \otimes v^j + v^j \otimes v^i)$ where it is understood that $v^i v^j$ and $v^j v^i$ are identical. Given an orthonormal basis f_1, ..., f_s of S as in Section 3 one arrives at a basis $\{v^{i_1} v^{i_2} \otimes f_j \mid 1 \le i_1 \le i_2 \le k, 1 \le j \le l\}$ of $V \otimes S$. In analogy to (2) define

$$\phi_\alpha^{ab} := v^a v^b \otimes f_\alpha = v^b v^a \otimes f_\alpha =: \phi_\alpha^{ba} \tag{25}$$

The invariant symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ induces a natural symmetric bilinear form on $\mathfrak{h}^* \otimes \mathfrak{h}^*$ which, by symmetry, factors through a symmetric bilinear form q_1 on $V = \mathrm{Sym}^2(\mathfrak{h}^*)$. If $(G^{ab})_{1 \le a,b \le k}$ as in Section 5 denotes the Gram matrix of $(\cdot|\cdot)$ with respect to the basis v^1, ..., v^k, the computation

$$\begin{aligned}
q_1\left(v^a v^b, v^c v^d\right) &= \frac{1}{4} q_1\left(v^a \otimes v^b + v^b \otimes v^a, v^c \otimes v^d + v^d \otimes v^c\right) \\
&= \frac{1}{4}\left[q_1\left(v^a \otimes v^b, v^c \otimes v^d\right) + q_1\left(v^a \otimes v^b, v^d \otimes v^c\right)\right] \\
&\qquad + \frac{1}{4}\left[q_1\left(v^b \otimes v^a, v^c \otimes v^d\right) + q_1\left(v^b \otimes v^a, v^d \otimes v^c\right)\right] \\
&= \frac{1}{4}\left(G^{ac}G^{bd} + G^{ad}G^{bc} + G^{bc}G^{ad} + G^{bd}G^{ac}\right) \\
&= \frac{1}{2}\left(G^{ac}G^{db} + G^{ad}G^{cb}\right) = \frac{1}{2}\left(G^{bd}G^{ca} + G^{bc}G^{da}\right) \\
&= \frac{1}{2}\left(G^{ca}G^{bd} + G^{cb}G^{ad}\right) = G^{a(c}G^{d)b} = G^{b(c}G^{d)a} = G^{b(d}G^{c)a} = G^{c(a}G^{b)d}
\end{aligned}$$

shows that the various symmetrizer brackets for all $a, b, c, d \in \{1, ..., k\}$ all describe the Gram matrix of q_1 with respect to the basis $\{v^{i_1} v^{i_2} \mid 1 \le i_1 \le i_2 \le k\}$. In analogy to Section 3 define a symmetric bilinear form on the tensor product $V \otimes S$ via

$$q(\phi_\alpha^{ab}, \phi_\beta^{cd}) := G^{a(c}G^{d)b}\delta_{\alpha\beta}$$

Remark 6.3. The above equality between various symmetrizer brackets makes it meaningful to define

$$\phi_\alpha^{\mathcal{A}} := \frac{1}{2}\phi_\alpha^{ab} + \frac{1}{2}\phi_\alpha^{ba} \qquad \phi_\beta^{\mathcal{B}} := \frac{1}{2}\phi_\beta^{cd} + \frac{1}{2}\phi_\beta^{dc}$$

and

$$G^{\mathcal{AB}} := G^{a(c}G^{d)b} = G^{c(a}G^{b)d} =: G^{\mathcal{BA}}$$

in order to make the formalism of Lemma 3.1 and Proposition 3.5 applicable also in the situation of $V = \mathrm{Sym}^2(\mathfrak{h}^*)$ by interpreting $\mathcal{A}$ and $\mathcal{B}$ as multi-indices whose constituents vary independently between 1 and k. For instance, expanding the commutator

$$[X, Y]_{\mathcal{AB}} = \sum_{\mathcal{C},\mathcal{D}=1}^{k} \left(X_{\mathcal{AC}} G^{\mathcal{CD}} Y_{\mathcal{DB}} - Y_{\mathcal{AC}} G^{\mathcal{CD}} X_{\mathcal{DB}}\right)$$

from (6) into the current setting with

$$\begin{array}{ll} \mathcal{A} \text{ corresponding to } a,\, b, & \mathcal{B} \text{ corresponding to } c,\, d, \\ \mathcal{C} \text{ corresponding to } e,\, f, & \mathcal{D} \text{ corresponding to } g,\, h \end{array}$$

then yields

$$[X,Y]_{ab\,cd} = \sum_{e,f,g,h=1}^{k} \left(X_{ab\,ef} G^{eg} G^{fh} Y_{gh\,cd} - Y_{ab\,ef} G^{eg} G^{fh} X_{gh\,cd} \right),$$

where the symmetrizer bracket can be dropped because of the symmetry properties of X and Y.

