

Representations Associated to Small Nilpotent Orbits for Real Spin Groups

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Abstract. The results in this paper provide a comparison between the K -structure of unipotent representations and regular sections of bundles on nilpotent orbits. Precisely, let $\widetilde{G}_0 = \widetilde{Spin}(a, b)$ with $a + b = 2n$, the nonlinear double cover of $Spin(a, b)$, and let $\widetilde{K} = Spin(a, \mathbb{C}) \times Spin(b, \mathbb{C})$ be the complexification of the maximal compact subgroup of \widetilde{G}_0 . We consider the nilpotent orbit \mathcal{O}_c parametrized by $[3 \ 2^{2k} \ 1^{2n-4k-3}]$ with $k > 0$. We provide a list of unipotent representations that are genuine, and prove that the list is complete using the coherent continuation representation. Separately we compute \widetilde{K} -spectra of the regular functions on certain real forms \mathcal{O} of \mathcal{O}_c transforming according to appropriate characters ψ under $C_{\widetilde{K}}(\mathcal{O})$, and then match them with the \widetilde{K} -types of the genuine unipotent representations. The results provide instances for the orbit philosophy.

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1. Introduction

In this paper we pursue the problems addressed in [6] for the real Spin groups. A major difference is the fact that we consider nonlinear covers of the real Spin groups.

We review the setup from [6]. Let G be a connected complex semisimple group (viewed as a real group) and let $\mathfrak{g}, \mathfrak{g}^\vee, \mathfrak{g}_{\mathbb{C}}$ be its Lie algebra, dual Lie algebra, and complexified Lie algebra, respectively. Special unipotent representations of G were introduced in [7]. To each nilpotent orbit $\mathcal{O}^\vee \subset \mathfrak{g}^\vee$ an infinitesimal character $\lambda_{\mathcal{O}^\vee}$ is associated via the Jacobson-Morozov theorem; the orbit is associated to a Lie triple $\{e^\vee, h^\vee, f^\vee\}$ and $\lambda_{\mathcal{O}^\vee} = h^\vee/2$ determines an infinitesimal character. *Special unipotent representations* are defined as the irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules Ξ satisfying

- (1) the infinitesimal character is $(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$, and
- (2) the annihilator of Ξ in the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, denoted by $Ann_{U(\mathfrak{g}_{\mathbb{C}})}\Xi$, is the unique maximal primitive ideal of $U(\mathfrak{g}_{\mathbb{C}})$ with infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$.

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Denote by $\mathcal{U}_G(\mathcal{O}^\vee, \lambda_{\mathcal{O}^\vee})$ the set of unipotent representations of G associated to \mathcal{O}^\vee . In [3], the unitarity of these representations is established for the case of classical groups, and the whole unitary dual for such groups is determined. In the process, a larger set of representations is introduced which are called unipotent. A finite set of infinitesimal characters $\lambda_{\mathcal{O}^\vee, s^\vee}$ is associated to each \mathcal{O}^\vee , and an irreducible module Ξ is *unipotent*, if it satisfies (1) and (2) with the more general $\lambda_{\mathcal{O}^\vee, s^\vee}$ instead. The results in [3] can be viewed as proving that the modules in $\mathcal{U}_G(\mathcal{O}^\vee, \lambda_{\mathcal{O}^\vee, s^\vee})$ are the building blocks of the unitary dual.

In [4] a different viewpoint is taken. Instead of parametrizing by \mathcal{O}^\vee , the unipotent representations are parametrized by nilpotent orbits $\mathcal{O} \subset \mathfrak{g}$. The precise setting is as follows. Let $G_0 \subset G$ be the real points of a complex linear reductive algebraic group G with Lie algebra \mathfrak{g} and maximal compact subgroup K_0 . Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ be the Cartan decomposition, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the complexification. Let K be the complexification of K_0 . Then K acts on \mathfrak{s} by the adjoint action. We denote by $C_K(e) = \{g \in K \mid Ad(g)e = e\}$, the centralizer of $e \in \mathfrak{s}$ in K .

For a general reductive real Lie group G_0 , [25] associates to each irreducible admissible representation Ξ the associated cycle $AC(\Xi)$, a union of nilpotent K -orbits in \mathfrak{s} , and an algebraic representation of each centralizer $C_K(e)$ of a representative e for each orbit in $AC(\Xi)$. The main results in [26] Chapter 7 and [25] Theorem 4.11 imply that when the complements of the orbits in $AC(\Xi)$ have codimension strictly greater than 1 in their closures,

$$\Xi|_{C_K} = R(\mathcal{O}, \psi) - Y$$

with ψ an algebraic representation, and Y an $S(\mathfrak{g}/\mathfrak{k})$ -module supported on orbits of strictly smaller dimension.

Definition 1.1. Let $e \in \mathfrak{s}$ be a nilpotent element, and let $\mathcal{O} := K \cdot e \subset \mathfrak{s}$ be the K -adjoint orbit of e . We say that an irreducible admissible representation Ξ is *associated* to \mathcal{O} , if \mathcal{O} occurs with nonzero multiplicity in the associated cycle. We will write $\mathcal{U}_{G_0}(\mathcal{O}, \lambda)$ for the set of irreducible representations of G_0 with maximal annihilator with infinitesimal character λ , and associated to \mathcal{O} .

Let $C_K(\mathcal{O}) := C_K(e)$ denote the centralizer of e in K , with e a representative of \mathcal{O} , and let $A_K(\mathcal{O}) := C_K(\mathcal{O})/C_K(\mathcal{O})^0$ be the component group. In the case of G_0 being a connected complex group viewed as a real group, it is conjectured that there exists an infinitesimal character $\lambda_{\mathcal{O}}$ such that in addition,

Conjecture 1.2. For each \mathcal{O} there is $\lambda_{\mathcal{O}}$ such that there is a 1-1 correspondence $\psi \in \widehat{A_K(\mathcal{O})} \longleftrightarrow \Xi(\mathcal{O}, \psi) \in \mathcal{U}_{G_0}(\mathcal{O}, \lambda_{\mathcal{O}})$ satisfying the additional condition

$$\Xi(\mathcal{O}, \psi)|_{C_K} \cong R(\mathcal{O}, \psi), \tag{1}$$

where $R(\mathcal{O}, \psi) = \text{Ind}_{C_K(\mathcal{O})}^K(\psi)$

$$= \{f : K \rightarrow V_\psi \mid f(gx) = \psi(x^{-1})f(g) \ \forall g \in K, x \in C_K(\mathcal{O})\} \tag{2}$$

is the ring of regular functions on \mathcal{O} transforming according to ψ .

It is also natural to conjecture that there is a choice of $\lambda_{\mathcal{O}}$ such that the representations are unitary.

Candidates for parameters $\lambda_{\mathcal{O}}$ satisfying the Conjecture above are listed in [4]. Essentially they coincide with the $\lambda_{\mathcal{O}^v, s^v}$ introduced in [3]. The validity of the conjecture is established for large classes of nilpotent orbits in the classical complex groups. Such parameters $\lambda_{\mathcal{O}}$ are available for the exceptional groups as well, [4] for F_4 , and to appear elsewhere for type E .

We now turn to the specifics of the real Spin groups. We use the same infinitesimal characters $\lambda_{\mathcal{O}_c}$ associated to the corresponding complex orbit, and consider representations with maximal annihilator.

Definition 1.3. A nilpotent K -orbit in \mathfrak{s} is called *small* if

$$\dim \mathcal{O} \leq \text{rank}(\mathfrak{k}) + |\Delta^+(\mathfrak{k}, \mathfrak{t})|, \tag{3}$$

where $\mathfrak{t} \subset \mathfrak{k}$ is a Cartan subalgebra, and $\Delta^+(\mathfrak{k}, \mathfrak{t})$ is a positive system.

These orbits have the property that there is a chance that the multiplicity of any $\mu \in \widehat{K}$ be uniformly bounded. The reason is as follows. Let (Π, X) be an admissible representation of G_0 , and μ be the highest weight of a representation $(\pi, V) \in \widehat{K}$ which is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Assume that $\dim \text{Hom}_K[\pi, \Pi] \leq C$, and Π has associated variety $\overline{\mathcal{O}}$ (cf. [25]). Then

$$\dim\{v: v \in X \text{ belongs to an isotypic component with } \|\mu\| \leq t\} \leq Ct^{|\Delta^+(\mathfrak{k}, \mathfrak{t})| + \dim \mathfrak{t}}.$$

The dimension of (π, V) grows like $t^{|\Delta^+(\mathfrak{k}, \mathfrak{t})|}$, the number of representations with highest weight $\|\mu\| \leq t$ grows like $t^{\dim \mathfrak{t}}$, and the multiplicities are assumed uniformly bounded. On the other hand, considerations involving primitive ideals imply that the dimension of this set grows like $t^{\dim G \cdot e/2}$ with $e \in \mathcal{O}$, and half the dimension of (the complex orbit) $G \cdot e$ is the dimension of the (K -orbit) $K \cdot e \in \mathfrak{s}$.

In the complex case, the appropriate condition for bounded multiplicities is that the orbit be *spherical*, i.e. the Borel subgroup has an open orbit, and results of [19] provide a classification. For the real Spin groups, there is no difference between small and spherical, and we do not elaborate on this.

More precisely, we choose infinitesimal characters corresponding to the complex case for which the orbits are minimal such that a (\mathfrak{g}, K) -module associated to such an orbit exists. We then compute the genuine representations for

$$(\widetilde{G}_0, \widetilde{K}) = (\widetilde{Spin}(a, b), Spin(a, \mathbb{C}) \times Spin(b, \mathbb{C}))$$

satisfying these conditions, compute the \widetilde{K} -spectrum and compare to the spectrum rational functions on the corresponding orbits. (Here we write $(\widetilde{G}_0, \widetilde{K})$ to emphasize that \widetilde{G}_0 is the nonlinear double cover of $Spin(a, b)$.)

Write $2n = a + b$, $a = 2p$, $a = 2p + 1$, $b = 2q$, and $b = 2q - 1$.

The representations are associated to real forms of the complex nilpotent orbit

$$\mathcal{O}_c = [3 \ 2^{2k} \ 1^{2n-4k-3}] \quad k > 0.$$

The condition $k > 0$ insures that these orbits are not special in the sense of Lusztig. So there are no representations with integral infinitesimal character associated to \mathcal{O}_c . The infinitesimal character is

$$\lambda = (n - k - 2, \dots, 1, 0; k + 1/2, \dots, 3/2, 1/2), \quad (4)$$

same as in [4]. – Here is a summary of the results.

In Section 2 we list the real forms of the nilpotent orbit and describe the (component groups) of their centralizers. In Section 3 we analyze the K -structure of certain $R(\mathcal{O}, \psi)$. In Section 4 we match them with a set of representations obtained by restriction from those listed in [17]. It is not clear that certain of these restrictions are irreducible. An alternative way to construct a set of representations with the required properties is to apply the derived functors construction to highest weight modules with the appropriate infinitesimal character and annihilator. The calculations are in the spirit of [13] and [21]. A comparison of the restrictions with the alternate construction shows that indeed certain of these restrictions are reducible.

Section 5 contains technicalities about *Spin* groups used to prove some of the results. Section 6 computes the coherent continuation representation and shows that the list of representations in Section 4 is complete; these are all the genuine representations with the given infinitesimal character associated to real forms of \mathcal{O}_c . Section 7 provides a construction of the representations using cohomological induction from highest weight modules.

A summary of the main results is in the following Theorem. The detailed results are in Sections 2 and 4.

Theorem 1.4. *Let $(\widetilde{G}_0, \widetilde{K}) = (\widetilde{Spin}(a, b), Spin(a, \mathbb{C}) \times Spin(b, \mathbb{C}))$ with $a + b = 2n$. Let \mathcal{O} be a certain real form (given in Section 2) of the complex nilpotent orbit $[3 \ 2^{2k} \ 1^{2n-4k-3}]$, $k > 0$, and $\lambda = (n - k - 2, \dots, 1, 0; k + 1/2, \dots, 3/2, 1/2)$. A list of genuine unipotent representations in $\mathcal{U}_{\widetilde{G}_0}(\mathcal{O}, \lambda)$ is completely provided for the eight cases (see Section 4). In each case, we compute $\pi|_{\widetilde{K}}$, the \widetilde{K} -types of π , for each $\pi \in \mathcal{U}_{\widetilde{G}_0}(\mathcal{O}, \lambda)$; separately we compute \widetilde{K} -spectra of the regular functions on \mathcal{O} transforming according to appropriate characters ψ under $C_{\widetilde{K}}(\mathcal{O})$, and then match them with the \widetilde{K} -types of the genuine unipotent representations.*

There is a significant difference from the complex case; condition (1) cannot hold in its stated form for the real case. This can already be seen for $SL(2, \mathbb{R})$. The spherical principal series with infinitesimal character zero is unipotent, and its associated cycle contains two nilpotent orbits. So its K -structure does not match any $R(\mathcal{O}, \psi)$. The phenomenon is analyzed in detail in [25]. A necessary condition for it to hold is that the complement of \mathcal{O} in its closure have codimension bigger than one. This is the case for the orbits studied in this paper. In particular this condition implies that the associated cycle only contains one orbit. Even so, because we are dealing with a nonlinear cover, the *correct* ψ turn out to be 1-dimensional characters of $C_{\widetilde{K}}(\mathcal{O})$ which are not trivial on the connected component.

Some of the results, particularly counting the representations and restricting from the odd Spin groups to the even ones, have their origin in [22]. There are relations to the work in [15] and [16] which we intend to pursue in future research.

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2. Preliminaries

2.1. Nilpotent orbits

We follow [8]. Nilpotent orbits in $\mathfrak{so}(a, b)$ are parametrized by orthogonal signed Young diagrams of signature (a, b) with numerals. We write a real orbit of the diagram $[3 \ 2^{2k} \ 1^{2n-4k-3}]$ as $[3^\epsilon 2^{2k} 1^{+,c} 1^{-,d}]$ (possibly with I, II), where 3^ϵ denotes the block of size 3 starting with sign ϵ ; $1^{+,c}$ denotes c blocks of size 1 labeled $+$, and c is omitted when $c = 1$; similarly for $1^{-,d}$.

The following eight cases of signed diagrams are treated in this paper.

- $Case\ 1 : a = 2p = 2k + 2, \quad \mathcal{O} = [3^+ 2^{2k} 1^-]_{I,II} \quad \mathcal{O} = [3^- 2^{2k} 1^+]_{I,II}$
 $b = 2p = 2k + 2,$
- $Case\ 2 : a = 2p = 2k + 2 + 2r_+, \quad \mathcal{O} = [3^- 2^{2k} 1^{+,2r_++1}]_{I,II} \quad \mathcal{O} = [3^+ 2^{2k} 1^- 1^{+,2r_+}]$
 $b = 2q = 2k + 2,$
- $Case\ 3 : a = 2p = 2k + 2, \quad \mathcal{O} = [3^+ 2^{2k} 1^{-,2r_-+1}]_{I,II} \quad \mathcal{O} = [3^- 2^{2k} 1^+ 1^{-,2r_-}]$
 $b = 2q = 2k + 2 + 2r_-,$
- $Case\ 4 : a = 2p + 1 = 2k + 1, \quad \mathcal{O} = [3^+ 2^{2k} 1^+ 1^{-,2}] \quad \mathcal{O} = [3^- 2^{2k} 1^- 1^{+,2}]$
 $b = 2q - 1 = 2k + 1,$
- $Case\ 5 : a = 2p + 1 = 2k + 3 + 2r_+, \quad \mathcal{O} = [3^+ 2^{2k} 1^{+,2r_++1}]$
 $b = 2q - 1 = 2k + 1,$
- $Case\ 6 : a = 2p + 1 = 2k + 1, \quad \mathcal{O} = [3^- 2^{2k} 1^{-,2r_-+1}]$
 $b = 2q - 1 = 2k + 3 + 2r_-,$
- $Case\ 7 : a = 2p + 1 = 2k + 1 + 2r_+, \quad \mathcal{O} = [3^- 2^{2k} 1^- 1^{+,2r_+}] \quad (\text{with } r_+ \geq 2)$
 $b = 2q - 1 = 2k + 3.$
- $Case\ 8 : a = 2p + 1 = 2k + 3, \quad \mathcal{O} = [3^+ 2^{2k} 1^+ 1^{-,2r_-}] \quad (\text{with } r_- \geq 2)$
 $b = 2q - 1 = 2k + 1 + 2r_-.$

As will become apparent at the end, these are the only \tilde{K} -orbits that are associated to genuine representations. Cases 1 and 4 are invariant under exchanging $+$ and $-$, Cases 2, 3, 5, 6 and 7, 8 correspond under exchanging $+$ and $-$. Nilpotent orbits I, II in Cases 1, 2, 3 are treated the same way. We will omit details for cases that match under these correspondences.

The proof of the next Proposition, and details about the nature of the component groups, are in Section 5.

Proposition 2.1.

Case 1: If $\mathcal{O} = [3^+2^{2k}1^-]_{I,II}$ or $[3^-2^{2k}1^+]_{I,II}$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2,3: If $\mathcal{O} = [3^-2^{2k}1^{+,2r_++1}]_{I,II}$, with $r_+ > 0$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

Case 2,3: If $\mathcal{O} = [3^+2^{2k}1^-1^{+,2r_+}]$, with $r_+ > 0$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

Case 4: If $\mathcal{O} = [3^+2^{2k}1^-1^{+,2}]$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

Case 5,6: If $\mathcal{O} = [3^+2^{2k}1^+]$, with $r_+ = 0$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

Case 5,6: If $\mathcal{O} = [3^+2^{2k}1^{+,2r_++1}]$ with $r_+ > 0$, then $A_{\tilde{K}}(\mathcal{O}) = 1$.

Case 7,8: If $\mathcal{O} = [3^-2^{2k}1^-1^{+,2r_+}]$, with $r_+ \geq 2$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

The cases are paired according to the $+$ and $-$ interchanged.

3. Regular sections

We compute the centralizers needed for $R(\mathcal{O}, \psi)$ in \mathfrak{k} and in \tilde{K} . We use the standard roots and basis for $\mathfrak{so}(a, b)$. The Cartan subalgebra is the fundamental one, a basis is given by $H(\epsilon_i)$, and the root vectors are $X(\pm\epsilon_i \pm \epsilon_j)$, $X(\pm\epsilon_i)$. Realizations in terms of the Clifford algebra, and explicit calculations are in Section 5.

Let $\{e, h, f\}$ with $e \in \mathcal{O}$ be a Lie triple such that $h \in \mathfrak{k}$ and $f \in \mathfrak{s}$. We denote by

- $C_{\mathfrak{k}}(h)_i$ the i -eigenspace of $ad(h)$ in \mathfrak{k} ,
- $C_{\mathfrak{k}}(e)_i$ the i -eigenspace of $ad(h)$ in the centralizer of e in \mathfrak{k} ,
- $C_{\mathfrak{k}}(h)^+ := \sum_{i>0} C_{\mathfrak{k}}(h)_i$, and $C_{\mathfrak{k}}(e)^+ := \sum_{i>0} C_{\mathfrak{k}}(e)_i$.

3.1. $\widetilde{\text{Spin}}(2p, 2q)$

These are Cases 1,2,3, so $p = k + 1$, $q = k + 1 + r_-$. The compact Cartan subalgebra has coordinates $(x_1, \dots, x_{k+1} \mid y_1, \dots, y_k, y_{k+1}, \dots, y_{k+r_-+1})$ with Cartan involution $\theta(x_i) = x_i$, $\theta(y_j) = y_j$.

We describe the centralizer for $[3^+2^{2k}1^{-,2r_-+1}]_I$ in \mathfrak{k} in detail. Representatives for e and h are

$$e = X(\epsilon_1 - \epsilon_{p+k+1}) + X(\epsilon_1 + \epsilon_{p+k+1}) + \sum_{2 \leq i \leq k+1} X(\epsilon_i + \epsilon_{p+i-1})$$

$$h = H(2\epsilon_1) + \sum_{2 \leq i \leq k+1} H(\epsilon_i + \epsilon_{p+i-1}) = (2, \underbrace{1, \dots, 1}_k \mid \underbrace{1, \dots, 1}_k, 0, \dots, 0).$$

Then

$$\begin{aligned}
 C_{\mathfrak{k}}(h)_0 &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_- + 2) \\
 C_{\mathfrak{k}}(h)_1 &= \text{Span}\{X(\epsilon_1 - \epsilon_i), 2 \leq i \leq k + 1, \\
 &\quad X(\epsilon_{p+j} \pm \epsilon_{p+l}), 1 \leq j \leq k < l \leq q\}, \\
 C_{\mathfrak{k}}(h)_2 &= \text{Span}\{X(\epsilon_i + \epsilon_j), 2 \leq i < j \leq k + 1, \\
 &\quad X(\epsilon_{p+i} + \epsilon_{p+j}), 1 \leq i < j \leq k\}, \\
 C_{\mathfrak{k}}(h)_3 &= \text{Span}\{X(\epsilon_1 + \epsilon_i), 2 \leq i \leq k + 1\}.
 \end{aligned}
 \tag{5}$$

Similarly $C_{\mathfrak{k}}(e)_0 \cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_- + 1)$
 $C_{\mathfrak{k}}(e)_1 = \text{Span}\{X(\epsilon_1 - \epsilon_i) + X(\epsilon_{p+i-1} \pm \epsilon_{p+k+1}), 2 \leq i \leq k + 1,$
 $X(\epsilon_{p+j} \pm \epsilon_{p+l}), 1 \leq j \leq k, k + 2 \leq l \leq q\},$ (6)
 $C_{\mathfrak{k}}(e)_2 = C_{\mathfrak{k}}(h)_2,$
 $C_{\mathfrak{k}}(e)_3 = C_{\mathfrak{k}}(h)_3.$

The $\mathfrak{gl}(k) \subset C_{\mathfrak{k}}(e)_0$ is embedded in $\mathfrak{gl}(k) \times \mathfrak{gl}(k) \subset C_{\mathfrak{k}}(h)_0$ via $x \mapsto (x, -x^t)$, and $\mathfrak{so}(2r_- + 1) \subset \mathfrak{so}(2r_- + 2)$ is the standard inclusion.

We denote by Det^χ a character of $C_{\mathfrak{k}}(e)$, a power of the determinant of $\mathfrak{gl}(p - 1) = \mathfrak{gl}(k)$. Assume \mathbf{p} is even throughout. This has the effect that for an irreducible representation, $V^* \cong V$, and details can easily be filled in for the other case. Because we are considering genuine representations of the nonlinear double cover, we need to compute regular functions for ψ which are not trivial on the connected component of the identity. So $\psi = Det^\chi$ where χ is a half-integer. This holds for all cases.

Case 1. As already noted, $p = k + 1$ and $q = k + 1$. We treat the orbit $\mathcal{O} = [3^+ 2^{2k} 1^-]_I$ only. The other orbits in this Case are related by outer automorphisms as follows.

Let ζ, η be the outer automorphisms determined by

$$\begin{aligned} \zeta : (x_1, \dots, x_p \mid y_1, \dots, y_p) &\mapsto (x_1, \dots, x_{p-1}, -x_p \mid y_1, \dots, y_{p-1}, -y_p), \\ \eta : (x_1, \dots, x_p \mid y_1, \dots, y_p) &\mapsto (y_1, \dots, y_p \mid x_1, \dots, x_p). \end{aligned}$$
 (7)

The other three orbits in Case 1 are conjugate to \mathcal{O} by an outer automorphism and are denoted by $\mathcal{O}^\zeta, \mathcal{O}^\eta, \mathcal{O}^{\zeta\eta}$.

The centralizer $C_{\mathfrak{k}}(h)$ is isomorphic to $\mathfrak{gl}(1) \times \mathfrak{gl}(p - 1) \times \mathfrak{gl}(p - 1) \times \mathfrak{so}(2)$.

