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A Note on the Construction of Second-Order Conformally Invariant Systems on Generalized Flag Manifolds

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Abstract. An automatic conformal invariance result is proved for systems of second-order differential operators on generalized flag manifolds. The result states that a purely algebraic datum (analogous to the symbol of a differential operator) that has the correct shape to have arisen from a conformally invariant system of second-order operators on a homogeneous line bundle does, in fact, arise from such a system on a suitable bundle.

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1. Introduction

Let G be a connected real semisimple Lie group, Q a parabolic subgroup, and $\mathcal{E} \to G/Q$ a homogeneous vector bundle. The Lie algebra $\mathfrak{g}$ acts on $\Gamma(\mathcal{E})$. If $X \in \mathfrak{g}$ then we write $\Pi_{\mathcal{E}}(X)\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ for the corresponding first-order differential operator. A system $D_1, \ldots, D_m$ of differential operators on $\Gamma(\mathcal{E})$ is *conformally invariant* if for all $X \in \mathfrak{g}$ we have

$$[\Pi_{\mathcal{E}}(X), D_j] = \sum_{i=1}^{m} C_{ij}(X) D_i$$

where the $C_{ij}(X)$ are smooth functions on G/Q. The solution space of a conformally invariant system is a $\mathcal{U}(\mathfrak{g})$-module via $\Pi_{\mathcal{E}}$. For this reason, as well as others connected with the study of the geometry and analysis of the space G/Q, it is interesting to find conformally invariant systems and to investigate their properties. We note, as the referee has pointed out, that the term "conformally invariant" is sometimes used in the literature to refer only to specific instances of the above setting that are more closely connected with conformal geometry.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be a Levi decomposition of $\mathfrak{q}$ and E be the representation of $\mathfrak{q}$ that corresponds to $\mathcal{E}$. Assume that E is irreducible. A *leading subspace* $U \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} E^*$ is an irreducible $\mathfrak{l}$-submodule such that $\mathfrak{n}U = \{0\}$. The theory of conformally invariant systems reveals that the equivalence classes of irreducible conformally invariant systems on $\mathcal{E}$ are in one-to-one correspondence

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with the leading subspaces. Each leading subspace has a degree, which corresponds to the order of the differential operators comprising the associated system. One always has the trivial leading subspace $\mathbb{C} \otimes E^*$ of degree zero; the corresponding conformally invariant system consists of the identity operator on $\Gamma(\mathcal{E})$. Leading subspaces of positive degree correspond to non-trivial conformally invariant systems on $\mathcal{E}$. (Barchini, Zierau, and the author [2] have developed the theory of conformally invariant systems in greater generality than is required for the claims of the present paragraph. The specific claims made here may be found elsewhere in the literature, beginning with the pioneering work of Kostant [5] for a system consisting of a single operator, but [2] may still prove a convenient reference because the definitions and notation used there are compatible with those used here.)

Since the algebra $\mathfrak{l}$ has a non-zero abelianization, the $\mathfrak{l}$-module E may be embedded into a continuous family of twists. We denote these twists generically by E^s, where s parametrizes the space $\mathbf{X}(\mathfrak{l})$ of characters of $\mathfrak{l}$. All the modules $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} (E^s)^*$ are realized on the same space and a subspace that is an $\mathfrak{l}$-submodule for one s is so for all s. Thus if $U \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} E^*$ is an irreducible $\mathfrak{l}$-submodule then we may regard U instead as a submodule of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} (E^s)^*$. We denote U regarded in this way by U^s. It now makes sense to ask whether there is some s such that U^s is a leading subspace.

If $U \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} E^*$ is an irreducible $\mathfrak{l}$-submodule of degree one then there is always a value of s such that U^s is a leading subspace. Indeed, the collection of such s is a codimension-one affine subspace of $\mathbf{X}(\mathfrak{l})$. This type of statement is what the author calls an automatic conformal invariance result. The fact that this result holds for degree-one systems has been proved several times in the literature from several different perspectives, sometimes in a disguised form. Xiao's work [12] is a recent entry-point to the literature on degree-one leading subspaces and the associated systems of operators. In addition to establishing the automatic conformal invariance result, Xiao describes how to construct the degree-one subspaces U to which the result may be applied. The work of Johnson, Korányi, and Reimann [3] is a significant earlier contribution that is not cited by Xiao. The works of Ørsted [10] and of Slovák and Souček [11] both contain proofs of the automatic conformal invariance result for first-order systems. There are, in addition, a number of earlier works that establish the result in special situations. The method of the present work could be used to give a short, self-contained proof of the automatic conformal invariance result for first-order systems. However, this is not done on account of the number of proofs already available in the literature.

The main result of the present work is roughly that if $\mathcal{E}$ is a line bundle and $U \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} E^*$ is an irreducible $\mathfrak{l}$-submodule of degree two then there is some s such that U^s is a leading subspace. That is, degree-two subspaces are automatically associated to conformally invariant systems in this setting. A precise statement of the result requires some more care and is given in Theorems 4.1 and 4.3. First, we must exclude certain trivial counterexamples from the statement. This is done implicitly below by giving a parameterization of the degree-two subspaces that are not counterexamples and stating the results in terms of this parameterization. Secondly, the suitable degree-two subspaces come in two kinds. For those of the first kind, the set of s such that U^s is a leading subspace is once again a codimension-one affine subspace of $\mathbf{X}(\mathfrak{l})$, which is precisely identified in Theorem 4.1. For those of the second kind, the set of s such that U^s is a leading subspace is a codimension-two affine subspace of $\mathbf{X}(\mathfrak{l})$. Note that subspaces of the second kind only occur when the dimension of $\mathbf{X}(\mathfrak{l})$ is at least two, so the set of suitable s is still guaranteed to be non-empty. In addition, a subspace of the second kind sometimes fits into a parametric family of such subspaces, which introduces a second parameter into the statement. We identify when this happens in Lemma 4.2.

The results described in the previous paragraph simplify the proofs of a number of earlier results and also guarantee the existence of many new conformally invariant systems of second-order operators. The explicit study of second-order conformally invariant systems on homogeneous line bundles has typically proceeded by constructing a candidate degree-two subspace U and then verifying by explicit computation that U^s is a leading subspace for suitable s. In light of the present results, the success of the second step was, in fact, inevitable in each of these cases. Among other results, Barchini, Zierau, and the author [1] constructed second-order conformally invariant systems on G/Q, with Q the Heisenberg parabolic, by this method. This work was extended to Q a quasi-Heisenberg maximal parabolic by Kubo [6, 8, 9] using the same method. The author [4] constructed second-order conformally invariant systems on G/Q, with Q the intersection of two maximal parabolics of abelian type, by a hybrid method that was a step towards the general results proved here.

It would be very interesting to have a uniform explicit description of all degree-two subspaces for a general parabolic in a semisimple group in the scalar case, perhaps along the lines of the description of all degree-one subspaces given by Xiao [12]. This would bring the existence theory for second-order systems on line bundles to a comparable degree of completeness with that of the existence theory for first-order systems on vector bundles. One possible complication for this goal is that all degree-one subspaces are standard (meaning that the associated homomorphism of generalized Verma modules is a standard homomorphism), whereas the results of Kubo [7] reveal that this is not the case for all degree-two subspaces. No automatic conformal invariance result is possible for general degree-three subspaces; Barchini, Zierau, and the author [2, Theorem 21] have given examples of degree-three subspaces in the scalar case no twist of which is a leading subspace. It does not appear to be known whether there could be an automatic conformal invariance result for degree-two subspaces in the non-scalar case (corresponding to second-order systems on vector bundles). It seems unlikely that such a result could be true, but examples are needed.

2. Background on graded Lie algebras

One purpose of this section is to assemble the notation and facts about simple $\mathbb{Z}$-graded Lie algebras that we shall require below. However, we shall prove more than we strictly require. For the applications in this paper, we need only the $j=1$ cases of Proposition 2.1, Corollary 2.2, Part (4) of Proposition 2.3, and Proposition 2.5. With the possible exception of Part (4) of Proposition 2.3, these special cases have simpler proofs and are also commonplaces, but the more general results seem to the author to be interesting and worth recording.

