

Monomial Bases and Pre-Lie Structure for Free Lie Algebras

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Abstract. We construct a pre-Lie structure on the free Lie algebra $\mathcal{L}(E)$ generated by a set E , giving an explicit presentation of $\mathcal{L}(E)$ as the quotient of the free pre-Lie algebra \mathcal{T}^E , generated by the (non-planar) E -decorated rooted trees, by some ideal I . The main result in this paper is a description of Gröbner bases in terms of trees.

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1. Introduction

In the spirit of F. Klein’s (1849–1925) “Erlangen Program”, any Lie group G is a group of symmetries of some class of differentiable manifolds. The corresponding infinitesimal transformations are given by the Lie algebra of G , which is the set of left-invariant vector fields on G . The problem of classification of groups of transformations has been considered by S. Lie (1842–1899) not only for subgroups of GL_n , but also for infinite dimensional groups [15].

The problem of classification of simple finite-dimensional Lie algebras over the field of complex numbers was solved at the end of the 19th century by W. Killing (1847–1923) and E. Cartan (1869–1951). The central figure of the origins of the theory of the structure of Lie algebras is W. Killing, whose paper in four parts laid the conceptual foundations of the theory. In 1884, W. Killing introduced the concept of Lie algebra independently of S. Lie and formulated the problem of determining all possible structures for real, finite dimensional Lie algebras. The joint work of W. Killing and E. Cartan establishes the foundations of the theory. W. Killing’s work contained many gaps which E. Cartan succeeded to fill [14], [15]. W. Killing, E. Cartan, S. Lie, and F. Engel are the main authors of the early development of the theory and some of its various applications.

The concept of pre-Lie algebra appeared in many works under various names. E. B. Vinberg and M. Gerstenhaber in 1963 independently presented the concept under two different names; “right symmetric algebras” and “pre-Lie algebras”

respectively [13, 25]. Other denominations, e.g. “Vinberg algebras”, appeared since then. “Chronological algebras” is the term used by A. Agrachev and R. V. Gamkrelidze in their work on *nonstationary vector fields* [1]. The term “pre-Lie algebras” is now the standard terminology. The Lie algebra of a real connected Lie group G admits a compatible pre-Lie structure if and only if G admits a left-invariant affine structure [5, Proposition 2.31], see also the work of J. L. Koszul [17] for more details about the pre-Lie structure, in a geometrical point of view.

In Sections 2, 3 of this paper, we recall some basics: trees, Lie and pre-Lie algebras, Gröbner bases. We construct, in Section 4, a structure of pre-Lie algebra on the free Lie algebra $\mathcal{L}(E)$ generated by a set E , and we give the explicit presentation of $\mathcal{L}(E)$ as the quotient of the free pre-Lie algebra \mathcal{T}^E by some ideal.

Recall that \mathcal{T}_{pl}^E is the linear span of the set T_{pl}^E of all planar E -decorated rooted trees, which forms together with the left Butcher product \circlearrowleft , and the left grafting \searrow respectively two magmatic algebras. In Section 5, we give a tree version of a monomial well-order on T_{pl}^E . We adapt the work of T. Mora [20] on Gröbner bases to a non-associative, magmatic context, using the descriptions of the free magmatic algebras $(\mathcal{T}_{pl}^E, \circlearrowleft)$ and $(\mathcal{T}_{pl}^E, \searrow)$ respectively, following [8]. We split the basis of E -decorated planar rooted trees into two parts $O(J')$ and $T(J')$, where J' is the ideal of \mathcal{T}_{pl}^E generated by the pre-Lie identity and by weighted anti-symmetry relations:

$$|\sigma|\sigma \circlearrowleft \tau + |\tau|\tau \circlearrowleft \sigma.$$

Here $T(J')$ is the set of maximal terms of elements of J' , and its complement $O(J')$ then defines a basis of $\mathcal{L}(E)$. We get one of the important results (Theorem 5.7), on the description of the set $O(J')$ in terms of trees.

In Section 6, we give a non-planar tree version of the monomial well-order above. We describe monomial bases for the pre-Lie (respectively free Lie) algebra $\mathcal{L}(E)$, using the procedures of Gröbner bases and our work described in [2], in the monomial basis for the free pre-Lie algebra \mathcal{T}^E .

2. Trees

In graph theory, a *tree* is a undirected connected finite graph, without cycles [10]. A *rooted tree* is defined as a tree with one designated vertex called the root. The other remaining vertices are partitioned into $k \geq 0$ disjoint subsets such that each of them in turn represents a rooted tree, and a subtree of the whole tree. This can be taken as a recursive definition for rooted trees, widely used in computer algorithms [16]. Rooted trees stand among the most important structures appearing in many branches of pure and applied mathematics.

In general, a tree structure can be described as a “branching” relationship between vertices, much like that found in the trees of nature. Many types of trees defined by all sorts of constraints on properties of vertices appear to be of interest in combinatorics and in related areas such as formal logic and computer science.

A *planar binary tree* is a finite oriented tree embedded in the plane, such that each internal vertex has exactly two incoming edges and one outgoing edge. One of the internal vertices, called the *root*, is a distinguished vertex with two incoming edges

and one edge, like a tail at the bottom, not ending at a vertex. The incoming edges in this type of trees are internal (connecting two internal vertices), or external (with one free end). The external incoming edges are called the *leaves*. We give here some examples of planar binary trees:



where the single edge “|” is the unique planar binary tree without internal vertices.

The *degree* of any planar binary tree is the number of its leaves. Denote by T_{pl}^{bin} (respectively \mathcal{T}_{pl}^{bin}) the set (respectively the linear span) of planar binary trees.

Define the *grafting operation* “ \vee ” on the space \mathcal{T}_{pl}^{bin} to be the operation that maps any planar binary trees t_1, t_2 into a new planar binary tree $t_1 \vee t_2$, which takes the Y -shaped tree \vee replacing its left (respectively right) branch by t_1 (respectively t_2), see the following examples:

$$| \vee | = \vee, \quad | \vee \vee = \vee\vee, \quad \vee \vee | = \vee\vee, \quad \vee \vee \vee = \vee\vee\vee, \quad | \vee \vee\vee = \vee\vee\vee.$$

The number of binary trees of degree n is given by the Catalan number

$$c_n = \frac{(2n)!}{(n+1)n!},$$

where the first ones are 1, 1, 2, 5, 14, 42, 132, This sequence of numbers is the sequence A000108 in [24].

Let E be a (non-empty) set. The *free magma* $M(E)$ generated by E can be described as the set of planar binary trees with leaves decorated by the elements of E , together with the “ \vee ” product described above [16, 11]. Moreover, the *linear span* $\mathcal{T}_{pl}^{bin,E}$, generated by the trees of the magma $M(E) = \mathcal{T}_{pl}^{bin,E}$ defined above, equipped with the grafting “ \vee ” is a description of the free magmatic algebra.

For any positive integer n , a *rooted tree of degree n* , or simply *n -rooted tree*, is a finite oriented tree together with n vertices. One of them, called the *root*, is a distinguished vertex without any outgoing edge. Any vertex can have arbitrarily many incoming edges, and any vertex distinct from the root has exactly one outgoing edge. Vertices with no incoming edges are called *leaves*.

A rooted tree is said to be *planar*, if it is endowed with an embedding in the plane. Otherwise, its called a (non-planar) rooted tree. Let E be a (non-empty) set. An E -decorated rooted tree is a pair (t, d) of a rooted tree t together with a map $d: V(t) \rightarrow E$, which decorates each vertex v of t by an element a of E , i.e. $d(v) = a$, where $V(t)$ is the set of all vertices of t . Here are the planar (undecorated) rooted trees up to five vertices:



From now on, we will consider that all our trees are decorated, except for some cases in which we will state the property explicitly. Denote by T_{pl}^E (respectively

T^E) the set of all planar (respectively non-planar) decorated rooted trees, and \mathcal{T}_{pl}^E (respectively \mathcal{T}^E) the linear space spanned by the elements of T_{pl}^E (respectively T^E). Any rooted tree σ with branches $\sigma_1, \dots, \sigma_k$ and a root \bullet , can be written as:

$$\sigma = B_{+,a}(\sigma_1 \cdots \sigma_k), \tag{1}$$

where $B_{+,a}$ is the operation which grafts a monomial $\sigma_1 \cdots \sigma_k$ of rooted trees on a common root decorated by an element a in E , which gives a new rooted tree. The planar rooted tree σ in formula (1) depends on the order of the branches, whereas this order is not important for the corresponding (non-planar) tree.

Define the (left) Butcher product \circlearrowleft of any planar rooted trees σ and τ by:

$$\sigma \circlearrowleft \tau := B_{+,a}(\sigma \tau_1 \cdots \tau_k), \tag{2}$$

where $\tau_1, \dots, \tau_k \in T_{pl}^E$, such that $\tau = B_{+,a}(\tau_1 \cdots \tau_k)$. It maps the pair of trees (σ, τ) into a new planar rooted tree induced by grafting the root of σ , on the left via a new edge, on the root of τ .