Proposition 6.4. *Let $\alpha = \sum_{i=1}^{k} \alpha_i v^i \in \mathfrak{h}^*$ be a real root. Then the matrices given by*

$$X(\alpha)_{ab\,cd} = \frac{1}{2}\alpha_a\alpha_b\alpha_c\alpha_d - \alpha_{(a}G_{b)(c}\alpha_{d)} + \frac{1}{4}G_{a(c}G_{d)b} \tag{26}$$

satisfy for all $\alpha, \beta \in \Delta^{\mathrm{re}}$ such that $(\alpha|\beta) = 0$

$$\begin{aligned} &[X(\alpha), X(\beta)]_{ab\,cd} = \\ &= \sum_{e,f,g,h=1}^{k} \left(X(\alpha)_{ab\,ef} G^{eg} G^{fh} X(\beta)_{gh\,cd} + X(\beta)_{ab\,ef} G^{eg} G^{fh} X(\alpha)_{gh\,cd} \right) = 0 \end{aligned}$$

and for all $\alpha, \beta \in \Delta^{\mathrm{re}}$ such that $(\alpha|\beta) = \mp 1$

$$\begin{aligned} &\{X(\alpha), X(\beta)\}_{ab\,cd} = \\ &= \sum_{e,f,g,h=1}^{k} \left(X(\alpha)_{ab\,ef} G^{eg} G^{fh} X(\beta)_{gh\,cd} + X(\beta)_{ab\,ef} G^{eg} G^{fh} X(\alpha)_{gh\,cd} \right) = \frac{1}{2} X(\alpha \pm \beta). \end{aligned}$$

In particular, the assigment $X_i \mapsto X(\boldsymbol{\alpha}_i)$ defines a finite-dimensional representation of $\mathfrak{k}$.

Remark 6.5. Formula (26) is [9, (5.4), p. 18].

Proof of Proposition 6.4. It suffices to establish the hypotheses of Proposition 3.5. By definition, $X(\alpha)$ is symmetric. Define

$$S_{ab\,cd} := \sum_{e,f,g,h=1}^{k} X(\alpha)_{ab\,ef} G^{eg} G^{fh} X(\beta)_{gh\,cd}$$

and calculate the following; for the sake of the exposition in the next calculation we use Einstein's summation convention, i.e., equal indices are summed over if one is upper and one is lower.

$$\begin{aligned} S_{ab\,cd} \;=\; & \left(\frac{1}{2}\alpha_a\alpha_b\alpha_e\alpha_f - \alpha_{(a}G_{b)(e}\alpha_{f)} + \frac{1}{4}G_{a(e}G_{f)b} \right) G^{eg}G^{fh} \\ & \left(\frac{1}{2}\beta_g\beta_h\beta_c\beta_d - \beta_{(g}G_{h)(c}\beta_{d)} + \frac{1}{4}G_{g(c}G_{d)h} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\alpha_a\alpha_b\alpha^g\alpha^h\beta_g\beta_h\beta_c\beta_d - \frac{1}{2}\alpha_a\alpha_b\alpha^g\alpha^h\beta_{(g}G_{h)(c}\beta_{d)} + \frac{1}{8}\alpha_a\alpha_b\alpha^g\alpha^h G_{g(c}G_{d)h} \\
&\quad -\frac{1}{2}\alpha_{(a}G_{b)(e}\alpha_{f)}\beta^e\beta^f\beta_c\beta_d + \alpha_{(a}G_{b)(e}\alpha_{f)}G^{eg}G^{fh}\beta_{(g}G_{h)(c}\beta_{d)} \\
&\quad -\frac{1}{4}\alpha_{(a}G_{b)(e}\alpha_{f)}G^{eg}G^{fh}G_{g(c}G_{d)h} + \frac{1}{8}G_{a(e}G_{f)b}\beta^e\beta^f\beta_c\beta_d \\
&\quad -\frac{1}{4}G_{a(e}G_{f)b}G^{eg}G^{fh}\beta_{(g}G_{h)(c}\beta_{d)} + \frac{1}{16}G_{a(e}G_{f)b}G^{eg}G^{fh}G_{g(c}G_{d)h} \\
&= \frac{1}{4}(\alpha|\beta)^2\alpha_a\alpha_b\beta_c\beta_d - \frac{1}{2}(\alpha|\beta)\alpha_a\alpha_b\alpha_{(c}\beta_{d)} + \frac{1}{8}\alpha_a\alpha_b\alpha_c\alpha_d \\
&\quad -\frac{1}{2}(\alpha|\beta)\alpha_{(a}\beta_{b)}\beta_c\beta_d + \frac{1}{2}\alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)} + \frac{1}{2}(\alpha|\beta)\alpha_{(a}G_{b)(c}\beta_{d)} \\
&\quad -\frac{1}{4}\alpha_{(a}G_{b)(c}\alpha_{d)} + \frac{1}{8}\beta_a\beta_b\beta_c\beta_d - \frac{1}{4}\beta_{(a}G_{b)(c}\beta_{d)} + \frac{1}{16}G_{a(c}G_{d)b}.
\end{aligned}$$