Let F be an algebraically closed field of characteristic zero, $\mathfrak{g}$ a simple Lie algebra over F, and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra. Let $R = R(\mathfrak{g}, \mathfrak{t})$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and

$$\mathfrak{t}_{\mathbb{Q}} = \{H \in \mathfrak{t} \mid \alpha(H) \in \mathbb{Q} \text{ for all } \alpha \in R\}$$

and

$$\mathfrak{t}^*_{\mathbb{Q}} = \{\lambda \in \mathfrak{t}^* \mid \lambda(H) \in \mathbb{Q} \text{ for all } H \in \mathfrak{t}_{\mathbb{Q}}\}$$

the associated $\mathbb{Q}$-forms of $\mathfrak{t}$ and $\mathfrak{t}^*$, respectively. Equip $\mathfrak{t}^*_{\mathbb{Q}}$ with a positive-definite bilinear form $(\cdot\,,\,\cdot)$ that is invariant under the Weyl group $W(R)$ of R and is normalized so that the square-length of the long roots is 2. This choice establishes an isomorphism between $\mathfrak{t}_{\mathbb{Q}}$ to $\mathfrak{t}^*_{\mathbb{Q}}$ under which $H \leftrightarrow \lambda$ when $\alpha(H) = (\alpha, \lambda)$ for all $\alpha \in R$. When this is so, we say that H *corresponds to* λ and vice versa. In particular, we shall regard the coroot system as a subset of $\mathfrak{t}^*_{\mathbb{Q}}$ via this correspondence. This done, the coroot of $\alpha \in R$ is $\check{\alpha} = 2\alpha/\|\alpha\|^2$.

Let $\Lambda \subset \mathfrak{t}^*_{\mathbb{Q}}$ denote the weight lattice of R. Let $R^+ \subset R$ be a positive system and R^s the corresponding set of simple roots. For each $\alpha \in R^s$ let $\varpi_\alpha \in \Lambda$ be the associated fundamental weight. These weights are characterized by the condition that $(\varpi_\alpha, \check{\beta}) = \delta_{\alpha\beta}$ for all $\alpha, \beta \in R^s$ and they make up a basis for Λ. For $\beta \in R^s$ let $Z_\beta \in \mathfrak{t}_{\mathbb{Q}}$ be the element that corresponds to $2\varpi_\beta/\|\beta\|^2$. These elements satisfy $\alpha(Z_\beta) = \delta_{\alpha\beta}$ for all $\alpha, \beta \in R^s$. Let $S \subset R^s$ and define

$$H_0 = \sum_{\alpha \in S} Z_\alpha,$$

so that

$$\beta(H_0) = \begin{cases} 1 & \text{if } \beta \in S, \\ 0 & \text{if } \beta \in R^s \setminus S. \end{cases} \tag{1}$$

The map $\mathrm{ad}(H_0)\colon \mathfrak{g} \to \mathfrak{g}$ is semisimple and has integral eigenvalues. Let $\mathfrak{g}(j)$ be the j-eigenspace of $\mathrm{ad}(H_0)$ for $j \in \mathbb{Z}$ and let $r \in \mathbb{N}$ be the largest value for which $\mathfrak{g}(r) \neq \{0\}$. Then

$$\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j) \tag{2}$$

is a $\mathbb{Z}$-grading of $\mathfrak{g}$. The subspace $\mathfrak{g}(0)$ is a reductive subalgebra of $\mathfrak{g}$ and we have a decomposition

$$\mathfrak{g}(0) = \mathfrak{z}(\mathfrak{g}(0)) \oplus \mathfrak{g}(0)^{ss},$$

where $\mathfrak{g}(0)^{ss} = [\mathfrak{g}(0), \mathfrak{g}(0)]$ is the semisimple part of $\mathfrak{g}(0)$. The ideal $\mathfrak{g}(0)^{ss}$ of $\mathfrak{g}(0)$ decomposes as the direct sum of simple ideals. These simple ideals are in one-to-one correspondence with the connected components of the graph obtained from the Dynkin diagram of $\mathfrak{g}$ by deleting the nodes corresponding to the elements of S and the edges incident on them. We call this graph the *S-deleted Dynkin diagram* of $\mathfrak{g}$. For $\gamma \in R$ let $H_\gamma \in \mathfrak{t}_{\mathbb{Q}}$ be the element that corresponds to $\check{\gamma}$. The set $\{H_\alpha \mid \alpha \in R^s \setminus S\}$ is a basis for $\mathfrak{t} \cap \mathfrak{g}(0)^{ss}$ and the set $\{Z_\alpha \mid \alpha \in S\}$ is a basis for $\mathfrak{z}(\mathfrak{g}(0))$. For $\alpha \in S$, the fundamental weight ϖ_α vanishes on $\mathfrak{t} \cap \mathfrak{g}(0)^{ss}$ and so extends to a character of $\mathfrak{g}(0)$. We denote this extension by the same symbol. The set $\{\varpi_\alpha \mid \alpha \in S\}$ is a basis for the space of characters of $\mathfrak{g}(0)$. If $s \in F^S$ then we let ϖ^s be the character of $\mathfrak{g}(0)$ that is defined by

$$\varpi^s = \sum_{\alpha \in S} s_\alpha \varpi_\alpha. \tag{3}$$

We shall frequently have to consider finite-dimensional $\mathfrak{g}(0)$-modules, so we make a few remarks about them. Let V be such a $\mathfrak{g}(0)$-module. If V is irreducible then $\mathfrak{z}(\mathfrak{g}(0))$ acts on V via a character and V remains irreducible when regarded as a $\mathfrak{g}(0)^{ss}$-module. In general, the module V need not be completely reducible; it will be so if and only if the action of $\mathfrak{z}(\mathfrak{g}(0))$ on V is semisimple. All the $\mathfrak{g}(0)$-modules we shall encounter will be of this type. Note that submodules, direct sums, contragredients, and tensor products of completely reducible $\mathfrak{g}(0)$-modules are also completely reducible.

Suppose now that V is a finite-dimensional irreducible $\mathfrak{g}(0)$-module. Then V has a one-dimensional highest weight space V_+ as $\mathfrak{g}(0)^{ss}$-module. The algebra $\mathfrak{t}$ acts on V_+ via some functional $\lambda_V \colon \mathfrak{t} \to F$ and we call λ_V the *extended highest weight* of V. It restricts to the usual highest weight of V as a $\mathfrak{g}(0)^{ss}$-module on $\mathfrak{t} \cap \mathfrak{g}(0)^{ss}$ and to the central character of V on $\mathfrak{z}(\mathfrak{g}(0))$. In particular, λ_V determines V up to isomorphism. It is worth noting that the central character ω_V of V is the element ω_V of the space of characters of $\mathfrak{g}(0)$ that satisfies $(\omega_V, \varpi_\alpha) = (\lambda_V, \varpi_\alpha)$ for all $\alpha \in S$. Now let V be any finite-dimensional completely reducible $\mathfrak{g}(0)$-module. Then V decomposes as the direct sum of various simultaneous eigenspaces for $\mathfrak{t}$. In keeping with the above terminology, we call these eigenspaces the *extended weight spaces* of V and the functionals by which $\mathfrak{t}$ acts on them the *extended weights* of V. If V is any $\mathfrak{g}(0)$-module and $s \in F^S$ then we let $V^s = \varpi^s \otimes V$ be the $\mathfrak{g}(0)$-module obtained by twisting V by ϖ^s. If V is completely reducible then the extended weights of V^s are obtained by adding ϖ^s to the extended weights of V. In particular, $\lambda_{V^s} = \lambda_V + \varpi^s$ and $\omega_{V^s} = \omega_V + \varpi^s$.

Let $j \in \mathbb{Z}$ with $1 \leq |j| \leq r$ and define

$$R(j) = \{\gamma \in R \mid \gamma(H_0) = j\}.$$

For $\gamma \in R$ let $\mathfrak{g}_\gamma$ denote the γ-root space of $\mathfrak{g}$. The subspace $\mathfrak{g}(j)$ of $\mathfrak{g}$ is a $\mathfrak{g}(0)$-module and we have

$$\mathfrak{g}(j) = \bigoplus_{\gamma \in R(j)} \mathfrak{g}_\gamma,$$

from which it follows that $\mathfrak{g}(j)$ is completely reducible. The set R is partially ordered by the relation $\beta_1 \leq \beta_2$ if and only if $(\beta_1, \varpi_\alpha) \leq (\beta_2, \varpi_\alpha)$ for all $\alpha \in R^s$. If $\beta_1, \beta_2 \in R(j)$ and $\beta_1 < \beta_2$ then there is a list $\gamma_1, \ldots, \gamma_k$ of simple roots such that $\beta_1 + \gamma_1 + \cdots + \gamma_l \in R$ for $0 \leq l \leq k$ and $\beta_2 = \beta_1 + \gamma_1 + \cdots + \gamma_k$. Since both β_1 and β_2 lie in $R(j)$ we must have $\gamma_l \in R^s \setminus S$ for all $1 \leq l \leq k$. It follows that the $\mathfrak{g}(0)^{ss}$-lowest weight spaces in $\mathfrak{g}(j)$ are precisely the weight spaces $\mathfrak{g}_\gamma$ with γ a minimal element of $R(j)$. Similarly, the $\mathfrak{g}(0)^{ss}$-highest weight spaces in $\mathfrak{g}(j)$ are precisely the weight spaces $\mathfrak{g}_\gamma$ with γ a maximal element of $R(j)$. Thus the irreducible $\mathfrak{g}(0)$-submodules of $\mathfrak{g}(j)$ may be expressed as