The left grafting \searrow is a bilinear operation defined on the vector space \mathcal{T}_{pl}^E , such that for any planar rooted trees σ and τ :

$$\sigma \searrow \tau = \sum_{v \text{ vertex of } \tau} \sigma \searrow_v \tau, \tag{3}$$

where $\sigma \searrow_v \tau$ is the tree obtained by grafting the tree σ , on the left, on the vertex v of the tree τ , such that σ becomes the leftmost branch, starting from v , of this new tree. For example:



The number of trees in T_{pl}^E is the same than in $T_{pl}^{bin,E}$: a one-to-one correspondence between them is given by D. Knuth's rotation correspondence [16] (see subsection 2.1 in [2]). On the other hand, for any homogeneous component T^n of (non-planar) undecorated rooted trees of degree "n", for $n \geq 1$, the number of trees in T^n is given by the sequence: 1, 1, 2, 4, 9, 20, 48, ..., which is sequence A000081 in [24].

The two graftings, defined by (2) and (3) above, provide the space \mathcal{T}_{pl}^E with structures of free magmatic algebras. K. Ebrahimi-Fard and D. Manchon showed, in their joint work (unpublished), that these two structures on \mathcal{T}_{pl}^E are isomorphic. The unique isomorphism $\Psi: (\mathcal{T}_{pl}^E, \circlearrowleft) \rightarrow (\mathcal{T}_{pl}^E, \searrow)$ extending the identity map on E admits an explicit matrix on the decorated tree basis:

$$\Psi(\tau) = \sum_{\sigma} c(\sigma, \tau) \sigma. \tag{4}$$

The coefficients $c(\sigma, \tau)$ are non-negative integers explicitly given in terms of three different partial orders on the vertices of σ and τ . For a detailed account of this work see [2, subsection 2.1].

In the non-planar case, the usual product $\circ\rightarrow$ given by the same formula (2), is known as the Butcher product. It is non-associative permutative (NAP), i.e. it satisfies the following identity:

$$s \circ\rightarrow (s' \circ\rightarrow t) = s' \circ\rightarrow (s \circ\rightarrow t),$$

for any (non-planar) trees s, s', t . The grafting product \rightarrow defined as a bilinear map on the vector space \mathcal{T}^E as follows:

$$s \rightarrow t = \sum_{v \in V(t)} s \rightarrow_v t, \tag{5}$$

for any $s, t \in \mathcal{T}^E$, where $s \rightarrow_v t$ is the (non-planar) decorated rooted tree obtained by grafting the tree s on the vertex v of the tree t . This product is pre-Lie (see Paragraph 3). For the case with one generator, we have:

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \circ\rightarrow \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \rightarrow \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array}.$$

3. Lie and pre-Lie algebras

A Lie algebra over a field K is a K -vector space \mathcal{L} , with a K -bilinear mapping $[\cdot, \cdot]: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ (the Lie bracket), satisfying the following properties:

$$[x, y] + [y, x] = 0 \text{ (anti-symmetry)} \tag{6}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \text{ (Jacobi identity)} \tag{7}$$

for all $x, y, z \in \mathcal{L}$.

A left pre-Lie algebra is a vector space \mathcal{A} over a field K , together with a bilinear operation " \triangleright " that satisfies:

$$(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z), \forall x, y, z \in \mathcal{A}. \tag{8}$$

The identity (8) is called the *left pre-Lie identity*, and it can be written as:

$$L_{[x,y]} = [L_x, L_y], \forall x, y \in \mathcal{A}, \tag{9}$$

where for every element x in \mathcal{A} , the linear transformation L_x of the vector space \mathcal{A} is defined by $L_x(y) = x \triangleright y, \forall y \in \mathcal{A}$, and $[x, y] = x \triangleright y - y \triangleright x$ is the commutator of the elements x and y in \mathcal{A} . The usual commutator $[L_x, L_y] = L_x L_y - L_y L_x$ of the linear transformations of \mathcal{A} defines a structure of Lie algebra over K on the vector space $L(\mathcal{A})$ of all linear transformations of \mathcal{A} . For any pre-Lie algebra \mathcal{A} , the bracket $[\cdot, \cdot]$ satisfies the Jacobi identity, hence induces a structure of Lie algebra on \mathcal{A} .

3.1. Free Lie algebras The Lie algebra of Lie polynomials, introduced by E. Witt (1911-1991), is actually the free Lie algebra. The first appearance of Lie polynomials was at the turn of the century in the work of Campbell, Baker and Hausdorff on the exponential mapping in a Lie group, when the well-known

result “Campbell-Baker-Hausdorff formula” appeared. For more details about a historical review of free Lie algebras, we refer the reader to the reference [21] and the references therein.

A *free Lie algebra* is a pair (\mathcal{L}, i) , of a Lie algebra \mathcal{L} together with a map $i: E \rightarrow \mathcal{L}$ from a (non-empty) set E into \mathcal{L} , satisfying the following universal property: for any Lie algebra \mathcal{L}' and any mapping $f: E \rightarrow \mathcal{L}'$, there is a unique Lie algebra homomorphism $\tilde{f}: \mathcal{L} \rightarrow \mathcal{L}'$ which makes the following diagram commute: It is

$$\begin{array}{ccc} E & \xrightarrow{i} & \mathcal{L} \\ & \searrow f & \downarrow \tilde{f} \\ & & \mathcal{L}' \end{array}$$

Figure 1: The universal property of the free Lie algebra.

unique up to an isomorphism. If \mathcal{L} is a K -Lie algebra and $E \subseteq \mathcal{L}$, then we say that E freely generates \mathcal{L} if (\mathcal{L}, i) is free, where i is the canonical injection from E to \mathcal{L} .

Recall from [7, 21] that the enveloping algebra $\mathcal{U}(\mathcal{L})$ of the free Lie algebra $\mathcal{L}(E)$ is the free unital associative algebra on E . The Lie algebra homomorphism $\varphi_0: \mathcal{L}(E) \rightarrow \mathcal{U}(\mathcal{L})$ is injective, and $\varphi_0(\mathcal{L}(E))$ is the Lie subalgebra of $\mathcal{U}(\mathcal{L})$ generated by $j(E)$, where $j := \varphi_0 \circ i$.

3.2. Gröbner bases The theory of Gröbner bases was introduced in 1965 by B. Buchberger for ideals in polynomial rings and an algorithm called Buchberger algorithm for their computation. This theory contributed, since the end of the Seventies, in the development of computational techniques for the symbolic solution of polynomial systems of equations and in the development of effective methods in Algebraic Geometry and Commutative Algebra. Moreover, this theory has been generalized to free non-commutative algebras and to various non-commutative algebras of interest in Differential Algebra, e.g. Weyl algebras, enveloping algebras of Lie algebras [20], and so on.

The attempt to imitate Gröbner basis theory for non-commutative algebras works fine up to the point where the termination of the analogue to the Buchberger algorithm can be proved. Gröbner bases and Buchberger algorithm have been extended, for the first time, to ideals in free non-commutative algebras by G. Bergman in 1978. Later, T. Mora in 1986 made precise in which sense Gröbner bases can be computed in free non-commutative algebras [20]. The construction of finite Gröbner bases for arbitrary finitely generated ideals in non-commutative rings is possible in the class of *solvable algebras*¹. This class comprises many algebras arising in mathematical physics such as: Weyl algebras, enveloping algebras of finite-dimensional Lie algebras, and iterated skew polynomial rings. Gröbner bases were studied, in these algebras, for special cases by J. Apel and W. Lassner in 1985, and in full generality by R. Kandri-Rody and V. Weispfenning in 1990 [3].

¹For more details about the solvable algebras see [3, Appendix: Non-Commutative Gröbner Bases, p. 526-528].

Recently, V.Drensky and R.Holtkamp used Gröbner theory in their work [8] for a non-associative, non-commutative case (the magmatic case). Whereas, L. A. Bokut, Y. Chen and Y. Li, in their work [4], give Gröbner-Shirshov basis for a right-symmetric algebra (pre-Lie algebra). The theory of Gröbner-Shirshov bases was invented by A. I. Shirshov for Lie algebras in 1962 [23].

We try in this paper, precisely in Section 5, to describe a monomial basis in tree version for the free Lie (respectively pre-Lie) algebras using the procedures of Gröbner bases, comparing with the one (i.e. the monomial basis) obtained for the free pre-Lie algebra in our preceding work [2]. We need here to review some basics for the theory of Gröbner bases.

Definition 3.1. Let $(M(E), \cdot)$ be the free magma generated by E . A total order $<$ on $M(E)$ is said to be *monomial* if it satisfies the following property:

$$\text{for any } x, y, z \in M(E), \text{ if } x < y, \text{ then } x \cdot z < y \cdot z \text{ and } z \cdot x < z \cdot y, \quad (10)$$

i.e. it is compatible with the product in $M(E)$.

This property, in (10), implies that for any $x, y \in M(E)$ then $x < x \cdot y$. An order is called a *well-ordering* if every strictly decreasing sequence of monomials is finite, or equivalently if every non-empty set of monomials has a minimal element.