Next one computes the commutator $C_{ab\,cd} := [X(\alpha), X(\beta)]_{ab\,cd}$ in the case of $(\alpha|\beta) = 0$ to be equal to

$$\begin{aligned}
C_{ab\,cd} &= \sum_{e,f,g,h=1}^{k}\left(X(\alpha)_{ab\,ef}G^{eg}G^{fh}X(\beta)_{gh\,cd} - X(\beta)_{ab\,ef}G^{eg}G^{fh}X(\alpha)_{gh\,cd}\right) \\
&= \frac{1}{8}\alpha_a\alpha_b\alpha_c\alpha_d + \frac{1}{2}\alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)} - \frac{1}{4}\alpha_{(a}G_{b)(c}\alpha_{d)} \\
&\quad +\frac{1}{8}\beta_a\beta_b\beta_c\beta_d - \frac{1}{4}\beta_{(a}G_{b)(c}\beta_{d)} + \frac{1}{16}G_{a(c}G_{d)b} - (\alpha \leftrightarrow \beta) \\
&= \frac{1}{8}\alpha_a\alpha_b\alpha_c\alpha_d + \frac{1}{2}\alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)} - \frac{1}{4}\alpha_{(a}G_{b)(c}\alpha_{d)} \\
&\quad +\frac{1}{8}\beta_a\beta_b\beta_c\beta_d - \frac{1}{4}\beta_{(a}G_{b)(c}\beta_{d)} + \frac{1}{16}G_{a(c}G_{d)b} \\
&\quad -\frac{1}{8}\beta_a\beta_b\beta_c\beta_d - \frac{1}{2}\beta_{(a}\alpha_{b)}\beta_{(c}\alpha_{d)} + \frac{1}{4}\beta_{(a}G_{b)(c}\beta_{d)} \\
&\quad -\frac{1}{8}\alpha_a\alpha_b\alpha_c\alpha_d + \frac{1}{4}\alpha_{(a}G_{b)(c}\alpha_{d)} - \frac{1}{16}G_{a(c}G_{d)b} = 0,
\end{aligned}$$

since $\alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)} = \beta_{(a}\alpha_{b)}\beta_{(c}\alpha_{d)}$. Here the symbol $(\alpha \leftrightarrow \beta)$ denotes a repetition of all previous terms with the roles of α and β interchanged. In a similar fashion one calculates the anti-commutator

$$A_{ab\,cd} := \sum_{e,f,g,h=1}^{k}\left(X(\alpha)_{ab\,ef}G^{eg}G^{fh}X(\beta)_{gh\,cd} + X(\beta)_{ab\,ef}G^{eg}G^{fh}X(\alpha)_{gh\,cd}\right)$$

for $(\alpha|\beta) = \pm 1$ to be

$$\begin{aligned}
A_{ab\,cd} &= \frac{1}{4}\alpha_a\alpha_b\beta_c\beta_d \mp \frac{1}{2}\alpha_a\alpha_b\alpha_{(c}\beta_{d)} + \frac{1}{8}\alpha_a\alpha_b\alpha_c\alpha_d \\
&\quad \mp \frac{1}{2}\alpha_{(a}\beta_{b)}\beta_c\beta_d + \frac{1}{2}\alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)} \pm \frac{1}{2}\alpha_{(a}G_{b)(c}\beta_{d)} \\
&\quad - \frac{1}{4}\alpha_{(a}G_{b)(c}\alpha_{d)} + \frac{1}{8}\beta_a\beta_b\beta_c\beta_d - \frac{1}{4}\beta_{(a}G_{b)(c}\beta_{d)} + \frac{1}{16}G_{a(c}G_{d)b} \\
&= \frac{1}{4}\beta_a\beta_b\alpha_c\alpha_d \mp \frac{1}{2}\beta_a\beta_b\beta_{(c}\alpha_{d)} + \frac{1}{8}\beta_a\beta_b\beta_c\beta_d
\end{aligned}$$