A representation of \tilde{K} will be denoted by its highest weight,

$$V = V(a_1, \dots, a_p \mid b_1, \dots, b_p), \quad a_1 \geq a_2 \geq \dots \geq |a_p|, \quad b_1 \geq b_2 \geq \dots \geq |b_p|.$$

All $a_i, b_j \in \mathbb{Z}$ or $a_i, b_j \in \mathbb{Z} + \frac{1}{2}$, but $a_i - b_j$ need not be integers; V is genuine precisely when $a_i - b_j \notin \mathbb{Z}$. We will compute

$$\text{Hom}_{C_{\mathfrak{k}}(e)}[V^*, \chi] = \text{Hom}_{C_{\mathfrak{k}}(e)_0} [V^*/(C_{\mathfrak{k}}(e)^+V^*), \chi] := (V^*/(C_{\mathfrak{k}}(e)^+V^*))^\chi \quad (8)$$

in two steps. In the first step we define a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ such that $e \in \mathfrak{n}$, $C_{\mathfrak{k}}(e) \in \mathfrak{p}$, and in addition $\mathfrak{n} \subset C_{\mathfrak{k}}(e)^+$. By Kostant's theorem $V^*/(\mathfrak{n}V^*)$ is known, and the computation of (8) reduces to a similar computation in \mathfrak{m} . This is done in step 2.

Step 1. Let $\xi := H(\epsilon_1 + \dots + \epsilon_p) = (1, \dots, 1 \mid 0, \dots, 0)$. It determines a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} \subset \mathfrak{k}$ where

$$\begin{aligned} \mathfrak{m} &= C_{\mathfrak{k}}(\xi) \cong \mathfrak{gl}(p) \times \mathfrak{so}(2p), \\ \mathfrak{n} &= \text{Span}\{X(\epsilon_i + \epsilon_j), 1 \leq i \neq j \leq p\} \subset C_{\mathfrak{k}}(e)_2 + C_{\mathfrak{k}}(e)_3. \end{aligned}$$

Kostant’s theorem on cohomology of finite dimensional representations implies that $V^*/(\mathfrak{n}V^*)$ is the irreducible \mathfrak{m} -module generated by its lowest weight. We denote it

$$\mathcal{W}(-a_p, -a_{p-1}, \dots, -a_1 \mid b_1, \dots, b_p). \tag{9}$$

The assumption p even implies $V^* \cong V$. Since we have $C_{\mathfrak{k}}(e)_0 + C_{\mathfrak{k}}(e)_1 \subset \mathfrak{m}$ and $C_{\mathfrak{k}}(e)^+ \cap \mathfrak{m} = C_{\mathfrak{m}}(e)^+$, it is enough to compute

$$[\mathcal{W}/(C_{\mathfrak{k}}(e)^+ \cap \mathfrak{m})\mathcal{W}]^X = [\mathcal{W}/(C_{\mathfrak{m}}(e)^+\mathcal{W})]^X.$$

Step 2. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{m}$ be the parabolic subalgebra in \mathfrak{m} determined by h , i.e.

$$\begin{aligned} \mathfrak{l} &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(p-1) \times \mathfrak{gl}(p-1) \times \mathfrak{so}(2) \\ \mathfrak{u} &= \text{Span}\{X(\epsilon_1 - \epsilon_i), X(\epsilon_{p-1+i} \pm \epsilon_{2p}), X(\epsilon_{p-1+i} + \epsilon_{p-1+j}), 2 \leq i \neq j \leq p\}, \end{aligned}$$

with $C_{\mathfrak{m}}(h) = \mathfrak{l}$, $C_{\mathfrak{m}}(h)^+ = \mathfrak{u}$. Then

$$\begin{aligned} C_{\mathfrak{m}}(e)_0 &= \text{Span}\{H(\epsilon_i - \epsilon_{p-1+i}), X(\epsilon_i - \epsilon_j) + X(-\epsilon_{p-1+i} + \epsilon_{p-1+j})\}, \\ C_{\mathfrak{m}}(e)^+ &= \text{Span}\{X(\epsilon_1 - \epsilon_i) - X(\epsilon_{p-1+i} - \epsilon_{2p}), X(\epsilon_1 - \epsilon_i) - X(\epsilon_{p-1+i} + \epsilon_{2p})\}. \end{aligned}$$

As in the case of \mathfrak{g} , $C_{\mathfrak{m}}(e)_0 \cong \mathfrak{gl}(p-1)$ embeds in $\mathfrak{gl}(p-1) \times \mathfrak{gl}(p-1) \subset \mathfrak{l}$ as $x \mapsto (0; x \mid -x^t; 0)$.

The module \mathcal{W} is a quotient of a (generalized) Verma module $M(\lambda) = U(\mathfrak{m}) \otimes_{U(\bar{\mathfrak{q}})} F_\lambda$ with λ the weight of \mathcal{W} made dominant for $\bar{\mathfrak{q}}$:

$$(-a_1; -a_p, \dots, -a_2 \mid -b_{p-1}, \dots, -b_1; -b_p).$$

The ; denotes the fact that this is a (highest) weight of

$$\mathfrak{l} \cong \mathfrak{gl}(1) \times \mathfrak{gl}(p-1) \times \mathfrak{gl}(p-1) \times \mathfrak{so}(2).$$

The positive system for $\Delta^+(\mathfrak{l})$ is the standard one for the Levi component. The nilradical decomposes $\mathfrak{u} = C_{\mathfrak{m}}(e)^+ \oplus \mathfrak{s}$ where $\mathfrak{s} = \text{Span}\{X(\epsilon_1 - \epsilon_i), 2 \leq i \leq p\}$ is a representation of $\mathfrak{gl}(1) \times \mathfrak{gl}(p-1) \times \mathfrak{so}(2p)$. The (generalized) Bernstein-Gelfand-Gelfand resolution is

$$0 \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=k}} M(w \cdot \lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=1}} M(w \cdot \lambda) \longrightarrow M(\lambda) \longrightarrow \mathcal{W} \longrightarrow 0, \tag{10}$$

with $w \cdot \lambda := w(\lambda + \rho(\mathfrak{m})) - \rho(\mathfrak{m})$, and $w \in W^+$, the $W(\mathfrak{l})$ -coset representatives that make $w \cdot \lambda$ dominant for $\Delta^+(\mathfrak{l})$. This is a free $C_{\mathfrak{m}}(e)^+$ -resolution so we can compute homology by considering

$$0 \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=k}} \overline{M(w \cdot \lambda)} \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=1}} \overline{M(w \cdot \lambda)} \longrightarrow \overline{M(\lambda)} \longrightarrow 0, \tag{11}$$

where for an \mathfrak{m} -module X , \overline{X} denotes $X/((C_{\mathfrak{m}}(e)^+)X)$.

As a module for $\mathfrak{gl}(1) \times \mathfrak{gl}(p-1) \times \mathfrak{so}(2p)$, \mathfrak{s} has highest weight $(1; 0, \dots, 0, -1 \mid 0, \dots, 0)$. Thus $S^m(\mathfrak{s}) \cong (m; 0, \dots, 0, -m \mid 0, \dots, 0; 0)$.

Let $\mu := (-\alpha_1; -\alpha_p, \dots, -\alpha_2 \mid -\beta_{p-1}, \dots, -\beta_1; -\beta_p)$ be the highest weight of an \mathfrak{l} -module. By the Littlewood-Richardson rule,

$$S^m(\mathfrak{s}) \otimes F_\mu = \sum \mathcal{W}(-\alpha_1 + m; -\alpha_p - m_p, \dots, -\alpha_2 - m_2 \mid -\beta_{p-1}, \dots, -\beta_1; -\beta_p). \tag{12}$$

The sum in (12) is taken over the set

$$\{m_i \mid m_i \geq 0, \sum_{i=2}^p m_i = m, m_i \leq \alpha_{i-1} - \alpha_i, 3 \leq i \leq p\}.$$

Lemma 3.1. $\text{Hom}_{C_m(e)_0}[S^m(\mathfrak{s}) \otimes F_\mu : \chi] \neq 0$ if and only if

$$\beta_1 \geq \alpha_2 + \chi \geq \beta_2 \geq \dots \geq \alpha_{p-1} + \chi \geq \beta_{p-1} \geq \alpha_p + \chi.$$

The multiplicity is 1.

Proof. The multiplicity of χ is nonzero precisely when

$$-\alpha_i - m_i + \beta_{i-1} = \chi \text{ for some } m_i \geq 0, \quad 2 \leq i \leq p.$$

The condition $0 \leq m_i \leq \alpha_{i-1} - \alpha_i$ implies $\beta_{i-1} \leq \alpha_{i-1} + \chi$ for $3 \leq i \leq p$. ■

Corollary 3.2. $\text{Hom}_{C_t(e)}[V, \chi] \neq 0$ only if

$$b_1 \geq a_2 + \chi \geq b_2 \geq \dots \geq a_{p-1} + \chi \geq b_{p-1} \geq a_p + \chi.$$

The multiplicity is ≤ 1 , and the action of $\text{ad } h$ is $-2 \sum_{1 \leq i \leq p} a_i$.

Proof. The first two statements follow from the surjection

$$\overline{M(\lambda)} \cong S(\mathfrak{s}) \otimes_{\mathbb{C}} F_\lambda \longrightarrow \overline{W} \longrightarrow 0. \tag{13}$$

The action of $\text{ad } h$ is computed from the module

$$\mathcal{W}(-a_1 + m; -b_{p-1}, \dots, -b_1 \mid -b_{p-1}, \dots, -b_1; -b_p)$$

with $m = \sum_{2 \leq i \leq p} (a_i - b_{i-1})$. The value is

$$2(-a_1 + \sum_{2 \leq i \leq p} -a_i + b_{i-1}) + 2(\sum_{1 \leq j \leq p-1} -b_j) = -2 \sum_{1 \leq i \leq p} a_i. \tag{14} \quad \blacksquare$$

We will need Lemma 3.3 in order to prove Proposition 3.4.

Let $\mathfrak{g} = \mathfrak{gl}(r, \mathbb{C})$, $V(a) = V(a_1, \dots, a_r)$ an irreducible finite dimensional representation with highest weight (a_1, \dots, a_r) with $a_i \in \mathbb{N}$. Let $\mathfrak{s} = \mathbb{C}^n$ be the standard representation with basis e_1, \dots, e_r . The Littlewood-Richardson rule implies that

$$S(\mathfrak{s}) \otimes V(a) = \sum V(a_1 + m_1, \dots, a_r + m_r)$$

with multiplicity 1, sum over $m_2 \leq a_1 - a_2, \dots, m_r \leq a_{r-1} - a_r$.

We need explicit information about the highest weights that occur in the decomposition. A typical weight of the tensor product will be denoted by $e^{\underline{m}'} \otimes v(\underline{a}')$ with $e^{\underline{m}'} := e_1^{m'_1} \dots e_r^{m'_r}$ and $v(\underline{a}')$ an eigenvector of V with weight (a'_1, \dots, a'_r) . The weight $\underline{m} + \underline{a} = (m_1 + a_1, \dots, m_r + a_r)$ is spanned by monomials of the form

$$e^{\underline{m}'} \otimes v(\underline{a}') \quad \underline{m}' + \underline{a}' = \underline{m} + \underline{a}.$$

Order the monomials by \underline{m}' lexicographically. The *lowest term* is $e^{\underline{m}} \otimes v(\underline{a})$.

Lemma 3.3. *The highest weight corresponding to $V(\underline{m} + \underline{a})$ has lowest term $e^{\underline{m}} \otimes v(\underline{a})$.*

Proof. Let $e^{\underline{m}'} \otimes v(\underline{a}')$ be a lowest monomial occurring in the highest weight. Since the factor occurs, a lowest term must occur. It must be annihilated by the action of all root vectors $X(\epsilon_i - \epsilon_j)$ with $i < j$. The formula is

$$X \cdot e^{\underline{m}'} \otimes v = X e^{\underline{m}'} \otimes v + e^{\underline{m}'} \otimes Xv.$$

The formula is $X(\epsilon_i - \epsilon_j)e^{\underline{m}'} = (m'_j)e_1^{m'_1} \dots e_i^{m'_i+1} \dots e_j^{m'_j-1} \dots e_r^{m'_r}$.

Thus each monomial is mapped into a term that is strictly higher (possibly zero) plus the term $e^{\underline{m}'} \otimes Xv$ which is lexicographically at the same level.

Applied to the highest weight expression, this implies that $Xv = 0$ for any $X = X(\epsilon_i - \epsilon_j)$. Thus the lowest term must be a multiple of the highest weight $v(\underline{a})$, which occurs with multiplicity one. It follows that $\underline{m}' = \underline{m}$ as well. ■

Proposition 3.4. $\text{Hom}_{C_t(e)}[V, \chi] \neq 0$ *only if*

$$a_1 + \chi \geq b_1 \geq a_2 + \chi \dots \geq a_p + \chi \geq |b_p|.$$

The multiplicities are ≤ 1 .

Proof. We need to prove three inequalities:

$$a_1 + \chi \geq b_1, \quad a_p + \chi \geq \pm b_p.$$

The module $M(w \cdot \lambda)$ decomposes according to the Baker-Campbell-Hausdorff formula

$$M(w \cdot \lambda) = S(\mathfrak{s}) \otimes F_{w \cdot \lambda} \oplus C_m(e)^+ U(\mathfrak{u}) \otimes F_{w \cdot \lambda},$$

and the summands are $C_m(e)_0$ -stable. We say the factors are transverse to each other. Suppose $b_1 < a_1 + \chi$, and w corresponds to the reflection $s_{\epsilon_1 - \epsilon_2}$. Then the complement of $\overline{M(w \cdot \lambda)}$ contains a $C_m(e)^+$ -component with weight

$$(-a_2 + 1 + m'_2; -a_p + m'_2, \dots - a_3 + m'_3, -a_1 + m'_2 \mid -b_{p-1}, \dots, -b_1; -b_p),$$

satisfying $m' = m'_2 + \dots + m'_p$ and $-a_1 - m'_2 = -b_1 + \chi$; similar equalities as in Lemma 3.1 hold for the other coordinates. The assumption $b_1 > a_1 + \chi$ guarantees that such a weight occurs. The vectors are of the form $S \otimes Av_{w \cdot \lambda}$ with A in the universal enveloping algebra of the first factor $gl(p - 1)$, and $X \in S^{m'}(\mathfrak{s})$. The

image under the differential is $ASX(\epsilon_1 - \epsilon_p)^{a_1 - a_2 + 1} \otimes v_\lambda$. This is nonzero and in the transverse to $C_m(e)^+U(\mathfrak{u}) \otimes F_{w,\lambda}$. The proof uses Lemma 3.3.

For the other two inequality, the analogous argument applies using the fact that the spaces

$$\mathfrak{s}_\pm = \text{Span}\{X(\epsilon_{p+i} \pm \epsilon_{2p})\}_{1 \leq i \leq p-1}$$

are also transverses. ■

To prove that the weights in Proposition 3.4 actually occur, it is enough to show that these weights do not occur in the term in the BGG resolution (13) with $\ell(w) = 1$. Recall

$$\begin{aligned} \rho = \rho(\mathfrak{m}) &= \left(-\frac{(p-1)}{2}; \frac{p-1}{2}, \frac{p-3}{2}, \dots, -\frac{(p-3)}{2} \mid -1, \dots, -(p-1); 0\right), \\ \lambda + \rho &= \left(-a_1 - \frac{(p-1)}{2}; -a_p + \frac{(p-1)}{2}, \dots, -a_2 - \frac{(p-3)}{2} \mid \right. \\ &\quad \left. -b_{p-1} - 1, -b_{p-2} - 2, \dots, -b_1 - (p-1); -b_p\right) \end{aligned} \tag{14}$$

For the case $\ell(w) = 1$, there are three elements. We enumerate them as w_1, w_2, w_3 , with $w_1 = s_{\epsilon_1 - \epsilon_p}, w_2 = s_{\epsilon_{p+1} - \epsilon_n}, w_3 = s_{\epsilon_{p+1} + \epsilon_n}$. Then

$$\begin{aligned} w_1 \cdot \lambda &= (-a_2 + 1; -a_p, \dots, -a_3, -a_1 - 1 \mid -b_{p-1}, \dots, -b_1; -b_p), \\ w_2 \cdot \lambda &= (-a_1; -a_p, \dots, -a_2 \mid -b_p + 1, \dots, -b_1; -b_{p-1} - 1), \\ w_3 \cdot \lambda &= (-a_1; -a_p, \dots, -a_2 \mid b_p + 1, \dots, -b_1; -b_{p-1} + 1). \end{aligned}$$

Lemma 3.5. $\overline{M(w_i \cdot \lambda)}$ has vectors transforming according to χ of $C_m(e)$ (trivial on $C_{\mathfrak{k}}(e)^+$), only if

$$\begin{aligned} w_1 : \quad & -a_1 - 1 - m_2 = -b_1 + \chi \quad \text{for some } m_2 \geq 0, \text{ i.e. } b_1 > a_1 + \chi \\ w_2 : \quad & -a_p - m_p = -b_p + 1 + \chi \quad \text{for some } m_p \geq 0, \text{ i.e. } b_p > a_p + \chi \\ w_3 : \quad & -a_p - m_p = b_p + 1 + \chi \quad \text{for some } m_p \geq 0, \text{ i.e. } -b_p > a_p + \chi. \end{aligned}$$

The multiplicities are 1, and the eigenvalue of $\text{ad } h$ is $-2 \sum_{1 \leq i \leq p-1} a_i$.

Proof. As in (12), the weights in $\overline{M(w_1 \cdot \lambda)}, \overline{M(w_2 \cdot \lambda)}$ and $\overline{M(w_3 \cdot \lambda)}$ are of the form

$$\begin{aligned} &(-a_2 + 1 + m; -a_p - m_p, \dots, -a_3 - m_3, -a_1 - 1 - m_2 \mid -b_{p-1}, \dots, -b_1; -b_p), \\ &(-a_1 + m; -a_p - m_p, \dots, -a_2 - m_2 \mid -b_p + 1, \dots - b_1; -b_{p-1} - 1), \\ &(-a_1 + m; -a_p - m_p, \dots, -a_2 - m_2 \mid b_p + 1, \dots - b_1; -b_{p-1} + 1), \end{aligned}$$

respectively. The proof is completed as in the case $\ell(w) = 0$. ■

Theorem 3.6. A representation $V(a_1, \dots, a_p \mid b_1, \dots, b_p)$ has vectors transforming as χ of $C_{\mathfrak{k}}(e)$ if and only if

$$a_1 + \chi \geq b_1 \geq \dots \geq a_p + \chi \geq |b_p|, \tag{15}$$

and the multiplicity is 1. In summary,

$$\text{Ind}_{C_{\tilde{K}}(e)_0}^{\tilde{K}}(\chi) = \bigoplus V(a_1, \dots, a_p \mid b_1, \dots, b_p),$$

satisfying $a_1 + \chi \geq b_1 \geq \dots \geq a_p + \chi \geq |b_p|$.

Proof. The proof is straightforward from the BGG resolution (13), Proposition 3.4, and Lemma 3.5. ■

Theorem 3.6 can be interpreted as computing regular functions on the universal cover $\tilde{\mathcal{O}}$ of \mathcal{O} transforming according to χ under $C_{\mathfrak{t}}(e)_0$. We decompose it further:

$$R(\tilde{\mathcal{O}}, \text{Det}^\chi) := \text{Ind}_{C_{\tilde{K}}(e)_0}^{\tilde{K}}(\text{Det}^\chi) = \text{Ind}_{C_{\tilde{K}}(e)}^{\tilde{K}} \left[\text{Ind}_{C_{\tilde{K}}(e)_0}^{C_{\tilde{K}}(e)}(\text{Det}^\chi) \right]. \tag{16}$$

The inner induced module splits into

$$\text{Ind}_{C_{\tilde{K}}(e)_0}^{C_{\tilde{K}}(e)}(\text{Det}^\chi) = \sum \psi$$

where ψ are the irreducible representations of $C_{\tilde{K}}(e)$ restricted to Det^χ on $C_{\tilde{K}}(e)_0$. We compute $R(\mathcal{O}, \psi) := \text{Ind}_{C_{\tilde{K}}(e)}^{\tilde{K}}(\psi)$ for \tilde{K} for $\chi = -1/2$; these are the cases matching representations. The formula in Theorem 3.6 specializes to

$$R(\tilde{\mathcal{O}}, \text{Det}^{-1/2}) = \bigoplus V(a_1, \dots, a_p \mid b_1, \dots, b_p)$$

satisfying $a_1 \geq b_1 + 1/2 \geq \dots \geq a_p \geq |b_p| + 1/2$.

Lemma 3.7. *Let $\nu_i, 1 \leq i \leq 4$, be the following \tilde{K} -types parametrized by their highest weights:*

$$\begin{aligned} \nu_1 &= (1/2, \dots, 1/2 \mid 0, \dots, 0), \nu_2 = (3/2, 1/2, \dots, 1/2 \mid 0, \dots, 0), \\ \nu_3 &= (1, \dots, 1 \mid 1/2, \dots, 1/2), \nu_4 = (1, \dots, 1 \mid 1/2, \dots, 1/2, -1/2). \end{aligned}$$

Let ψ_i be the restriction of the highest weight of ν_i to $C_{\tilde{K}}(e)$. Then

$$\text{Ind}_{C_{\tilde{K}}(e)_0}^{C_{\tilde{K}}(e)}(\text{Det}^{-1/2}) = \sum_{i=1}^4 \psi_i.$$

We write $C_{\tilde{K}}(\mathcal{O}) := C_{\tilde{K}}(e)$.

Proposition 3.8. *The induced representation (16) decomposes as*

$$\text{Ind}_{C_{\tilde{K}}(\mathcal{O})_0}^{\tilde{K}}(\text{Det}^{-1/2}) = \sum_{i=1}^4 R(\mathcal{O}, \psi_i),$$

where $\bar{V} := V(\beta_1 + 1/2, \dots, \beta_p + 1/2 \mid \delta_1, \dots, \delta_p)$ and

$$\begin{aligned} R(\mathcal{O}, \psi_1) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_1) = \bigoplus \bar{V}, \quad \sum (\beta_i + \delta_j) \in 2\mathbb{Z}, \\ R(\mathcal{O}, \psi_2) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_2) = \bigoplus \bar{V}, \quad \sum (\beta_i + \delta_j) \in 2\mathbb{Z} + 1, \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \dots, \geq \beta_p \geq |\delta_p|$ and $\beta_i, \delta_j \in \mathbb{Z}$,

$$R(\mathcal{O}, \psi_3) = \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_3) = \bigoplus \bar{V}, \quad \sum (\beta_i + \delta_j) \in 2\mathbb{Z},$$

$$R(\mathcal{O}, \psi_4) = \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_4) = \bigoplus \bar{V}, \quad \sum (\beta_i + \delta_j) \in 2\mathbb{Z} + 1,$$

satisfying $\beta_1 \geq \delta_1 \geq \dots, \geq \beta_p \geq |\delta_p|$ and $\beta_i, \delta_j \in \mathbb{Z} + 1/2$.