Let \mathcal{M}_E be the K -linear span of the free magma $M(E)$, and I be any magmatic (two-sided) ideal of \mathcal{M}_E . For any element $f = \sum_{x \in M(E)} \lambda_x x$ (finite sum) in I , define $T(f)$ to be the maximal term of f with respect to a given monomial order defined on $M(E)$, namely $T(f) = \lambda_{x_0} x_0$, with $x_0 = \max\{x \in M(E), \lambda_x \neq 0\}$. Denote $T(I) := \{T(f) : f \in I\}$ the set of all maximal terms of elements of I . Note that the set $T(I)$ forms a (two-sided) ideal of the magma $M(E)$ [20]. Define the set $O(I) := M(E) \setminus T(I)$. We have that the magma $M(E) = T(I) \cup O(I)$ is the disjoint union of $T(I), O(I)$ respectively. As a consequence, we get that:

$$\mathcal{M}_E = \text{Span}_K(T(I)) \oplus \text{Span}_K(O(I)). \quad (11)$$

Define a linear mapping φ from I into $\text{Span}_K(T(I))$, which makes the following diagram commute, where P is the projection map.

$$\begin{array}{ccc} I & \xrightarrow{i} & \mathcal{M}_E \xrightarrow{\cong} \text{Span}_K(T(I)) \oplus \text{Span}_K(O(I)) \\ & \searrow \varphi & \downarrow P \\ & & \text{Span}_K(T(I)) \end{array}$$

Figure 2: Definition of φ .

Then the mapping φ is defined by:

$$\varphi(f) = \sum_{x \in T(I)} \alpha_x x, \text{ for } f \in I, \quad (12)$$

where $f = \sum_{x \in T(I)} \alpha_x x + \text{corrective term in } \text{Span}_K(O(I))$, and $\alpha_x \in K$ for all $x \in T(I)$. The map φ is obviously injective. Indeed, for any $f \in I$ and $\varphi(f) = 0$, then $f \in \text{Span}_K(O(I))$, and from Theorem 5.2, $\text{Span}_K(O(I)) \cap I = \{0\}$. Also, according to Theorem 5.2 and by the definition of φ in (12), we note that φ is surjective. Hence, φ is an isomorphism of vector spaces. Thus, we can deduce from the formula (11):

$$\mathcal{M}_E = I \oplus \text{Span}_K(O(I)). \quad (13)$$

In Section 5, we will give a tree version of the monomial well-ordering with a review of T. Mora's work [20], in the case of rooted trees.

3.3. Free pre-Lie algebras

As a particular example of pre-Lie algebras, take the linear space of the set of all (non-planar) E -decorated rooted trees \mathcal{T}^E which has a structure of pre-Lie algebra together with the product " \rightarrow " defined in (5).

Free pre-Lie algebras have been handled in terms of rooted trees by F. Chapoton and M. Livernet [6], who also described the pre-Lie operad explicitly, and by A. Dzhumadil'daev and C. Löfwall independently [9]. For an elementary version of the approach by Chapoton and Livernet without introducing operads, see e.g. [19, Paragraph 6.2]:

Theorem 3.2. *Let k be a positive integer. The free pre-Lie algebra with k generators is the vector space \mathcal{T} of (non-planar) rooted trees with k colors, endowed with grafting.*

4. A pre-Lie structure on free Lie algebras

Let $\mathcal{L}(E)$ be the free Lie algebra generated by a (non-empty) disjoint union of subsets $E = \bigsqcup_{i \in \mathbb{N}} E_i$, where E_i is the subset of elements $a_1^i, \dots, a_{d_i}^i$ of degree i , and $\#E_i = d_i$. The free Lie algebra $\mathcal{L}(E)$ can be graded, using the grading of E :

$$\mathcal{L}(E) = \bigoplus_{i \in \mathbb{N}} \mathcal{L}_i, \quad (14)$$

where \mathcal{L}_i is the subspace of all elements of $\mathcal{L}(E)$ of degree i . In particular $E_i \subset \mathcal{L}_i$. Define an operation \triangleright for $x, y \in \mathcal{L}(E)$ by:

$$x \triangleright y := \frac{1}{|x|} [x, y]. \quad (15)$$

Proposition 4.1. *The operation \triangleright defined by (15) is a bilinear product which satisfies the pre-Lie identity.*

Proof. For $x, y, z \in \mathcal{L}(E)$, we have:

$$\begin{aligned}
(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) &= \frac{1}{|x|} [x, y] \triangleright z - \frac{1}{|y|} x \triangleright [y, z] \\
&= \frac{1}{|x|(|x| + |y|)} [[x, y], z] - \frac{1}{|x||y|} [x, [y, z]] \\
&= \frac{1}{|x|(|x| + |y|)} [[x, y], z] - \frac{1}{|x||y|} ([[x, y], z] - [y, [z, x]]), \\
&\quad \text{since } [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \text{ (the Jacobi identity)} \\
&= \left(\frac{1}{|x|} \right) \left(\frac{|y| - (|x| + |y|)}{|y|(|x| + |y|)} \right) [[x, y], z] + \frac{1}{|x||y|} [y, [z, x]] \\
&= \frac{1}{|y|(|x| + |y|)} [[y, x], z] - \frac{1}{|x||y|} [y, [x, z]] = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z).
\end{aligned}$$

Then $\mathcal{L}(E)$ together with \triangleright forms a graded pre-Lie algebra generated by E . \blacksquare

A pre-Lie algebra structure can be put this way on any \mathbb{N} -graded Lie algebra \mathcal{L} such that $\mathcal{L}_0 = \{0\}$. Another pre-Lie bracket, proposed on \mathcal{L} by T. Schedler [22]², is given by:

$$x \blacktriangleright y = \frac{|y|}{|x| + |y|} [x, y], \quad \text{for any } x, y \in \mathcal{L}. \quad (16)$$

These two constructions are isomorphic, via the linear map:

$$\alpha : (\mathcal{L}, \blacktriangleright) \longrightarrow (\mathcal{L}, \triangleright), \quad x \longmapsto |x|x. \quad (17)$$

Indeed, α is a bijection, and for any $x, y \in \mathcal{L}$ we have:

$$\begin{aligned}
\alpha(x \blacktriangleright y) &= \alpha\left(\frac{|y|}{|x| + |y|} [x, y]\right), \quad (\text{by the definition of } \blacktriangleright \text{ in (16)}), \\
&= \frac{|y|}{|x| + |y|} \alpha([x, y]) = \frac{|y|}{|x| + |y|} (|x| + |y|) [x, y], \quad (\text{by the definition of } \alpha \text{ above}), \\
&= |y|[x, y] = \frac{|x||y|}{|x|} [x, y] = (|x||y|)x \triangleright y, \quad (\text{by the definition of } \triangleright \text{ in (15)}), \\
&= |x|x \triangleright |y|y = \alpha(x) \triangleright \alpha(y).
\end{aligned}$$

Denote by $[\cdot, \cdot]_{\triangleright}$ the underlying Lie bracket induced by the pre-Lie product \triangleright , which is defined by:

$$[x, y]_{\triangleright} = x \triangleright y - y \triangleright x, \quad \text{for } x, y \in \mathcal{L}. \quad (18)$$

Then the two Lie structures defined on \mathcal{L} by the Lie brackets $[\cdot, \cdot]$, $[\cdot, \cdot]_{\triangleright}$ respectively, are also isomorphic via α . Indeed, by substituting the pre-Lie product \triangleright , described in (15), by the Lie bracket $[\cdot, \cdot]$ in the definition of the Lie bracket $[\cdot, \cdot]_{\triangleright}$ in (18), we get:

²For more details about this construction of pre-Lie algebra see [22, Proposition 3.3.3] and [12].

$$[x, y]_{\triangleright} = \frac{1}{|x|}[x, y] - \frac{1}{|y|}[y, x] = \frac{|x| + |y|}{|x||y|}[x, y], \text{ for any } x, y \in \mathcal{L}, \tag{19}$$

but,

$$\begin{aligned} \alpha([x, y]) &= |[x, y]| [x, y] = (|x| + |y|)[x, y] = |x||y|[x, y]_{\triangleright} \quad (\text{by (19)}) \\ &= [|x|x, |y|y]_{\triangleright} = [\alpha(x), \alpha(y)]_{\triangleright} \quad (\text{by (17)}). \end{aligned}$$

For any (non-planar) rooted tree t , we can decorate the vertices of t by elements of E , by means of a map $d: V(t) \rightarrow E$, where $V(t)$ is the set of vertices of t . Denote by T^E the set of all (non-planar) rooted trees decorated by the elements of E , define the degree $|t|$ of a decorated tree t in T^E by:

$$|t| := \sum_{v \in V(t)} |d(v)| \tag{20}$$

In particular, there is a unique pre-Lie homomorphism Φ from $(\mathcal{T}^E, \rightarrow)$ onto $(\mathcal{L}(E), \triangleright)$, such that:

$$\Phi(\bullet) = a \text{ for any } a \in E. \tag{21}$$

If we take $t = t_1 \rightarrow (t_2 \rightarrow (\dots \rightarrow (t_k \rightarrow \bullet) \dots)) \in \mathcal{T}^E$, then:

$$\Phi(t) = x_1 \triangleright (x_2 \triangleright (\dots \triangleright (x_k \triangleright a) \dots)), \tag{22}$$

with $x_i = \Phi(t_i)$, and $|t_i| = |x_i|, \forall i = 1, \dots, k$. Let I be the two-sided ideal of \mathcal{T}^E generated by all elements on the form:

$$|s|(s \rightarrow t) + |t|(t \rightarrow s), \text{ for } s, t \in \mathcal{T}^E. \tag{23}$$