$$\mp \frac{1}{2}\beta_{(a}\alpha_{b)}\alpha_c\alpha_d + \frac{1}{2}\beta_{(a}\alpha_{b)}\beta_{(c}\alpha_{d)} \pm \frac{1}{2}\beta_{(a}G_{b)(c}\alpha_{d)}$$
$$- \frac{1}{4}\beta_{(a}G_{b)(c}\beta_{d)} + \frac{1}{8}\alpha_a\alpha_b\alpha_c\alpha_d - \frac{1}{4}\alpha_{(a}G_{b)(c}\alpha_{d)} + \frac{1}{16}G_{a(c}G_{d)b}$$
$$= \frac{1}{4}\alpha_a\alpha_b\alpha_c\alpha_d \mp \frac{1}{2}\alpha_a\alpha_b\alpha_{(c}\beta_{d)} \mp \frac{1}{2}\beta_{(a}\alpha_{b)}\alpha_c\alpha_d + \frac{1}{4}\alpha_a\alpha_b\beta_c\beta_d$$
$$+ \frac{1}{4}\beta_a\beta_b\alpha_c\alpha_d + \alpha_{(a}\beta_{b)}\alpha_{(c}\beta_{d)} \mp \frac{1}{2}\beta_a\beta_b\beta_{(c}\alpha_{d)} \mp \frac{1}{2}\alpha_{(a}\beta_{b)}\beta_c\beta_d + \frac{1}{4}\beta_a\beta_b\beta_c\beta_d$$
$$- \frac{1}{2}\alpha_{(a}G_{b)(c}\alpha_{d)} \pm \frac{1}{2}\alpha_{(a}G_{b)(c}\beta_{d)} \pm \frac{1}{2}\beta_{(a}G_{b)(c}\alpha_{d)} - \frac{1}{2}\beta_{(a}G_{b)(c}\beta_{d)} + \frac{1}{8}G_{a(c}G_{d)b}$$
$$= \frac{1}{4}\alpha_a\alpha_b\alpha_c\alpha_d \mp \frac{1}{4}\left(\alpha_a\alpha_b\alpha_c\beta_d + \alpha_a\alpha_b\beta_c\alpha_d + \alpha_a\beta_b\alpha_c\alpha_d + \beta_a\alpha_b\alpha_c\alpha_d\right)$$
$$+ \frac{1}{4}\left(\alpha_a\alpha_b\beta_c\beta_d + \beta_a\beta_b\alpha_c\alpha_d + \alpha_a\beta_b\alpha_c\beta_d + \beta_a\alpha_b\alpha_c\beta_d + \alpha_a\beta_b\beta_c\alpha_d + \beta_a\alpha_b\beta_c\alpha_d\right)$$
$$\mp \frac{1}{4}\left(\beta_a\beta_b\beta_c\alpha_d + \beta_a\beta_b\alpha_c\beta_d + \beta_a\alpha_b\beta_c\beta_d + \alpha_a\beta_b\beta_c\beta_d\right) + \frac{1}{4}\beta_a\beta_b\beta_c\beta_d$$
$$- \frac{1}{2}\Big(\alpha_{(a}G_{b)(c}\alpha_{d)} \mp \alpha_{(a}G_{b)(c}\beta_{d)} \mp \beta_{(a}G_{b)(c}\alpha_{d)} + \beta_{(a}G_{b)(c}\beta_{d)}\Big) + \frac{1}{8}G_{a(c}G_{d)b}$$
$$= \frac{1}{4}(\alpha \mp \beta)_a(\alpha \mp \beta)_b(\alpha \mp \beta)_c(\alpha \mp \beta)_d - \frac{1}{2}(\alpha \mp \beta)_{(a}G_{b)(c}(\alpha \mp \beta)_{d)} + \frac{1}{8}G_{a(c}G_{d)b}$$
$$= \frac{1}{2}X(\alpha \mp \beta)_{ab\,cd}.$$

This proves the claim. ∎

Again, we conclude this section with a coordinate-free version of Proposition 6.4.