$$\mathfrak{g}(j,[\beta,\infty)) = \bigoplus_{\gamma\in R(j),\ \gamma\geq\beta} \mathfrak{g}_\gamma$$

with $\beta \in R(j)_{\min}$ or as

$$\mathfrak{g}(j,(-\infty,\beta]) = \bigoplus_{\gamma\in R(j),\ \gamma\leq\beta} \mathfrak{g}_\gamma$$

with $\beta \in R(j)_{\max}$. In these two displays the standard interval notation associated with a partially ordered set is being used. The $\mathfrak{g}(0)$-module $\mathfrak{g}(-j)$ is isomorphic to the contragredient of $\mathfrak{g}(j)$, since the Killing form of $\mathfrak{g}$ restricts to a non-degenerate $\mathfrak{g}(0)$-invariant pairing between $\mathfrak{g}(-j)$ and $\mathfrak{g}(j)$. More precisely, if $\beta \in R(j)_{\min}$ then we have

$$\mathfrak{g}(j,[\beta,\infty))^* \cong \mathfrak{g}(-j,(-\infty,-\beta])$$

via the pairing induced by the Killing form of $\mathfrak{g}$. Similarly, if $\beta \in R(j)_{\max}$ then we have

$$\mathfrak{g}(j,(-\infty,\beta])^* \cong \mathfrak{g}(-j,[-\beta,\infty))$$

in the same way. The $\mathfrak{g}(0)$-module $\mathfrak{g}(j)$ is extended-weight-multiplicity-free, since the extended weight spaces are precisely the root spaces and the root and extended weight coincide. In particular, $\mathfrak{g}(j)$ is multiplicity-free as a $\mathfrak{g}(0)$-module. It follows from this and Schur's Lemma that

$$\mathrm{End}_{\mathfrak{g}(0)}\big(\mathfrak{g}(j)\big) \cong F^{n_j}$$

where n_j is the cardinality of $R(j)_{\min}$, which coincides with the cardinality of $R(j)_{\max}$ and also with the number of irreducible constituents of $\mathfrak{g}(j)$.

Proposition 2.1. *Let $j \in \mathbb{Z}$ with $1 \leq |j| \leq r$. Then the map $\mathfrak{z}(\mathfrak{g}(0)) \to \mathrm{End}_{\mathfrak{g}(0)}(\mathfrak{g}(j))$ given by $Z \mapsto \mathrm{ad}(Z)\big|_{\mathfrak{g}(j)}$ is surjective. In particular, the number of irreducible constituents of the $\mathfrak{g}(0)$-module $\mathfrak{g}(j)$ is*

$$n_j = |S| - \dim\{Z \in \mathfrak{z}(\mathfrak{g}(0)) \mid [Z,X] = 0 \textit{ for all } X \in \mathfrak{g}(j)\}.$$

Proof. Let $\mathfrak{r}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}(-j)$, $\mathfrak{g}(0)$, and $\mathfrak{g}(j)$. Then $\mathfrak{t} \subset \mathfrak{r}$ and the set of roots γ such that $\mathfrak{g}_\gamma \subset \mathfrak{r}$ is closed under negation. It follows that $\mathfrak{r}$ is reductive in $\mathfrak{g}$ and, in particular, reductive. We have the decomposition $\mathfrak{r} = \mathfrak{z}(\mathfrak{r}) \oplus \mathfrak{r}^{ss}$, and $\mathfrak{z}(\mathfrak{r}) \subset \mathfrak{t}$ since $\mathfrak{t}$ is self-centralizing. Given this, the fact that $\mathfrak{t} \subset \mathfrak{g}(0) \subset \mathfrak{r}$ implies that $\mathfrak{z}(\mathfrak{r}) \subset \mathfrak{z}(\mathfrak{g}(0))$. In order that $Z \in \mathfrak{z}(\mathfrak{g}(0))$ should lie in $\mathfrak{z}(\mathfrak{r})$ it is necessary and sufficient that $[Z,X] = 0$ for all $X \in \mathfrak{g}(j)$. The necessity follows from the definition and the sufficiency from the duality between $\mathfrak{g}(j)$ and $\mathfrak{g}(-j)$ that was observed above. We conclude that

$$\mathfrak{z}(\mathfrak{r}) = \{Z \in \mathfrak{z}(\mathfrak{g}(0)) \mid [Z,X] = 0 \text{ for all } X \in \mathfrak{g}(j)\}. \tag{4}$$

Observe that $\mathfrak{t} \cap \mathfrak{r}^{ss}$ is a Cartan subalgebra of $\mathfrak{r}^{ss}$. The orthogonal projection $\mathfrak{t} \to \mathfrak{t} \cap \mathfrak{r}^{ss}$ with respect to the Killing form induces an inclusion of $(\mathfrak{t} \cap \mathfrak{r}^{ss})^*$ into

$\mathfrak{t}^*$ and identifies the root system $R(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$ with a subsystem of $R(\mathfrak{g}, \mathfrak{t})$. In particular, the positive system $R^+(\mathfrak{g}, \mathfrak{t})$ that has already been chosen induces a positive system $R^+(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$. We wish to determine the corresponding simple system $R^s(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$. Suppose that $\gamma \in R^s(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$. Then $\mathfrak{r}^{ss}_\gamma = \mathfrak{g}_\gamma$ must be contained either in $\mathfrak{g}(0)$ or in $\mathfrak{g}(|j|)$. The reason is that for all other root spaces $\mathfrak{g}_\gamma$ that appear in $\mathfrak{r}^{ss}$, the root γ is either visibly negative or visibly a non-trivial sum of positive roots. The root system $R(\mathfrak{g}(0)^{ss}, \mathfrak{t}\cap\mathfrak{g}(0)^{ss})$ is contained in $R(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$ and it follows that the set of simple roots $\gamma \in R^s(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$ such that $\mathfrak{g}_\gamma \subset \mathfrak{g}(0)$ is precisely $R^s \setminus S$. Now suppose that $\gamma \in R^s(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$ is such that $\mathfrak{g}_\gamma \subset \mathfrak{g}(|j|)$. Then $\mathfrak{g}_\gamma$ is necessarily a $\mathfrak{g}(0)^{ss}$-lowest weight space for $\mathfrak{g}(|j|)$, for otherwise γ could be expressed as the sum of an element of $R^+(\mathfrak{g}(0)^{ss}, \mathfrak{t}\cap\mathfrak{g}(0)^{ss})$ and some other root whose root space lies in $\mathfrak{g}(|j|)$, contradicting the assumed simplicity of γ. Conversely, if γ is a $\mathfrak{g}(0)^{ss}$-lowest weight for $\mathfrak{g}(|j|)$ then $\gamma \in R^s(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss})$, because no such non-trivial decomposition of γ is possible. Let $\{\gamma_1, \ldots, \gamma_{n_j}\}$ be the set of roots that are $\mathfrak{g}(0)^{ss}$-lowest weights in $\mathfrak{g}(|j|)$. The argument we have just given shows that

$$R^s(\mathfrak{r}^{ss}, \mathfrak{t}\cap\mathfrak{r}^{ss}) = (R^s \setminus S) \cup \{\gamma_1, \ldots, \gamma_{n_j}\}. \tag{5}$$

We now have two evaluations of the rank of $\mathfrak{r}^{ss}$. By (4),

$$\begin{aligned}\operatorname{rank}(\mathfrak{r}^{ss}) &= \dim(\mathfrak{t}) - \dim(\mathfrak{z}(\mathfrak{r}))\\ &= |R^s| - \dim\{Z \in \mathfrak{z}(\mathfrak{g}(0)) \mid [Z, X] = 0 \text{ for all } X \in \mathfrak{g}(j)\}\end{aligned}$$

and, by (5),

$$\operatorname{rank}(\mathfrak{r}^{ss}) = |R^s| - |S| + n_j.$$

By equating these, we obtain

$$n_j = |S| - \dim\{Z \in \mathfrak{z}(\mathfrak{g}(0)) \mid [Z, X] = 0 \text{ for all } X \in \mathfrak{g}(j)\},$$

as claimed. Since $\dim(\mathfrak{z}(\mathfrak{g}(0))) = |S|$, this equation also implies that the dimension of the algebra $\operatorname{End}_{\mathfrak{g}(0)}(\mathfrak{g}(j))$, which is n_j, is equal to the dimension of the image of the map $Z \mapsto \operatorname{ad}(Z)\big|_{\mathfrak{g}(j)}$. It follows that the map is surjective. ∎

Note that $R(1)_{\min} = S$, so that $\mathfrak{g}(1)$ always has $|S|$ irreducible constituents, the maximum number of constituents that can appear in any $\mathfrak{g}(j)$. Also note that $R(r)_{\max}$ consists of the highest root alone, so that $\mathfrak{g}(r)$ is always irreducible.