The analogous results for the other orbits in case 1 follow by applying the outer automorphisms in (7)

Case 2, 3. It is enough to consider the three nilpotent orbits,

$$(a) \quad \mathcal{O} = [3^+ 2^{2k} 1^{-, 2r_- + 1}]_{I, II} \quad h = H(\underbrace{2, 1, \dots, 1, \pm 1}_k \mid \underbrace{1, \dots, 1}_k, \underbrace{0, 0, \dots, 0}_{r_-}),$$

$$(b) \quad \mathcal{O} = [3^- 2^{2k} 1^{+, 2r_-}] \quad h = H(\underbrace{1, \dots, 1, 0}_k \mid \underbrace{2, 1, \dots, 1}_k, \underbrace{0, \dots, 0}_{r_-}).$$

Case (a). $\mathcal{O} = [3^+ 2^{2k} 1^{-, 2r_- + 1}]_{I, II}$

We assume $p = k + 1, q = k + 1 + 2r_-$ with $r_- > 0$. Denote $\mathcal{O}_{I, II} = [3^+ 2^{2k} 1^{-, 2r_- + 1}]_{I, II}$ according to the semisimple elements in the Lie triple

$$h_{I, II} = (2, \underbrace{1, \dots, 1, \pm 1}_k \mid \underbrace{1, \dots, 1}_k, \underbrace{0, 0, \dots, 0}_{r_-}).$$

The orbits are conjugate by the outer automorphism

$$\zeta : (x_1, \dots, x_p \mid y_1, \dots, y_q) \mapsto (x_1, \dots, -x_p \mid y_1, \dots, -y_q).$$

We only treat the case $\mathcal{O} = \mathcal{O}_I$. Similar result holds for $\mathcal{O}^\zeta = \mathcal{O}_{II}$.

Proposition 3.9. *A representation $V(a_1, \dots, a_p \mid b_1, \dots, b_q)$ has invariant vectors under $C_{\mathfrak{k}}(e)^+$ which transform according to Det^χ under $C_{\mathfrak{k}}(e)_0 \cong \mathfrak{gl}(k) \times \mathfrak{so}(2r_- - 1)$ if and only if $b_{k+2} = \dots = b_q = 0$ and*

$$a_1 + \chi \geq b_1 \geq a_2 + \chi \geq \dots \geq a_p + \chi \geq b_p \geq 0,$$

and the multiplicity is 1.

Proof. The representation $\mathcal{W}(a_1, \dots, a_{k+1} \mid b_1, \dots, b_k; b_{k+1}, \dots, b_q)$ has $\mathfrak{so}(2r_- + 1)$ -fixed vectors only if $b_{k+2} = \dots = b_q = 0$ by Helgason's theorem; in that case, the fixed vector is the highest weight $(b_{k+1}, 0, \dots, 0)$. Otherwise the proof is identical to Case 1. ■

As in equation (16), we compute $R(\mathcal{O}, \psi)$ for \tilde{K} and $\chi = -1/2 - r_-$; these are the cases matching representations. From Proposition 3.9,

$$R(\widetilde{\mathcal{O}}_I, \text{Det}^{-1/2 - r_-}) = \text{Ind}_{C_{\tilde{K}}(e)^0}^{\tilde{K}}(\text{Det}^{-1/2 - r_-})$$

$$= \bigoplus V(a_1, \dots, a_p \mid b_1, \dots, b_p, 0, \dots, 0) \tag{17}$$

satisfying $a_1 \geq b_1 + 1/2 + r_- \geq \dots \geq a_p \geq b_p + 1/2 + r_- \geq 0$.

Proposition 3.10. *Let*

$$\begin{aligned} \psi_1 &= (r_- + 1/2, \dots, r_- + 1/2 \mid 0, \dots, 0) \mid_{C_{\tilde{K}}(e)}, \\ \psi_2 &= (r_- + 3/2, r_- + 1/2, \dots, r_- + 1/2 \mid 0, \dots, 0) \mid_{C_{\tilde{K}}(e)}. \end{aligned}$$

The induced representation (17) decomposes as

$$\text{Ind}_{C_{\tilde{K}}(\mathcal{O}_I)^0}^{\tilde{K}}(\text{Det}^{-1/2-r_-}) = R(\mathcal{O}_I, \psi_1) + R(\mathcal{O}_I, \psi_2),$$

where $\bar{V} = V(\beta_1 + r_- + 1/2, \dots, \beta_p + r_- + 1/2 \mid \delta_1, \dots, \delta_p, 0, \dots, 0)$, and

$$\begin{aligned} R(\mathcal{O}_I, \psi_1) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O}_I)}^{\tilde{K}}(\psi_1) = \bigoplus \bar{V} \quad \text{with } \sum(\beta_i + \delta_j) \in 2\mathbb{Z}, \\ R(\mathcal{O}_I, \psi_2) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O}_I)}^{\tilde{K}}(\psi_2) = \bigoplus \bar{V} \quad \text{with } \sum(\beta_i + \delta_j) \in 2\mathbb{Z} + 1, \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$ and $\beta_i, \delta_j \in \mathbb{Z}$. The corresponding results for $R(\mathcal{O}_{II}, \psi^\zeta)$ follow by applying the automorphism ζ .

Case (b). $\mathcal{O} = [3^+ 2^{2k} 1^{-1+}, 2r_+]$

We assume $p = k + 1 + r_+$, $q = k + 1$ with $r_+ > 0$. A representative of the orbit is

$$\begin{aligned} e &= X(\epsilon_1 - \epsilon_{p+q}) + X(\epsilon_1 + \epsilon_{p+q}) + \sum_{2 \leq j \leq k+1} X(\epsilon_j + \epsilon_{p+j-1}), \\ h &= (2, \underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{r_+} \mid \underbrace{1, \dots, 1}_k, 0). \end{aligned}$$

Proposition 3.11. *A representation $V(a_1, \dots, a_p \mid b_1, \dots, b_q)$ has invariant vectors under $C_{\mathfrak{k}}(e)^+$ transforming according to Det^χ under $C_{\mathfrak{k}}(e)_0 \cong \mathfrak{gl}(k) \times \mathfrak{so}(2r_+)$ if and only if $a_{k+2} = \dots = a_p = 0$ and*

$$a_1 + \chi \geq b_1 \geq a_2 + \chi \geq b_2 \geq \dots \geq a_q + \chi \geq |b_q|.$$

Proof. The proof follows Case 1. The fact that $a_{k+2} = \dots = a_p = 0$ follows from the requirement that the character be trivial on the $\mathfrak{so}(2r_+)$ -factor. \blacksquare

As before, $\chi = r_+ - 1/2$ is the case corresponding to representations. Proposition 3.11 specializes to

$$\begin{aligned} R(\tilde{\mathcal{O}}, \text{Det}^{r_+-1/2}) &= \text{Ind}_{C_{\tilde{K}}(e)^0}^{\tilde{K}}(\text{Det}^{r_+-1/2}) \\ &= \bigoplus V(a_1, \dots, a_q, 0, \dots, 0 \mid b_1, \dots, b_p) \end{aligned} \tag{18}$$

satisfying $a_1 + r_+ - 1/2 \geq b_1 \geq \dots \geq a_q + r_+ - 1/2 \geq |b_q|$.

Proposition 3.12. *Let*

$$\begin{aligned} \phi_1 &= (0, \dots, 0 \mid r_+ - 1/2, \dots, r_+ - 1/2) \mid_{C_{\tilde{K}}(e)}, \\ \phi_2 &= (0, \dots, 0 \mid r_+ - 1/2, \dots, -(r_+ - 1/2)) \mid_{C_{\tilde{K}}(e)} \end{aligned}$$

Then the induced representation (18) decomposes as

$$\text{Ind}_{C_{\tilde{K}}(\mathcal{O})^0}^{\tilde{K}}(\text{Det}^{r_+-1/2}) = R(\mathcal{O}, \phi_1) + R(\mathcal{O}, \phi_2),$$

where $\tilde{V} = V(\beta_1, \dots, \beta_q, 0, \dots, 0 \mid \delta_1, \dots, \delta_q)$ and

$$\begin{aligned} R(\mathcal{O}, \phi_1) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\phi_1) = \bigoplus \tilde{V} \quad \text{with } \sum(\beta_i + \delta_j) \in 2\mathbb{Z}, \\ R(\mathcal{O}, \phi_2) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\phi_2) = \bigoplus \tilde{V} \quad \text{with } \sum(\beta_i + \delta_j) \in 2\mathbb{Z} + 1, \end{aligned}$$

satisfying $\beta_1 + r_+ - 1/2 \geq \delta_1 \geq \dots \geq \beta_q + r_+ - 1/2 \geq |\delta_q|$ and $\beta_i \in \mathbb{Z}, \delta_j \in \mathbb{Z} + 1/2$.

3.2. $\widetilde{\text{Spin}}(2p + 1, 2q - 1)$

These are Cases 4–8. The two orbits in Case 4 are obtained from Case 7 and Case 8 by putting $r_+ = 1$ and $r_- = 1$, and they are related by the automorphisms in (7). So we deal with Cases 5, 6 and Cases 7, 8.

Case 5, 6. The orbit is $\mathcal{O} = [3 + 2^{2k} 1^{+, 2r_+ + 1}]$ and $2p + 1 = 2k + 3 + 2r_+, 2q - 1 = 2k + 1$. The fundamental Cartan subalgebra has coordinates

$$(x_1, \dots, x_{k+1}, x_{k+2}, \dots, x_{k+1+r_+} \mid y_1, \dots, y_k; z)$$

with Cartan involution $\theta(x_i) = x_i, \theta(y_j) = y_j, \theta(z) = -z$.

Representatives for e and h are

$$\begin{aligned} e &= X(\epsilon_1)_n + \sum_{2 \leq i \leq k+1} X(\epsilon_i + \epsilon_{p+i-1}) \\ h &= H(2\epsilon_1) + \sum_{2 \leq i \leq k+1} H(\epsilon_i + \epsilon_{p+i-1}) = (2, \underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{r_+} \mid \underbrace{1, \dots, 1}_k; 0) \end{aligned}$$

where the last coordinate after the “;” is the z . Then

$$\begin{aligned} C_{\mathfrak{k}}(h)_0 &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_+ + 1) \times \mathfrak{gl}(k), \\ C_{\mathfrak{k}}(h)_1 &= \text{Span}\{X(\epsilon_1 - \epsilon_i), X(\epsilon_i)_c, X(\epsilon_{p+i-1})_c, 2 \leq i \leq k + 1, \\ &\quad X(\epsilon_j \pm \epsilon_l), 1 \leq j \leq k + 1 < l \leq p\}, \\ C_{\mathfrak{k}}(h)_2 &= \text{Span}\{X(\epsilon_1)_c, X(\epsilon_i + \epsilon_j), X(\epsilon_{p+i-1} + \epsilon_{p+j-1}), 2 \leq i < j \leq k + 1\}, \\ C_{\mathfrak{k}}(h)_3 &= \text{Span}\{X(\epsilon_1 + \epsilon_i), 2 \leq i \leq k + 1\}. \end{aligned} \tag{19}$$

Similarly

$$\begin{aligned} C_{\mathfrak{k}}(e)_0 &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_+ + 1) \\ C_{\mathfrak{k}}(e)_1 &= \text{Span}\{X(\epsilon_1 - \epsilon_i) + X(\epsilon_{p+i-1})_c, 2 \leq i \leq k + 1, X(\epsilon_i \pm \epsilon_l), \\ &\quad 2 \leq i \leq k + 1 < l \leq p, X(\epsilon_{p-1+j})_c, 2 \leq i \leq k + 1, 1 \leq j \leq k\}, \\ C_{\mathfrak{k}}(e)_2 &= C_{\mathfrak{k}}(h)_2, \\ C_{\mathfrak{k}}(e)_3 &= C_{\mathfrak{k}}(h)_3. \end{aligned} \tag{20}$$

The $\mathfrak{gl}(k)$ embeds in $\mathfrak{gl}(k) \times \mathfrak{gl}(k)$ as before $x \mapsto (x, -x^t)$.

Proposition 3.13. *A representation $V(a_1, \dots, a_p \mid b_1, \dots, b_{q-1})$ has vectors invariant for $C_{\mathfrak{k}}(e)^+$ which transform according to Det^χ under $C_{\mathfrak{k}}(e)_0 \cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_+ + 1)$ if and only if $a_{k+2} = \dots = a_p = 0$, and*

$$a_1 + \chi \geq b_1 \geq \dots \geq a_k + \chi \geq b_k \geq a_{k+1} + \chi. \tag{21}$$

Proof. Step 1. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} \subset \mathfrak{k}$ be the parabolic subalgebra determined by

$$\xi = (\underbrace{1, \dots, 1}_{k+1}, \underbrace{0, \dots, 0}_{r_+} \mid \underbrace{0, \dots, 0}_k; 0).$$

Then $\mathfrak{n} \subset C_{\mathfrak{k}}(e)^+$, and we can apply Kostant’s theorem to reduce the computation to $\mathfrak{m} \cong \mathfrak{gl}(k + 1) \times \mathfrak{so}(2r_+ + 1) \times \mathfrak{gl}(k)$. The \mathfrak{n} -coinvariants are the module

$$\mathcal{W}(-a_1, \dots, -a_{k+1}; a_{k+2}, \dots, a_p \mid b_1, \dots, b_k).$$

We have assumed k even for simplicity. By Helgason’s theorem, $a_{k+2} = \dots = a_p = 0$. We must compute the multiplicity of a character χ trivial on the nilradical of $C_{\mathfrak{m}}(e)$.

Step 2. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{m}$ be the parabolic subalgebra determined by (the restriction of) h :

$$\mathfrak{l} \cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{gl}(k) \times \mathfrak{so}(2k + 1).$$

The proof proceeds as in Case 1 and Cases 2,3 ; see also Cases 7, 8. ■

When $r_+ > 0$, $a_i \in \mathbb{Z}$ and $b_j \in \mathbb{Z} + 1/2$. When $r_+ = 0$, $A_{\tilde{K}}(\mathcal{O})$ has two components. The character χ is not determined by its differential; there are two possibilities, corresponding to $a_i \in \mathbb{Z}, b_j \in \mathbb{Z} + 1/2$ and $a_i \in \mathbb{Z} + 1/2, b_j \in \mathbb{Z}$.

Case 7, 8. The fundamental Cartan subalgebra has coordinates

$$(x_1, \dots, x_{k+1} \mid y_1, \dots, y_k, y_{k+1}, \dots, y_{k+r_-}, z)$$

with Cartan involution $\theta(x_i) = x_i, \theta(y_j) = y_j, \theta(z) = -z$. We describe the centralizer for $[3^+2^{2k}1^{+1-}, 2r_-]$ with $r_- \geq 2$ in \mathfrak{k} in detail. Representatives for e and h from the previous section are

$$e = X(\epsilon_1) + \sum_{2 \leq i \leq k+1} X(\epsilon_i + \epsilon_{p+i-1})$$

$$h = H(2\epsilon_1) + \sum_{2 \leq i \leq k+1} H(\epsilon_i + \epsilon_{p+i-1}) = (2, \underbrace{1, \dots, 1}_k \mid \underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{r_-}; 0)$$

where the last coordinate after the ”;” is the z . Then

$$\begin{aligned} C_{\mathfrak{k}}(h)_0 &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_- + 1), \\ C_{\mathfrak{k}}(h)_1 &= \text{Span}\{X(\epsilon_1 - \epsilon_i), X(\epsilon_i), X(\epsilon_{p+i-1}), 2 \leq i \leq k + 1, \\ &\quad X(\epsilon_{p+j} \pm \epsilon_{p+l}), 1 \leq j \leq k < l \leq q\}, \\ C_{\mathfrak{k}}(h)_2 &= \text{Span}\{X(\epsilon_1), X(\epsilon_i + \epsilon_j), X(\epsilon_{p+i-1} + \epsilon_{p+j-1}), 2 \leq i < j \leq k + 1\}, \\ C_{\mathfrak{k}}(h)_3 &= \text{Span}\{X(\epsilon_1 + \epsilon_i), 2 \leq i \leq k + 1\}. \end{aligned} \tag{22}$$

Similarly

$$\begin{aligned}
 C_{\mathfrak{k}}(e)_0 &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_-) \\
 C_{\mathfrak{k}}(e)_1 &= \text{Span}\{X(\epsilon_1 - \epsilon_i) + X(\epsilon_{p+i-1} \pm \epsilon_{p+k+1}), 2 \leq i \leq k+1, \\
 &\quad X(\epsilon_{p+j} \pm \epsilon_{p+k+l}), 1 \leq j \leq k, k+2 \leq l \leq q-1\}, \\
 C_{\mathfrak{k}}(e)_2 &= C_{\mathfrak{k}}(h)_2 \\
 C_{\mathfrak{k}}(e)_3 &= C_{\mathfrak{k}}(h)_3.
 \end{aligned}
 \tag{23}$$

The $\mathfrak{gl}(k) \subset C_{\mathfrak{k}}(e)_0$ is embedded in $\mathfrak{gl}(k) \times \mathfrak{gl}(k) \subset C_{\mathfrak{k}}(h)_0$ as before, $x \mapsto (x, -x^t)$, and $\mathfrak{so}(2r_-) \subset \mathfrak{so}(2r_- + 1)$ in the standard way.

Proposition 3.14. *A representation $V(a_1, \dots, a_p \mid b_1, \dots, b_{q-1})$ has vectors invariant under $C_{\mathfrak{k}}(e)^+$ and transforming according to Det^χ under $C_{\mathfrak{k}}(e)_0 \cong \mathfrak{gl}(k) \times \mathfrak{so}(2r_-)$ if and only if $b_{k+2} = \dots = b_{q-1} = 0$, and*

$$a_1 + \chi \geq b_1 \geq a_2 + \chi \geq \dots \geq b_k \geq a_{k+1} + \chi \geq b_{k+1}.
 \tag{24}$$

Proof. The proof is essentially the same as for the other cases.

Step 1. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} \subset \mathfrak{k}$ be the parabolic subalgebra determined by

$$\xi = (\underbrace{1, \dots, 1}_{k+1} \mid 0, \dots, 0; 0).$$

Then $\mathfrak{n} \subset C_{\mathfrak{k}}(e)^+$, and we can apply Kostant's theorem to reduce the computation to $\mathfrak{m} \cong \mathfrak{gl}(k+1) \times \mathfrak{so}(2k+1+2r_-)$. Let $\mathcal{W}(a_1, \dots, a_{k+1} \mid b_1, \dots, b_k, b_{k+1}, \dots, b_{k+r_-})$ ($q-1 = k+r_-$) be an irreducible representation of \mathfrak{m} parametrized by its highest weight, and χ be a character of $C_{\mathfrak{m}}(e)$ trivial on the nilradical. We will compute

$$[\mathcal{W}(a_1, \dots, a_{k+1} \mid b_1, \dots, b_k, b_{k+1}, \dots, b_{k+r_-})]^\chi.$$

Step 2. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{m}$ be the parabolic subalgebra determined by (the restriction of) h :

$$\mathfrak{l} \cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{gl}(k) \times \mathfrak{so}(2r_- + 1).$$

Then $C_{\mathfrak{m}}(e)_0 \cong \mathfrak{gl}(k) \times \mathfrak{so}(2r_-)$, $C_{\mathfrak{m}}(e)_2 = C_{\mathfrak{m}}(h)_2$ and $C_{\mathfrak{m}}(e)_3 = C_{\mathfrak{m}}(h)_3$. $C_{\mathfrak{m}}(e)_1 \subset C_{\mathfrak{m}}(h)_1$ has complements $\mathfrak{s}_0, \mathfrak{s}_\pm$ spanned by

$$\begin{aligned}
 \mathfrak{s} &= \text{Span}\{X(\epsilon_1 - \epsilon_i), 2 \leq i \leq k+1\}, \\
 \mathfrak{s}_- &= \text{Span}\{X(\epsilon_{p+i} - \epsilon_{p+q-1}), 1 \leq i \leq k\} \\
 \mathfrak{s}_+ &= \text{Span}\{X(\epsilon_{p+i} + \epsilon_{p+q-1}), 1 \leq i \leq k\}.
 \end{aligned}$$

Then $S^m(\mathfrak{s}) = V(m; 0, \dots, 0, -m \mid \underbrace{0, \dots, 0}_k; 0, \underbrace{0, \dots, 0}_{r_- - 1})$ as before. The (generalized)

Bernstein-Gelfand-Gelfand resolution, using $\bar{\mathfrak{q}}$, is

$$0 \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=k}} M(w \cdot \lambda) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=1}} M(w \cdot \lambda) \longrightarrow M(\lambda) \longrightarrow \mathcal{W} \longrightarrow 0,
 \tag{25}$$

with $w \cdot \lambda := w(\lambda + \rho(\mathfrak{m})) - \rho(\mathfrak{m})$, and $w \in W^+$, the $W(\mathfrak{l})$ -coset representatives that make $w \cdot \lambda$ dominant for $\Delta^+(\mathfrak{l})$. This is a free $C_{\mathfrak{m}}(e)^+$ -resolution so we can compute homology by considering

$$0 \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=k}} \overline{M(w \cdot \lambda)} \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{w \in W^+ \\ \ell(w)=1}} \overline{M(w \cdot \lambda)} \longrightarrow \overline{M(\lambda)} \longrightarrow 0, \quad (26)$$

where for an \mathfrak{m} -module X , \overline{X} denotes $X/((C_{\mathfrak{m}}(e)^+)X)$. The weight λ is

$$(-a_1, -a_{k+1}, \dots, -a_k \mid -b_k, \dots, -b_1, b_{k+1}, \dots, b_{q-1}).$$

The fact that $b_{k+2} = \cdots = b_{q-1} = 0$ follows from Helgason’s theorem for the pair $\mathfrak{so}(2r_-) \subset \mathfrak{so}(2r_- + 1)$. The fixed vector is the highest weight. ■

The χ relevant to matching with representations are

<i>Case 4,</i>	$\chi = -1/2,$
<i>Case 5, 6 with $r_+ = 0,$</i>	$\chi = 1/2,$
<i>Case 5, 6 with $r_+ > 0,$</i>	$\chi = r_+ + 1/2$
<i>Case 7, 8,</i>	$\chi = -r_- + 1/2.$

Proposition 3.15. *The \tilde{K} - structure in Cases 4–8 is as follows.*

Case 4: $\mathcal{O} = [3^+ 2^{2k} 1^{-1+,2}]$. In this case $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$. Let

$$\psi_1 = (1/2, \dots, 1/2 \mid 0, \dots, 0)|_{C_{\tilde{K}}(e)}, \quad \psi_2 = (3/2, 1/2, \dots, 1/2 \mid 0, \dots, 0)|_{C_{\tilde{K}}(e)}$$

Then
$$R(\tilde{\mathcal{O}}, Det^{1/2}) = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2),$$

with $\bar{V} = V(\beta_1 + 1/2, \dots, \beta_p + 1/2 \mid \delta_1, \dots, \delta_p)$ and

$$R(\mathcal{O}, \psi_1) = \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_1) = \bigoplus \bar{V} \quad \text{with } \sum(\beta_i + \delta_j) \in 2\mathbb{Z},$$

$$R(\mathcal{O}, \psi_2) = \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_2) = \bigoplus \bar{V} \quad \text{with } \sum(\beta_i + \delta_j) \in 2\mathbb{Z} + 1,$$

satisfying $\beta_1 \geq \delta_1 \geq \dots, \geq \beta_p \geq \delta_p \geq 0$ and $\beta_i, \delta_j \in \mathbb{Z}$.

The automorphism η in (7) relates the result for $R(\mathcal{O}, \psi)$, with $\mathcal{O}^\eta = [3^- 2^{2k} 1^{+1-,2}]$ and the corresponding ψ^η .

Case 5, 6: $\mathcal{O} = [3^+ 2^{2k} 1^+]$. In this case $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$. Let

$$\psi_1 = (0, \dots, 0 \mid 1/2, \dots, 1/2)|_{C_{\tilde{K}}(e)}, \quad \psi_2 = (1/2, \dots, 1/2 \mid 1, \dots, 1)|_{C_{\tilde{K}}(e)}. \quad (27)$$

Then
$$R(\tilde{\mathcal{O}}, Det^{1/2}) = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2)$$

with $\tilde{V} = V(\beta_1, \dots, \beta_p \mid \delta_1, \dots, \delta_{p-1})$ and

$$R(\mathcal{O}, \psi_1) = \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_1) = \bigoplus \tilde{V}, \quad \beta_i \in \mathbb{Z}, \quad \delta_j \in \mathbb{Z} + 1/2,$$

$$R(\mathcal{O}, \psi_2) = \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_2) = \bigoplus \tilde{V}, \quad \beta_i \in \mathbb{Z} + 1/2, \quad \delta_j \in \mathbb{Z},$$

satisfying $\beta_1 + 1/2 \geq \delta_1 \geq \dots, \geq \beta_{p-1} + 1/2 \geq \delta_{p-1} \geq \beta_p + 1/2$.