The ideal I satisfies the following properties:

Proposition 4.2. *The quotient $\mathcal{L}'(E) := \mathcal{T}^E/I$ has structures of pre-Lie algebra and Lie algebra, respectively.*

Proof. Using the pre-Lie grafting \rightarrow defined on \mathcal{T}^E , we can define the following operations on $\mathcal{L}'(E)$:

$$\bar{s} \triangleright^* \bar{t} := \bar{s} \rightarrow \bar{t} := \overline{s \rightarrow t}, \tag{24}$$

$$[\bar{s}, \bar{t}] := \overline{[s, t]} := |s|\overline{s \rightarrow t}, \tag{25}$$

for any $s, t \in \mathcal{T}^E$, where the bar stands for the class modulo I . The product in (24) is pre-Lie by definition. The bracket defined in (25) is well-defined and satisfies the following identities:

1. The anti-symmetry identity: for any $s, t \in \mathcal{T}^E$, we have

$$[\bar{s}, \bar{t}] = -[\bar{t}, \bar{s}], \text{ since, } |s|(s \rightarrow t) + |t|(t \rightarrow s) \in I.$$

2. The Jacobi identity: for any $s, t, t' \in \mathcal{T}^E$, then

$$\begin{aligned} [\bar{s}, [\bar{t}, \bar{t}']] + [[\bar{s}, \bar{t}'], \bar{t}] &= |s||t| (\overline{(s \rightarrow (t \rightarrow t'))}) + |s|(|s| + |t'|) (\overline{(s \rightarrow t') \rightarrow t}) \\ &= |s||t| ((\overline{(s \rightarrow (t \rightarrow t'))}) - \overline{(t \rightarrow (s \rightarrow t'))}) \text{ (using the anti-symmetry identity)} \\ &= |s||t| (((\overline{(s \rightarrow t) \rightarrow t'}) - \overline{(t \rightarrow s) \rightarrow t'})) \text{ (using the pre-Lie identity)} \\ &= |s||t| (\overline{(s \rightarrow t - t \rightarrow s) \rightarrow t'}) \\ &= |s||t| (\overline{((s \rightarrow t + \frac{|s|}{|t|} s \rightarrow t) \rightarrow t')}) \text{ (using the anti-symmetry identity)} \\ &= |s||t| \frac{|s| + |t|}{|t|} (\overline{(s \rightarrow t) \rightarrow t'}) = |s|(|s| + |t|) (\overline{(s \rightarrow t) \rightarrow t'}) = [[\bar{s}, \bar{t}], \bar{t}']. \quad \blacksquare \end{aligned}$$

Proposition 4.3. $I = Ker \Phi$.

Let $(M(E), \cdot)$ be the free magma generated by E , and let \mathcal{M}_E be the free magmatic algebra generated by E , i.e. the linear span of the magma $M(E)$. Define a new magmatic product $*$ on $M(E)$ by:

$$x * y := |x|x \cdot y \tag{26}$$

for any $x, y \in M(E)$, and extend bilinearly. We need, to prove Proposition 4.3, to introduce the following lemmas.

Lemma 4.4. *The two magmatic algebras (\mathcal{M}_E, \cdot) and $(\mathcal{M}_E, *)$ are isomorphic.*

Proof. By the universal property of the free magmatic algebra, there is a unique morphism $\gamma: (\mathcal{M}_E, \cdot) \rightarrow (\mathcal{M}_E, *)$ such that $\gamma(a) = a$, for any $a \in E$. For any $x, y \in \mathcal{M}_E$, we have:

$$\gamma(x \cdot y) = \gamma(x) * \gamma(y) = |\gamma(x)| \gamma(x) \cdot \gamma(y). \tag{27}$$

Hence one can see, by induction on the degree of elements of the magma $M(E)$, that we have for any $z \in M(E)$:

$$\gamma(z) = f(z)z, \tag{28}$$

where $f: M(E) \rightarrow \mathbb{N}$ is recursively given by: $f(a) = 1$, for any $a \in E$, and $f(x \cdot y) = |x|f(x)f(y)$ for $x, y \in M(E)$ (for more details about this mapping see Example 4.6 below). Hence γ is an isomorphism. \blacksquare

Now, let J be the two-sided ideal generated by the the anti-symmetry and the Jacobi identities on $(\mathcal{M}_E, *)$, and let J' be the two-sided ideal of (\mathcal{M}_E, \cdot) generated by the pre-Lie identity and the elements on the form:

$$|x|x \cdot y + |y|y \cdot x, \text{ for } x, y \in M(E). \tag{29}$$

Lemma 4.5. $J = J'$.

Proof. Let J'_1 be the ideal generated by the elements (29). Equivalently, J'_1 is generated by the elements $x * y + y * x$, for $x, y \in M(E)$. We have:

$$\begin{aligned} &x \cdot (y \cdot z) - (x \cdot y) \cdot z - y \cdot (x \cdot z) + (y \cdot x) \cdot z = \\ &= \frac{1}{|x||y|}x * (y * z) - \frac{1}{|x|(|x| + |y|)}(x * y) * z - \frac{1}{|x||y|}y * (x * z) \\ &\quad + \frac{1}{|y|(|x| + |y|)}(y * x) * z \\ &= \frac{1}{|x||y|(|x| + |y|)}((|x| + |y|)x * (y * z) - |y|(x * y) * z - (|x| + |y|)y * (x * z) \\ &\quad + |x|(y * x) * z) \\ &= \frac{1}{|x||y|}(-y * (x * z) + (y * x) * z + x * (y * z)) - \frac{1}{|x|(|x| + |y|)}(y * x + x * y) * z \\ &= \frac{1}{|x||y|}(x * (y * z) + y * (z * x) + z * (x * y)) \text{ modulo } J'_1, \end{aligned}$$

hence $x \cdot (y \cdot z) - (x \cdot y) \cdot z - y \cdot (x \cdot z) + (y \cdot x) \cdot z \in J$. This means $J' \subset J$.

Conversely,

$$\begin{aligned} &x * (y * z) + y * (z * x) + z * (x * y) = \\ &= |x||y|(x \cdot (y \cdot z) - (x \cdot y) \cdot z - y \cdot (x \cdot z) + (y \cdot x) \cdot z) \text{ modulo } J'_1, \end{aligned} \tag{30}$$

hence the left-hand side of (30) belongs to J' , proving the inverse inclusion. ■

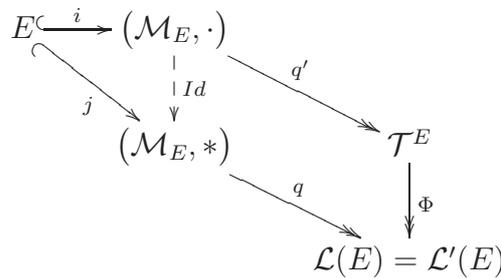


Figure 3: The quotient maps q and q' .

Proof of Proposition 4.3. The free pre-Lie algebra generated by E is given by \mathcal{T}^E [6], [9]. Hence, the quotient $\mathcal{L}'(E) = (\mathcal{M}_E, \cdot)/J' = \mathcal{T}^E/I$ is a pre-Lie (respectively Lie) algebra. The Lie algebra $\mathcal{L}(E) = (\mathcal{M}_E, *)/J$ carries a pre-Lie algebra structure induced by the product defined in (15), such that the free pre-Lie algebra $\mathcal{PL}(E) := \mathcal{M}_E/J'_2 = \mathcal{T}^E$, where J'_2 is the two-sided ideal generated by the pre-Lie identity on (\mathcal{M}_E, \cdot) , is homomorphic to $\mathcal{L}(E)$ by Φ described in (21) and (22), as pre-lie algebras, as in the commutative diagram in Figure 3, where q, q' therein are quotient maps.

From Figure 3 and Lemmas 4.4, 4.5, we get that:

$$\text{Ker}(\Phi \circ q') = J' = J = \text{Ker } q, \text{ and then } \text{Ker } \Phi = q'(J') = q'(J) = I,$$

therefore Proposition 4.3 is proved. ■

Note that the Lie product on $\mathcal{L}(E)$ is the image of $*$ by q . The pre-Lie product \triangleright is the image of “ \cdot ” by $\Phi \circ q'$. Hence, we recover Proposition 4.1 this way.

Example 4.6. The free magma $M(E)$ can also be identified with the set of all planar binary rooted trees, with leaves decorated by the elements of E , together with the product \vee defined in section 2. For instance,

$$\begin{aligned}
 a \cdot b &= \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \end{array}, \quad (a \cdot b) \cdot c = \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagup \\ \end{array}, \quad a \cdot (b \cdot c) = \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagdown \quad \diagup \\ \end{array}, \quad \text{and} \\
 z = x \cdot y &= ((a \cdot b) \cdot c) \cdot (d \cdot e) = \begin{array}{c} a \quad b \quad c \quad d \quad e \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ x \quad \quad \quad y \end{array}, \tag{31}
 \end{aligned}$$

with $x := (a \cdot b) \cdot c$, and $y := d \cdot e$. Then:

$$\begin{aligned}
 f(z) &= f(x \cdot y) = |x|f(x)f(y) = |x|(|a \cdot b|f(a \cdot b)f(c)) (|d|f(d)f(e)) \\
 &= |x|((|a| + |b|) (|d|(|a|f(a)f(b)f(c)f(d)f(e)))) \\
 &= |a| |d| (|a| + |b|) (|a| + |b| + |c|) \quad (\text{since, } f(a)f(b)f(c)f(d)f(e) = 1).
 \end{aligned}$$

There is another description of f , detailed as follows: in a planar binary tree, there are two types of edges, going on the left (from bottom to top) or going on the right. Consequently, except the root, there are two types of vertices, the left ones (the incoming edge on the left) and the right ones. Let t be a planar binary tree, with leaves decorated by elements of E , then $f(t)$ is the product over all left vertices v of the sums of the degree of the decorations of the leaves l with a path from v to l .