Corollary 2.2. *Let $j \in \mathbb{Z}$ with $1 \leq |j| \leq r$. If $\beta_1, \beta_2 \in R(j)_{\min}$ are distinct then*

$$\big[\mathfrak{g}(j, [\beta_1, \infty)), \mathfrak{g}(-j, (-\infty, -\beta_2])\big] = \{0\}.$$

Proof. Let $X \in \mathfrak{g}(j, [\beta_1, \infty))$ and $Y \in \mathfrak{g}(-j, (-\infty, -\beta_2])$. By Proposition 2.1, we may find $Z \in \mathfrak{z}(\mathfrak{g}(0))$ such that $[Z, X] = X$ and $[Z, Y] = 0$. Then $[X, Y] = [[Z, X], Y] = [Z, [X, Y]] = 0$ because $[X, Y] \in \mathfrak{g}(0)$. ∎

Let $1 \leq j \leq r$. For $\beta \in R(j)_{\min}$ let $\mathfrak{a}_\beta$ be the subspace of $\mathfrak{g}(0)$ spanned by the commutators $[X, Y]$ with $X \in \mathfrak{g}(j, [\beta, \infty))$ and $Y \in \mathfrak{g}(-j, (-\infty, -\beta])$.

Proposition 2.3. *Let $1 \le j \le r$ and $\beta \in R(j)_{\min}$. Then*

(1) $\mathfrak{a}_\beta$ *is an ideal of* $\mathfrak{g}(0)$*;*

(2) $\mathfrak{a}_\beta$ *is reductive;*

(3) $\mathfrak{a}_\beta^{ss} = \mathfrak{a}_\beta \cap \mathfrak{g}(0)^{ss}$ *is the direct sum of the simple ideals in* $\mathfrak{g}(0)^{ss}$ *that correspond to those components of the* S*-deleted Dynkin diagram of* $\mathfrak{g}$ *that contain a simple root* α *such that* $(\alpha, \beta) \neq 0$*;*

(4) $\mathfrak{z}(\mathfrak{a}_\beta) = \mathfrak{a}_\beta \cap \mathfrak{z}(\mathfrak{g}(0))$ *is one-dimensional. A functional* ϖ *in the span of* $\{\varpi_\alpha \mid \alpha \in S\}$ *vanishes on* $\mathfrak{z}(\mathfrak{a}_\beta)$ *if and only if* $(\varpi, \beta) = 0$.

Proof. The fact that $[\mathfrak{g}(0), \mathfrak{a}_\beta] \subset \mathfrak{a}_\beta$ follows from the definition of $\mathfrak{a}_\beta$ and the fact that $\mathfrak{g}(j, [\beta, \infty))$ and $\mathfrak{g}(-j, (-\infty, -\beta])$ are $\mathfrak{g}(0)$-modules. Thus Part (1) is verified. Part (2) and the equalities $\mathfrak{a}_\beta^{ss} = \mathfrak{a}_\beta \cap \mathfrak{g}(0)^{ss}$ and $\mathfrak{z}(\mathfrak{a}_\beta) = \mathfrak{a}_\beta \cap \mathfrak{z}(\mathfrak{g}(0))$ are general facts about ideals in reductive algebras.

Let $\alpha \in R^s \setminus S$. Since $\beta \in R(j)_{\min}$, the α-string through β has the form $\beta, \beta + \alpha, \ldots, \beta + q\alpha$ with $q = 2(\alpha, \beta)/\|\alpha\|^2$. If $(\alpha, \beta) \neq 0$ then $\beta + \alpha \in R(j)$ and we conclude that $[X_{\beta+\alpha}, X_{-\beta}]$ lies in $\mathfrak{a}_\beta$. This commutator is a non-zero multiple of X_α and so the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{a}_\beta$. It follows from this that the simple ideal corresponding to the component of the S-deleted Dynkin diagram containing α is contained in $\mathfrak{a}_\beta$, as claimed. Now suppose that C is a component of the S-deleted Dynkin diagram such that $(\alpha, \beta) = 0$ for all $\alpha \in C$ and let $\mathfrak{c}$ be the corresponding simple ideal of $\mathfrak{g}(0)$. If $\alpha \in C$ then $\beta - \alpha$ is not a root, because $\beta \in R(j)_{\min}$, and the fact that $(\alpha, \beta) = 0$ then implies that $\beta + \alpha$ is not a root either. Thus $X_{-\alpha}$ and X_α annihilate X_β for all $\alpha \in C$. These elements generate $\mathfrak{c}$ and so $\mathfrak{c} \cdot X_\beta = \{0\}$. Since $\mathfrak{c}$ is an ideal in $\mathfrak{g}(0)$ it follows that $\mathfrak{c}$ annihilates the $\mathfrak{g}(0)$-module generated by X_β, which is $\mathfrak{g}(j, [\beta, \infty))$. In particular, if $\gamma \in R(j) \cap [\beta, \infty)$ and $\alpha \in C$ then $\gamma + \alpha$ is not a root. Now the roots whose root spaces lie in $\mathfrak{a}_\beta$ are those of the form $\gamma_2 - \gamma_1$ with $\gamma_1, \gamma_2 \in R(j) \cap [\beta, \infty)$. It follows from what we have just done that no α in C is of this form and so $\mathfrak{c}$ is not contained in $\mathfrak{a}_\beta$. This completes the proof of Part (3).

There is a $\mathfrak{g}(0)$-module homomorphism

$$\mathfrak{g}(j, [\beta, \infty)) \otimes \mathfrak{g}(-j, (-\infty, -\beta]) \to \mathfrak{a}_\beta$$

given on simple tensors by $X \otimes Y \mapsto [X, Y]$ and this homomorphism is surjective by definition. Since $\mathfrak{g}(j, [\beta, \infty))$ and $\mathfrak{g}(-j, (-\infty, -\beta])$ are irreducible, the multiplicity of the trivial module in the tensor product is at most one and so the multiplicity of the trivial module in $\mathfrak{a}_\beta$ is also at most one. This is the same as saying that $\dim\big(\mathfrak{a}_\beta \cap \mathfrak{z}(\mathfrak{g}(0))\big) \le 1$. The element $H_\beta = [X_\beta, X_{-\beta}]$ lies in $\mathfrak{a}_\beta$ and we may express it as $H_\beta = Z + W$ where $Z \in \mathfrak{z}(\mathfrak{a}_\beta) \subset \mathfrak{z}(\mathfrak{g}(0))$ and $W \in \mathfrak{a}_\beta^{ss} \subset \mathfrak{g}(0)^{ss}$. If ϖ lies in the span of $\{\varpi_\alpha \mid \alpha \in S\}$ then ϖ extends to a character of $\mathfrak{g}(0)$; in particular, ϖ vanishes on $\mathfrak{g}(0)^{ss}$ and so $\varpi(H_\beta) = \varpi(Z)$. On the other hand, $\varpi(H_\beta) = (\varpi, \check{\beta}) = 2(\varpi, \beta)/\|\beta\|^2$. If we choose $\varpi = \varpi_\alpha$ where $\alpha \in S$ is one of the simple roots that appear in β then $(\varpi, \beta) \neq 0$. From this we conclude that $Z \neq 0$ and hence that $\mathfrak{z}(\mathfrak{a}_\beta)$ has dimension exactly one. Moreover, Z is a basis for $\mathfrak{z}(\mathfrak{a}_\beta)$ and so the identity $\varpi(Z) = 2(\varpi, \beta)/\|\beta\|^2$ implies the last statement in Part (4). ■

Let $j \in \mathbb{Z}$ with $1 \le |j| \le r$. Although $\mathfrak{g}(j)$ is extended-weight-multiplicity-free, it is not necessarily weight-multiplicity-free as a $\mathfrak{g}(0)^{ss}$-module. However, we do at least have the following observation.