Case 5, 6: $\mathcal{O} = [3^+2^{2k}1^{+,2r_++1}]$ with $r_+ > 0$. In this case $A_{\tilde{K}}(\mathcal{O}) \cong 1$. Then

$$\begin{aligned} R(\tilde{\mathcal{O}}, Det^{r_++1/2}) &= R(\mathcal{O}, Det^{r_++1/2}) \\ &= \bigoplus V(\beta_1, \dots, \beta_q, 0, \dots, 0 \mid \delta_1, \dots, \delta_{q-1}) \end{aligned}$$

satisfying $\beta_1 + r_+ + 1/2 \geq \delta_1 \geq \dots \geq \beta_{q-1} + 1/2 \geq \delta_{q-1} \geq \beta_q$, $\beta_i \in \mathbb{Z}$, and $\delta_j \in \mathbb{Z} + 1/2$.

Case 7, 8: $\mathcal{O} = [3^+2^{2k}1^{+,2r_-}]$. Let

$$\begin{aligned} \psi_1 &= (r_- - 1/2, \dots, r_- - 1/2 \mid 0, \dots, 0)|_{C_{\tilde{K}}(e)}, \\ \psi_2 &= (r_- + 1/2, r_- - 1/2, \dots, r_- - 1/2 \mid 0, \dots, 0)|_{C_{\tilde{K}}(e)}. \end{aligned}$$

In this case $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$. Then

$$R(\tilde{\mathcal{O}}, Det^{-r_-+1/2}) = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2),$$

with $\hat{V} = V(\beta_1 + r_-1/2, \dots, \beta_p + r_- - 1/2 \mid \delta_1, \dots, \delta_p, 0, \dots, 0)$ and

$$\begin{aligned} R(\mathcal{O}, \psi_1) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_1) = \bigoplus \hat{V} \quad \text{with } \sum (\beta_i + \delta_j) \in 2\mathbb{Z}, \\ R(\mathcal{O}, \psi_2) &= \text{Ind}_{C_{\tilde{K}}(\mathcal{O})}^{\tilde{K}}(\psi_2) = \bigoplus \hat{V} \quad \text{with } \sum (\beta_i + \delta_j) \in 2\mathbb{Z} + 1, \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq \delta_p \geq 0$ and $\beta_i, \delta_j \in \mathbb{Z}$.

Proof. The calculations of $R(\tilde{\mathcal{O}}, Det^x)$ are essentially the same for all Cases 4, 5, 6 and 7, 8. The calculations of the $R(\mathcal{O}, \psi)$ are different. In Cases 4 and 7, 8 the disconnectedness of the centralizer is already present for $K = SO(2p + 1, 2q - 1)$. More precisely, $A_{\tilde{K}}(\mathcal{O}) = A_K(\mathcal{O})$ is nontrivial. In Cases 5, 6 with $r_- = 0$ and $A_{\tilde{K}}(\mathcal{O}) \neq A_K(\mathcal{O})$. Finally in Cases 5, 6 with $r_- > 0$, $A_{\tilde{K}}(\mathcal{O}) = 1$, and there is nothing further to prove.

For $R(\tilde{\mathcal{O}}, Det^x)$ in Cases 5, 6, we give details for $\mathcal{O} = [3^+2^{2k}1^+]$. Then

$$e = X(\epsilon_1)_n + \sum_{2 \leq j \leq k+1} X(\epsilon_j + \epsilon_{p+j-1}), \quad h = (2, \underbrace{1, \dots, 1}_k \mid \underbrace{1, \dots, 1}_k; 0).$$

The centralizers $C_{\mathfrak{k}}(h)$ and $C_{\mathfrak{k}}(e)$ are as in (22) and (23) with $r_- = 0$.

Let $V(a_1, \dots, a_{k+1} \mid b_1, \dots, b_k)$ be a \tilde{K} -type. Then Steps 1 and 2 imply that V has a $C_{\mathfrak{k}}(e)^+$ -fixed vector transforming according to Det^x if and only if

$$a_1 \geq b_1 + \chi \geq \dots \geq b_k + \chi \geq a_{k+1}.$$

The genuine \tilde{K} -types must satisfy $a_i \in \mathbb{Z}, b_j \in \mathbb{Z} + \frac{1}{2}$ or $a_i \in \mathbb{Z} + \frac{1}{2}, b_j \in \mathbb{Z}$ and χ a half-integer. The case $\chi = -1/2$ is relevant to the representations.

The element $(-I, -I)$ acts by -1 on the representation. The two elements $(I, -I)$ and $(-I, I)$ therefore act by opposite signs. Then

$$R(\tilde{\mathcal{O}}, Det^x) = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2),$$

where $\psi_1 = (0, \dots, 0 \mid 1/2, \dots, 1/2)|_{C_{\tilde{K}}(e)}$, $\psi_2 = (1/2, \dots, 1/2 \mid 1, \dots, 1)|_{C_{\tilde{K}}(e)}$, and where ψ_1, ψ_2 are again the corresponding \tilde{K} -types restricting to $C_{\tilde{K}}(e)$. This coincides with the result in the statement if replacing $+$ by $-$.

For Cases 4 and 7,8 we give details for $R(\tilde{\mathcal{O}}, Det^X)$ with $\mathcal{O} = [3^+ 2^{2k} 1^{-1+}, 2]$ in Case 4. Other cases are similar. The component group satisfies $A_{\tilde{K}}(\mathcal{O}) = A_K(\mathcal{O}) \cong \mathbb{Z}_2$. We use the realization

$$\begin{aligned}
 e &= X(\epsilon_1 - \epsilon_{p+k+1}) + X(\epsilon_1 + \epsilon_{p+k+1}) + \sum_{2 \leq i \leq k+1} X(\epsilon_i + \epsilon_{p+i-1}) \\
 h &= 2H(\epsilon_1) + \sum_{2 \leq i \leq k+1} H(\epsilon_i + \epsilon_{p-1+i}) = (2, \underbrace{1, \dots, 1}_k \mid \underbrace{1, \dots, 1}_k, 0; 0).
 \end{aligned}
 \tag{28}$$

$$\begin{aligned}
 C_{\mathfrak{t}}(h)_0 &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{gl}(k) \times \mathfrak{so}(3), \\
 C_{\mathfrak{t}}(h)_1 &= \text{Span}\{X(\epsilon_1 - \epsilon_i), X(\epsilon_{p+i-1} \pm \epsilon_{p+k+1}), 2 \leq i \leq k+1, \\
 &\quad X(\epsilon_i)_c, X(\epsilon_{p+i-1})_c, 2 \leq i \leq k+1\}, \\
 C_{\mathfrak{t}}(h)_2 &= \text{Span}\{X(\epsilon_1)_c, X(\epsilon_i + \epsilon_j), 2 \leq i < j \leq k+1, \\
 &\quad X(\epsilon_{p+i-1} + \epsilon_{p+j-1}), 2 \leq i < j \leq k+1\}, \\
 C_{\mathfrak{t}}(h)_3 &= \text{Span}\{X(\epsilon_1 + \epsilon_i), 2 \leq i \leq k+1\}.
 \end{aligned}
 \tag{29}$$

Similarly

$$\begin{aligned}
 C_{\mathfrak{t}}(e)_0 &\cong \mathfrak{gl}(1) \times \mathfrak{gl}(k) \times \mathfrak{so}(2), \\
 C_{\mathfrak{t}}(e)_1 &= \text{Span}\{X(\epsilon_1 - \epsilon_i) + X(\epsilon_{p+i-1} \pm \epsilon_{p+k+1}), 2 \leq i \leq k+1, \\
 &\quad X(\epsilon_i)_c, X(\epsilon_{p+i-1})_c, 2 \leq i \leq k+1\}, \\
 C_{\mathfrak{t}}(e)_2 &= C_{\mathfrak{t}}(h)_2, \\
 C_{\mathfrak{t}}(e)_3 &= C_{\mathfrak{t}}(h)_3.
 \end{aligned}
 \tag{30}$$

The $\mathfrak{gl}(k) \subset C_{\mathfrak{t}}(e)_0$ is embedded in $\mathfrak{gl}(k) \times \mathfrak{gl}(k) \subset C_{\mathfrak{t}}(h)_0$ as before, $x \mapsto (x, -x^t)$. The element $e^{\pi i H(\epsilon_1 \pm \epsilon_6)}$ represents the nontrivial element in the component group. The vector in $V(a_1, \dots, a_{k+1} \mid b_1, \dots, b_k, b_{k+1})$ which is $C_{\mathfrak{t}}(e)^+$ -invariant and transforms according to Det^X , has weight

$$(-a_1 + k\chi - \sum_{2 \leq i \leq k+1} a_i + \sum_{1 \leq j \leq k} b_j, -b_2 + \chi, \dots, -b_k + \chi \mid -b_1, \dots, -b_k, b_{k+1}).$$

The nontrivial element of $A_{\tilde{K}}(\mathcal{O})$ acts by

$$e^{\pi i (\sum_{1 \leq i \leq k+1} a_i + \sum_{1 \leq j \leq k+1} b_j + k\chi)},$$

and has different values according to the *parity* of the sum in the exponent. This accounts for the decomposition

$$R(\tilde{\mathcal{O}}, Det^X) = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2).$$

■

4. Representations

We will obtain representations associated to the various \mathcal{O} by restricting the representations of $(\mathfrak{g}' = \mathfrak{so}(p', q'), \widetilde{K}' = Spin(p') \times Spin(q'))$ constructed in [17]. They are unitary, associated to the orbit $\mathcal{O}' = [2^{2k+2} 1^{2n-4k-3}]$, and have infinitesimal character

$$\lambda' = (n - k - 1 - 1/2, \dots, 1/2; k + 1, \dots, 1).$$

We recall their \widetilde{K}' -spectrum from [17]. Let $\widetilde{G}' = \widetilde{Spin}(p', q')$ be such that p' is odd and q' even.

$p' - 1 = q'$: There are four representations

$$\begin{aligned} & (\lambda_1, \dots, \lambda_{q'/2} \mid \lambda_1 + 1/2, \dots, \lambda_{q'/2} + 1/2), & \lambda_i \in \mathbb{Z}, \\ & (\lambda_1, \dots, \lambda_{q'/2} \mid \lambda_1 + 1/2, \dots, -\lambda_{q'/2} - 1/2), & \lambda_i \in \mathbb{Z}, \\ & (\lambda_1, \dots, \lambda_{q'/2} \mid \lambda_1 + 1/2, \dots, \lambda_{q'/2} + 1/2), & \lambda_i \in \mathbb{Z} + 1/2, \\ & (\lambda_1, \dots, \lambda_{q'/2} \mid \lambda_1 + 1/2, \dots, -\lambda_{q'/2} - 1/2), & \lambda_i \in \mathbb{Z} + 1/2. \end{aligned}$$

$p' - 1 > q'$: There are two representations,

$$\begin{aligned} & \left(\lambda_1, \dots, \lambda_{q'/2}, 0, \dots, 0 \mid \lambda_1 + \frac{p' - q'}{2}, \dots, \lambda_{q'/2} + \frac{p' - q'}{2} \right), \\ & \left(\lambda_1, \dots, \lambda_{q'/2}, 0, \dots, 0 \mid \lambda_1 + \frac{p' - q'}{2}, \dots, -\lambda_{q'/2} - \frac{p' - q'}{2} \right). \end{aligned}$$

$p' - 1 < q'$: One representation,

$$\left(\lambda_1 + \frac{q' - p'}{2}, \dots, \lambda_{(p'-1)/2} + \frac{q' - p'}{2} \mid \lambda_1, \dots, \lambda_{(p'-1)/2}, 0, \dots, 0 \right).$$

Theorem 4.1. *The representations attached to \mathcal{O} have the following \widetilde{K} -structure.*

Case 1. $\widetilde{G}_0 = \widetilde{Spin}(2p, 2p)$, $2p = 2k + 2$, $(r_+ = 0)$

There are eight representations obtained by restriction from $\widetilde{Spin}(2p + 1, 2p)$, with \widetilde{K} -structure:

$$\begin{aligned} \pi_3 : & (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \beta_p + 1/2) & \text{with } \sum (\delta_i + \beta_j) \in 2\mathbb{Z}, \\ \pi_4 : & (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \beta_p + 1/2) & \text{with } \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1, \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z}$;

$$\begin{aligned} \tau_3 : & (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, -\beta_p - 1/2) & \text{with } \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1, \\ \tau_4 : & (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, -\beta_p - 1/2) & \text{with } \sum (\delta_i + \beta_j) \in 2\mathbb{Z}, \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z}$;

$$\begin{aligned} \sigma_1 : & (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \beta_p + 1/2) & \text{with } \sum (\delta_i + \beta_j) \in 2\mathbb{Z}, \\ \sigma_2 : & (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \beta_p + 1/2) & \text{with } \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1, \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z} + 1/2$;

$$\xi_1 : (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, -\beta_p - 1/2) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1,$$

$$\xi_2 : (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, -\beta_p - 1/2) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z},$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z} + 1/2$.

Another eight representations are obtained by restriction from $\widetilde{Spin}(2p, 2p + 1)$, with \widetilde{K} -structure:

$$\pi_1 : (\beta_1 + 1/2, \dots, \beta_p + 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z},$$

$$\pi_2 : (\beta_1 + 1/2, \dots, \beta_p + 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1,$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z}$;

$$\tau_1 : (\beta_1 + 1/2, \dots, -\beta_p - 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1,$$

$$\tau_2 : (\beta_1 + 1/2, \dots, -\beta_p - 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z},$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z}$;

$$\sigma_3 : (\beta_1 + 1/2, \dots, \beta_p + 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z},$$

$$\sigma_4 : (\beta_1 + 1/2, \dots, \beta_p + 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1,$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z} + 1/2$;

$$\xi_3 : (\beta_1 + 1/2, \dots, -\beta_p - 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z} + 1,$$

$$\xi_4 : (\beta_1 + 1/2, \dots, -\beta_p - 1/2 \mid \delta_1, \dots, \delta_p) \quad \text{with} \quad \sum (\delta_i + \beta_j) \in 2\mathbb{Z},$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$, $\beta_i, \delta_j \in \mathbb{Z} + 1/2$.

The representations with the same subscripts have the same central character.

Case 2. $\widetilde{G}_0 = \widetilde{Spin}(2p, 2q)$, $2p = 2k + 2 + 2r_+$, $2q = 2k + 2$, $r_+ = p - q > 0$, (Case 3 is corresponding to Case 2 with $+$ replaced by $-$.)

There are four representations obtained by restriction from $\widetilde{Spin}(2p + 1, 2q)$, with \widetilde{K} -structure, using $S := \sum (\delta_i + \beta_j)$:

$$\pi_1 : (\delta_1, \dots, \delta_q, 0, \dots, 0 \mid \beta_1 + r_+ + 1/2, \dots, \beta_q + r_+ + 1/2) \quad \text{with} \quad S \in 2\mathbb{Z},$$

$$\pi_2 : (\delta_1, \dots, \delta_q, 0, \dots, 0 \mid \beta_1 + r_+ + 1/2, \dots, \beta_q + r_+ + 1/2) \quad \text{with} \quad S \in 2\mathbb{Z} + 1,$$

$$\sigma_1 : (\delta_1, \dots, \delta_q, 0, \dots, 0 \mid \beta_1 + r_+ + 1/2, \dots, -(\beta_q + r_+ + 1/2)) \quad \text{with} \quad S \in 2\mathbb{Z} + 1,$$

$$\sigma_2 : (\delta_1, \dots, \delta_q, 0, \dots, 0 \mid \beta_1 + r_+ + 1/2, \dots, -(\beta_q + r_+ + 1/2)) \quad \text{with} \quad S \in 2\mathbb{Z},$$

satisfying $\beta_1 \geq \delta_1 \geq \beta_1 \geq \dots \geq \beta_q \geq \delta_q \geq 0$, $\beta_i, \delta_j \in \mathbb{Z}$.

There are two representations obtained by restriction from $\widetilde{Spin}(2p, 2q + 1)$, with \widetilde{K} -structure:

$$\tau_1 : (\beta_1, \dots, \beta_q, 0, \dots, 0 \mid \delta_1, \dots, \delta_q) \quad \text{with} \quad S \in 2\mathbb{Z},$$

$$\tau_2 : (\beta_1, \dots, \beta_q, 0, \dots, 0 \mid \delta_1, \dots, \delta_q) \quad \text{with} \quad S \in 2\mathbb{Z} + 1,$$

satisfying $\beta_1 + r_+ - 1/2 \geq \delta_1 \geq \dots \geq \beta_q + r_+ - 1/2 \geq |\delta_q|$, $\beta_i \in \mathbb{Z}, \delta_j \in \mathbb{Z} + 1/2$.

The representations π_i, σ_i, τ_i have the same central character for $i = 1, 2$.

Case 4. $\widetilde{G}_0 = \widetilde{Spin}(2p + 1, 2p + 1)$, $2p + 1 = 2k + 1$, $r_+ = 1$ or $r_- = 1$.

There is one representation (which may decompose further) obtained by restriction from $\widetilde{Spin}(2p + 2, 2p + 1)$, with \widetilde{K} -structure

$$\pi_1 : (\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \beta_p + 1/2)$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq \delta_p \geq 0$, and $\beta_i, \delta_j \in \mathbb{Z}$.

There is another representation (which may decompose further) obtained by restriction from $\widetilde{Spin}(2p + 1, 2p + 2)$, with \widetilde{K} -structure

$$\pi_2 : (\beta_1 + 1/2, \dots, \beta_p + 1/2 \mid \delta_1, \dots, \delta_p)$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq \delta_p \geq 0$, and $\beta_i, \delta_j \in \mathbb{Z}$. The representations π_1 and π_2 have different central characters.

Case 5. $\widetilde{G}_0 = \widetilde{Spin}(2p + 1, 2q - 1)$, $2p + 1 = 2k + 3 + 2r_+$, $2q - 1 = 2k + 1$, $(r_+ = p - q)$. (Case 6 is corresponding to Case 5 with $+$ replaced by $-$.)

When $r_+ = 0$, there are two representations obtained by restriction from the group $\widetilde{Spin}(2p + 1, 2p)$, with K -structure:

$$\begin{aligned} \pi_1 &: (\beta_1, \dots, \beta_p \mid \delta_1, \dots, \delta_{p-1}) \quad \text{with } \beta_i \in \mathbb{Z}, \delta_j \in \mathbb{Z} + 1/2 \\ \pi_2 &: (\beta_1, \dots, \beta_p \mid \delta_1, \dots, \delta_{p-1}) \quad \text{with } \beta_i \in \mathbb{Z} + 1/2, \delta_j \in \mathbb{Z}, \end{aligned}$$

satisfying $\beta_1 + 1/2 \geq \delta_1 \geq \dots \geq \beta_{p-1} + 1/2 \geq \delta_{p-1} \geq \beta_p + 1/2$. The representations π_1 and π_2 have different central characters.

When $r_+ > 0$, there is a representation obtained by restriction from the group $\widetilde{Spin}(2p + 1, 2q)$, with K -structure:

$$\pi : (\beta_1, \dots, \beta_q, \underbrace{0, \dots, 0}_{p-q} \mid \delta_1, \dots, \delta_{q-1})$$

satisfying $\beta_1 + r_+ + 1/2 \geq \delta_1 \geq \dots \geq \beta_{q-1} + r_+ + 1/2 \geq \delta_{q-1} \geq \beta_q + r_+ + 1/2$, and $\beta_i \in \mathbb{Z}, \delta_j \in \mathbb{Z} + 1/2$.

Case 7. $\widetilde{G}_0 = \widetilde{Spin}(2p + 1, 2q - 1)$, $2p + 1 = 2k + 1 + 2r_+$, $2q - 1 = 2k + 3$, $(r_+ = p - q + 2)$ (Case 8 is corresponding to Case 7 with $+$ replaced by $-$.)

There is one representation (which may decompose further) obtained by restriction from $\widetilde{Spin}(2p + 2, 2q - 1)$, with \widetilde{K} -structure

$$\pi : (\delta_1, \dots, \delta_{q-1}, \underbrace{0, \dots, 0}_{p-q+1} \mid \beta_1 + r_+ - 1/2, \dots, \beta_{q-1} + r_+ - 1/2)$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_{q-1} \geq \delta_{q-1} \geq 0$, and $\beta_i, \delta_j \in \mathbb{Z}$.

4.1. Proof of Theorem 4.1

Case 1. Let $p' = 2p + 1, q' = 2p$, so $p' - 1 = q'$, and $\frac{p'-q'}{2} = \frac{1}{2}$. The restrictions of the four representations of $\widetilde{Spin}(2p + 1, 2p)$ are:

$$\begin{aligned} &(\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \pm(\beta_p + 1/2)), \quad \beta_i, \delta_j \in \mathbb{Z}, \\ &(\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \pm(\beta_p + 1/2)), \quad \beta_i, \delta_j \in \mathbb{Z} + 1/2, \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \dots \geq \beta_p \geq |\delta_p|$. Similarly we get another four representations by restricting from $\widetilde{Spin}(2p, 2p + 1)$.

The center of $\widetilde{Spin}(2p, 2p)$ does not act by a scalar, so these representations decompose further into the sixteen listed in the theorem. Also, the highest weights of the \tilde{K} -types of an irreducible representation must differ by the root lattice.

Case 2, 3. We consider $a = 2p = 2k + 2 + 2r_+, b = 2q = 2k + 2, r_+ = p - q > 0$ only. Let $p' = 2p + 1, q' = 2q$. This is the case $p' - 1 > q'$, and so $\frac{p'-q'}{2} = r_+ + 1/2$. The restrictions of the two representations of $\widetilde{Spin}(p', q')$ are

$$\begin{aligned} &(\delta_1, \dots, \delta_q, 0, \dots, 0 \mid \beta_1 + r_+ + 1/2, \dots, \beta_q + r_+ + 1/2) \\ &(\delta_1, \dots, \delta_q, 0, \dots, 0 \mid \beta_1 + r_+ + 1/2, \dots, -(\beta_q + r_+ + 1/2)) \end{aligned}$$

satisfying $\beta_1 \geq \delta_1 \geq \beta_1 \geq \dots \geq \beta_q \geq \delta_q \geq 0, \beta_i, \delta_j \in \mathbb{Z}$.

Let $p' = 2k + 3 = 2q + 1, q' = 2k + 2 + 2r_+ = 2p$. This is the case $p' - 1 < q'$, and $\frac{q'-p'}{2} = r_+ - 1/2$. The restriction of the single representation of $\widetilde{Spin}(p', q')$ is

$$(\beta_1, \dots, \beta_q, 0, \dots, 0 \mid \delta_1, \dots, \delta_q)$$

satisfying $\beta_1 + r_+ - 1/2 \geq \delta_1 \geq \dots \geq \beta_q + r_+ - 1/2 \geq |\delta_q|, \beta_i \in \mathbb{Z}, \delta_j \in \mathbb{Z} + 1/2$.

The center of $\widetilde{Spin}(2p, 2q)$ does not act by a scalar, so these representations decompose further into the six listed in the theorem. Also, the highest weights of the \tilde{K} -types of an irreducible representation must differ by the root lattice.

Case 4. Thus $a = 2p + 1 = 2k + 1$ and $b = 2q - 1 = 2k + 1$. Let $p' = 2p + 2$ and $q' = 2p + 1$. There are two representations, and they restrict to the same

$$(\delta_1, \dots, \delta_p \mid \beta_1 + 1/2, \dots, \beta_p + 1/2)$$

satisfying $\beta_1 \geq \delta_1 \dots \beta_p \geq \delta_p \geq 0$.