Consequently, from Propositions 4.1, 4.2 and 4.3, we get the following result.

Corollary 4.7. *There is a unique pre-Lie (respectively Lie) isomorphism between $\mathcal{L}'(E)$ and $\mathcal{L}(E)$, such that $\Phi(a \text{ mod. } J') = a \text{ mod. } J$, for any $a \in E$.*

5. A monomial well-order on the planar rooted trees, and applications

Let $E := \bigsqcup_{n \geq 1} E_n$ be a disjoint union of finite subsets $E_n = \{a_1^n, \dots, a_{d_n}^n\}$, where E_n is the subset of all elements of E of degree n . We order the elements of E by:

$$a_1^1 < \dots < a_{d_1}^1 < a_1^2 < \dots < a_{d_2}^2 < \dots < a_1^i < \dots < a_{d_i}^i < \dots \tag{32}$$

Some particular sets E of generators can be considered:

1. $E = \bigsqcup_{n \geq 1} E_n$, where $\#E_i = 0$ or 1. A particular situation is:
 - (a) take $E = \{a_1, \dots, a_s\}$, with $a_i \in E_i$, and $|a_i| = i$, for $i = 1, \dots, s$.
2. $E = E_1$, where $\#E_1 = d_1 = d$, as a special case:
 - (a) take $d_1 = d = 2$.

Theorem 5.2. *If I is a (two-sided) ideal of \mathcal{T}_{pl}^E , then:*

1. $\mathcal{T}_{pl}^E = I \oplus \text{Span}_K(O(I))$.
2. $\mathcal{T}_{pl}^{E,*} := \mathcal{T}_{pl}^E/I$ is isomorphic, as a K -vector space, to $\text{Span}_K(O(I))$.
3. For each $f \in \mathcal{T}_{pl}^E$ there is a unique $g := \text{Can}(f, I) \in \text{Span}_K(O(I))$, such that $f - g \in I$. Moreover:
 - (a) $\text{Can}(f, I) = \text{Can}(g, I)$ if and only if $f - g \in I$.
 - (b) $\text{Can}(f, I) = 0$ if and only if $f \in I$.

The symbol $\text{Can}(f, I)$, which satisfies the identities above, is called the canonical form of f in $\text{Span}_K(O(I))$.

Proof. The proof is detailed in [20, Theorem 1.1] in the associative case. The procedure followed in the proof of (1) consists in the following algorithm:

```

 $f_0 := f, \phi_0 := 0, h_0 := 0, i := 0$ 
while  $f_i \neq 0$  do
  if  $T(f_i) \notin T(I)$  then
     $\phi_{i+1} := \phi_i, h_{i+1} := h_i + lc(f_i)T(f_i), f_{i+1} := f_i - lc(f_i)T(f_i)$ 
    else  $\%T(f_i) \in T(I)\%$ 
    choose  $g_i \in I$ , such that  $T(g_i) = T(f_i), lc(g_i) = 1$ 
     $\phi_{i+1} := \phi_i + lc(f_i)g_i, h_{i+1} := h_i, f_{i+1} := f_i - lc(f_i)g_i$ 
     $i := i + 1$ 
 $\phi := \phi_i, h := h_i$ 
    
```

The correctness of this algorithm is based on the following observations: $\forall i : \phi_i \in I, h_i \in \text{Span}_K(O(I)), f_i + \phi_i + h_i = f$. Termination is guaranteed by the easy observation that if $f_n \neq 0$ then $T(f_n) \preceq T(f_{n-1})$ and by the fact that \preceq is a well-ordering. ■

Let J' be the two-sided ideal of \mathcal{T}_{pl}^E generated by the pre-Lie identity and all elements on the form:

$$|\sigma|\sigma \circlearrowleft \tau + |\tau|\tau \circlearrowright \sigma, \text{ for any (non-empty) trees } \sigma, \tau \in T_{pl}^E. \tag{36}$$

Example 5.3. In this example we calculate $\text{Can}(f, J')$, where $f = \begin{matrix} \bullet & & \bullet & & \bullet & & \bullet \\ | & & | & & | & & / \backslash \\ \bullet & & \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix}$ and J' is the ideal defined by (36), using the algorithm described in the proof of Theorem 5.2 above:

$$\begin{aligned}
 f_0 &= \begin{matrix} \bullet & & \bullet & & \bullet & & \bullet \\ | & & | & & | & & / \backslash \\ \bullet & & \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix}, \quad \phi_0 = 0, \quad h_0 = 0 \\
 T(f_0) &= \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} \in T(J'), \text{ choose } g_0 = 3 \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} \in J', \quad lc(g_0) = 1 \\
 \phi_1 &= 3 \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix}, \quad h_1 = 0, \quad f_1 = \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} + \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} - 3 \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} \\
 T(f_1) &= \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} \in T(J'), \text{ choose } g_1 = \frac{1}{2} \left(\begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} + 2 \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} \right) \circlearrowleft \begin{matrix} \bullet & & \bullet & & \bullet \\ | & & | & & | \\ \bullet & & \bullet & & \bullet \end{matrix} \in J', \quad lc(g_1) = 1
 \end{aligned}$$

$$\phi_2 = 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad h_2 = 0, \quad f_2 = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

$$T(f_2) = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \notin T(J'), \text{ then:}$$

$$\phi_3 = 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad h_3 = \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad f_3 = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

$$T(f_3) = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \in T(J'), \text{ choose } g_3 = \frac{1}{3} \left(\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right) \in J'$$

$$\phi_4 = 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad h_4 = \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad f_4 = \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

$$T(f_4) = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \notin T(J'), \text{ then:}$$

$$\phi_5 = 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad h_5 = \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad f_5 = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

$$T(f_5) = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \in T(J'), \text{ choose } g_5 = \frac{1}{2} \left(\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + 2 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right) \in J'$$

$$\phi_6 = 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad h_6 = \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad f_6 = -\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

$$T(f_6) = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \notin T(J'), \text{ then:}$$

$$\phi_7 = 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad h_7 = \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \quad f_7 = 0,$$

then we obtain that $Can(f, J') = \frac{3}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$.

One can note that choosing different g 's at each step in the procedures above while changing the intermediate computations would not change the final result.

Theorem 5.2 does not describe the contents of each of $T(I)$ and $O(I)$. We try here to get a description of them, using the magma of planar rooted trees T_{pl}^E with its K -linear span \mathcal{T}_{pl}^E . Let J be the (two-sided) ideal of \mathcal{T}_{pl}^E generated by the pre-Lie identity with respect to the magmatic product \circlearrowleft . By Theorem 5.2, we have:

$$\mathcal{T}_{pl}^E = J \oplus Span_K(O(J)). \tag{37}$$

Proposition 5.4. *$O(J)$ is the set of $\sigma \in T_{pl}^E$ such that for any $v \in V(\sigma)$, the branches starting from v are displayed in nondecreasing order from left to right.*

The following lemma will help us to prove Proposition 5.4:

Lemma 5.5. *Any tree σ in T_{pl}^E which does not verify the condition of Proposition 5.4 belongs to $T(J)$.*

2. The case $k = 2$. This corresponds to the case $\tilde{\sigma} = \sigma \circlearrowleft l_m$, where l_m is a ladder in T_{pl}^E , with m vertices for $m \geq 2$. If σ is an element of $T(J')$ then so is $\tilde{\sigma}$. If not, σ is a ladder by the induction hypothesis. See the discussion below.
3. The case $k \geq 3$. These are trees $\tilde{\sigma} = (\sigma \circlearrowleft \tau)$ where $\tau \in T_{pl}^E$, with $k - 1$ branches, is not a ladder. We have then $\tilde{\sigma} \in T(J')$ by induction hypothesis.