Lemma 2.4. *Let $j \in \mathbb{Z}$ with $1 \le |j| \le r$ and $\beta \in R(j)_{\min}$. Then $\mathfrak{g}(j,[\beta,\infty))$ is weight-multiplicity-free as a $\mathfrak{g}(0)^{ss}$-module.*

Proof. Let λ_1 and λ_2 be distinct extended weights that appear in the extended weight space decomposition of $\mathfrak{g}(j,[\beta,\infty))$. On restriction to $\mathfrak{z}(\mathfrak{g}(0))$, λ_1 and λ_2 agree with β and hence with each other. Thus λ_1 and λ_2 do not agree on restriction to $\mathfrak{t} \cap \mathfrak{g}(0)^{ss}$. This means that distinct extended weights that appear in $\mathfrak{g}(j,[\beta,\infty))$ remain distinct as weights when $\mathfrak{g}(j,[\beta,\infty))$ is regarded as a $\mathfrak{g}(0)^{ss}$-module. The conclusion follows from this. ■

Proposition 2.5. *Let $j \in \mathbb{Z}$ with $1 \le |j| \le r$ and W be an irreducible $\mathfrak{g}(0)$-module. Then the $\mathfrak{g}(0)$-module $\mathfrak{g}(j) \otimes W$ is multiplicity-free.*

Proof. List $R(j)_{\min}$ as $\beta_1, \ldots, \beta_{n_j}$ so that

$$\mathfrak{g}(j) = \bigoplus_{i=1}^{n_j} \mathfrak{g}(j,[\beta_i,\infty))$$

is the decomposition of $\mathfrak{g}(j)$ into irreducible $\mathfrak{g}(0)$-modules. Note that $\mathfrak{z}(\mathfrak{g}(0))$ acts on the summand $\mathfrak{g}(j,[\beta_i,\infty))$ via the restriction of the functional β_i to $\mathfrak{z}(\mathfrak{g}(0))$. It follows from Proposition 2.1 that the list $\beta_1\big|_{\mathfrak{z}(\mathfrak{g}(0))}, \ldots, \beta_{n_j}\big|_{\mathfrak{z}(\mathfrak{g}(0))}$ is linearly independent. In particular, the list contains no repetition. By Schur's Lemma there is a functional $\lambda\colon \mathfrak{z}(\mathfrak{g}(0)) \to F$ such that $\mathfrak{z}(\mathfrak{g}(0))$ acts on W via λ. We have

$$\mathfrak{g}(j) \otimes W = \bigoplus_{i=1}^{n_j} \mathfrak{g}(j,[\beta_i,\infty)) \otimes W \tag{6}$$

and $\mathfrak{z}(\mathfrak{g}(0))$ acts on the summand $\mathfrak{g}(j,[\beta_i,\infty)) \otimes W$ via $\beta_i + \lambda$. Thus the action of $\mathfrak{z}(\mathfrak{g}(0))$ on $\mathfrak{g}(j) \otimes W$ separates the summands in (6). It follows from this that it is sufficient to establish that $\mathfrak{g}(j,[\beta_i,\infty)) \otimes W$ is multiplicity-free as a $\mathfrak{g}(0)^{ss}$-module. This is true, by a standard fact about tensor products, because $\mathfrak{g}(j,[\beta_i,\infty))$ is weight-multiplicity-free as a $\mathfrak{g}(0)^{ss}$-module by Lemma 2.4, and W is irreducible as a $\mathfrak{g}(0)^{ss}$-module. ■

3. Background on generalized Verma modules

In this section we introduce the notation that we shall use for generalized Verma modules and recall a few basic facts about them. We continue with the assumptions and notation of Section 2. Let

$$\mathfrak{q} = \bigoplus_{j \ge 0} \mathfrak{g}(j)$$

be the parabolic subalgebra of $\mathfrak{g}$ associated to the subset S of simple roots and

$$\mathfrak{n} = \bigoplus_{j>0} \mathfrak{g}(j)$$

be its nilradical. The opposite parabolic subalgebra is

$$\bar{\mathfrak{q}} = \bigoplus_{j\leq 0} \mathfrak{g}(j)$$

and its nilradical is

$$\bar{\mathfrak{n}} = \bigoplus_{j<0} \mathfrak{g}(j).$$

Let V be an irreducible $\mathfrak{g}(0)$-module. We may, and always shall, extend consider V as a $\mathfrak{q}$-module by making $\mathfrak{n}$ act trivially on V. The generalized Verma module associated to V is then

$$\mathcal{M}_{\mathfrak{q}}(V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} V.$$

By the PBW Theorem, $\mathcal{M}_{\mathfrak{q}}(V)$ may be identified with $\mathcal{U}(\bar{\mathfrak{n}})\otimes V$ as a $\mathcal{U}(\bar{\mathfrak{n}})$-module. A subspace $U \subset \mathcal{M}_{\mathfrak{q}}(V)$ that is invariant under $\mathfrak{g}(0)$, irreducible as a $\mathfrak{g}(0)$-module, and satisfies $\mathfrak{n}U = \{0\}$ will be called a *leading subspace*. The subspace $F \otimes V$ is always a leading subspace, which we call the *tautological leading subspace*, but there may be others. If U is a leading subspace of $\mathcal{M}_{\mathfrak{q}}(V)$ then we may form the generalized Verma module $\mathcal{M}_{\mathfrak{q}}(U)$. The map $\mathcal{M}_{\mathfrak{q}}(U) \to \mathcal{M}_{\mathfrak{q}}(V)$ given on simple tensors by $a \otimes u \mapsto au$ is a homomorphism of $\mathcal{U}(\mathfrak{g})$-modules. The image of the tautological leading subspace in $\mathcal{M}_{\mathfrak{q}}(U)$ under this map is U, so that U may be recovered from the associated homomorphism.

The derivation $\mathrm{ad}(H_0)\colon \mathfrak{g} \to \mathfrak{g}$ extends to a derivation $x \mapsto H_0x - xH_0$ of $\mathcal{U}(\mathfrak{g})$, which we shall denote by the same symbol. This derivation is locally finite, semisimple, and has integral eigenvalues. If $x \in \mathcal{U}(\mathfrak{g})$ and $v \in V$ then

$$H_0(x \otimes v) = \big(\mathrm{ad}(H_0)x + \lambda_V(H_0)x\big) \otimes v$$

and it follows that multiplication by H_0 is a locally finite, semisimple map of $\mathcal{M}_{\mathfrak{q}}(V)$. By identifying $\mathcal{M}_{\mathfrak{q}}(V)$ with $\mathcal{U}(\bar{\mathfrak{n}})\otimes V$, we see that this map has eigenvalues in the set $-\mathbb{N} + \lambda_V(H_0)$. We write $\mathcal{M}^k_{\mathfrak{q}}(V)$ for the $(k + \lambda_V(H_0))$-eigenspace of the map. Let $U \subset \mathcal{M}_{\mathfrak{q}}(V)$ be an irreducible $\mathfrak{g}(0)$-submodule. By Schur's Lemma, we necessarily have $U \subset \mathcal{M}^{-d}_{\mathfrak{q}}(V)$ for some $d \in \mathbb{N}$, which we call the *degree* of U. Note that $d = \lambda_V(H_0) - \lambda_U(H_0)$. The tautological leading subspace is the unique subspace of degree zero.

Let $s \in F^S$. The module $\mathcal{M}_{\mathfrak{q}}(V^s)$ is realized on the same underlying space as the module $\mathcal{M}_{\mathfrak{q}}(V)$. Moreover, the identity map of the underlying spaces is an isomorphism between the $\mathfrak{g}(0)$-modules $\mathcal{M}_{\mathfrak{q}}(V^s)$ and $\mathcal{M}_{\mathfrak{q}}(V)^s$. In particular, if $U \subset \mathcal{M}_{\mathfrak{q}}(V)$ is a $\mathfrak{g}(0)$-invariant subspace then the twist $U^s \subset \mathcal{M}_{\mathfrak{q}}(V^s)$ is realized on the same underlying space as U. If U is an irreducible $\mathfrak{g}(0)$-module then so also is U^s. In contrast, the property of being a leading subspace is not invariant under twisting.