Proposition 6.6. *For $V = \mathfrak{h}^*$ let $(\cdot|\cdot)$ denote the induced invariant bilinear form on $\mathfrak{h}^*$. Moreover, for $\alpha \in \Delta^{\mathrm{re}}$ let $\pi_\alpha := \alpha\,(\alpha|\cdot) \in \mathrm{End}\,(\mathfrak{h}^*)$. Define $X\colon \Delta^{\mathrm{re}} \to \mathrm{End}\left(\mathrm{Sym}^2(\mathfrak{h}^*)\right)$ via*

$$\alpha \mapsto X(\alpha) \;\; := \;\; \pi_\alpha \otimes \pi_\alpha - (\pi_\alpha \otimes id_{\mathfrak{h}^*} + id_{\mathfrak{h}^*} \otimes \pi_\alpha) + \frac{1}{2} id_{\mathfrak{h}^*} \otimes id_{\mathfrak{h}^*}. \qquad (27)$$

Then X satisfies (15) *and* (16) *for all real roots α, β with $(\alpha|\beta) \in \{0, \pm 1\}$ and thus provides a representation σ of $\mathfrak{k}$ by sending*

$$X_i \mapsto \sigma\left(X_i\right) := X\left(\boldsymbol{\alpha}_i\right) \otimes \Gamma\left(\boldsymbol{\alpha}_i\right).$$

Proof. Observe

$$\pi_\alpha\pi_\beta = \begin{cases} 0, & \text{if } (\alpha|\beta) = 0, \\ \pm\alpha\,(\beta|\cdot)\,, & \text{if } (\alpha|\beta) = \pm 1, \end{cases}$$

and abbreviate ${}_\alpha\pi_\beta := \alpha\,(\beta|\cdot)$ and $1 \equiv id_{\mathfrak{h}^*}$. One computes for $(\alpha|\beta) = 0$ that

$$[X(\alpha), X(\beta)] = \pi_\alpha\pi_\beta \otimes \pi_\alpha\pi_\beta - \pi_\alpha\pi_\beta \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha\pi_\beta + \frac{1}{2}\pi_\alpha \otimes \pi_\alpha$$
$$- \pi_\alpha\pi_\beta \otimes \pi_\beta + \pi_\alpha\pi_\beta \otimes 1 + \pi_\alpha \otimes \pi_\beta - \frac{1}{2}\pi_\alpha \otimes 1$$

$$
\begin{aligned}
&- \pi_\beta \otimes \pi_\alpha\pi_\beta + \pi_\beta \otimes \pi_\alpha + 1 \otimes \pi_\alpha\pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \\
&+ \frac{1}{2}\pi_\beta \otimes \pi_\beta - \frac{1}{2}\left(\pi_\beta \otimes 1 + 1 \otimes \pi_\beta\right) + \frac{1}{4} \cdot 1 \otimes 1 - (\alpha \leftrightarrow \beta) \\
= &\frac{1}{2}\pi_\alpha \otimes \pi_\alpha + \pi_\alpha \otimes \pi_\beta - \frac{1}{2}\pi_\alpha \otimes 1 + \pi_\beta \otimes \pi_\alpha \\
&- \frac{1}{2} \cdot 1 \otimes \pi_\alpha + \frac{1}{2}\pi_\beta \otimes \pi_\beta - \frac{1}{2}\left(\pi_\beta \otimes 1 + 1 \otimes \pi_\beta\right) + \frac{1}{4} \cdot 1 \otimes 1 - (\alpha \leftrightarrow \beta) = 0
\end{aligned}
$$

because the first part is symmetric in α and β. (Here the symbol $(\alpha \leftrightarrow \beta)$ again denotes a repetition of all previous terms with the roles of α and β interchanged.) Before evaluating the anti-commutator consider for $(\alpha|\beta) = \mp 1$

$$
\pi_{\alpha\pm\beta} = (\alpha \pm \beta)\left(\alpha \pm \beta|\cdot\right) = \pi_\alpha \pm {}_\alpha\pi_\beta \pm {}_\beta\pi_\alpha + \pi_\beta = \pi_\alpha - \pi_\alpha\pi_\beta - \pi_\beta\pi_\alpha + \pi_\beta
$$