Let us discuss the case (2) when σ is a ladder and the ladder l_m does not belong to $T(J')$. Let l_1, l_2 be ladders in T_{pl}^E with n_1, n_2 vertices respectively, where $n_1, n_2 < n$, and let:

$$\tilde{\sigma} = \begin{array}{c} l_1 \bullet \quad l_2 \bullet \\ \diagdown \quad \diagup \\ r \end{array} = l_1 \circlearrowleft (l_2 \circlearrowleft \bullet), \quad \sigma' = \begin{array}{c} l_2 \bullet \quad l_1 \bullet \\ \diagdown \quad \diagup \\ r \end{array} = l_2 \circlearrowleft (l_1 \circlearrowleft \bullet). \tag{40}$$

By the pre-Lie identity, with respect to the left Butcher product \circlearrowleft , we find the following element:

$$f_0 = \begin{array}{c} l_1 \bullet \quad l_2 \bullet \\ \diagdown \quad \diagup \\ r \end{array} - \begin{array}{c} \bullet \\ | \\ l_1 \\ | \\ l_2 \\ | \\ r \end{array} + \begin{array}{c} \bullet \\ | \\ l_2 \\ | \\ l_1 \\ | \\ r \end{array} - \begin{array}{c} l_2 \bullet \quad l_1 \bullet \\ \diagdown \quad \diagup \\ r \end{array} \tag{41}$$

in J' , such that $\tilde{\sigma}, \sigma'$ are bigger trees, with respect to the order \preceq defined in (33), than the two other trees in f_0 . Let $|l_i| = p_i$, where $p_i > 0$, for $i = 1, 2$. We have the following cases for p_i :

1. Either $p_1 = p_2$, then in this case we take the elements:

$$g = p_2 \begin{array}{c} l_2 \bullet \quad l_1 \bullet \\ \diagdown \quad \diagup \\ r \end{array} + (p_1 + |r|) \begin{array}{c} \bullet \\ | \\ l_1 \\ | \\ l_2 \\ | \\ r \end{array}, \quad f_1 = \begin{array}{c} l_1 \bullet \quad l_2^{(1)} \bullet \\ \diagdown \quad \diagup \\ r_1 \end{array} - \begin{array}{c} \bullet \\ | \\ l_2^{(1)} \\ | \\ l_1 \\ | \\ r_1 \end{array} + \begin{array}{c} l_2^{(1)} \bullet \quad l_1 \bullet \\ \diagdown \quad \diagup \\ r_1 \end{array} - \begin{array}{c} \bullet \\ | \\ l_1 \\ | \\ l_2^{(1)} \\ | \\ r_1 \end{array}, \tag{42}$$

in J' , where $l_2 = l_2^{(1)} \circlearrowleft \bullet$. Then we get the element:

$$f = p_2 f_0 + g - (p_1 + |r|) f_1 \in J', \tag{43}$$

such that $T(f) = \tilde{\sigma}$, since: $\begin{array}{c} \bullet \\ | \\ l_2^{(1)} \\ | \\ l_1 \\ | \\ r_1 \end{array} \prec \begin{array}{c} l_1 \bullet \quad l_2 \bullet \\ \diagdown \quad \diagup \\ r \end{array}$, for the order \preceq .

2. Or, $p_2 < p_1$, then $\tilde{\sigma} = T(f_0)$, where f_0 is the element described in (41), hence $\tilde{\sigma} \in T(J')$.
3. Or, $p_1 < p_2$, here we have that $\tilde{\sigma} \prec \sigma'$ and the element f_0 described in (41) is an element in J' such that $T(f_0) = \sigma'$, hence $\sigma' \in T(J')$. Now, for $\tilde{\sigma}$ we can get an element in J' such that $\tilde{\sigma}$ becomes the leading term of this element, as follows: we replace the tree $\sigma' = l_2 \circlearrowleft (l_1 \circlearrowleft \bullet)$ in f_0 by the tree:

$$\sigma'' := (l_1 \circlearrowleft \bullet) \circlearrowleft l_2 = \begin{array}{c} \bullet \\ | \\ l_1 \\ | \\ l_2 \\ | \\ r \end{array}, \tag{44}$$

using the element g described in (42). This new tree σ'' is also greater than $\tilde{\sigma}$ with respect to the order \preceq . By the pre-Lie identity, we can get the element f described in (43) such that $\tilde{\sigma}$ and

$$\sigma'_1 := l_2^{(1)} \circlearrowleft ((l_1 \circlearrowleft \bullet) \circlearrowleft \bullet) = \begin{array}{c} \bullet \\ | \\ l_2^{(1)} \\ | \\ l_1 \\ | \\ r_1 \end{array}$$

are the two biggest trees appearing in this element.

We verify whether $p_1 = |l_1| > |l_2^{(1)}| = p_2 - |r_1|$, i.e. $\sigma'_1 \preceq \tilde{\sigma}$, or not. If so, then $\tilde{\sigma} \in T(J')$. If not, we replace σ'_1 in f by the tree:

$$\sigma''_1 := ((l_1 \circlearrowleft \bullet) \circlearrowleft \bullet) \circlearrowleft l_2^{(1)} = \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_1 \\ | \\ \bullet \\ | \\ l_2^{(1)} \end{array}. \tag{45}$$

If $n_2 = 1$, the tree σ''_1 is a ladder. If $n_2 \geq 2$, then σ''_1 is not a ladder and is greater than $\tilde{\sigma}$. Then we need to apply the pre-Lie identity once again to the tree σ''_1 in (45), and replace it by:

$$\sigma'_2 := l_2^{(2)} \circlearrowleft (((l_1 \circlearrowleft \bullet) \circlearrowleft \bullet) \circlearrowleft \bullet) = \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_1 \\ | \\ \bullet \\ | \\ r_2 \end{array}, \text{ where } l_2^{(2)} \circlearrowleft \bullet = l_2^{(1)}.$$

Let $p_2^{(i)} = |l_2^{(i)}|$, where $l_2 = (\dots((l_2^{(i)} \circlearrowleft \bullet) \circlearrowleft \bullet) \dots) \circlearrowleft \bullet$, for $i \geq 1$. After a finite number s of steps applying the pre-Lie identity in the expression:

$$\sigma''_s := (((\dots((l_1 \circlearrowleft \bullet) \circlearrowleft \bullet) \dots) \circlearrowleft \bullet) \circlearrowleft \bullet) \circlearrowleft l_2^{(s)} = \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_{s-1} \\ | \\ \bullet \\ | \\ r_s \end{array} \tag{46}$$

where $\sigma'_s = \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_{s-1} \\ | \\ \bullet \\ | \\ r_s \end{array}$, which can be formulated as:

$$f_s = \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_{s-1} \\ | \\ \bullet \\ | \\ r_s \end{array} - \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_{s-1} \\ | \\ \bullet \\ | \\ r_s \end{array} + \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_{s-1} \\ | \\ \bullet \\ | \\ r_s \end{array} - \begin{array}{c} \bullet \\ | \\ r \\ | \\ \bullet \\ | \\ r_{s-1} \\ | \\ \bullet \\ | \\ r_s \end{array} \in J', \tag{47}$$

we can find an element $f \in J'$, such that $\tilde{\sigma}$ and σ'_s become bigger trees of f with $p_2^{(s)} < p_1$, i.e. $\sigma'_s \prec \tilde{\sigma}$. Hence, $\tilde{\sigma}$ described in (40) is in $T(J')$. Then, Theorem 5.7 is proved. ■

6. A monomial basis for the free Lie algebra

The set T^E forms the free Non-Associative Permutive (NAP) magma generated by the set $\{\bullet^a : \text{for } a \in E\}$, under the usual Butcher product \circlearrowleft . Corresponding to the total order defined in (33), we can define a non-planar version \preceq of this order, as follows:

$$\text{for any } s, t \in T^E, \text{ then } s \preceq t \text{ if and only if} \tag{48}$$

1. $|s| < |t|$, or :
2. $|s| = |t|$ and $b(s) < b(t)$, or:
3. $|s| = |t|, b(s) = b(t) = k$ and $s = B_{+,r}(s_1 \dots s_k), t = B_{+,r'}(t_1 \dots t_k)$ such that $\exists j \leq k$, with $s_i = t_i$, for $i < j, s_j \preceq t_j$ where $s_1 \preceq \dots \preceq s_k, t_1 \preceq \dots \preceq t_k$ are the branches of s, t respectively, or:
4. $|s| = |t|, b(s) = b(t) = k, s_l = t_l$, for all $l = 1, \dots, k$ and $r \leq r'$, where \bullet^r (respectively $\bullet^{r'}$) is the root of s (respectively t).

By the same way as in Proposition 5.1, we observe that the order \preceq defined in (48) is a monomial well order. The space \mathcal{T}^E forms with the Butcher product the free NAP algebra generated by E [18]. The first author introduced in [2]³ a section S from the NAP algebra $(\mathcal{T}^E, \circ \rightarrow)$ into the magmatic algebra $(\mathcal{T}_{pl}^E, \circ \searrow)$:

$$(\mathcal{T}_{pl}^E, \circ \searrow) \xrightleftharpoons[S]{\pi} (\mathcal{T}^E, \circ \rightarrow).$$

Here, we choose $S(t) = S_{min}(t) := Min_{\preceq} \{ \tau \in T_{pl}^E : \pi(\tau) = t \}$, for any $t \in T^E$, where $Min_{\preceq} \{ - \}$ means that we choose the minimal element τ in T_{pl}^E with respect to the order " \preceq " with $\pi(\tau) = t$.