Let ρ denote half the sum of the positive roots of $\mathfrak{g}$. If U and V are irreducible $\mathfrak{g}(0)$-modules then we define an inhomogeneous linear form in $s \in F^S$ by

$$\ell_{UV}(s) = 2(\lambda_V - \lambda_U, \varpi^s + \rho) + \|\lambda_V\|^2 - \|\lambda_U\|^2.$$

If $U \subset \mathcal{M}_{\mathfrak{q}}(V)$ has degree $d \geq 1$ then ℓ_{UV} is non-constant and, in particular, non-zero. To establish this, it suffices to show that $(\lambda_V - \lambda_U, \varpi^s)$ is not identically zero. If we choose $s_0 \in F^S$ such that ϖ^{s_0} corresponds to the element H_0 then $(\lambda_V - \lambda_U, \varpi^{s_0}) = (\lambda_V - \lambda_U)(H_0) = d \geq 1$, as required.

Lemma 3.1. *Let V be an irreducible $\mathfrak{g}(0)$-module and $U \subset \mathcal{M}_{\mathfrak{q}}(V)$ an irreducible $\mathfrak{g}(0)$-submodule. If $s \in F^S$ is such that $U^s \subset \mathcal{M}_{\mathfrak{q}}(V^s)$ is a leading subspace then $\ell_{UV}(s) = 0$.*

Proof. Since U^s is a leading subspace in $\mathcal{M}_{\mathfrak{q}}(V^s)$ there is a non-zero homomorphism from the module $\mathcal{M}_{\mathfrak{q}}(U^s)$ to the module $\mathcal{M}_{\mathfrak{q}}(V^s)$ and so these two modules must have the same infinitesimal character. If we regard the infinitesimal character as an element of $\mathfrak{t}^*/W(R)$, where $W(R)$ denotes the Weyl group of R, in the usual way then this means that there is some $\sigma \in W(R)$ such that $\sigma(\lambda_U + \varpi^s + \rho) = \lambda_V + \varpi^s + \rho$. In particular, we have $\|\lambda_U + \varpi^s + \rho\|^2 = \|\lambda_V + \varpi^s + \rho\|^2$. The identity $\ell_{UV}(s) = 0$ results from this by expanding and simplifying. ∎

4. Second-order systems

In this section we prove our main results, Theorems 4.1 and 4.3.

Let $\beta \in S$. For brevity, we shall write $\mathfrak{g}(1, [\beta, \infty))$ as $\mathfrak{g}(1, \beta)$ and, similarly, $\mathfrak{g}(-1, (-\infty, -\beta])$ as $\mathfrak{g}(-1, -\beta)$ in this section. Let $S = \{\beta_1, \ldots, \beta_n\}$. We have the decomposition

$$\mathrm{sym}^2(\mathfrak{g}(-1)) \cong \bigoplus_{1 \leq i \leq n} \mathrm{sym}^2(\mathfrak{g}(-1, -\beta_i)) \oplus \bigoplus_{1 \leq i < j \leq n} \mathfrak{g}(-1, -\beta_i) \otimes \mathfrak{g}(-1, -\beta_j). \quad (7)$$

The center of $\mathfrak{g}(0)$ acts on $\mathfrak{g}(-1, -\beta_i)$ via the restriction of the functional $-\beta_i$ to the center. There are elements $Z_1, \ldots, Z_n$ in $\mathfrak{z}(\mathfrak{g}(0))$ such that $\beta_i(Z_j) = \delta_{ij}$. It follows from this that the various summands in (7) have distinct central characters. Thus if E is an irreducible $\mathfrak{g}(0)$-submodule of $\mathrm{sym}^2(\mathfrak{g}(-1))$ then E must be contained in one of the summands. If E is contained in $\mathrm{sym}^2(\mathfrak{g}(-1, -\beta_i))$ for some i then we say that E is of the *first kind.* If E is contained in $\mathfrak{g}(-1, -\beta_i) \otimes \mathfrak{g}(-1, -\beta_j)$ for some $i < j$ then we say that E is of the *second kind.*

There is a $\mathfrak{g}(0)$-homomorphism $\mathbf{s}\colon \mathrm{sym}^2(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ that satisfies $\mathbf{s}(XY) = \frac{1}{2}(XY + YX)$ on monomials. By mild abuse of notation, let F denote the trivial $\mathfrak{g}(0)$-module. The homomorphism $\mathbf{s}$ gives rise to a $\mathfrak{g}(0)$-module homomorphism $\mathrm{sym}^2(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}}) \otimes F$ which sends the monomial XY to $\mathbf{s}(XY) \otimes 1$. As we have observed, $\mathcal{U}(\bar{\mathfrak{n}}) \otimes F$ may be identified with $\mathcal{M}_{\mathfrak{q}}(F)$ and so we may regard this map as a $\mathfrak{g}(0)$-module homomorphism $\theta\colon \mathrm{sym}^2(\mathfrak{g}(-1)) \to \mathcal{M}_{\mathfrak{q}}(F)$. The image of θ lies in $\mathcal{M}_{\mathfrak{q}}^{-2}(F)$. Let $E \subset \mathrm{sym}^2(\mathfrak{g}(-1))$ be an irreducible submodule of the first kind. Then $U = \theta(E)$ is a degree-two irreducible $\mathfrak{g}(0)$-submodule of $\mathcal{M}_{\mathfrak{q}}(F)$. We say that U *corresponds* to E. Note that $\ell_{UF}(s) = \ell_{EF}(s)$.

If M is a $\mathfrak{g}(0)$-module then all the $\mathfrak{g}(0)$-modules M^s are realized on the same underlying space. If $v \in M$ then we shall write v^s for the vector v regarded as an element of the module M^s.

Theorem 4.1. *Let $E \subset \operatorname{sym}^2(\mathfrak{g}(-1))$ be an irreducible submodule of the first kind and $U \subset \mathcal{M}_{\mathfrak{q}}(F)$ be the corresponding degree-two submodule. Then U^s is a leading subspace in $\mathcal{M}_{\mathfrak{q}}(F^s)$ if and only if $\ell_{EF}(s) = 0$.*

Proof. One direction follows from Lemma 3.1. It remains to show that if $\ell_{EF}(s) = 0$ then $\mathfrak{n}U^s = \{0\}$. To do this it suffices to show that if $\ell_{EF}(s) = 0$ then $\mathfrak{g}(1)U^s = \{0\}$. We have $E \subset \operatorname{sym}^2(\mathfrak{g}(-1, -\beta_i))$ for some i. If $X, Y \in \mathfrak{g}(-1)$ and $W \in \mathfrak{g}(1)$ then

$$\begin{aligned} &W\big(\tfrac{1}{2}(XY + YX) \otimes 1^s\big) \\ &\quad = \big(\varpi^s([W,Y])X + \varpi^s([W,X])Y + \tfrac{1}{2}[[W,X],Y] + \tfrac{1}{2}[[W,Y],X]\big) \otimes 1^s \qquad (8) \end{aligned}$$

in $\mathcal{M}_{\mathfrak{q}}(F^s)$. It follows from this identity and Corollary 2.2 that if $W \in \mathfrak{g}(1, \beta_j)$ with $j \neq i$ and $\xi \in U$ then $W\xi^s = 0$. Thus we are reduced to showing that if $\ell_{EF}(s) = 0$ then $\mathfrak{g}(1, \beta_i)U^s = \{0\}$. By (8), there is a $\mathfrak{g}(0)$-module homomorphism

$$\Psi^s \colon \mathfrak{g}(1, \beta_i) \otimes \theta\big(\operatorname{sym}^2(\mathfrak{g}(-1, -\beta_i))\big)^s \to \mathfrak{g}(-1, -\beta_i) \otimes F^s$$

that satisfies $W \otimes \xi^s \mapsto W\xi^s$ on simple tensors. We wish to show that if $\ell_{EF}(s) = 0$ then

$$\Psi^s(\mathfrak{g}(1, \beta_i) \otimes U^s) = \{0\}.$$

Let ψ^s denote the restriction of Ψ^s to $\mathfrak{g}(1, \beta_i) \otimes U^s$.
By Proposition 2.5 the dimension of the space

$$\operatorname{Hom}_{\mathfrak{g}(0)}\big(\mathfrak{g}(1, \beta_i) \otimes U^s, \mathfrak{g}(-1, -\beta_i) \otimes F^s\big) \qquad (9)$$

is at most one. If it were zero then we would have $\psi^s = 0$ for all s and this would contradict Lemma 3.1 since $\ell_{EF}(s)$ is not identically zero. Thus the dimension of the space (9) is exactly one. Moreover, the space (9) is independent of s in the sense that the underlying spaces of the two modules do not vary with s and a map between these spaces that belongs to (9) for one value of s belongs to it for all values of s. Thus we may choose a basis φ for (9). There is a function $c \colon F^S \to F$ such that $\psi^s = c(s)\varphi$ for all $s \in F^S$.
An appropriately scaled highest weight vector in E has the form