Similarly for $p' = 2p + 1$ and $q' = 2p + 2$. These representations decompose further, not detected by the action of the center; see Conjecture 4.2 and the introduction. Their \tilde{K} -structure differs by whether $\sum \delta_i + \sum \beta_j$ is in the root lattice or not. We write $\pi = \pi^e + \pi^o$.

Case 5, 6. We consider $a = 2p + 1 = 2k + 3 + 2r_+, b = 2q - 1 = 2k + 1, r_+ = p - q \geq 0$ only. Let $p' = 2p + 1 = 2k + 3 + 2r_+, q' = 2q = 2k + 2$. When $r_+ > 0, p' - 1 > q'$ and $\frac{p'-q'}{2} = r_+ + 1/2$. The restrictions of the two representations that occur for $\widetilde{Spin}(p', q')$ coincide:

$$(\beta_1, \dots, \beta_q, \underbrace{0, \dots, 0}_{p-q} \mid \delta_1, \dots, \delta_{q-1})$$

such that $\beta_1 + r_+ + 1/2 \geq \delta_1 \geq \dots \geq \beta_{q-1} + r_+ + 1/2 \geq \delta_{q-1} \geq \beta_q + r_+ + 1/2$, and $\beta_i \in \mathbb{Z}, \delta_j \in \mathbb{Z} + 1/2$.

The case when $r_+ = 0$ satisfies $p' - 1 = q'$. In addition to the representation above, there are two more representations. Their restriction has \widetilde{K} -structure

$$(\beta_1, \dots, \beta_p \mid \delta_1, \dots, \delta_{p-1})$$

satisfying $\beta_1 + 1/2 \geq \delta_1 \geq \beta_2 + 1/2 \geq \dots \geq \delta_{p-1} \geq \beta_p + 1/2$.

Case 7, 8. We consider $a = 2p + 1 = 2k + 1 + 2r_+$, $b = 2q - 1 = 2k + 3$, $r_+ = p - q + 2 \geq 0$ only. Let $p' = 2q - 1 = 2k + 3$, $q' = 2p + 2 = 2k + 2 + 2r_+$. In this case, $p' - 1 < q'$, and $\frac{q' - p'}{2} = r_+ - 1/2$. The representation of $\widetilde{Spin}(p', q')$ restricts to

$$(\delta_1, \dots, \delta_{q-1}, \underbrace{0, \dots, 0}_{p-q+1} \mid \beta_1 + r_+ - 1/2, \dots, \beta_{q-1} + r_+ - 1/2)$$

satisfying $\beta_1 \geq \delta_1 \geq \beta_1 \geq \dots \geq \beta_{q-1} \geq \delta_{q-1} \geq 0$, $\beta_i, \delta_j \in \mathbb{Z}$. As in Case 4, this representation decomposes further, not detected by the action of the center; see Conjecture 4.2 and the introduction. We write $\pi = \pi^e + \pi^o$.

Conjecture 4.2. Each representation in Case 4, Cases 7 and 8 decomposes into two irreducible factors; we write $\pi = \pi^e + \pi^o$.

The derived functors construction of the representations verifies this conjecture. Since we have omitted the details of this alternate construction, we list the above as a conjecture.

4.2. Infinitesimal character and restriction. Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}) \subset \mathfrak{g}' = \mathfrak{so}(2n + 1, \mathbb{C})$, and $G = SO(2n, \mathbb{C})$ and $G' = SO(2n + 1, \mathbb{C})$ the corresponding groups sharing a $(\theta$ -stable) Cartan subgroup $H = TA$. Let \mathcal{I}' be the unique maximal primitive ideal with infinitesimal character

$$\lambda' = (n - k - 1 - 1/2, \dots, 3/2, 1/2; k + 1, \dots, 1).$$

There is a unique (\mathfrak{g}', K') -module π' with these properties, and it is spherical unitary. In particular $\pi' = U(\mathfrak{g}')/\mathcal{I}'$. Let π be any module with annihilator \mathcal{I}' . Then \mathfrak{g} acts via the map $X \in \mathfrak{g} \mapsto X \cdot 1 \in U(\mathfrak{g}')/\mathcal{I}'$. Write $\pi' = \pi_0 + \pi_1$ where π_0 is the unique spherical irreducible (\mathfrak{g}, K) -submodule. The image of $U(\mathfrak{g})$ is contained in π_0 . We aim to show that π_0 has infinitesimal character $\lambda = (n - k - 2, \dots, 0; k + 1/2, \dots, 3/2, 1/2)$. Then all the factors of the restriction of π to \mathfrak{g} have this infinitesimal character as well. In particular this is true for the factors in the restrictions of the modules of $\mathfrak{so}(p', q')$ considered in Theorem 4.1.

It is enough to check the action of \mathfrak{g} on the spherical function corresponding to π' . By [10] pages 31-32, its restriction to A is (up to a multiple),

$$\phi' = \frac{\sum_{w \in W(B_n)} \epsilon(w) e^{w\lambda'}}{\prod (e^{\epsilon_i/2} - e^{-\epsilon_i/2}) \cdot \Delta}$$

with $\Delta = \prod_{\alpha \in R(D_n)} (e^{\alpha/2} - e^{-\alpha/2})$ and $R(D_n)$ the standard positive roots for type D_n . The claim follows if we show that the restriction of ϕ' to G is the spherical function

$$\phi = \frac{\sum_{s \in W(D_n)} \epsilon(s) e^{s\lambda}}{\Delta}.$$

The next Lemma completes the proof.

Lemma 4.3.
$$\sum_{s \in W(D_n)} \epsilon(s) e^{s\lambda} \cdot \prod (e^{\epsilon_i/2} - e^{-\epsilon_i/2}) = \sum_{w \in W(B_n)} \epsilon(w) e^{w\lambda'}$$

Proof. Both sides are skew invariant under $W(D_n)$. It is enough to count the occurrences of the dominant regular weights on the right. On the left there are only two such weights, $(n - k - 1/2, n - k - 3/2, \dots, k + 3/2, k + 1, \dots, 3/2, \pm 1/2)$ occurring with opposite signs. On the left, the weights are of the form

$$(n - k - 2, \dots, k + 1, k + 1/2, \dots, 1, 0) + (\pm 1, 2, \dots, \pm 1/2).$$

The parity of the number of $-1/2$ in the weight being added determines the sign. The only weights that give a dominant regular sum are $(1/2, \dots, 1/2, 1/2)$ and $(1/2, \dots, 1/2, -1/2)$. ■

4.3. Matchup between regular sections on orbits and representations.

We match the \tilde{K} -spectra of the representations in Theorem 4.1 and the regular sections on nilpotent orbits computed in Section 3. We do this for Cases 1, 2, 4, 5, 7, and use the notation from Section 3 (with possible change from $-$ to $+$) and Theorem 4.1. The notation χ_i distinguishes different central characters. In each table, the representations in the same row have the same central character; the representations in the same column are attached to the same orbit.

Case 1 (where $k = p - 1$).

	$\mathcal{O} = [3^+ 2^{2k} 1^-]_I$	$\mathcal{O}^\zeta = [3^+ 2^{2k} 1^-]_{II}$	$\mathcal{O}^\eta = [3^- 2^{2k} 1^+]_I$	$\mathcal{O}^{\zeta\eta} = [3^- 2^{2k} 1^+]_{II}$
χ_1	$\pi_1 _{\tilde{K}} = R(\mathcal{O}, \psi_1)$	$\tau_1 _{\tilde{K}} = R(\mathcal{O}^\zeta, \psi_2^\zeta)$	$\sigma_1 _{\tilde{K}} = R(\mathcal{O}^\eta, \psi_3^\eta)$	$\xi_1 _{\tilde{K}} = R(\mathcal{O}^{\zeta\eta}, \psi_4^{\zeta\eta})$
χ_2	$\pi_2 _{\tilde{K}} = R(\mathcal{O}, \psi_2)$	$\tau_2 _{\tilde{K}} = R(\mathcal{O}^\zeta, \psi_1^\zeta)$	$\sigma_2 _{\tilde{K}} = R(\mathcal{O}^\eta, \psi_4^\eta)$	$\xi_2 _{\tilde{K}} = R(\mathcal{O}^{\zeta\eta}, \psi_3^{\zeta\eta})$
χ_3	$\sigma_3 _{\tilde{K}} = R(\mathcal{O}, \psi_3)$	$\xi_3 _{\tilde{K}} = R(\mathcal{O}^\zeta, \psi_4^\zeta)$	$\pi_3 _{\tilde{K}} = R(\mathcal{O}^\eta, \psi_1^\eta)$	$\tau_3 _{\tilde{K}} = R(\mathcal{O}^{\zeta\eta}, \psi_2^{\zeta\eta})$
χ_4	$\sigma_4 _{\tilde{K}} = R(\mathcal{O}, \psi_4)$	$\xi_4 _{\tilde{K}} = R(\mathcal{O}^\zeta, \psi_3^\zeta)$	$\pi_4 _{\tilde{K}} = R(\mathcal{O}^\eta, \psi_2^\eta)$	$\tau_4 _{\tilde{K}} = R(\mathcal{O}^{\zeta\eta}, \psi_1^{\zeta\eta})$

Case 2.

	$\mathcal{O}_I = [3^- 2^{2k} 1^{+, 2r_+ + 1}]_I$	$\mathcal{O}_{II} = \mathcal{O}_I^\zeta = [3^- 2^{2k} 1^{+, 2r_+ + 1}]_{II}$	$\mathcal{O} = [3^+ 2^{2k} 1^- 1^{+, 2r_+}]$
χ_1	$\pi_1 _{\tilde{K}} = R(\mathcal{O}_I, \psi_1)$	$\sigma_1 _{\tilde{K}} = R(\mathcal{O}_{II}, \psi_2^\zeta)$	$\tau_1 _{\tilde{K}} = R(\mathcal{O}, \phi_1)$
χ_2	$\pi_2 _{\tilde{K}} = R(\mathcal{O}_I, \psi_2)$	$\sigma_2 _{\tilde{K}} = R(\mathcal{O}_{II}, \psi_1^\zeta)$	$\tau_2 _{\tilde{K}} = R(\mathcal{O}, \phi_2)$

Case 4.

	$\mathcal{O} = [3^+ 2^{2k} 1^{+1-, 2}]$	$\mathcal{O}^\eta = [3^- 2^{2k} 1^- 1^{+, 2}]$
χ_1	$\pi_1 _{\tilde{K}} = (\pi_1^e + \pi_1^o) _{\tilde{K}} = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2)$	
χ_2	$\pi_2 _{\tilde{K}} = (\pi_2^e + \pi_2^o) _{\tilde{K}} = R(\mathcal{O}^\eta, \psi_1^\eta) + R(\mathcal{O}^\eta, \psi_2^\eta)$	

Case 5 with $r_+ = 0$.

	$\mathcal{O} = [3^+ 2^{2k} 1^+]$
χ_1	$\pi_1 _{\tilde{K}} = R(\mathcal{O}, \psi_1)$
χ_2	$\pi_2 _{\tilde{K}} = R(\mathcal{O}, \psi_2)$

Case 5 with $r_+ > 0$.

$$\frac{\mathcal{O} = [3^+ 2^{2k} 1^{+, 2r_+ + 1}]}{\chi_1 \mid \pi_1|_{\tilde{K}} = R(\mathcal{O}, \text{Det}^{r_+ + 1/2})}$$

Case 7.

$$\frac{\mathcal{O} = [3^- 2^{2k} 1^{-1+, 2r_+}]}{\chi_1 \mid \pi|_{\tilde{K}} = (\pi^e + \pi^o)|_{\tilde{K}} = R(\mathcal{O}, \psi_1) + R(\mathcal{O}, \psi_2)}$$

5. Clifford algebras and Spin groups

Since the main interest is in the case of $Spin(V)$, the simply connected groups of type D , we realize everything in the context of the Clifford algebra.

5.1. Structure Let (V, Q) be a quadratic space of even dimension $2n$, with a basis $\{e_i, f_i\}$ with $1 \leq i \leq n$, satisfying $Q(e_i, f_j) = \delta_{ij}$, $Q(e_i, e_j) = Q(f_i, f_j) = 0$. Occasionally we will replace e_j, f_j by two orthogonal vectors v_j, w_j satisfying $Q(v_j, v_j) = Q(w_j, w_j) = 1$, and orthogonal to the e_i, f_i for $i \neq j$. Precisely they will satisfy $v_j = (e_j + f_j)/\sqrt{2}$ and $w_j = (e_j - f_j)/(i\sqrt{2})$ (where $i := \sqrt{-1}$, not an index). Let $C(V)$ be the Clifford algebra with automorphisms α defined by $\alpha(x_1 \cdots x_r) = (-1)^r x_1 \cdots x_r$ and \star given by $(x_1 \cdots x_r)^\star = (-1)^r x_r \cdots x_1$, subject to the relation $xy + yx = 2Q(x, y)$ for $x, y \in V$. The double cover of $O(V)$ is

$$Pin(V) := \{x \in C(V) \mid x \cdot x^\star = 1, \alpha(x)Vx^\star \subset V\}.$$

The double cover $Spin(V)$ of $SO(V)$ is given by the elements in $Pin(V)$ which are in $C(V)^{even}$, i.e. $Spin(V) := Pin(V) \cap C(V)^{even}$. For $Spin$, α can be suppressed from the notation since it is the identity.

The action of $Pin(V)$ on V is given by $\rho(x)v = \alpha(x)vx^\star$. The element $-I \in SO(V)$ is covered by

$$\pm \mathcal{E}_{2n} = \pm i^{n-1}vw \prod_{1 \leq j \leq n-1} [1 - e_j f_j] = \pm i^n \prod_{1 \leq j \leq n} [1 - e_j f_j]. \tag{31}$$

These elements satisfy $\mathcal{E}_{2n}^2 = \begin{cases} +Id & \text{if } n \in 2\mathbb{Z}, \\ -Id & \text{otherwise.} \end{cases}$

The center of $Spin(V)$ is

$$Z(Spin(V)) = \{\pm I, \pm \mathcal{E}_{2n}\} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n \text{ is even,} \\ \mathbb{Z}_4 & \text{if } n \text{ is odd.} \end{cases}$$

The Lie algebra of $Pin(V)$ as well as $Spin(V)$ is formed of elements of even order ≤ 2 satisfying $x + x^\star = 0$.

The adjoint action is $\text{ad } x(y) = xy - yx$. A Cartan subalgebra and the root vectors corresponding to the usual basis in Weyl normal form are formed of the elements

$$\begin{aligned}
 (1 - e_i f_i)/2 &\longleftrightarrow H(\epsilon_i) \\
 e_i e_j/2 &\longleftrightarrow X(-\epsilon_i - \epsilon_j), \\
 e_i f_j/2 &\longleftrightarrow X(-\epsilon_i + \epsilon_j), \\
 f_i f_j/2 &\longleftrightarrow X(\epsilon_i + \epsilon_j).
 \end{aligned}
 \tag{32}$$

Root Structure. We use $1 \leq i, l \leq p$ and $1 \leq j, m \leq q - 1$ consistently. We give a realization of the Lie algebra for $Spin(2p + 1, 2q - 1)$. The case $Spin(2p, 2q)$, is (essentially) obtained by suppressing the short roots.

Compact	Noncompact
$\mathfrak{t} = \{(1 - e_i f_i), (1 - e_{p+j} f_{p+j})\}$	$\mathfrak{a} = \{v^+ v^-\}$
$H(\epsilon_i), H(\epsilon_{p+j})$	$H(\epsilon_{p+q})$
$f_i v^+, e_i v^+, f_{p+j} v^-, e_{p+j} v^-$	$f_i v^-, e_i v^-, v^+ f_{p+j}, v^+ e_{p+j}$
$X(\epsilon_i)_c, X(-\epsilon_i)_c, X(\epsilon_{p+j})_c, X(-\epsilon_{p+j})_c$	$X(\epsilon_i)_n, X(-\epsilon_i)_n, X(\epsilon_{p+j})_n, X(-\epsilon_{p+j})_n$
$f_i f_l, f_i e_l, e_i e_l, e_i f_l$	$f_i f_{p+j}, f_i e_{p+j}, e_i e_{p+l}, e_i f_{p+l}$
$f_{p+j} f_{p+m}, f_{p+j} e_{p+m}, e_{p+j} f_{p+m}, e_{p+j} e_{p+m}$	
$X(\epsilon_i + \epsilon_l), X(\epsilon_i - \epsilon_l), X(-\epsilon_i - \epsilon_l), X(-\epsilon_i + \epsilon_l)$	$X(\epsilon_i + \epsilon_{p+j}), X(\epsilon_i - \epsilon_{p+j}),$ $X(-\epsilon_i - \epsilon_{p+j}), X(-\epsilon_i + \epsilon_{p+j}),$
$X(\epsilon_{p+j} + \epsilon_{p+m}), X(\epsilon_{p+j} - \epsilon_{p+m}),$ $X(-\epsilon_{p+j} - \epsilon_{p+m}), X(-\epsilon_{p+j} + \epsilon_{p+m}).$	

Nilpotent orbits, complex case. In this case, we write $\tilde{K} = Spin(V) = Spin(2n, \mathbb{C})$, $K = SO(V) = SO(2n, \mathbb{C})$. A nilpotent orbit of an element e will have Jordan blocks denoted by

$$\begin{aligned}
 e_1 &\longrightarrow e_2 \longrightarrow \dots \longrightarrow e_k \longrightarrow v \longrightarrow -f_k \longrightarrow f_{k-1} \longrightarrow -f_{k-2} \longrightarrow \dots \longrightarrow \pm f_1 \longrightarrow 0 \\
 e_1 &\longrightarrow e_2 \longrightarrow \dots \longrightarrow e_{2\ell} \longrightarrow 0 \\
 f_{2\ell} &\longrightarrow -f_{2\ell-1} \longrightarrow \dots \longrightarrow -f_1 \longrightarrow 0
 \end{aligned}
 \tag{33}$$

with the conventions about the e_i, f_j, v as before. There is an even number of odd sized blocks, and any two blocks of equal odd size $2k + 1$ can be replaced by a pair of blocks of the form as the even ones. A realization is given by

$$\frac{1}{2} \left(\sum_{i=1}^{k-1} e_{i+1} f_i + v f_k \right) \text{ for the odd block, and } \frac{1}{2} \left(\sum_{i=1}^{2l-1} e_{i+1} f_i \right) \text{ for the even.}$$

When there are only even blocks, there are two orbits; one block of the form $(\sum_{1 \leq i < \ell-1} e_{i+1} f_i + e_\ell f_{\ell-1})/2$ is replaced by $(\sum_{1 \leq i < \ell-1} e_{i+1} f_i + f_\ell f_{\ell-1})/2$.

The centralizer of e in $\mathfrak{so}(V)$ has Levi component isomorphic to a product of $\mathfrak{so}(r_{2k+1})$ and $\mathfrak{sp}(2r_{2\ell})$ where r_j is the number of blocks of size j . The centralizer of e in $SO(V)$ has Levi component $\prod Sp(2r_{2\ell}) \times S[\prod O(r_{2k+1})]$. For each odd sized block define

$$\mathcal{E}_{2k+1} = i^k v \prod (1 - e_j f_j).
 \tag{34}$$

This is an element in $Pin(V)$, and acts by $-Id$ on the block. Even products of $\pm \mathcal{E}_{2k+1}$ belong to $Spin(V)$, and represent the connected components of $C_{\tilde{K}}(e)$.

Proposition 5.1. *Let m be the number of distinct odd blocks. Then*

$$A_K(\mathcal{O}) \cong \begin{cases} \mathbb{Z}_2^{m-1} & \text{if } m > 0 \\ 1 & \text{if } m = 0. \end{cases}$$

Furthermore,

- (1) *If E has an odd block of size $2k + 1$ with $r_{2k+1} > 1$, then $A_{\tilde{K}}(\mathcal{O}) \cong A_K(\mathcal{O})$.*
- (2) *If all $r_{2k+1} \leq 1$, then there is an exact sequence*

$$1 \longrightarrow \{\pm I\} \longrightarrow A_{\tilde{K}}(\mathcal{O}) \longrightarrow A_K(\mathcal{O}) \longrightarrow 0.$$

Proof. Assume that there is an $r_{2k+1} > 1$. Let

$$\begin{array}{ccccccc} e_1 & \rightarrow & \dots & \rightarrow & e_{2k+1} & \rightarrow & 0 \\ f_{2k+1} & \rightarrow & \dots & \rightarrow & -f_1 & \rightarrow & 0 \end{array}$$

be two of the blocks. In the Clifford algebra this element is

$$e = (e_2 f_1 + \dots + e_{2k+1} f_{2k})/2.$$

The element $\sum_{j=1}^{2k+1} (1 - e_j f_j)$ in the Lie algebra commutes with e . So its exponential

$$\prod \exp(i\theta(1 - e_j f_j)/2) = \prod [\cos \theta/2 + i \sin \theta/2 (1 - e_j f_j)] \tag{35}$$

also commutes with e . At $\theta = 0$, the element in (35) is I ; at $\theta = 2\pi$, it is $-I$. Thus $-I$ is in the connected component of the identity of $A_{\tilde{K}}(\mathcal{O})$ (when $r_{2k+1} > 1$), and therefore $A_{\tilde{K}}(\mathcal{O}) = A_K(\mathcal{O})$.

Assume there are no blocks of odd size. Then $C_K(\mathcal{O}) \cong \prod Sp(r_{2l})$ is simply connected, so $C_{\tilde{K}}(\mathcal{O}) \cong C_K(\mathcal{O}) \times \{\pm I\}$. Therefore $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

Assume there are m distinct odd blocks with $m \in 2\mathbb{Z}_{>0}$ and $r_{2k_1+1} = \dots = r_{2k_m+1} = 1$. In this case,

$$C_K(\mathcal{O}) \cong \prod Sp(r_{2l}) \times S[\underbrace{O(1) \times \dots \times O(1)}_m],$$

and hence $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2^{m-1}$. Even products of $\{\pm \mathcal{E}_{2k_j+1}\}$ are representatives of elements in $A_{\tilde{K}}(\mathcal{O})$. They satisfy

$$\mathcal{E}_{2k+1} \cdot \mathcal{E}_{2\ell+1} = \begin{cases} -\mathcal{E}_{2\ell+1} \cdot \mathcal{E}_{2k+1} & k \neq \ell, \\ (-1)^k I & k = \ell. \end{cases} \quad \blacksquare$$

Corollary 5.2.

- (1) *If $\mathcal{O} = [3 \ 2^{n-2} \ 1]$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\pm \mathcal{E}_3 \cdot \mathcal{E}_1, \pm I\}$.*
- (2) *If $\mathcal{O} = [3 \ 2^{2k} \ 1^{2n-4k-3}]$ with $2n - 4k - 3 > 1$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.*
- (3) *If $\mathcal{O} = [2^n]_{I,II}$ (n even), then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.*
- (4) *If $\mathcal{O} = [2^{2k} \ 1^{2n-4k}]$ with $2k < n$, then $A_{\tilde{K}}(\mathcal{O}) \cong 1$.*

In all cases $C_{\tilde{K}}(\mathcal{O}) = Z(\tilde{K}) \cdot C_{\tilde{K}}(\mathcal{O})^0$.