and, thus,

$$
\begin{aligned}
\pi_{\alpha\pm\beta} \otimes \pi_{\alpha\pm\beta} &= \left(\pi_\alpha \pm {}_\alpha\pi_\beta \pm {}_\beta\pi_\alpha + \pi_\beta\right) \otimes \left(\pi_\alpha \pm {}_\alpha\pi_\beta \pm {}_\beta\pi_\alpha + \pi_\beta\right) \\
&= \left(\pi_\alpha - \pi_\alpha\pi_\beta - \pi_\beta\pi_\alpha + \pi_\beta\right) \otimes \left(\pi_\alpha - \pi_\alpha\pi_\beta - \pi_\beta\pi_\alpha + \pi_\beta\right) \\
&= \pi_\alpha \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha\pi_\beta - \pi_\alpha \otimes \pi_\beta\pi_\alpha + \pi_\alpha \otimes \pi_\beta \\
&\quad -\pi_\alpha\pi_\beta \otimes \pi_\alpha + \pi_\alpha\pi_\beta \otimes \pi_\alpha\pi_\beta + \underline{\pi_\alpha\pi_\beta \otimes \pi_\beta\pi_\alpha} - \pi_\alpha\pi_\beta \otimes \pi_\beta \\
&\quad -\pi_\beta\pi_\alpha \otimes \pi_\alpha + \underline{\pi_\beta\pi_\alpha \otimes \pi_\alpha\pi_\beta} + \pi_\beta\pi_\alpha \otimes \pi_\beta\pi_\alpha - \pi_\beta\pi_\alpha \otimes \pi_\beta \\
&\quad +\pi_\beta \otimes \pi_\alpha - \pi_\beta \otimes \pi_\alpha\pi_\beta - \pi_\beta \otimes \pi_\beta\pi_\alpha + \pi_\beta \otimes \pi_\beta. \qquad (28)
\end{aligned}
$$

For the anti-commutator one computes

$$
\begin{aligned}
\{X(\alpha), X(\beta)\} &= \pi_\alpha\pi_\beta \otimes \pi_\alpha\pi_\beta - \pi_\alpha\pi_\beta \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha\pi_\beta + \frac{1}{2}\pi_\alpha \otimes \pi_\alpha \\
&- \pi_\alpha\pi_\beta \otimes \pi_\beta + \pi_\alpha\pi_\beta \otimes 1 + \pi_\alpha \otimes \pi_\beta - \frac{1}{2}\pi_\alpha \otimes 1 \\
&- \pi_\beta \otimes \pi_\alpha\pi_\beta + \pi_\beta \otimes \pi_\alpha + 1 \otimes \pi_\alpha\pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \\
&+ \frac{1}{2}\pi_\beta \otimes \pi_\beta - \frac{1}{2}\left(\pi_\beta \otimes 1 + 1 \otimes \pi_\beta\right) + \frac{1}{4} \cdot 1 \otimes 1 + (\alpha \leftrightarrow \beta) \\
= &\ \pi_\alpha\pi_\beta \otimes \pi_\alpha\pi_\beta - \pi_\alpha\pi_\beta \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha\pi_\beta - \pi_\alpha\pi_\beta \otimes \pi_\beta \qquad (29) \\
&+ \pi_\alpha \otimes \pi_\beta - \pi_\beta \otimes \pi_\alpha\pi_\beta + \pi_\beta \otimes \pi_\alpha + \frac{1}{2}\pi_\beta \otimes \pi_\beta + \frac{1}{2}\pi_\alpha \otimes \pi_\alpha \qquad (30) \\
&+ \pi_\beta\pi_\alpha \otimes \pi_\beta\pi_\alpha - \pi_\beta\pi_\alpha \otimes \pi_\beta - \pi_\beta \otimes \pi_\beta\pi_\alpha - \pi_\beta\pi_\alpha \otimes \pi_\alpha \qquad (31) \\
&+ \pi_\beta \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\beta\pi_\alpha + \pi_\alpha \otimes \pi_\beta + \frac{1}{2}\pi_\alpha \otimes \pi_\alpha + \frac{1}{2}\pi_\beta \otimes \pi_\beta \qquad (32) \\
&+ \pi_\alpha\pi_\beta \otimes 1 - \frac{1}{2}\pi_\alpha \otimes 1 + 1 \otimes \pi_\alpha\pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha - \frac{1}{2}\pi_\beta \otimes 1 - \frac{1}{2}1 \otimes \pi_\beta \qquad (33) \\
&+ \pi_\beta\pi_\alpha \otimes 1 - \frac{1}{2}\pi_\beta \otimes 1 + 1 \otimes \pi_\beta\pi_\alpha - \frac{1}{2}1 \otimes \pi_\beta - \frac{1}{2}\pi_\alpha \otimes 1 - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \qquad (34) \\
&+ \frac{2}{4} \cdot 1 \otimes 1.
\end{aligned}
$$