Proposition 6.1. *The section map S_{min} defined above is an increasing map.*

Proof. Take two trees s and t in T^E with $s \preceq t$. The section S_{min} , obviously, respects the degree and the number of branches of the trees. Hence, we can suppose $|s| = |t|$ and $b(s) = b(t) = l$. We have then:

$$s = B_{+,r}(s_1, \dots, s_l), \quad t = B_{+,r'}(t_1, \dots, t_l), \quad \text{with } s_1 \preceq \dots \preceq s_l, \quad t_1 \preceq \dots \preceq t_l. \quad (49)$$

Condition (3) of the definition, in (48), of the order \preceq exactly means that the l -tuple of branches of $S_{min}(s)$ is lexicographically smaller than the l -tuple of branches of $S_{min}(t)$. If s and t have the same branches and $s \preceq t$, we also have $S_{min}(s) \preceq S_{min}(t)$, as one can see by comparing the roots. This proves Proposition 6.1. ■

Proposition 6.2. *The section map S_{min} on T^E is a bijection onto $O(J)$, where J is the (two-sided) ideal generated by the pre-Lie identity in $(\mathcal{T}_{pl}^E, \circ \searrow)$.*

Proof. Clear from Proposition 5.4. ■

Define a relation R on T^E as follows:

$$\begin{aligned} sRs' \text{ if and only if there are } t, t' \in T^E \text{ and } v, w \in V(t') \\ \text{such that } s = t \rightarrow_v t', s' = t \rightarrow_w t' \end{aligned} \quad (50)$$

for $s, s' \in T^E$, and w is related with v by an edge $\begin{matrix} \bullet \\ \nearrow \\ v \\ \bullet \end{matrix}^w$ with w above v . Let $\#$ be the transitive closure of the relation R defined in (50), i.e. for $s, s' \in T^E$, we say that $s\#s'$ if and only if there is $s_1, \dots, s_l \in T^E$ such that $sRs_1R \dots Rs_lRs'$.

Lemma 6.3. *Let $s, s', t \in T^E$, if $s' \preceq s$ then $s' \rightarrow_v t \preceq s \rightarrow_v t$, for $v \in V(t)$.*

Proof. Immediate from the definition (48) of the order \preceq . ■

Lemma 6.4. *Let $s, s' \in T^E$, if $s\#s'$ then $s' \prec s$.*

Proof. For $s, s' \in T^E$, if sRs' , then by definition of the relation R in (50), there are $\begin{matrix} \bullet \\ \nearrow \\ v \\ \bullet \end{matrix}^w, t, t' \in T^E$ and $v, w \in V(t)$ such that $s = t \rightarrow_v t', s' = t \rightarrow_w t'$, and an edge $\begin{matrix} \bullet \\ \nearrow \\ v \\ \bullet \end{matrix}$ in t' . Obviously, the tree obtained by grafting t on the tree t' at v is greater, with respect to the order \preceq , than the tree deduced by grafting t on t' at w , i.e $s' \prec s$. The passage from R to $\#$ is obvious. ■

³See subsection 2.2 in [2].

Proposition 6.5. *The Butcher product $\circ\rightarrow$ is compatible with the relation R , i.e. for $s, s', t \in T^E$, if sRs' then $(s\circ\rightarrow t)R(s'\circ\rightarrow t)$ and $(t\circ\rightarrow s)R(t\circ\rightarrow s')$. Also, if sRs' and tRt' then $(s\circ\rightarrow t)\#(s'\circ\rightarrow t')$, for $t' \in T^E$.*

Proof. For any $s, s', t, t' \in T^E$, if sRs' and tRt' , by definition of R we have: $s = s_1 \rightarrow_v s_2$, $s' = s_1 \rightarrow_w s_2$, for $v, w \in V(s_2)$ with $\begin{smallmatrix} \bullet \\ \nearrow \\ v \end{smallmatrix}$ in s_2 , and $t = t_1 \rightarrow_u t_2$, $t' = t_1 \rightarrow_{w'} t_2$, for $u, w' \in V(t_2)$ with $\begin{smallmatrix} \bullet \\ \nearrow \\ u \end{smallmatrix}$ in t_2 .

Let $s\circ\rightarrow t = (s_1 \rightarrow_v s_2)\circ\rightarrow(t_1 \rightarrow_u t_2) = s_1 \rightarrow_v s''$, for $v \in V(s'')$, where $s'' = s_2\circ\rightarrow t$ and $s'\circ\rightarrow t = (s_1 \rightarrow_w s_2)\circ\rightarrow(t_1 \rightarrow_u t_2) = s_1 \rightarrow_w s''$ for $\begin{smallmatrix} \bullet \\ \nearrow \\ v \end{smallmatrix}$ in s'' . Then:

$$s\circ\rightarrow t = (s_1 \rightarrow_v s'')R(s_1 \rightarrow_w s'') = s'\circ\rightarrow t. \tag{51}$$

Also, for $s'\circ\rightarrow t' = (s_1 \rightarrow_w s_2)\circ\rightarrow(t_1 \rightarrow_{w'} t_2) = t_1 \rightarrow_{w'} s'''$, where $s''' = s'\circ\rightarrow t_2$, with $w' \in V(t_2) \subset V(s''')$, and $s'\circ\rightarrow t = (s_1 \rightarrow_w s_2)\circ\rightarrow(t_1 \rightarrow_u t_2) = t_1 \rightarrow_u s'''$, for $u \in V(s''')$. Then we have:

$$s'\circ\rightarrow tRs'\circ\rightarrow t'. \tag{52}$$

One can verify that $s\circ\rightarrow tRs\circ\rightarrow t'$ by following the same steps as above. So, from (51) and (52), we obtain that $s\circ\rightarrow t\#s'\circ\rightarrow t'$. ■

For any $t \in T^E$, define a class of t with respect to $\#$ by:

$$[t]_{\#} := \{s \in T^E : t\#s\}. \tag{53}$$

This class has the following properties:

1. t is maximal among the representative elements in the class $[t]_{\#}$, i.e. for any $s \in [t]_{\#}$ then $s \preceq t$. This property is deduced from Lemma 6.4.
2. For any $s \in [t]_{\#}$, then $[s]_{\#} \subset [t]_{\#}$.

Lemma 6.6. *Let $\tilde{\Psi}_{S_{min}} : \mathcal{T}^E \rightarrow \mathcal{T}^E$ be defined by the following diagram:*

$$\begin{array}{ccc} (\mathcal{T}_{pl}^E, \circlearrowleft) & \xrightarrow{\Psi} & (\mathcal{T}_{pl}^E, \searrow) \\ \uparrow S_{min} \quad \downarrow \pi & & \downarrow \pi \\ (\mathcal{T}^E, \circ\rightarrow) & \xrightarrow{\tilde{\Psi}_{S_{min}}} & (\mathcal{T}^E, \rightarrow) \end{array}$$

and let $\beta_{S_{min}}(s, t)_{s, t \in T^E}$ be the coefficients defined by

$$\tilde{\Psi}_{S_{min}}(t) = \sum_{s \in T^E} \beta_{S_{min}}(s, t)s. \tag{54}$$

Then $\beta_{S_{min}}(s, t) \neq 0$ if and only if $s \in [t]_{\#}$.

Proof. The map $\tilde{\Psi}_{S_{min}}$ and the coefficients $\beta_{S_{min}}(s, t)$ are described in [2, Corollary 5]. In particular, the coefficients $\beta_{S_{min}}(s, t)$ are non-negative integers. We prove this Lemma by the induction on the number of vertices of the tree.

Suppose that (54) is realized for any tree in T^E with a number of vertices less than or equal to n . Take $t \in T^E$ be a tree, such that $\#V(t) = n + 1$ and $t = t_1 \circ \rightarrow t_2$, where t_1 is the minimal branch of t with respect to the order \preceq . Then we have:

$$\begin{aligned} \tilde{\Psi}_{S_{min}}(t) &= \tilde{\Psi}_{S_{min}}(t_1 \circ \rightarrow t_2) = \bar{\Psi} \circ S_{min}(t_1 \circ \rightarrow t_2) = \bar{\Psi}(S_{min}(t_1) \circ \searrow S_{min}(t_2)) \\ &= \pi(\Psi \circ S_{min}(t_1) \searrow \Psi \circ S_{min}(t_2)) = \tilde{\Psi}_{S_{min}}(t_1) \rightarrow \tilde{\Psi}_{S_{min}}(t_2) \\ &= \left(\sum_{s' \in [t_1]_{\#}} \beta_{S_{min}}(s', t_1) s' \right) \rightarrow \left(\sum_{s'' \in [t_2]_{\#}} \beta_{S_{min}}(s'', t_2) s'' \right) \\ &= \sum_{\substack{s' \in [t_1]_{\#} \\ s'' \in [t_2]_{\#}}} \beta_{S_{min}}(s', t_1) \beta_{S_{min}}(s'', t_2) s' \rightarrow s''. \end{aligned}$$

From Proposition 6.5, we have that:

$$t = t_1 \circ \rightarrow t_2 \# s := s' \circ \rightarrow s'' \# s' \rightarrow_v s'', \text{ for } v \in V(s''). \tag{55}$$

Let s^v be the smallest branch of the tree s , defined above in (55), starting from v , and s_v be the corresponding trunk (what remains when the branch s^v is removed). Then we have:

$$\beta_{S_{min}}(s, t) = \sum_{v \in V(s)} \beta_{S_{min}}(s^v, t_1) \beta_{S_{min}}(s_v, t_2). \tag{56}$$

Hence, we get: $\tilde{\Psi}_{S_{min}}(t) = \sum_{s \in [t]_{\#}} \beta_{S_{min}}(s, t) s.$ ■