$$X_{-\beta_i}X_{-\nu} + \sum_{\substack{\mu \in (\beta_i, \infty) \cap R(1) \\ \mu - \beta_i - \nu \in R,\ \mu \neq \nu}} a_\mu X_{-\mu} X_{\mu - \beta_i - \nu}$$

for some $\nu \in [\beta_i, \infty) \cap R(1)$ and some scalars $a_\mu \in F$. Note that the restriction $\mu \neq \nu$ in the second term is present because $E \subset \operatorname{sym}^2(\mathfrak{g}(-1, -\beta_i))$ and so $X_{-\nu}X_{-\beta_i} = X_{-\beta_i}X_{-\nu}$. It follows that an appropriately scaled highest weight vector in U^s has the form

$$\xi^s = \frac{1}{2}(X_{-\beta_i}X_{-\nu} + X_{-\nu}X_{-\beta_i}) \otimes 1^s + \sum_{\substack{\mu\in(\beta_i,\infty)\cap R(1)\\ \mu-\beta_i-\nu\in R,\ \mu\neq\nu}} \frac{a_\mu}{2}(X_{-\mu}X_{\mu-\beta_i-\nu} + X_{\mu-\beta_i-\nu}X_{-\mu}) \otimes 1^s.$$

It follows from this and (8) that

$$\psi^s(X_{\beta_i} \otimes \xi^s) = (1 + \delta_{\beta_i\nu})\varpi^s(H_{\beta_i})X_{-\nu} \otimes 1^s + \zeta \otimes 1^s$$

where $\zeta \in \mathfrak{g}(-1,-\beta_i)$ is independent of s. Since $\beta_i \in S$, $H_{\beta_i} \notin \mathfrak{g}(0)^{ss}$ and so $\varpi^s(H_{\beta_i})$ is a non-constant linear form. We conclude from this that $c(s)$ is a non-constant inhomogeneous linear form on F^S.
If $c(s) = 0$ for some $s \in F^S$ then $\psi^s = 0$ and so U^s is a leading subspace in $\mathcal{M}_{\mathfrak{q}}(F^s)$. Lemma 3.1 then implies that $\ell_{EF}(s) = 0$. That is, the vanishing of the non-constant inhomogeneous form $c(s)$ entails the vanishing of the non-constant inhomogeneous linear form $\ell_{EF}(s)$. It follows that there is some $k \in F^\times$ such that $c(s) = k\ell_{EF}(s)$. Thus $\ell_{EF}(s) = 0$ implies that $c(s) = 0$ and hence that U^s is a leading subspace in $\mathcal{M}_{\mathfrak{q}}(F^s)$, as required. ■

Suppose now that E is an irreducible $\mathfrak{g}(0)$-submodule of $\mathrm{sym}^2(\mathfrak{g}(-1))$ of the second kind. Then there are $i < j$ such that E lies in the image of the summand $\mathfrak{g}(-1,-\beta_i)\otimes\mathfrak{g}(-1,-\beta_j)$ under the isomorphism (7). The image of this summand is spanned by the monomials XY with $X \in \mathfrak{g}(-1,-\beta_i)$ and $Y \in \mathfrak{g}(-1,-\beta_j)$ and so we denote it by $\mathfrak{g}(-1,-\beta_i)\mathfrak{g}(-1,-\beta_j)$. Let $z \in F$. There is a $\mathfrak{g}(0)$-homomorphism $\theta_z\colon \mathfrak{g}(-1,-\beta_i)\mathfrak{g}(-1,-\beta_j) \to \mathcal{M}_{\mathfrak{q}}(F)$ that satisfies

$$\theta_z(XY) = \big(\frac{1}{2}(XY + YX) + z[X,Y]\big) \otimes 1$$

on monomials. The image of θ_z lies in $\mathcal{M}_{\mathfrak{q}}^{-2}(F)$. It is possible that θ_z does not in fact depend on z. This will be the case when $\mathfrak{g}(-1,-\beta_i)$ and $\mathfrak{g}(-1,-\beta_j)$ commute with one another in $\mathfrak{g}$. We take a momentary detour to identify the condition under which this does not happen. If the condition of Lemma 4.2 holds then there must be at least one irreducible constituent E of $\mathfrak{g}(-1,-\beta_i)\mathfrak{g}(-1,-\beta_j)$ such that the restriction of θ_z to E genuinely depends on z.

Lemma 4.2. *Let $1 \le i < j \le n$. We have $[\mathfrak{g}(-1,-\beta_i),\mathfrak{g}(-1,-\beta_j)] \neq \{0\}$ if and only if the shortest path from β_i to β_j in the Dynkin diagram does not pass through any element of $S \setminus \{\beta_i,\beta_j\}$.*

Proof. Suppose that $[\mathfrak{g}(-1,-\beta_i),\mathfrak{g}(-1,-\beta_j)] \neq \{0\}$. As a $\mathfrak{g}(0)$-module, the space $[\mathfrak{g}(-1,-\beta_i),\mathfrak{g}(-1,-\beta_j)]$ is a quotient of $\mathfrak{g}(-1,-\beta_i)\otimes\mathfrak{g}(-1,-\beta_j)$ and so the highest weights of its irreducible constituents have the form $-\beta_i - \nu_j$ with $-\nu_j$ a weight of $\mathfrak{g}(-1,-\beta_j)$. These weights have the form $-\nu_j = -\beta_j - \mu$ with μ a linear combination of elements of $R^s \setminus S$. It follows that $\beta_i + \beta_j + \mu \in R$ for some μ which is a linear combination of elements of $R^s \setminus S$. The support of a root in the

Dynkin diagram is necessarily connected and so $\{\beta_i, \beta_j\} \cup T$ is connected, where $T \subset R^s \setminus S$ is the support of μ. The shortest path from β_i to β_j therefore involves only β_i, β_j, and certain elements of T, and therefore does not pass through any other element of S, as required.

Now suppose that the shortest path from β_i to β_j passes through no element of $S \setminus \{\beta_i, \beta_j\}$. Let $T \subset R^s \setminus S$ be set of nodes besides β_i and β_j that occur on the path. Then T is a connected subset of the Dynkin diagram and so $\mu = \sum T$ is a root in $\mathfrak{g}(0)^{ss}$. Moreover, $T \cup \{\beta_j\}$ and $T \cup \{\beta_i, \beta_j\}$ are connected subsets of the Dynkin diagram and so $\beta_j + \mu$ and $\beta_i + \beta_j + \mu$ are both roots. Thus

$$\mathfrak{g}_{-\beta_i - \beta_j - \mu} = [\mathfrak{g}_{-\beta_i}, \mathfrak{g}_{-\beta_j - \mu}] \subset [\mathfrak{g}(-1, -\beta_i), \mathfrak{g}(-1, -\beta_j)]$$

and so $[\mathfrak{g}(-1, -\beta_i), \mathfrak{g}(-1, -\beta_j)] \neq \{0\}$, as required. ■

By an *affine subspace* of F^S we mean a subset of the form $t + A$ where $t \in F^S$ and $A \subset F^S$ is a subspace. The codimension of the affine subspace $t + A$ is the codimension of A.

Theorem 4.3. *Let $1 \leq i < j \leq n$ and $E \subset \mathfrak{g}(-1, -\beta_i)\mathfrak{g}(-1, -\beta_j)$ be an irreducible submodule. Let $z \in F$ and $U_z = \theta_z(E) \subset \mathcal{M}_{\mathfrak{q}}(F)$ be the corresponding degree-two submodule. Then there is an affine subspace $\Gamma \subset F^S$ of codimension two such that U_z^s is a leading subspace if and only if $s \in \Gamma$.*
Moreover, $\Gamma \subset \{s \in F^S \mid \ell_{EF}(s) = 0\}$.

Proof. It suffices to prove that there exists an affine subspace Γ of the required kind such that $s \in \Gamma$ if and only if $\mathfrak{g}(1)U_z^s = \{0\}$. Let $X \in \mathfrak{g}(-1, -\beta_i)$ and $Y \in \mathfrak{g}(-1, -\beta_j)$. If $W \in \mathfrak{g}(1, \beta_k)$ with $k \notin \{i, j\}$ then it follows from Corollary 2.2 that $W\theta_z(XY) = 0$ in $\mathcal{M}_{\mathfrak{q}}(F^s)$ regardless of the value of s. Thus the condition $\mathfrak{g}(1)U_z^s = \{0\}$ is equivalent to the conjunction of the conditions $\mathfrak{g}(1, \beta_i)U_z^s = \{0\}$ and $\mathfrak{g}(1, \beta_j)U_z^s = \{0\}$.