The lines (29)–(32) equal the term

$$\begin{array}{l}\pi_\alpha \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha\pi_\beta - \pi_\alpha \otimes \pi_\beta\pi_\alpha + \pi_\alpha \otimes \pi_\beta \\ -\pi_\beta\pi_\alpha \otimes \pi_\alpha - \pi_\beta\pi_\alpha \otimes \pi_\beta + \pi_\beta\pi_\alpha \otimes \pi_\beta\pi_\alpha + \underline{\pi_\beta \otimes \pi_\alpha} \\ -\pi_\alpha\pi_\beta \otimes \pi_\alpha + \pi_\alpha\pi_\beta \otimes \pi_\alpha\pi_\beta - \pi_\alpha\pi_\beta \otimes \pi_\beta + \underline{\pi_\alpha \otimes \pi_\beta} \\ \pi_\beta \otimes \pi_\beta - \pi_\beta \otimes \pi_\beta\pi_\alpha - \pi_\beta \otimes \pi_\alpha\pi_\beta + \pi_\beta \otimes \pi_\alpha\end{array}$$

which is almost identical to the expression for $\pi_{\alpha\pm\beta} \otimes \pi_{\alpha\pm\beta}$ derived in formula (28) if it were not for the underlined terms. Nevertheless, for $h \in \mathfrak{h}^*$ one evaluates

$$\begin{aligned}(\pi_\alpha\pi_\beta \otimes \pi_\beta\pi_\alpha + \pi_\beta\pi_\alpha \otimes \pi_\alpha\pi_\beta)(h,h) &= (\mp\alpha\,(\beta|h)) \otimes (\mp\beta\,(\alpha|h)) \\ &\quad + (\mp\beta\,(\alpha|h)) \otimes (\mp\alpha\,(\beta|h)) \\ &= (\alpha|h)\,(\beta|h) \cdot (\alpha \otimes \beta + \beta \otimes \alpha)\end{aligned}$$

whereas

$$\begin{aligned}(\pi_\alpha \otimes \pi_\beta + \pi_\beta \otimes \pi_\alpha)(h,h) &= \alpha\,(\alpha|h) \otimes (\beta|h)\,\beta \\ &\quad + (\beta|h)\,\beta \otimes \alpha\,(\alpha|h) \\ &= (\alpha|h)\,(\beta|h) \cdot (\alpha \otimes \beta + \beta \otimes \alpha) \\ &= (\pi_\alpha\pi_\beta \otimes \pi_\beta\pi_\alpha + \pi_\beta\pi_\alpha \otimes \pi_\alpha\pi_\beta)(h,h)\end{aligned}$$

for arbitrary $h \in \mathfrak{h}^*$. Thus, using the diagonalizability of real symmetric tensors of degree two, the first four lines of the anti-commutator are equal to $\pi_{\alpha\pm\beta} \otimes \pi_{\alpha\pm\beta}$. The lines (33)–(34) are evaluated to be

$$\begin{aligned}&+ \pi_\alpha\pi_\beta \otimes 1 - \frac{1}{2}\pi_\alpha \otimes 1 + 1 \otimes \pi_\alpha\pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha - \frac{1}{2}\pi_\beta \otimes 1 - \frac{1}{2}1 \otimes \pi_\beta \\ &+ \pi_\beta\pi_\alpha \otimes 1 - \frac{1}{2}\pi_\beta \otimes 1 + 1 \otimes \pi_\beta\pi_\alpha - \frac{1}{2}1 \otimes \pi_\beta - \frac{1}{2}\pi_\alpha \otimes 1 - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \\ = &\left(\pi_\alpha\pi_\beta - \frac{1}{2}\pi_\alpha - \frac{1}{2}\pi_\beta + \pi_\beta\pi_\alpha - \frac{1}{2}\pi_\beta - \frac{1}{2}\pi_\alpha\right) \otimes 1 \\ &+ 1 \otimes \left(\pi_\alpha\pi_\beta - \frac{1}{2}\pi_\alpha - \frac{1}{2}\pi_\beta + \pi_\beta\pi_\alpha - \frac{1}{2}\pi_\beta - \frac{1}{2}\pi_\alpha\right) \\ = &-(\pi_\alpha - \pi_\alpha\pi_\beta - \pi_\beta\pi_\alpha + \pi_\beta) \otimes 1 \\ &- 1 \otimes (\pi_\alpha - \pi_\alpha\pi_\beta - \pi_\beta\pi_\alpha + \pi_\beta) \\ = &-\pi_{\alpha\pm\beta} \otimes 1 - 1 \otimes \pi_{\alpha\pm\beta}.\end{aligned}$$