Corollary 6.7. *Let $t \in T^E$, then the maximal term $T(\tilde{\Psi}_{S_{min}}(t))$, with respect to the order defined in (48), of $\tilde{\Psi}_{S_{min}}(t)$ is the tree t itself.*

From [2], we have that the set $\mathcal{B} = \{ \tilde{\Psi}_{S_{min}}(t) : t \in T^E \}$ forms a monomial basis for the free pre-Lie algebra $(\mathcal{T}^E, \rightarrow)$. Let I be the (two-sided) ideal generated by the elements on the form described in (23), then we have the following commutative diagram:

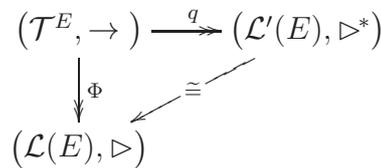


Figure 4

where $\mathcal{L}'(E) = \mathcal{T}^E/I$, and the product \triangleright^* is the pre-Lie product defined in (24). $\mathcal{L}(E)$ is the free Lie algebra generated by E which carries the pre-Lie algebra structure by the product \triangleright defined in (15). The restriction of Φ to $Span_K(O(I))$ is an injective map. Indeed, for any $h_1, h_2 \in Span_K(O(I))$,

$$\begin{aligned}
\Phi(h_1) = \Phi(h_2) &\Rightarrow \Phi(h_1 - h_2) = 0 \Rightarrow (h_1 - h_2) \in \text{Ker } \Phi = I \\
&\Rightarrow (h_1 - h_2) \in \text{Span}_K(O(I)) \cap I = \{0\} \\
&\Rightarrow h_1 - h_2 = 0 \Rightarrow h_1 = h_2.
\end{aligned}$$

Also, since $\Phi: \mathcal{T}^E \rightarrow \mathcal{L}(E)$ is a surjective map, then we have:

$$\begin{aligned}
\mathcal{L}(E) &= \Phi(\mathcal{T}^E) = \Phi(I \oplus \text{Span}_K(O(I))) \quad (\text{by Theorem 5.2}) \\
&= \Phi(\text{Span}_K(O(I))), \quad \text{since } \text{Ker } \Phi = I \text{ and } \Phi(I) = \{0\}.
\end{aligned}$$

Hence, $\Phi: \text{Span}_K(O(I)) \rightarrow \mathcal{L}(E)$ is a surjective and an injective map. Then it is an isomorphism of vector spaces.

Theorem 6.8. *For any $t \in O(I)$, we have:*

$$\tilde{\Psi}_{S_{min}}(t) = \text{Can}(\tilde{\Psi}_{S_{min}}(t), I) = t. \quad (57)$$

Moreover, the set $\tilde{\mathcal{B}} := \{\Phi(t) : t \in O(I)\}$ is a monomial basis for the pre-Lie algebra $(\mathcal{L}(E), \triangleright)$.

Proof. The property (57) is induced from Theorem 5.7 and the definition of $\tilde{\Psi}_{S_{min}}$. We obviously have that the set $\mathcal{B}' = O(I)$ is a basis for $\text{Span}_K(O(I))$. Therefore, as $\Phi: \text{Span}_K(O(I)) \rightarrow \mathcal{L}(E)$ is an isomorphism of vector spaces, $\tilde{\mathcal{B}} := \Phi(\mathcal{B}')$ forms a basis for the pre-Lie algebra $(\mathcal{L}(E), \triangleright)$. This basis is monomial thanks to (57), such that:

$$\Phi(t) = \Phi(\tilde{\Psi}_{S_{min}}(t)), \quad \text{for all } t \in O(I),$$

this proves Theorem 6.8. ■

Consequently, we get the following immediate result.

Corollary 6.9. *The set $\tilde{\mathcal{B}} := \{\Phi(t) : t \in O(I)\}$ is a monomial basis for the free Lie algebra $(\mathcal{L}(E), [\cdot, \cdot])$.*

Examples 6.10. Here, we calculate few first bases $\tilde{\mathcal{B}}_n$ for homogeneous components \mathcal{L}_n of the free Lie algebra $\mathcal{L}(E)$ up to $n = 4$, using Corollary 6.9, as follows:

1. As a particular case, take $E = \{a_i : i \in \mathbb{N}\}$ with $|a_i| = i$, for all $i \in \mathbb{N}$, with total order $a_1 < a_2 < \dots < a_s < \dots$ on the generators. From [2], we have:

$$\mathcal{B}(\mathcal{T}_4^E) = \left\{ \underbrace{\begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \\ | \\ \circ \\ | \\ \circ \end{array}, \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \\ | \\ \circ \\ | \\ \circ \end{array}, \dots, \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\}, \underbrace{\begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \circ \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array}, \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \dots, \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \circ \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array}, \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad / \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\},$$

$$\underbrace{\begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \circ \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array}, \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \dots, \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad / \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad / \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\},$$

$$\underbrace{\begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \circ \end{array} + 3 \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array} + \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \circ \end{array} + \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array}, \dots, \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad / \\ \bullet \end{array} + 3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad / \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right\}.$$

Hence, we have:

$$\tilde{\mathcal{B}}_1 = E, \quad \tilde{\mathcal{B}}_2 = \{ [x, y] : x, y \in E \}, \quad \tilde{\mathcal{B}}_3 = \{ [[x, y], x], [[x, y], y] : x, y \in E \},$$

$$\tilde{\mathcal{B}}_4 = \{ [[[x, y], x], x], [[[x, y], x], y], [[[x, y], y], y] : x, y \in E \}.$$

Remark 6.11. In the monomial basis $\tilde{\mathcal{B}}_4$ for \mathcal{L}_4 , calculated in (2.) above, we observe the following:

the tree $\begin{array}{c} \circ \\ | \\ \bullet \end{array}$ is not in $O(I)$, since there is an element $f = \begin{array}{c} \circ \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \circ \end{array}$ that belongs to

I such that $T(f) = \begin{array}{c} \circ \\ | \\ \bullet \end{array} \in T(I)$. Indeed, from the pre-Lie identity, and the so-called weighted anti-symmetry identity described in (23), we have, drawing non-planar trees explicitly:

$$f_1 = \overline{\Psi} \left(\left(\begin{array}{c} \circ \\ | \\ \circ \end{array} \curvearrowright \begin{array}{c} \circ \\ | \\ \bullet \end{array} \right) \curvearrowright \bullet - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \curvearrowright \begin{array}{c} \circ \\ | \\ \bullet \end{array} - \left(\begin{array}{c} \circ \\ | \\ \bullet \end{array} \curvearrowright \begin{array}{c} \circ \\ | \\ \bullet \end{array} \right) \curvearrowright \bullet + \begin{array}{c} \circ \\ | \\ \bullet \end{array} \curvearrowright \left(\begin{array}{c} \circ \\ | \\ \bullet \end{array} \curvearrowright \bullet \right) \right) =$$

$$= \overline{\Psi} \left(\begin{array}{c} \circ \\ | \\ \bullet \end{array} - \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} - \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} + \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} \right),$$

$$f_2 = \overline{\Psi} \left(\left(\begin{array}{c} \circ \\ | \\ \bullet \end{array} \curvearrowright \begin{array}{c} \circ \\ | \\ \bullet \end{array} + 2 \begin{array}{c} \circ \\ | \\ \bullet \end{array} \curvearrowright \begin{array}{c} \circ \\ | \\ \bullet \end{array} \right) \curvearrowright \bullet \right) = \overline{\Psi} \left(\begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} + 2 \begin{array}{c} \circ \\ | \\ \bullet \end{array} \right), \text{ and}$$

$$f_3 = \overline{\Psi} \left(\begin{array}{c} \circ \\ | \\ \bullet \end{array} \curvearrowright \begin{array}{c} \circ \\ | \\ \bullet \end{array} + 3 \begin{array}{c} \circ \\ | \\ \bullet \end{array} \curvearrowright \begin{array}{c} \circ \\ | \\ \bullet \end{array} \right) = \overline{\Psi} \left(\begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} + 3 \begin{array}{c} \circ \\ | \\ \bullet \end{array} \right)$$

are elements in I , hence

$$f_4 = f_1 + f_2 - f_3 = \overline{\Psi} \left(3 \begin{array}{c} \circ \\ | \\ \bullet \end{array} - 3 \begin{array}{c} \bullet \\ | \\ \circ \end{array} - \begin{array}{c} \circ \quad \circ \\ \diagdown \quad / \\ \bullet \end{array} \right) \in I.$$

But, $f_5 = \overline{\Psi} \left(\begin{array}{c} \circ \\ \diagdown \quad \diagup \\ \bullet \end{array} \right) \in I$, hence $f = f_4 + f_5 = 3 \begin{array}{c} \circ \\ \bullet \\ \bullet \end{array} - 3 \begin{array}{c} \circ \\ \bullet \\ \circ \end{array} \in I$. Then, we have:

$$\begin{aligned} \Phi(f) &= 3 \left((x \triangleright y) \triangleright x \right) \triangleright y - 3 \left((x \triangleright y) \triangleright y \right) \triangleright x - (x \triangleright y) \triangleright (y \triangleright x) \\ &= [[x, y], x], y - [[[x, y], y], x] + [[[x, y], [x, y]] \\ &= 0, \end{aligned}$$

and then,

$$[[[x, y], x], y] = [[[x, y], y], x].$$

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