Let $X \in \mathfrak{g}(-1, -\beta_i)$, $Y \in \mathfrak{g}(-1, -\beta_j)$, and $W \in \mathfrak{g}(1, \beta_i)$. A calculation using Corollary 2.2 and the Jacobi identity shows that

$$\begin{aligned} &W\Big(\big(\frac{1}{2}(XY + YX) + z[X, Y]\big) \otimes 1^s\Big) \\ &\quad = \big(\varpi^s([W, X])Y + (z + \frac{1}{2})[[W, X], Y]\big) \otimes 1^s. \end{aligned} \tag{10}$$

It follows that there is a $\mathfrak{g}(0)$-module homomorphism

$$\Psi_i^s \colon \mathfrak{g}(1, \beta_i) \otimes \theta_z\big(\mathfrak{g}(-1, -\beta_i)\mathfrak{g}(-1, \beta_j)\big)^s \to \mathfrak{g}(-1, -\beta_j) \otimes F^s$$

that satisfies $\Psi_i^s(W \otimes \xi^s) = W\xi^s$ on simple tensors. Let ψ_i^s denote the restriction of Ψ_i^s to $\mathfrak{g}(1, \beta_i) \otimes U_z^s$. Then $\mathfrak{g}(1, \beta_i)U_z^s = \{0\}$ if and only if $\psi_i^s = 0$. Similarly, if $X \in \mathfrak{g}(-1, -\beta_i)$, $Y \in \mathfrak{g}(-1, -\beta_j)$, and $W \in \mathfrak{g}(1, \beta_j)$ then

$$\begin{aligned} &W\Big(\big(\frac{1}{2}(XY + YX) + z[X, Y]\big) \otimes 1^s\Big) \\ &\quad = \big(\varpi^s([W, Y])X + (-z + \frac{1}{2})[[W, Y], X]\big) \otimes 1^s. \end{aligned} \tag{11}$$

It follows that there is a $\mathfrak{g}(0)$-module homomorphism

$$\Psi_j^s\colon \mathfrak{g}(1,\beta_j)\otimes\theta_z\big(\mathfrak{g}(-1,-\beta_i)\mathfrak{g}(-1,\beta_j)\big)^s\to\mathfrak{g}(-1,-\beta_i)\otimes F^s$$

that satisfies $\Psi_j^s(W\otimes\xi^s)=W\xi^s$ on simple tensors. Let ψ_j^s denote the restriction of Ψ_j^s to $\mathfrak{g}(1,\beta_j)\otimes U_z^s$. Then $\mathfrak{g}(1,\beta_j)U_z^s=\{0\}$ if and only if $\psi_j^s=0$.

By Proposition 2.5 the spaces

$$\operatorname{Hom}_{\mathfrak{g}(0)}\big(\mathfrak{g}(1,\beta_i)\otimes U_z^s,\mathfrak{g}(-1,-\beta_j)\otimes F^s\big)$$

and

$$\operatorname{Hom}_{\mathfrak{g}(0)}\big(\mathfrak{g}(1,\beta_j)\otimes U_z^s,\mathfrak{g}(-1,-\beta_i)\otimes F^s\big)$$

both have dimension at most one. In fact, since $\mathfrak{g}(1,\beta_i)$ and $\mathfrak{g}(-1,-\beta_i)$ are contragredient modules as are $\mathfrak{g}(1,\beta_j)$ and $\mathfrak{g}(-1,-\beta_j)$, the two spaces are isomorphic to one another. If both had dimension zero then it would follow that U_z^s is a leading subspace for all s, contrary to Lemma 3.1. Thus both spaces have dimension one and we may choose bases φ_i and φ_j for them. The construction of the previous paragraph implies that there are functions $c_i\colon F^S\to F$ and $c_j\colon F^S\to F$ such that $\psi_i^s=c_i(s)\varphi_i$ and $\psi_j^s=c_j(s)\varphi_j$ for all $s\in F^S$.

A highest weight vector in E has the form

$$a_{\beta_i}X_{-\beta_i}X_{-\beta_j-\nu}+a_{\beta_j}X_{-\beta_i-\nu}X_{-\beta_j}+\sum_{\substack{\mu\in(\beta_i,\infty)\cap R(1)\\ \tau\in(\beta_j,\infty)\cap R(1)\\ \mu+\tau=\beta_i+\beta_j+\nu}}a_\mu X_{-\mu}X_{-\tau}$$

where either $\nu=0$ or ν is a positive root in $R(0)$. If $\nu=0$ then the third term is zero and we may assume after rescaling that $a_{\beta_i}=a_{\beta_j}=1/2$. If $\nu\neq0$ then we at least have $a_{\beta_i}\neq0$ and $a_{\beta_j}\neq0$. Let ξ_z be the image of this highest weight vector under θ_z. Then ξ_z^s is a highest weight vector in U_z^s for all s and by considering $\psi_i^s(X_{\beta_i}\otimes\xi_z^s)$ and $\psi_j^s(X_{\beta_j}\otimes\xi_z^s)$ as we did in the proof of Theorem 4.1 we conclude again that $c_i(s)$ and $c_j(s)$ are non-constant inhomogeneous linear forms on F^S.

The map φ_i is non-zero and so we may choose a vector $v\in\mathfrak{g}(1,\beta_i)\otimes U_z^s$ such that $\varphi_i(v)\neq0$. We have $\psi_i^s(v)=c_i(s)\varphi_i(v)$. Inspection of (10) shows that

$$\psi_i^s(v)=\sum_{p=1}^m\varpi^s(X_p)Y_p\otimes1^s+\zeta\otimes1^s \tag{12}$$

where $X_p\in[\mathfrak{g}(1,\beta_i),\mathfrak{g}(-1,-\beta_i)]=\mathfrak{a}_{\beta_i}$, $Y_p\in\mathfrak{g}(-1,-\beta_j)$, and ζ is independent of s. The value of $\varpi^s(X_p)$ depends only on the the component of X_p in $\mathfrak{z}(\mathfrak{a}_{\beta_i})$. By Part (4) of Proposition 2.3, $\mathfrak{z}(\mathfrak{a}_{\beta_i})$ is one-dimensional. Let us choose a basis Z_i for $\mathfrak{z}(\mathfrak{a}_{\beta_i})$. We may rewrite (12) as

$$\psi_i^s(v)=\varpi^s(Z_i)\sum_{p=1}^m x_pY_p\otimes1^s+\zeta\otimes1^s$$

with suitable $x_p\in F$. It follows from this and the fact that $c_i(s)$ is non-constant that there are $k_i\in F^\times$ and $d_i\in F$ such that $c_i(s)=k_i\varpi^s(Z_i)+d_i$. A parallel argument shows that $c_j(s) = k_j\varpi^s(Z_j)+d_j$ where Z_j is a basis for $\mathfrak{z}(\mathfrak{a}_{\beta_j})$, $k_j \in F^\times$, and $d_j \in F$.

By Part (4) of Proposition 2.3, for ϖ in the span of $\{\varpi_\alpha \mid \alpha \in S\}$ we have $\varpi(Z_i) = 0$ if and only if $(\varpi, \beta_i) = 0$ and $\varpi(Z_j) = 0$ if and only if $(\varpi, \beta_j) = 0$. Since β_i and β_j are distinct simple roots they are linearly independent. It follows that Z_i and Z_j are linearly independent and hence that the homogeneous linear forms $\varpi^s(Z_i)$ and $\varpi^s(Z_j)$ are linearly independent. Thus

$$\Gamma = \{s \in F^S \mid c_i(s) = 0 \text{ and } c_j(s) = 0\}$$

is a codimension two affine subspace of F^S. The construction of $c_i(s)$ and $c_j(s)$ shows that U_z^s is a leading subspace if and only if $s \in \Gamma$. Finally, Lemma 3.1 implies that if $s \in \Gamma$ then $\ell_{EF}(s) = 0$. This completes the proof. ■

Note that the proof of Theorem 4.3 shows that the affine subspace Γ has the form $\Gamma = s_0 + A$ where

$$A = \{s \in F^S \mid \varpi^s(Z_i) = 0 \text{ and } \varpi^s(Z_j) = 0\}$$

with Z_i a basis for $\mathfrak{z}(\mathfrak{a}_{\beta_i})$ and Z_j a basis for $\mathfrak{z}(\mathfrak{a}_{\beta_j})$. The space A is independent of E and of z. The dependence of Γ on these objects is through the value of s_0.

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