So one finds that for $(\alpha|\beta) = \mp 1$ one has

$$\begin{aligned}\{X(\alpha), X(\beta)\} &= \pi_{\alpha\pm\beta} \otimes \pi_{\alpha\pm\beta} - \pi_{\alpha\pm\beta} \otimes 1 - 1 \otimes \pi_{\alpha\pm\beta} + \frac{2}{4} \cdot 1 \otimes 1 \\ &= \pi_{\alpha\pm\beta} \otimes \pi_{\alpha\pm\beta} - \pi_{\alpha\pm\beta} \otimes 1 - 1 \otimes \pi_{\alpha\pm\beta} + \frac{1}{2} \cdot 1 \otimes 1 \\ &= X(\alpha \pm \beta)\end{aligned}$$

as desired. ∎

Remark 6.7. Again note that the canonical Weyl group representation $\rho\colon W \to \mathrm{GL}(\mathrm{Sym}^2(\mathfrak{h}^*))$ yields $X(\alpha) = \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$. Therefore Remark 4.2 applies and the statement of Proposition 6.6 in fact follows from the observation that ρ (restricted to any standard subgroup Sym_3) does not contain the sign representation as an irreducible component.

7. Outlook and comments

We stress that the representations we construct here very likely are not irreducible so that instead of calling the whole module a $\frac{n}{2}$-spin representation ($n \in \{3, 5\}$) one should actually rather reserve this name for one (or several) appropriately chosen submodules. Also note that the amalgamation methods from [6, Section 11] allow one to integrate the constructed Lie algebra representations to group level and that the resulting group representations are afforded by the so-called spin cover of the maximal compact subgroup of the ambient Kac-Moody group, but not by the maximal compact subgroup itself.

We arrive at the first open problem for further research:

Problem 7.1. Classify the submodules of the two representations constructed in the Theorem.

We firmly believe higher $\frac{n}{2}$-spin representations ($n \geq 7$) to exist. A machine-based construction of a $\frac{7}{2}$-spin representation can be found in [9]. There are at least two viable strategies for the systematic construction of higher spin representations. The first one is to emulate the Clebsch–Gordan formula once Problem 7.1 is solved: study tensor products (and their submodule structure) of an odd number of copies of the existing $\frac{1}{2}$-, $\frac{3}{2}$-, $\frac{5}{2}$-spin representations. The second strategy is to make use of the observation by Paul Levy alluded to in the introduction of this note and discussed in more detail in Remark 4.2; that is, develop a systematic representation theory for simply-laced Coxeter groups (notably of the submodules of $\mathrm{Sym}^n(\mathfrak{h}^*)$), identify those that do not contain the sign representation, and as discussed in Remark 4.2 make use of the ansatz $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$. We point out that indeed the third symmetric power of the natural reflection representation of the Coxeter group Sym_3 does contain a sign representation and, thus, one will have to pass to a properly chosen submodule of $\mathrm{Sym}^3(\mathfrak{h}^*)$ in order to have any chance of constructing a $\frac{7}{2}$-spin representation for $\mathfrak{k}$ using Levy's observation.

Altogether, one may formulate the following problem:

Problem 7.2. Approach higher $\frac{n}{2}$-spin representations ($n \geq 7$)

- by studying the submodules of tensor products of an odd number of copies of the existing $\frac{1}{2}$-, $\frac{3}{2}$-, $\frac{5}{2}$-spin representations (i.e., by emulating the Clebsch–Gordan formula),
- by exhibiting sign representation-free submodules of the Coxeter group-module $\mathrm{Sym}^n(\mathfrak{h}^*)$ and making use of the ansatz $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\,\mathrm{id}$ (i.e., by a systematic use of Levy's observation).

Furthermore, compare the results of the two approaches in order to check whether one may classify the representations resulting from the ansatz $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\mathrm{id}$ and whether a Clebsch–Gordan-type formula might hold in this context.

We currently do not have any intuition concerning the possible existence of finite-dimensional representations of $\mathfrak{k}$ that do not come from the ansatz $X(\alpha) := \rho(s_\alpha) - \frac{1}{2}\,\mathrm{id}$.

The non-simply laced situation is likely to be very wild. Already the $\frac{1}{2}$-spin representations afford many interdependencies stemming from various covering and folding techniques of diagrams, allowing some fixed $\mathfrak{k}$ to act on $\frac{1}{2}$-spin modules corresponding to various diagrams. (See [6, Section 17] for the corresponding phenomenon on group level.) Moreover, in the non-simply laced case the spin cover need not be universal, as can already be seen in the finite-dimensional situation for the diagram C_n since $\pi_1(U(n)) = \mathbb{Z}$, which potentially complicates things even further as the natural dichotomy "integrates to SO(3)" vs. "integrates to Spin(3) only" of $\mathfrak{so}_3(\mathbb{R})$-modules is likely to give way to something much more involved.

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