Nilpotent orbits, real case. Write $V = V^+ \oplus V^-$ a sum of two (complex) spaces, each endowed with a nondegenerate quadratic form. Recall the notation in Section 2. The spaces V^\pm have dimensions a and b . We use bases as in the complex case, e_j^\pm, f_j^\pm , and plus v^\pm when a, b are both odd. In this case, we write $\tilde{K} = Spin(V^+) \times Spin(V^-) = Spin(a, \mathbb{C}) \times Spin(b, \mathbb{C})$, $\tilde{K} = SO(V^+) \times SO(V^-) = SO(a, \mathbb{C}) \times SO(b, \mathbb{C})$. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ for the (complexification of the) Cartan decomposition. Nilpotent orbits of \tilde{K} (as well as K) in \mathfrak{s} are parametrized by signed partitions where the basis elements alternate between V^+ and V^- but otherwise as in Equation (33). The centralizer of e in \mathfrak{k} is a product

$$\prod \mathfrak{sp}(2r_{2\ell}) \times \prod [\mathfrak{so}(r_{2k+1}^+) \times \mathfrak{so}(r_{2k+1}^-)]$$

where $r_{2\ell}$ is the number of blocks of even size 2ℓ , r_{2k+1}^\pm is the number of blocks of odd size $2k + 1$ starting with \pm . We compute the centralizer of e in \tilde{K} , and its component group.

The even sized blocks do not contribute to the component group. They can however be used to deduce that $(-I, -I) \in C_{\tilde{K}}(\mathcal{O})^0$ as in the complex case.

For the odd sized blocks,

$$\begin{aligned} \mathcal{E}_{2k+1}^+ &= i^k (v^+ \prod (1 - e_j^+ f_j^+), \prod (1 - e_j^- f_j^-)) \\ \mathcal{E}_{2k+1}^- &= i^k (\prod (1 - e_j^+ f_j^+), v^- \prod (1 - e_j^- f_j^-)) \end{aligned} \tag{36}$$

Products with an even number of both \pm of such elements give representatives of the component group.

Lemma 5.3.

(1) If $r_{2k+1}^+ > 1$ for some $2k + 1 \geq 1$, then

$$(-I, I) \in C_{\tilde{K}}(\mathcal{O})^0, \text{ if } k \text{ even, and } (I, -I) \in C_{\tilde{K}}(\mathcal{O})^0, \text{ if } k \text{ odd.}$$

Similarly for r_{2k+1}^- with k even and odd interchanged.

(2) If $r_{2\ell} > 1$ for some $2\ell > 0$, then $(-I, -I) \in C_{\tilde{K}}(\mathcal{O})^0$.

Proof. Assume that $r_{2k+1}^+ > 0$.

$$\begin{array}{ccccccc} e_1^+ & \rightarrow & e_2^- & \rightarrow & \dots & \rightarrow & e_{2k+1}^+ & \rightarrow & 0 \\ f_{2k+1}^+ & \rightarrow & -f_{2k}^- & \rightarrow & \dots & \rightarrow & -f_1^+ & \rightarrow & 0 \end{array}$$

represent two equal size blocks starting with the same sign $+$. The corresponding element in the Clifford algebra is $e = (e_2^- f_1^+ + \dots + e_{2k+1}^+ f_{2k}^-)/2$. Similar to the case $r_{2\ell} > 0$ below,

$$\left(\prod_{j=0}^k [\cos \theta + i \sin \theta (1 - e_{2j+1}^+ f_{2j+1}^+)], \prod_{j=0}^k [\cos \theta + i \sin \theta (1 - e_{2j}^- f_{2j}^-)] \right)$$

belongs to $Spin(V^+) \times Spin(V^-)$ and centralizes e for all θ . This gives a continuous path between (I, I) and $(I, -I)$ when k is odd; and a continuous path between (I, I) and $(-I, I)$ when k is even.

Assume that some $r_{2\ell} > 1$. Let

$$\begin{aligned} e_1^+ &\rightarrow e_2^- &\rightarrow \dots &\rightarrow e_{2\ell}^- &\rightarrow 0 \\ f_{2\ell}^- &\rightarrow -f_{2\ell-1}^+ &\rightarrow \dots &\rightarrow -f_1^+ &\rightarrow 0 \end{aligned}$$

represent two blocks of size 2ℓ . Again, let $e = (e_2^- f_1^+ + e_3^+ f_2^- + \dots + e_{2\ell}^- f_{2\ell-1}^+)/2$ be a representative. Then e is centralized by $[(1 - e_1^+ f_1^+) + (1 - e_2^- f_2^-) + \dots + (1 - e_{2\ell}^- f_{2\ell}^-)]/2$ in the Lie algebra. Exponentiating,

$$\left(\prod_{j=1}^{\ell} [\cos \theta/2 + i \sin \theta/2 (1 - e_{2j-1}^+ f_{2j-1}^+)], \prod_{j=1}^{\ell} [\cos \theta/2 + i \sin \theta/2 (1 - e_{2j}^- f_{2j}^-)] \right)$$

belongs to $Spin(V^+) \times Spin(V^-)$ and centralizes e for all θ . This is (I, I) when $\theta = 0$, and is $(-I, -I)$ when $\theta = \pi$, and hence it gives a continuous path between (I, I) and $(-I, -I)$. ■

We apply the Lemma to the orbit of the diagram $[3 \ 2^{2k} \ 1^{2n-4k-3}]$ with $k > 0$.

Proposition 5.4 (Proposition 2.1).

- (1) If $\mathcal{O} = [3^+ 2^{2k} 1^-]_{I,II}$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
- (2) If $\mathcal{O} = [3^+ 2^{2k} 1^+]$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.
- (3) If $\mathcal{O} = [3^+ 2^{2k} 1^{+,2r_++1}]$ with $r_+ > 0$, then $A_{\tilde{K}}(\mathcal{O}) = 1$.
- (4) If $\mathcal{O} = [3^+ 2^{2k} 1^- 1^{+,2r_+}]$, with $r_+ > 0$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.
- (5) If $\mathcal{O} = [3^- 2^{2k} 1^{+,2r_++1}]$, with $r_+ > 0$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.
- (6) If $\mathcal{O} = [3^- 2^{2k} 1^- 1^{+,2r_+}]$, with $r_+ \geq 2$, then $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

Similarly for the nilpotent orbits with the $+$ and $-$ interchanged.

Proof. In case (1), the odd blocks can be represented by

$$\begin{aligned} e_1^+ &\longrightarrow v_2^- \longrightarrow -f_1^+ \longrightarrow 0, \\ w_2^- &\longrightarrow 0. \end{aligned}$$

The corresponding element in the Clifford algebra is $e = (v_2^- f_1^+)/2$. The element $(i(1 - e_1^+ f_1^+), v_2^- w_2^-) = (i(1 - e_1^+ f_1^+), i(1 - e_2^- f_2^-))$ is in $Spin(V^+) \times Spin(V^-)$, acts by $-Id$ on the blocks and centralizes e .

Note that $A_K(\mathcal{O}) \cong \mathbb{Z}_2$. The inverse image in $C_{\tilde{K}}(\mathcal{O})$ of $C_K(\mathcal{O})$ contains the element $\{(\pm I, \pm I), (\pm i(1 - e_1^+ f_1^+), \pm i(1 - e_2^- f_2^-))\}$. By Lemma 5.3(2) we have $(-I, -I) \in C_{\tilde{K}}(\mathcal{O})^0$, so $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

In case (2), there is only one orbit with this signed partition. The odd blocks are represented by

$$\begin{aligned} e_1^+ &\longrightarrow v_3^- \longrightarrow -f_1^+ \longrightarrow 0, \\ v_2^+ &\longrightarrow 0. \end{aligned}$$

The corresponding element in the Clifford algebra is $e = (v_3^- f_1^+)/2$. The element $(iv_2^+(1 - e_1^+ f_1^+), v_1^-)$ acts by $-Id$ on the blocks and centralizes e , but is in $Pin(V^+) \times Pin(V^-)$, so cannot contribute to the centralizer $C_{\tilde{K}}(e)$. As in the

previous case, (I, I) and $(-I, -I)$ are in the same connected component. Thus $A_{\tilde{K}}(\mathcal{O}) \cong \mathbb{Z}_2$.

In cases (3)–(6), Lemma 5.3 implies $(\pm I, \pm I) \in C_{\tilde{K}}(\mathcal{O})^0$. So $A_{\tilde{K}}(\mathcal{O}) = A_K(\mathcal{O})$. ■

Remark 5.5. In Proposition 2.1, the generators of $A_{\tilde{K}}(\mathcal{O})$ can be chosen as follows: (1) $(-I, -I)$, $(i(1 - e_1^+ f_1^+), i(1 - e_2^- f_2^-))$; (2) $(I, -I)$; (3) (I, I) ; (4) $(i(1 - e_1^+ f_1^+), i(1 - e_2^- f_2^-))$; (5) $(i(1 - e_1^+ f_1^+), i(1 - e_2^- f_2^-))$; (6) $(i(1 - e_1^+ f_1^+), i(1 - e_2^- f_2^-))$. Furthermore, in cases (1), (2), (4), (5), nontrivial representatives of $A_{\tilde{K}}(\mathcal{O})$ can be chosen to be elements in $Z(\tilde{K})$.

Example 5.6. Let $\mathcal{O} = [3^{-2} 2^2 1^{-1} 1^{+2}]$, i.e. $k = 1$, $r_+ = 1$. Then $A_{\tilde{K}}(\mathcal{O}) \cong A_K(\mathcal{O})$ has two connected components. The Jordan blocks are

$$\begin{aligned} e_1^- &\longrightarrow v^+ \longrightarrow -f_1^- \longrightarrow 0 \\ e_2^+ &\longrightarrow e_2^- \longrightarrow 0 \\ f_2^- &\longrightarrow -f_2^+ \longrightarrow 0 \\ v^- &\longrightarrow 0 \\ v_1^+ &\longrightarrow 0 \\ w_1^+ &\longrightarrow 0. \end{aligned}$$

The elements of group $C_{\tilde{K}}(\mathcal{O})$ can be written as products of three factors

$$\begin{aligned} &(\pm v^+ v_1^+, \pm(1 - e_1^- f_1^-)) \cdot \\ &\left(\cos \frac{\theta_1}{2} + i \sin \frac{\theta_1}{2} v_1^+ w_1^+, I \right) \cdot \left(\cos \frac{\theta_2}{2} (I, I) + i \sin \frac{\theta_2}{2} (1 - e_2^+ f_2^+, 1 - e_2^- f_2^-) \right). \end{aligned}$$

6. Counting representations

In this section, we will count the number of representations attached to the complex nilpotent orbit of the form $\mathcal{O}_c = [3 \ 2^{2k} \ 1^{2n-4k-3}]$, with $k > 0$. This orbit is not special in the sense of Lusztig. The infinitesimal character cannot be integral. Considerations coming from primitive ideals imply that the infinitesimal character must be regular, and have integrality given by the system $D_{k+1} \times D_{n-k-1}$. The infinitesimal character with minimal length satisfying the above conditions, and corresponding to \mathcal{O}_c must be

$$\lambda = (n - k - 2, \dots, 1, 0; k + 1/2, \dots, 3/2, 1/2) \tag{37}$$

with $k + 1 \leq n - k - 1$, and hence $0 < k \leq n/2 - 1$. These are the infinitesimal characters which (conjecturally) admit unitary unipotent representations attached to \mathcal{O}_c .

The setting is as in the previous sections. For convenience, in this section we use a slightly different notation. We write $(\tilde{G}, \tilde{K}) = (\widetilde{Spin}(c, d), Spin(c) \times Spin(d))$ with $2n = c + d$, and

$$c = 2p, \quad c = 2p + 1, \quad d = 2q, \quad d = 2q + 1.$$

We assume that $c \geq d$ in this section. We will classify all groups \tilde{G} that admit an admissible representation with infinitesimal character λ . It turns out that Cases 1, 2, 4, 5, 7 in Section 2 cover such groups. Let $n_{\mathcal{O}} := |\mathcal{U}_{\tilde{G}}(\mathcal{O}_c, \lambda)|$ be the number of unipotent representations of \tilde{G} attached to \mathcal{O}_c and λ . We ultimately calculate $n_{\mathcal{O}}$ for each case.

Before getting further, we need some structure theory.

6.1. Cartan subalgebras of \mathfrak{g}_0 . Conjugacy classes of Cartan subalgebras have the following representatives, with the given θ :

$$\mathfrak{h}^{r^+, r^-, m, s} = \{(x_1^+, \dots, x_{r^+}^+, x_1^-, \dots, x_{r^-}^-, y_1, \dots, y_m, y_{m+1}, \dots, y_{2m}, z_1, \dots, z_s)\},$$

$$\theta(x_i^\pm) = x_i^\pm, \theta(y_j) = y_{j+m}, \theta(z_k) = -z_k.$$

When $\mathfrak{g}_0 = \mathfrak{so}(2p, 2q)$, s is even, when $\mathfrak{g}_0 = \mathfrak{so}(2p+1, 2q+1)$, s is odd. We write $s = 2s' + \epsilon$, where $\epsilon = 0$ when n is even, and $\epsilon = 1$ when n is odd. Furthermore, $m + r^+ + s' = p$ and $m + r^- + s' = q$. The orthogonal space $V = V^+ \oplus V^-$ has basis

$$e_i, v^+, f_i, \quad e_j, v^-, f_j \quad 1 \leq i \leq p, p+1 \leq j \leq p+q \tag{38}$$

with e_i, f_i, v^+ a basis of V^+ and e_j, f_j, v^- a basis of V^- . The e, f are isotropic and in duality, the v^\pm unit vectors orthogonal to the e, f .

We will use $1 \leq i, l \leq p$ and $p+1 \leq j, k \leq p+q$ consistently. When convenient, we denote $c = 2p, 2p+1$ and $d = 2q, 2q+1$.

The Lie algebra with respect to the fundamental Cartan subalgebra is realized as follows. For $\mathfrak{so}(2p, 2q)$, v^\pm and the corresponding terms are missing.

Recall the basis of \mathfrak{g} formed of the (complexification of the) fundamental Cartan subalgebra and its root vectors:

Compact	Noncompact
$\mathfrak{t} = \{\sqrt{-1}(1 - e_i f_i)/2, \sqrt{-1}(1 - e_j f_j)/2\}$	$\mathfrak{a} = \{v^+ v^-\}$
$\sqrt{-1}H(\epsilon_i), \sqrt{-1}H(\epsilon_j)$	$H(\epsilon_{p+q+1})$
$f_i v^+, e_i v^+, f_j v^-, e_j v^-$	$f_i v^-, e_i v^-, v^+ f_j, v^+ e_j$
$X(\epsilon_i)_c, X(-\epsilon_i)_c, X(\epsilon_j)_c, X(-\epsilon_j)_c$	$X(\epsilon_i)_n, X(-\epsilon_i)_n, X(\epsilon_j)_n, X(-\epsilon_j)_n$
$f_i f_l, f_i e_l, e_i e_l, e_i f_l$	$f_i f_j, f_i e_j, e_i e_j, e_i f_j$
$f_j f_k, f_j e_k, e_j f_k, e_j e_k$	
$X(\epsilon_i + \epsilon_l), X(\epsilon_i - \epsilon_l), X(-\epsilon_i - \epsilon_l), X(-\epsilon_i + \epsilon_l)$	$X(\epsilon_i + \epsilon_j), X(\epsilon_i - \epsilon_j),$ $X(-\epsilon_i - \epsilon_j), X(-\epsilon_i + \epsilon_j),$
$X(\epsilon_j + \epsilon_k), X(\epsilon_j - \epsilon_k),$ $X(-\epsilon_j - \epsilon_k), X(-\epsilon_j + \epsilon_k).$	

Realizations of the other Cartan subalgebras $\mathfrak{h}^{r^+, r^-, m, s}$ are

$$\sqrt{-1}(1 - e_i f_i)/2, \sqrt{-1}(1 - e_j f_j)/2, \quad 1 \leq i \leq r^+, p+1 \leq j \leq p+r^-,$$

$$\sqrt{-1}H(\epsilon_i), \sqrt{-1}H(\epsilon_j),$$

$$\sqrt{-1}(e_{r^++t} f_{r^++t} - e_{p+r^-+t} f_{p+r^-+t})/2, e_{r^++t} e_{p+r^-+t} + f_{r^++t} f_{p+r^-+t}, \quad 1 \leq t \leq m$$

$$\sqrt{-1}H(\epsilon_{r^++t} - \epsilon_{p+r^-+t}), X(\epsilon_{r^++t} + \epsilon_{p+r^-+t}) + X(-\epsilon_{r^++t} - \epsilon_{p+r^-+t})$$

$$e_{r^++m+l} e_{p+r^-+m+l} + f_{r^++m+l} f_{p+r^-+m+l}, e_{r^++m+l} f_{p+r^-+m+l} + f_{r^++m+l} e_{p+r^-+m+l},$$

$$X(-\epsilon_{r^++m+l} - \epsilon_{p+r^-+m+l}) + X(\epsilon_{r^++m+l} + \epsilon_{p+r^-+m+l}),$$

$$X(\epsilon_{r^++m+l} - \epsilon_{p+r^-+m+l}) + X(-\epsilon_{r^++m+l} + \epsilon_{p+r^-+m+l}), \quad 1 \leq l \leq s.$$

When $p = q = m$, there are two nonconjugate Cartan subalgebras, denoted by $\mathfrak{h}_I^{0,0,m,0}$ and $\mathfrak{h}_{II}^{0,0,m,0}$. For $1 \leq t \leq p - 1$ the subalgebra $\mathfrak{h}_I^{0,0,m,0}$ is generated by

$$H(\epsilon_t - \epsilon_{p+t}), H(\epsilon_p - \epsilon_{2p}), X(\epsilon_t + \epsilon_{p+t}) + X(-\epsilon_t - \epsilon_{p+t}), X(\epsilon_p + \epsilon_{2p}) + X(-\epsilon_p - \epsilon_{2p}),$$

and the subalgebra $\mathfrak{h}_{II}^{0,0,m,0}$ is generated by

$$H(\epsilon_t - \epsilon_{p+t}), H(\epsilon_p + \epsilon_{2p}), X(\epsilon_t + \epsilon_{p+t}) + X(-\epsilon_t - \epsilon_{p+t}), X(\epsilon_p - \epsilon_{2p}) - X(-\epsilon_p + \epsilon_{2p}).$$

6.2. Center of \tilde{G} . The center of \tilde{G} is contained in the maximal compact subgroup \tilde{K} .

Lemma 6.1.

- (1) When $c = 2p + 1, d = 2q + 1$, $Z(\tilde{G}) = \{(\pm I, \pm I)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
- (2) When $c = 2p, d = 2q$,

$$Z(\tilde{G}) = \left\{ (\pm I, \pm I), \left(\pm i^p \prod_{i=1}^p (1 - e_i f_i), \pm i^q \prod_{j=1}^q (1 - e_{p+j} f_{p+j}) \right) \right\} \\ \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_4 & \text{if at least one of } p \text{ and } q \text{ is odd,} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{otherwise.} \end{cases} \quad (39)$$

Lemma 6.2. Let $\mu = (a_1, \dots, a_p \mid b_1, \dots, b_q)$ be a \tilde{K} -type parametrized by its highest weight, and let χ be the restriction of the highest weight of μ to $Z(\tilde{G})$. Then, for $\epsilon_j = \pm 1$,

$$\chi(\epsilon_1 I, \epsilon_2 I) = \epsilon_1^{2a_1} \epsilon_2^{2b_1}, \\ \chi \left(\epsilon_1 i^p \prod_{i=1}^p (1 - e_i f_i), \epsilon_2 i^q \prod_{j=1}^q (1 - e_{p+j} f_{p+j}) \right) = \epsilon_1^{2a_1} \epsilon_2^{2b_1} i^{2(a_1 + \dots + a_p + b_1 + \dots + b_q)}. \quad (40)$$

Proof. As a linear functional of the fundamental Cartan subalgebra, μ acts by

$$\mu : \frac{\theta_1 \sqrt{-1}(1 - e_1 f_1)}{2} + \dots + \frac{\theta_{p+q} \sqrt{-1}(1 - e_{p+q} f_{p+q})}{2} \\ \mapsto \sqrt{-1}(a_1 \theta_1 + \dots + a_p \theta_p + b_1 \theta_{p+1} + \dots + b_q \theta_{p+q}).$$

Setting $\theta_1 = 2\pi$ and $\theta_j = 0$ for all $j \neq 0$, $\chi : (-I, I) \mapsto e^{2\pi i a_1} = (-1)^{2a_1}$, since

$$e^{\sqrt{-1} \theta_i (1 - e_i f_i) / 2} = \cos \frac{\theta_i}{2} + \sqrt{-1} \sin \frac{\theta_i}{2} (1 - e_i f_i).$$

The action of χ on $(\pm I, \pm I)$ is similar. Similarly, setting $\theta_i = \pi$,

$$\mu : \left(\epsilon_1 i^p \prod_{i=1}^p (1 - e_i f_i), \epsilon_2 i^q \prod_{j=1}^q (1 - e_{p+j} f_{p+j}) \right) \\ \mapsto e^{i\pi(a_1 + \dots + a_p + b_1 + \dots + b_q)} = i^{2(a_1 + \dots + a_p + b_1 + \dots + b_q)}. \quad \blacksquare$$

Then it is clear that

- χ factors through $SO(c, d)$ iff $a_i, b_j \in \mathbb{Z}$;
- χ factors through $Spin(c, d)$ if $a_i, b_j \in \mathbb{Z}$ or $a_i, b_j \in \mathbb{Z} + \frac{1}{2}$;
- χ is genuine for $\widetilde{Spin}(c, d)$ if and only if $a_i \in \mathbb{Z}$ and $b_j \in \mathbb{Z} + \frac{1}{2}$, or $a_i \in \mathbb{Z} + \frac{1}{2}$ and $b_j \in \mathbb{Z}$.

Denote the set of genuine central characters of $Z(\widetilde{G})$ by $\prod_g(Z(\widetilde{G}))$. The next lemma characterizes $\prod_g(Z(\widetilde{G}))$.

Lemma 6.3. *Let μ_j be the \widetilde{K} -type parametrized by its highest weight:*

$$\begin{aligned} \mu_1 &= (\underbrace{1/2, \dots, 1/2}_p \mid \underbrace{0, \dots, 0}_q), & \mu_2 &= (\underbrace{0, \dots, 0}_p \mid \underbrace{1/2, \dots, 1/2}_q), \\ \mu_3 &= (\underbrace{1/2, \dots, 1/2, -1/2}_p \mid \underbrace{0, \dots, 0}_q), & \mu_4 &= (\underbrace{0, \dots, 0}_p \mid \underbrace{1/2, \dots, 1/2, -1/2}_q). \end{aligned} \tag{41}$$

Let χ_j be the restriction of the highest weight of μ_j to $Z(\widetilde{G})$.

(a) If $c = 2p + 1, d = 2q + 1$, then $\prod_g(Z(\widetilde{G})) = \{\chi_1, \chi_2\}$.

(a) If $c = 2p, d = 2q$, then $\prod_g(Z(\widetilde{G})) = \{\chi_1, \chi_2, \chi_3, \chi_4\}$.

Moreover, given any \widetilde{K} -type μ , the central character of μ is χ_j iff $\mu - \mu_j$ is in the root lattice of D_n .

Proof. The proof easily follows from (40). ■

6.3. Cartan subgroups of \widetilde{G} . The θ -stable Cartan subgroup

$$\widetilde{H}^{r^+, r^-, m, s} = \widetilde{T}^{r^+, r^-, m, s} \cdot A^{r^+, r^-, m, s}$$

of \widetilde{G} is the centralizer of $\mathfrak{h}^{r^+, r^-, m, s}$ in \widetilde{G} . We write $\mathfrak{h}^{r^+, r^-, m, s} = \mathfrak{t}^{r^+, r^-, m, s} + \mathfrak{a}^{r^+, r^-, m, s}$. We have that $\widetilde{H}^{r^+, r^-, m, s} = \widetilde{T}^{r^+, r^-, m, s} \exp_{\widetilde{G}} \mathfrak{a}^{r^+, r^-, m, s}$. The goal is to compute $\widetilde{T}^{r^+, r^-, m, s} = Z_{\widetilde{K}}(\mathfrak{h}^{r^+, r^-, m, s})$.

We first do this for $\widetilde{G} = \widetilde{Spin}(2, 2)$ in detail. As before, the orthogonal space $V = V^+ \oplus V^-$ has basis $e_1, f_1; e_2, f_2$ with $e_1, f_1 \in V^+, e_2, f_2 \in V^-$.

There are four conjugacy classes of Cartan subalgebras of $\mathfrak{so}(2, 2)$, denoted

$$\mathfrak{h}^{1,1,0,0}, \mathfrak{h}_I^{0,0,1,0}, \mathfrak{h}_{II}^{0,0,1,0}, \mathfrak{h}^{0,0,0,2}.$$

Note that an element in $\widetilde{K} = Spin(2) \times Spin(2)$ is of the form $(e^{i\theta_1 h(\epsilon_1)}, e^{i\theta_2 h(\epsilon_2)})$, where

$$e^{i\theta_j h(\epsilon_j)} = \exp[i\frac{\theta_j}{2}(1 - e_j f_j)] = \cos \frac{\theta_j}{2} + i \sin \frac{\theta_j}{2}(1 - e_j f_j). \tag{42}$$

(1) $\widetilde{H}^{1,1,0,0} = Z_{\widetilde{K}}(h(\epsilon_1)) \cap Z_{\widetilde{K}}(h(\epsilon_2)) = \{(e^{i\theta_1 h(\epsilon_1)}, e^{i\theta_2 h(\epsilon_2)})\} \cong (S^1)^2$.

(2) The cases $\widetilde{H}_I^{0,0,1,0}$ and $\widetilde{H}_{II}^{0,0,1,0}$ are similar, so we do the former one only. We shall calculate $Z_{\widetilde{K}}(\mathfrak{a})$. Let $(e^{i\theta_1 h(\epsilon_1)}, e^{i\theta_2 h(\epsilon_2)}) \in Z_{\widetilde{K}}(\mathfrak{a})$. Then

$$\begin{aligned} X(\epsilon_1 + \epsilon_2) + X(-\epsilon_1 - \epsilon_2) &= \text{Ad } e^{\theta_1 h(\epsilon_1) + \theta_2 h(\epsilon_2)} [X(\epsilon_1 + \epsilon_2) + X(-\epsilon_1 - \epsilon_2)] \\ &= e^{i(\theta_1 + \theta_2)} X(\epsilon_1 + \epsilon_2) + e^{i(-\theta_1 - \theta_2)} X(-\epsilon_1 - \epsilon_2). \end{aligned}$$

This gives that $\theta_1 + \theta_2 = 2k\pi$, $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} (e^{i\theta_1 h(\epsilon_1)}, e^{i\theta_2 h(\epsilon_2)}) &= (e^{i\theta_1 h(\epsilon_1)}, e^{i(2k\pi - \theta_1)h(\epsilon_2)}) \\ &= \left(\cos \frac{\theta_1}{2} + i \sin \frac{\theta_1}{2} (1 - e_1 f_1), \cos \frac{\theta_2}{2} - i \sin \frac{\theta_2}{2} (1 - e_2 f_2) \right) \cdot (I, e^{i2k\pi h(\epsilon_2)}) \\ &= \left(\cos \frac{\theta_1}{2} + i \sin \frac{\theta_1}{2} (1 - e_1 f_1), \cos \frac{\theta_2}{2} - i \sin \frac{\theta_2}{2} (1 - e_2 f_2) \right) \cdot (I, \pm I) \end{aligned}$$

(3) For $\widetilde{H}^{0,0,0,2}$, let $(e^{i\theta_1 h(\epsilon_1)}, e^{i\theta_2 h(\epsilon_2)}) \in Z_{\widetilde{K}}(\mathfrak{a})$. We have the relations

$$\begin{aligned} X(\epsilon_1 + \epsilon_2) + X(-\epsilon_1 - \epsilon_2) &= \text{Ad } e^{\theta_1 h(\epsilon_1) + \theta_2 h(\epsilon_2)} [X(\epsilon_1 + \epsilon_2) + X(-\epsilon_1 - \epsilon_2)] \\ &= e^{i(\theta_1 + \theta_2)} X(\epsilon_1 + \epsilon_2) + e^{i(-\theta_1 - \theta_2)} X(-\epsilon_1 - \epsilon_2), \end{aligned}$$

and

$$\begin{aligned} X(\epsilon_1 - \epsilon_2) + X(-\epsilon_1 + \epsilon_2) &= \text{Ad } e^{\theta_1 h(\epsilon_1) + \theta_2 h(\epsilon_2)} [X(\epsilon_1 - \epsilon_2) + X(-\epsilon_1 + \epsilon_2)] \\ &= e^{i(\theta_1 - \theta_2)} X(\epsilon_1 - \epsilon_2) + e^{i(-\theta_1 + \theta_2)} X(-\epsilon_1 + \epsilon_2). \end{aligned}$$

This gives that $\theta_1 - \theta_2 = 2k\pi$, $\theta_1 + \theta_2 = 2l\pi$, $k, l \in \mathbb{Z}$ and hence we write

$$\theta_1 = (k + l)\pi, \theta_2 = (k - l)\pi.$$

Therefore,

$$\begin{aligned} (e^{i\theta_1 h(\epsilon_1)}, e^{i\theta_2 h(\epsilon_2)}) &= (e^{i(k+l)\pi h(\epsilon_1)}, e^{i(k-l)\pi h(\epsilon_2)}) \\ &= \left(\cos \frac{(k+l)\pi}{2} + i \sin \frac{(k+l)\pi}{2} (1 - e_1 f_1), \cos \frac{(k-l)\pi}{2} + i \sin \frac{(k-l)\pi}{2} (1 - e_2 f_2) \right). \end{aligned}$$

This gives eight elements in $Z_{\widetilde{K}}(\mathfrak{a})$:

$$(\pm I, \pm I), (\pm i(1 - e_1 f_1), \pm i(1 - e_2 f_2)).$$

Recall $\widetilde{G} = \widetilde{Spin}(c, d)$, $c = 2p, d = 2q$ or $c = 2p + 1, d = 2q + 1$. We change the orthonormal basis to

$$v_i, w_i, v^+, v_j, w_j, v^-, \quad 1 \leq i \leq p, p + 1 \leq j \leq p + q$$

for $V = V^+ \oplus V^-$, where $v_i = \frac{e_i + f_i}{\sqrt{2}}$, $w_i = \frac{e_i - f_i}{\sqrt{-2}}$, and the same relations hold for v_j, w_j, e_j, f_j . Again, when c, d are even, the corresponding terms of v^+ and v_- are missing.

Define a finite subgroup $F^{r^+, r^-, m, s}$ of \tilde{K} as follows.

- (1) When $m = 0$ and $s = 0$ or 1 , define $F^{r^+, r^-, m, s} = 1$.
- (2) When $s > 1$, define $F^{r^+, r^-, m, s}$ to be the subgroup generated by the elements of order four of \tilde{K} ,

$$\begin{aligned} &\{(\pm v_i v_k, \pm v_{i+p} v_{k+p}), (\pm w_i w_k, \pm w_{i+p} w_{k+p}), \quad r^+ + m + 1 \leq i < k \leq p \\ &\quad (\pm v_l w_k, \pm v_{l+p} w_{k+p}), \quad r^+ + m + 1 \leq l, k \leq p \\ &\quad (\pm v_i v^+, \pm v_{i+p} v^-), (\pm w_i v^+, \pm w_{i+p} v^-), \quad r^+ + m + 1 \leq i \leq p\}. \end{aligned}$$

In this case, $|F^{r^+, r^-, m, s}| = 2^{s-1} \cdot 4$. It is an extension of \mathbb{Z}_2^{s-1} of $\mathbb{Z}_2 \times \mathbb{Z}_2$, the center of $Spin(c) \times Spin(d)$.

- (3) When $m \neq 0$ and $s = 0$ or 1 , define $F^{r^+, r^-, m, s} = \{(I, \pm I)\}$. In this case $|F^{r^+, r^-, m, s}| = 2$.

Lemma 6.4. *The Cartan subgroup $\tilde{H}^{r^+, r^-, s, m}$ has the direct product decomposition*

$$\tilde{H}^{r^+, r^-, m, s} = F^{r^+, r^-, m, s} \times \exp_{\tilde{G}}(\mathfrak{h}^{r^+, r^-, m, s}).$$

Thus, the number of components of $\tilde{H}^{r^+, r^-, m, s}$ is $|F^{r^+, r^-, m, s}|$, listed above.

Corollary 6.5. *$\tilde{H}^{r^+, r^-, m, s}$ is abelian iff $s < 3$.*

Proof. Since $\exp_{\tilde{G}}(\mathfrak{h}^{r^+, r^-, m, s}) \subset \tilde{H}_0 \subset Z(\tilde{H}^{r^+, r^-, m, s})$, to determine whether $\tilde{H}^{r^+, r^-, m, s}$ is abelian, we just need to look at $F^{r^+, r^-, m, s}$ by Lemma 6.4.

When $s = 0$, $|F^{r^+, r^-, m, s}| = 1$ or 2 . So $F^{r^+, r^-, m, s}$ is obviously abelian.

When $s = 2$, $F^{r^+, r^-, m, s} = \{(\pm I, \pm I), (\pm v_1 w_1, \pm v_2 w_2)\}$, and is clearly abelian.

When $s \geq 3$, there exist sets of orthonormal vectors $\{v^+, w^+, u^+\} \subset V^+$ and $\{v^-, w^-, u^-\} \subset V^-$ such that $(v^+ w^+, v^- w^-), (w^+ u^+, w^- u^-) \in \tilde{H}^{r^+, r^-, m, s}$. Then $v^+ w^+ w^+ u^+ = v^+ u^+$, whereas $w^+ u^+ v^+ w^+ = u^+ v^+ = -v^+ u^+$. Therefore $\tilde{H}^{r^+, r^-, m, s}$ is not abelian. ■

6.4. Regular characters. See [2] or [20] for more detail in this section. Suppose G is a real reductive group (possibly nonlinear).

Definition 6.6. A regular character of G is a triple $\gamma = (H, \Gamma, \lambda)$ consisting of a θ -stable Cartan subgroup H , an irreducible representation Γ of H , and $\lambda \in \mathfrak{h}^*$, satisfying the following conditions.

- (a) $\langle \lambda, \alpha^\vee \rangle \in \mathbb{R}^\times$ for all imaginary roots α ;
- (b) $d\Gamma = \lambda + \rho_i(\lambda) - 2\rho_{i,c}(\lambda)$;
- (c) $\langle \lambda, \alpha^\vee \rangle \neq 0 \forall \alpha \in \Delta$.

The group under consideration is $\tilde{G} = \widetilde{Spin}(c, d)$. In this case we will write $\gamma = (\tilde{H}, \Gamma, \lambda)$. We say that γ is genuine if Γ is a genuine representation of \tilde{H} .

Let $I(\gamma)$ denote that standard module corresponding to the parameter γ , and let $J(\gamma)$ denote the unique irreducible quotient of $I(\gamma)$.

Proposition 6.7. ([2])

- (a) Let \tilde{H} be a Cartan subgroup of \tilde{G} . Every representation Γ of \tilde{H} is parametrized by a genuine character of $Z(\tilde{H})$, i.e. $\Gamma|_{Z(\tilde{H})}$, the restriction of Γ to $Z(\tilde{H})$.
- (b) $Z(\tilde{H}) = Z(\tilde{G})\tilde{H}^0$, so a genuine character of $Z(\tilde{H})$ is determined by its restriction to $Z(\tilde{G})$ and its differential. Moreover, if $\gamma = (\tilde{H}, \Gamma, \lambda)$ is a genuine character of \tilde{G} , then γ is determined by λ and $\Gamma|_{Z(\tilde{G})}$.

Let Λ be a genuine representation of the fundamental Cartan subgroup \tilde{H} with $d\Lambda = \lambda$. When $c = 2p, d = 2q$, $\tilde{H} = \tilde{H}^{p,q,0,0}$; when $c = 2p + 1, d = 2q + 1$, $\tilde{H} = \tilde{H}^{p,q,0,1}$.

Lemma 6.8. Given $\tilde{G} = \widetilde{Spin}(c, d)$, $c + d = 2n$.

- (a) Suppose that $n \in 2\mathbb{Z}$ ($c = 2p, d = 2q$). Then the infinitesimal character of any genuine discrete series representation of \tilde{G} is conjugate to the form

$$(a_1, \dots, a_p \mid b_1, \dots, b_q),$$

with $a_i \in \mathbb{Z}$, $b_j \in \mathbb{Z} + \frac{1}{2}$, or $a_i \in \mathbb{Z} + \frac{1}{2}$, $b_j \in \mathbb{Z}$, and $a_1 > \dots > |a_p| \geq 0$, $b_1 > \dots > |b_q| \geq 0$.

- (b) Suppose that $n \in 2\mathbb{Z} + 1$ ($c = 2p + 1, d = 2q + 1$). Then the infinitesimal character of any genuine fundamental series representation of \tilde{G} is conjugate to the form

$$(a_1, \dots, a_p \mid b_1, \dots, b_q \mid x),$$

with $a_i \in \mathbb{Z}$, $b_j \in \mathbb{Z} + \frac{1}{2}$ or $a_i \in \mathbb{Z} + \frac{1}{2}$, $b_j \in \mathbb{Z}$ and $a_1 > \dots > a_p \geq 0$, $b_1 > \dots > b_q \geq 0$, and x is either in \mathbb{Z} or $\mathbb{Z} + \frac{1}{2}$.

Then the following corollary easily follows.

Corollary 6.9. The following groups are the only ones which admit a representation with infinitesimal character defined in (37).

- (a) $\tilde{G} = \widetilde{Spin}(2p, 2q)$, with $p = n - k - 1$, $q = k + 1$;
- (b) $\tilde{G} = \widetilde{Spin}(2p + 1, 2q + 1)$, with $p + 1 = n - k - 1$, $q = k + 1$;
- (c) $\tilde{G} = \widetilde{Spin}(2p + 1, 2q + 1)$, with $p = n - k - 1$, $q + 1 = k + 1$.

By Corollary 6.9, given λ in (37), we deal with the eight cases listed in Section 2.

6.5. Coherent continuation action. The number of representations with associated cycle \mathcal{O} equals the multiplicity of the *sgn* representation of $W(\lambda)$ in the coherent continuation representation. The orbit \mathcal{O} is the minimal orbit which can occur for the given infinitesimal character, and this corresponds to the *sgn* representation. So we first study the coherent continuation action for the group \tilde{G} . The formulas of the coherent continuation action can be derived from those of the action of Hecke operators. As in [20], given λ as in (37), we define a family

of infinitesimal character $\mathcal{F}(\lambda)$ including λ . Note that every $\lambda' \in \mathcal{F}(\lambda)$ can be indexed by some $w \in W/W(\lambda)$. Write $\mathcal{B}_{\lambda',\chi}$ for the set of equivalence classes of standard representation parameters with infinitesimal character $\lambda' \in \mathcal{F}(\lambda)$ and a fixed central character χ of \widetilde{G} , and

$$\mathcal{B} := \coprod_{\lambda' \in \mathcal{F}(\lambda), \chi \in \widehat{Z(\widetilde{G})}} \mathcal{B}_{\lambda',\chi}.$$

As we will see later the coherent continuation action is closely related to the cross action. We may use $\mathcal{F}(\lambda)$ to define the cross action of W on \mathcal{B} , denoted $w \times \gamma$ for $w \in W, \gamma \in \mathcal{B}$, as shown in [20]. In fact, fixing an infinitesimal character $\lambda' \in \mathcal{F}(\lambda)$ and a central character χ , $W(\lambda)$ acts on $\mathcal{B}_{\lambda',\chi}$ by the cross action.

We set $\mathcal{M} = \mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}][\mathcal{B}]$. We fix the abstract infinitesimal character $\lambda_a \in \mathcal{F}(\lambda)$ corresponding to the positive root system $\Delta^+ := \Delta_a^+(\mathfrak{g}, \mathfrak{h}^a)$ (where \mathfrak{h}^a is an abstract Cartan subalgebra of \mathfrak{g}) and the set of simple roots $\Pi_a \subset \Delta_a^+$. For $s = s_\alpha$ with $\alpha \in \Pi_a$, the action of T_s on $\gamma \in \mathcal{M}$ is defined in Section 9 of [20].

On the other hand, we consider $\Delta(\lambda)$, the integral root system for λ and the integral Weyl group $W(\lambda)$. As we take the infinitesimal character λ in (37), the integral root system for λ is $\Delta(D_{n-k-1} \times D_{k+1})$, and due to Corollary 6.9, this is

$$\Delta(\lambda) = \begin{cases} \Delta(D_p) \times \Delta(D_q) & \text{if } c = 2p, d = 2q, \\ \Delta(D_{p+1}) \times \Delta(D_q) \text{ or } \Delta(D_p) \times \Delta(D_{q+1}) & \text{if } c = 2p+1, d = 2q+1. \end{cases} \tag{43}$$

Also we choose $\Pi(\lambda)$ to be a set of simple roots for $\Delta(\lambda)$.

Given $\alpha \in \Pi(\lambda)$, we decompose $s_\alpha = s_{\alpha_1} \cdots s_{\alpha_m}$ with $\alpha_j \in \Pi_a$. Replace T_{s_β} with T_β for each root β for simplicity. Then

$$\begin{aligned} T_\alpha(\gamma) &= T_{\alpha_1} \cdot T_{\alpha_2} \cdots T_{\alpha_m}(\gamma) \\ &= p_1(u) \cdots p_m(u) s_{\alpha_1} \times (s_{\alpha_2} \times \cdots (s_{\alpha_m} \times \gamma)) \\ &\quad + (\text{terms from more split Cartan subgroups}), \end{aligned} \tag{44}$$

where $p_j(u) \in \mathbb{Z}[u, u^{-1}]$. By [24], we can define the coherent continuation action of $W(\lambda)$ on $\mathbb{Z}[\mathcal{B}]$, denoted $w \cdot \gamma$, with $w \in W(\lambda), \gamma \in \mathcal{B}$, as follows. For $s_\alpha \in W(\lambda)$ with $\alpha \in \Pi(\lambda)$,

$$s_\alpha \cdot \gamma := -T_{s_\alpha}(\gamma)|_{u=1}, \text{ with each term } \delta \text{ on the right side multiplied by } (-1)^{l(\gamma)-l(\delta)},$$

where l is a length function defined on parameters and it can be looked up in [24]. Therefore, from each step T_{α_j} in (44), we may define

$$s_{\alpha_j} \cdot \delta = -T_{\alpha_j}(\delta)|_{u=1}, \text{ if } \alpha_j \text{ is real or imaginay for } \delta$$

and
$$s_{\alpha_j} \cdot \delta = T_{\alpha_j}(\delta)|_{u=1}, \text{ if } \alpha_j \text{ is complex for } \delta.$$

Let $m(\gamma, s_\alpha)$ be the number of occurrences of imaginary roots in $\{\alpha_j, 1 \leq j \leq m\}$. An easy calculation shows that

$$s_\alpha \cdot \gamma = (-1)^{m(\gamma, s_\alpha)} s_\alpha \times \gamma + (\text{terms from more split Cartan subgroups}). \tag{45}$$

Now fix a block $\mathcal{B}_{\lambda, \chi}$ of regular characters of \tilde{G} , then $W(\lambda)$ acts on $\mathbb{Z}[\mathcal{B}_{\lambda, \chi}]$ by the coherent continuation action, since $w \times \gamma \in \mathcal{B}_{\lambda, \chi}$ for all $w \in W(\lambda), \gamma \in \mathcal{B}$. Due to the reason stated in the beginning of the section, the goal is to compute $[sgn_{W(\lambda)} : \mathbb{Z}[\mathcal{B}_{\lambda, \chi}]]$, the multiplicity of the sign representation in $\mathbb{Z}[\mathcal{B}_{\lambda, \chi}]$ when considered as $W(\lambda)$ -representations.

Notice that two λ -regular characters $\gamma_i = (\widetilde{H}_i, \Gamma_i, \overline{\gamma}_i)$ and $\gamma_j = (\widetilde{H}_j, \Gamma_j, \overline{\gamma}_j)$ from \mathcal{D} are in the same cross action orbit if and only if $\widetilde{H}_i = \widetilde{H}_j$. We enumerate the Cartan subgroups of \tilde{G} as $\{\widetilde{H}_1, \dots, \widetilde{H}_l\}$, and pick a regular character γ_j specified by \widetilde{H}_j , then $\{\gamma_1, \dots, \gamma_l\}$ is a set of representatives of the cross action orbits of $W(\lambda)$ on $\mathbb{Z}[\mathcal{B}_{\lambda, \chi}]$.

Let $W_{\gamma_j} = \{w \in W(\lambda) \mid w \times \gamma_j = \gamma_j\}$ be the cross stabilizer of γ_j in $W(\lambda)$. Then we have the following Proposition.

Proposition 6.10. $\mathbb{Z}[\mathcal{B}_{\lambda, \chi}] \simeq \bigoplus_j Ind_{W_{\gamma_j}}^{W(\lambda)}(\epsilon_j)$, where ϵ_j is a one-dimensional representation of W_{γ_j} such that for $w \in W_{\gamma_j}$, $w \cdot \gamma_j = \epsilon_j(w)\gamma_j +$ other terms from more split Cartan subgroups.

Proof. This can be easily proved by the formulas given in [20] and (45). ■

By Proposition 6.10 and Frobenius reciprocity, the multiplicity of $sgn_{W(\lambda)}$ in $\mathbb{Z}[\mathcal{B}_{\lambda, \chi}]$ is $[sgn_{W(\lambda)} : \mathbb{Z}[\mathcal{B}_{\lambda, \chi}]] = [sgn_{W(\lambda)}|_{W_{\gamma_j}} : \epsilon_j]$, which is equal to 0 or 1, since $sgn_{W(\lambda)}|_{W_{\gamma_j}}$ is one-dimensional. This means that we have reduced our goal to count the number of γ_j 's making $[sgn_{W(\lambda)}|_{W_{\gamma_j}} : \epsilon_j] = 1$. Equivalently, we calculate the number of γ_j such that

$$sgn_{W(\lambda)}|_{W_{\gamma_j}} = \epsilon_j. \tag{46}$$

Due to Proposition 6.10 and (46), we have to analyze W_{γ_j} and ϵ_j for each γ_j .

Some notation for Weyl group elements. Let $\rho' = (\rho_1, \dots, \rho_n) = w\rho$ for some $w \in W = W(D_n)$. We define the notations for elements in W as follows. For $i \neq j$, write $s_{i,j} = s_{\epsilon_i - \epsilon_j}$ and $s_{\overline{i,j}} = s_{\epsilon_i + \epsilon_j}$ to be the reflections with respect to the corresponding roots. Moreover, for $1 \leq i < j \leq n$, let $t_{i,j}$ and $t_{\overline{i,j}}$ denote the corresponding Weyl group elements such that (writing “ic.” for “interchanging”)

$$t_{i,j}(\rho') = \begin{cases} \text{ic. the numbers } i \text{ and } j \text{ in } \rho' & \text{if } ij > 0 \\ \text{ic. and changing both signs of the numbers } i \text{ and } j \text{ in } \rho' & \text{if } ij < 0 \end{cases}$$

$$t_{\overline{i,j}}(\rho') = \begin{cases} \text{ic. the numbers } i \text{ and } j \text{ in } \rho' & \text{if } ij < 0 \\ \text{ic. and changing both signs of the numbers } i \text{ and } j \text{ in } \rho' & \text{if } ij > 0 \end{cases}$$

For $0 < j \leq n - 1$, let $t_{0,j}$ and $t_{\overline{0,j}}$ denote the corresponding Weyl group elements such that

$$t_{0,j}(\rho') = \begin{cases} \text{ic. the numbers } 0 \text{ and } j \text{ in } \rho' & \text{if } j > 0 \\ \text{ic. and changing the sign of the number } j \text{ in } \rho' & \text{if } j < 0 \end{cases}$$

$$t_{\overline{0,j}}(\rho') = \begin{cases} \text{ic. the numbers } 0 \text{ and } j \text{ in } \rho' & \text{if } j < 0 \\ \text{ic. and changing the sign of the number } j \text{ in } \rho' & \text{if } j > 0 \end{cases}$$

Note that $s_{i,j}$ and $s_{\bar{i},\bar{j}}$, $1 \leq i \neq j \leq n$, are fixed Weyl groups elements, whereas $t_{i,j}$ and $t_{\bar{i},\bar{j}}$, $0 \leq i \neq j \leq n - 1$, are dependent of ρ' . We have the following advantages of using the notation for t 's:

- (a) $t_{i,i+1}$ (or $t_{\bar{i},\bar{i}+1}$) is a simple reflection no matter what ρ' it is acting on.
- (b) Let $\delta \in \mathcal{B}$ be a parameter in the chamber of $\rho_\delta = w_\delta \rho$ for some $w_\delta \in W$. Then $t_{i,j}$ (or $t_{\bar{i},\bar{j}}$) is integral for δ if and only if i is in the k -th position of ρ_δ and j is in the l -th position in ρ_δ , and $k - l \in 2\mathbb{Z}$.

Given a parameter $\gamma = (\tilde{H}^{r^+,r^-,m,s}, \Gamma, \lambda) \in \mathcal{B}_{\lambda,\chi}$. Write $s = 2s'$ if n is even; $s = 2s' + 1$ if n is odd. Then γ can be expressed as follows:

$$\underbrace{(a_1, \dots, a_{r^+}; b_1, \dots, b_{r^-})}_{r^++r^-} \mid \underbrace{(a_{r^++1}, b_{r^-+1}, \dots, a_{r^++m}, b_{r^-+m})}_{2m} \mid \underbrace{(a_{r^++m+1}, \dots, a_{r^++m+s'}, b_{r^-+m+1}, \dots, b_{r^-+m+s'}, x)}_s, \tag{47}$$

where $a_i \in \mathbb{Z}$, $b_i \in \mathbb{Z} + \frac{1}{2}$ (or the other way round), $\epsilon_i \pm \epsilon_j$ are imaginary for $1 \leq i < j \leq r^+ + r^-$; $\epsilon_i \pm \epsilon_j$ are real for $r^+ + r^- + 2m + 1 \leq i < j \leq n$; for $r^+ + r^- + 1 \leq i < j \leq r^+ + r^- + 2m$, $\epsilon_i - \epsilon_j$ is imaginary and $\epsilon_i + \epsilon_j$ is real. The coordinate x is missing when n is even, and it is either integral or half integral.

We compute the cross stabilizer for such parameters.

Lemma 6.11. *Let γ be a parameter given as in (47). Then from [2],*

$$W_\gamma = W^C(\lambda)^\theta \rtimes (W^r(\lambda) \times W^i(\lambda)). \tag{48}$$

Furthermore, each group in (48) is explicitly expressed as follows:

$$W^C(\lambda)^\theta = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } m = 0 \\ W(\Delta(D_m \times D_m)) \times [\mathbb{Z}_2^* \times \mathbb{Z}_2^\bullet \times \mathbb{Z}_2^\diamond] & \text{if } m \neq 0 \end{cases} \tag{49}$$

$$W^r(\lambda) = \begin{cases} W(D_{s'}) \times W(D_{s'}) & \text{if } n = p + q \\ W(D_{s'}) \times W(D_{s'+1}) & \text{if } n = p + q + 1 \end{cases} \tag{50}$$

$$W^i(\lambda) = W(D_{r^+}) \times W(D_{r^-}). \tag{51}$$

In (49),

$$\mathbb{Z}_2^* = \begin{cases} \mathbb{Z}_2 & \text{if } r^+, r^- \text{ are both nonzero,} \\ 1 & \text{otherwise;} \end{cases} \quad \mathbb{Z}_2^\bullet = \begin{cases} \mathbb{Z}_2 & \text{if } s \geq 2. \\ 1 & \text{otherwise.} \end{cases} \tag{52}$$

$$\mathbb{Z}_2^\diamond = \begin{cases} \mathbb{Z}_2 & \text{if } r^- = 0, r^+ \geq 1, s = 1 \text{ with the coordinate } x \text{ being half-integral.} \\ 1 & \text{otherwise.} \end{cases}$$

The generators for $W^C(\lambda)^\theta$ can be chosen as follows:

- (a) When $m = 0$, the generators for $W^C(\lambda)^\theta = \mathbb{Z}_2 \times \mathbb{Z}_2$ can be taken to be $s_{1,r^++r^-+2m+1} s_{\bar{1},r^++r^-+2m+1}$ and $s_{r^++1,r^++r^-+2m+s'+1} s_{\bar{r^++1},r^++r^-+2m+s'+1}$;
- (b) when $\mathbb{Z}_2^* = \mathbb{Z}_2$, the generator can be taken to be $s_{1,r^++r^-+1} s_{\bar{1},r^++r^-+1} s_{r^++1,r^++r^-+2} s_{\bar{r^++1},r^++r^-+2}$;

- (c) when $\mathbb{Z}_2^\bullet = \mathbb{Z}_2$, the generator can be taken to be $s_{r^++r^--1, r^++r^--2m+1} s_{r^++r^--1, r^++r^--2m+1} s_{r^++r^--2, r^++r^--2m+2} s_{r^++r^--2, r^++r^--2m+2}$;
- (d) when $\mathbb{Z}_2^\diamond = \mathbb{Z}_2$, the generator can be taken to be $s_{1, r^++1} s_{1, r^++1} s_{r^++2, r^++2m+1} s_{r^++2, r^++2m+1}$.

Remark 6.12. (1) $W^r(\lambda)$ is nontrivial iff $s \geq 3$.
 (2) $W^i(\lambda)$ is nontrivial iff $r^+ \geq 2$ and $r^- \geq 2$.

Now we are ready to prove the following lemma.

Lemma 6.13. Let $w \in W_{\gamma_j}$. Retain the notation in Proposition 6.10 and Lemma 6.11. We have the analysis for ϵ_j corresponding to γ_j :

- (a) If $w \in W^i(\lambda)$, then $\epsilon_j(w) = \text{sgn}(w)$.
- (b) If $w \in W^r(\lambda)$, then $\epsilon_j(w) = 1$.
- (c) If $w \in W(\Delta(D_m \times D_m)) \subset W^C(\lambda)^\theta$, then $\epsilon_j(w) = 1$.
- (d) In the case that $\mathbb{Z}_2^* = \mathbb{Z}_2$, if w is a generator of one of the \mathbb{Z}_2 factors, then $\epsilon_j(w) = -1$; the same for the cases that $\mathbb{Z}_2^\bullet = \mathbb{Z}_2$ and $\mathbb{Z}_2^\diamond = \mathbb{Z}_2$.

Proof. By the comment given before Proposition 6.10, we may choose any γ_j specified by \tilde{H}_j to simplify the computation. In each case, we take a generator $w \in W_{\gamma_j}$ and decompose $s_\alpha = s_{\alpha_1} \cdots s_{\alpha_l}$ with $\alpha_i \in \Pi_a$. By (45), we just need to count $m(\gamma, w)$, the number of occurrences of imaginary roots in $\{\alpha_1, \dots, \alpha_l\}$ (with respect to γ).

In case (a), we may choose the parameter γ_j to be

$$(0, 1, 2, \dots, r^+ - 1; \frac{1}{2}, \frac{3}{2}, \dots, \frac{1}{2} + r^- - 1 \mid \dots \mid \dots),$$

which is in the chamber of $\rho = (0, 2, 4, \dots; 1, 3, 5, \dots \mid \dots \mid \dots)$. A generator in $W^i(\lambda)$ is of the form $s_{i, i+1}$ with $1 \leq i \leq r^+ - 1$ or $r^+ + 1 \leq i \leq r^+ + r^- - 1$, $s_{1, 2}$, or s_{r^++1, r^++2} . We treat $w = s_{i, i+1}$ only, since the rest will be similar. Decompose

$$\begin{aligned} s_{i, i+1} &= t_{k, k+2} && \text{for some } 0 \leq k \leq r^+ + r^- - 1 \\ &= t_{k+1, k+2} t_{k, k+1} t_{k+1, k+2}. \end{aligned}$$

It is clear that each $t_{i, l}$ is a simple reflection through an imaginary root, and hence $m(\gamma_j, s_\alpha) = 3$. Therefore we conclude that $\epsilon_j(w) = \text{sgn}(w)$ for $w \in W^i(\lambda)$.

Case (b) is similar. We choose the parameter γ_j to be

$$(\dots \mid \dots \mid 0, 1, \dots, \frac{1}{2}, \frac{3}{2}, \dots),$$

which is in the chamber of $\rho = (\dots \mid \dots \mid 0, 2, \dots, 1, 3, \dots)$. Everything is the same as in case (a), except that each t_{il} is real for the parameter which is acted, and hence $m(\gamma_j, s_\alpha) = 0$. Therefore we conclude that $\epsilon_j(w) = 1$ for $w \in W^r(\lambda)$.

In case (c), we choose the parameter γ_j to be

$$(\cdots \mid \underline{0 \frac{1}{2}} \quad \underline{1 \frac{3}{2}} \cdots \mid \cdots),$$

which is in the chamber of $\rho = (\cdots \mid 0, 1, 2, 3, \cdots \mid \dots)$. Take w to be the Weyl group element such that

$$w : \gamma_j \mapsto (\cdots \mid \underline{1 \frac{3}{2}} \quad \underline{0 \frac{1}{2}} \cdots \mid \cdots),$$

then w is one of the generators of $W(\Delta(D_m \times D_m)) \subset W^C(\lambda)^\theta$. We decompose w in terms of t 's:

$$w = t_{02}t_{13} = t_{12}t_{01}t_{12}t_{23}t_{12}t_{23}.$$

It is easy to check that $m(\gamma_j, w) = 2$.

Another kind of generators in $W(\Delta(D_m \times D_m))$ is of the form

$$\bar{w} : \gamma_j \mapsto (\cdots \mid \underline{-1 \quad -\frac{3}{2}} \quad \underline{-0 \quad -\frac{1}{2}} \cdots \mid \cdots).$$

We decompose \bar{w} in terms of t 's:

$$\bar{w} = t_{02}t_{13} = t_{12}t_{01}t_{12}t_{10}t_{10}t_{12}t_{23}t_{12}t_{10}t_{10},$$

and get $m(\gamma_j, s_{\bar{\alpha}}) = 2$. Therefore, we conclude that $\epsilon_j(w) = 1$ for $w \in W(\Delta(D_m \times D_m)) \subset W^C(\lambda)^\theta$.

In case (d), we choose the parameter γ_j to be

$$(\dots, 0; \frac{1}{2} \cdots \mid \cdots \mid 1 \dots \frac{3}{2} \dots).$$

The generators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ factor can be chosen to be

$$w_1 : \gamma_j \mapsto (\dots, -0; \frac{1}{2} \cdots \mid \cdots \mid -1 \dots \frac{3}{2} \dots)$$

and $w_2 : \gamma_j \mapsto (\dots, 0; -\frac{1}{2} \cdots \mid \cdots \mid 1 \dots -\frac{3}{2} \dots).$

We treat w_1 only. Decompose

$$w_1 = t_{02}t_{02} = t_{12}t_{01}t_{01}t_{12}.$$

It can be checked that $m(\gamma_j, w_1) = 0$, and hence $\epsilon_j(w_1) = 1$. Similarly for w_2 . We conclude that $\epsilon(w) = 1$ for w in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ factor.

In case (e), we choose the parameter γ_j to be

$$(\dots, 0; \frac{1}{2} \cdots \mid \underline{1 \frac{3}{2}} \cdots \mid \cdots) \quad \text{or} \quad (\cdots \mid \underline{1 \frac{3}{2}} \mid 0, \frac{1}{2} \dots).$$

We treat the first one only. The generator of the \mathbb{Z}_2 factor can be chosen to be

$$w : (\dots, 0; \frac{1}{2} \cdots \mid \underline{1 \frac{3}{2}} \cdots \mid \cdots) \mapsto (\dots, -0; -\frac{1}{2} \cdots \mid \underline{-1 \quad -\frac{3}{2}} \cdots \mid \cdots).$$

Decompose

$$w = t_{02}t_{02}t_{13}t_{13} = t_{12}t_{01}t_{12}t_{23}t_{12}t_{23}t_{12}t_{01}t_{12}t_{10}t_{10}t_{12}t_{23}t_{12}t_{10}t_{10},$$

and get $m(\gamma_j, w) = 1$. So $\epsilon_j(w) = -1$ for w in the \mathbb{Z}_2 factor. ■

By Lemma 6.13, γ_j does not satisfy (46) if and only if $W^r(\lambda)$ is nontrivial or $\mathbb{Z}_2^\bullet = \mathbb{Z}_2$ is contained in W_{γ_j} . Therefore, we have to rule out γ_j specified by $\widetilde{H}^{r^+, r^-, m, s}$ satisfying either of the following.

- (1) $s \geq 3$;
- (2) $m \geq 1$ and $r^+ \geq 1, r^- \geq 1$;
- (3) $m \geq 1$ and $s \geq 2$.
- (4) $m \geq 1$ and $s = 1, r^- = 0, r^+ \geq 1$, and $k = (n - 3)/2$.

Consequently, we have the following lemma.

Lemma 6.14. *Let $n_{\mathcal{O}}^\chi$ be the number of irreducible representations of \widetilde{G} with central character χ attached to $\mathcal{O}_{\mathbb{C}}$ and λ .*

- (a) *In Case 1, $n_{\mathcal{O}}^\chi = 4$.*
- (b) *In Case 2 and 3, $n_{\mathcal{O}}^\chi = 3$.*
- (c) *In Case 4, $n_{\mathcal{O}}^\chi = 2$.*
- (d) *In Case 5 and 6 with $\widetilde{G} = \widetilde{Spin}(2p + 1, 2p - 1)$, $n_{\mathcal{O}}^\chi = 2$.*
- (e) *In Case 5 and 6 with $\widetilde{G} = \widetilde{Spin}(2p + 1, 2q + 1)$, $q < p - 1$, $n_{\mathcal{O}}^\chi = 1$.*
- (f) *In Case 7 and 8, $n_{\mathcal{O}}^\chi = 2$.*

Example 6.15. Let $k = 1$ and consider the infinitesimal character

$$\lambda = (n - 3, \dots, 1, 0; 3/2, 1/2).$$

\widetilde{G} admits an admissible representation in the following cases as we fix a genuine central character χ of \widetilde{G} .

- (1) $k = \frac{n}{2} - 1 = 1, \widetilde{G} = Spin(4, 4), \lambda = (1, 0; 3/2, 1/2)$. The counting argument gives the parameters from $\widetilde{H}^{2,2,0,0}, \widetilde{H}^{1,1,0,2}, \widetilde{H}_I^{0,0,2,0}, \widetilde{H}_{II}^{0,0,2,0}$, so $n_{\mathcal{O}}^\chi = 4$.
- (2) $\widetilde{G} = Spin(2n - 4, 4)$ with $2n - 4 > 4$. The counting argument gives the parameters from $\widetilde{H}^{n-2,2,0,0}, \widetilde{H}^{n-3,1,0,2}, \widetilde{H}^{n-4,0,2,0}$, so $n_{\mathcal{O}}^\chi = 3$.
- (3) $\widetilde{G} = \widetilde{Spin}(2n - 3, 3)$. The counting argument gives the parameter from $\widetilde{H}^{n-2,1,0,1}$, so $n_{\mathcal{O}}^\chi = 1$.
- (4) $\widetilde{G} = \widetilde{Spin}(2n - 5, 5)$. The counting argument gives the parameters from $\widetilde{H}^{n-3,2,0,1}, \widetilde{H}^{n-5,0,2,1}$, so $n_{\mathcal{O}}^\chi = 2$.

Lemma 6.16. *Given λ in (37), let*

$$\Pi_{g,\lambda}(Z(\widetilde{G})) = \{\chi \in \Pi_g(Z(\widetilde{G})) \mid \exists \pi \in \mathcal{U}_{\widetilde{G}}(\mathcal{O}_{\mathbb{C}}, \lambda) \text{ s.t. } \pi|_{Z(\widetilde{G})} = \chi\}.$$

Then we have

- (a) *in Case 1, $|\Pi_{g,\lambda}(Z(\widetilde{G}))| = 4$;*
- (b) *in Case 2 and 3, $|\Pi_{g,\lambda}(Z(\widetilde{G}))| = 2$;*
- (c) *in Case 4, $|\Pi_{g,\lambda}(Z(\widetilde{G}))| = 2$;*
- (d) *in Case 5 and 6 with $\widetilde{G} = \widetilde{Spin}(2p + 1, 2p - 1)$, $|\Pi_{g,\lambda}(Z(\widetilde{G}))| = 2$;*
- (e) *in Case 5 and 6 with $\widetilde{G} = \widetilde{Spin}(2p + 1, 2q + 1)$, $q < p - 1$, $|\Pi_{g,\lambda}(Z(\widetilde{G}))| = 1$;*
- (f) *in Case 7 and 8, $|\Pi_{g,\lambda}(Z(\widetilde{G}))| = 1$.*

Below is the main theorem of the section and it follows from Lemma 6.14 and 6.16 since

$$|\mathcal{U}_{\tilde{G}}(\mathcal{O}_c, \lambda)| = \sum_{\chi \in \Pi_{g, \lambda}(Z(\tilde{G}))} n_{\mathcal{O}}^{\chi}.$$

Theorem 6.17. *Let $n_{\mathcal{O}} := |\mathcal{U}_{\tilde{G}}(\mathcal{O}_c, \lambda)|$ be the number of unipotent representations of \tilde{G} attached to \mathcal{O}_c and λ . Then*

Case 1: $n_{\mathcal{O}} = 16$;

Case 2, 3: $n_{\mathcal{O}} = 6$;

Case 4: $n_{\mathcal{O}} = 4$;

Case 5, 6: When $\tilde{G} = \widetilde{Spin}(2p + 1, 2p - 1)$ $n_{\mathcal{O}} = 2$;

Case 5, 6: When $\tilde{G} = \widetilde{Spin}(2p + 1, 2q + 1)$, $q < p - 1$, $n_{\mathcal{O}} = 1$;

Case 7, 8: $n_{\mathcal{O}} = 1$.

7. A construction

7.1. Littlewood Rule. $F_{\mathfrak{h}}(\lambda)$ will denote the finite dimensional representation with highest weight λ of the Lie algebra \mathfrak{h} .

We will use the following result, a generalization of the Littlewood rule as in [9]. Let $(V, \langle \cdot, \cdot \rangle)$ be an orthogonal space of dimension m with positive definite inner product, and let $O(m)$ be the corresponding (compact) orthogonal group. A representation is parametrized by a partition/tableau such that there are at most m rows, and the sum of the lengths of the first two columns is $\leq m$. An irreducible representation was parametrized earlier by its highest weight as $(\tau_1, \dots, \tau_{\lfloor m/2 \rfloor}, \epsilon)$ with $\epsilon = \pm 1$. The corresponding partition is $(\tau_1, \dots, \tau_{\lfloor m/2 \rfloor}, \underbrace{1, \dots, 1}_{m-2(1-\epsilon)}, 0, \dots, 0)$.

Let $Sp(2n, \mathbb{R})$ be the symplectic group of rank n . Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k} + \mathfrak{p}^+ + \mathfrak{p}^-$, in the standard coordinates. The oscillator correspondence matches $W(\tau)$ with an irreducible highest weight module $\Theta(W) = E_{\tau}$ as follows. For a partition $(\lambda_1, \dots, \lambda_m)$, define

$$\lambda^{\sharp} = (-m/2 - \tau_n, \dots, -m/2 - \tau_1),$$

where the tableau of τ with at most m parts has possibly been padded with 0 's to make n parts. Then $\Theta(W) = E_{\tau}$ has highest weight λ^{\sharp} . The length $\ell(\lambda)$ of a tableau $(\lambda_1, \dots, \lambda_m)$ is defined to be the number of nonzero λ_i .

Proposition 7.1. $[W_{O(m)}(\tau) : F(\lambda)] = [F(\lambda^{\sharp}) : E_{\tau}]$

Under the assumption $\ell(\lambda) \leq (m + 1)/2$, let $n = m/2$ if m is even, $n = (m + 1)/2$ if m is odd. Then E_{τ} is an irreducible generalized Verma module for $\mathfrak{q} = \mathfrak{k} + \mathfrak{p}^+$, and the formula can be written as

$$[W(\tau) : F(\lambda)] = [F(\lambda) : W_{gl(n)}(\tau) \otimes S(\mathfrak{p}^+)],$$

where $S(\mathfrak{p}^+) = \sum F(2m_1, \dots, 2m_n), \quad 2m_j \in 2\mathbb{Z}.$

Proof. The first formula is standard for the Θ -correspondence; see [9] for further explanations, references to the original result, and generalizations. When $\ell(\lambda) \leq m/2$, this is the classical Littlewood rule; we use $n = \ell(\lambda) \leq m/2$. When $\ell(\lambda) = (m + 1)/2$, necessarily m is odd, and we use $n = (m + 1)/2$. We need to show that the generalized Verma module for $\mathfrak{k} + \mathfrak{p}^+$ with highest weight λ^\sharp is irreducible. The infinitesimal character is

$$\lambda^\sharp + \rho = (1/2, -1/2 - \tau_1, \dots, -m/2 + 1 - \tau_{n-1}).$$

The first term is greater than 0, the rest are negative. The only possible factor would be the highest weight module with weight

$$(-1/2, -1/2 - \tau_1, \dots, -m/2 + 1 - \tau_{n-1}).$$

But this weight does not differ from λ^\sharp by an element in the root lattice, so the generalized Verma module is irreducible. ■

We treat the case $a = 2p = 2k + 2$, $b = 2q = 2k + 2 + 2r_-$ with $r_- \geq 0$ (Cases 1 and 3 in Section 2) in detail, and note the necessary modifications for $a = 2p + 1$, $b = 2q - 1$ with $r_- \geq 0$ (Cases 4, 6, 8 in Section 2). Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra determined by $\xi = (\underbrace{1, \dots, 1}_p \mid 0, \dots, 0)$.

The Levi component is $\mathfrak{l} = \mathfrak{gl}(p) \times \mathfrak{so}(b)$. The real forms of the factors are $\mathfrak{u}(p) \times \mathfrak{so}(b)$ for $a = 2p$, and $\mathfrak{u}(p) \times \mathfrak{so}(1, b - 1)$ for $a = 2p + 1$. The nilradical is $\mathfrak{u} = \mathfrak{u}_1 + \mathfrak{u}_2$ with $\mathfrak{u}_2 \subset \mathfrak{k}$ in all cases. Furthermore

$$\mathfrak{u}_1 \cap \mathfrak{s} = \begin{cases} \text{Span}\{X(\epsilon_i \pm \epsilon_{p+j})\}_{1 \leq i \leq p, 1 \leq j \leq q} & \text{for } a = 2p, \\ \text{Span}\{X(\epsilon_i \pm \epsilon_{p+j}), X(\epsilon_i)\}_{1 \leq i \leq p, 1 \leq j \leq q-1} & \text{for } a = 2p + 1. \end{cases}$$

We consider $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} [F(\mu_L \mid \mu_R)]$, (53)

the generalized Verma module where $F(\mu_L \mid \mu_R) := F_{\mathfrak{gl}(p)}(\mu_L) \boxtimes F_{\mathfrak{so}(b)}(\mu_R)$ is the module of $\mathfrak{gl}(p) \times \mathfrak{so}(b)$ with highest weights μ_L for $\mathfrak{gl}(p)$ and μ_R for $\mathfrak{so}(b)$. We use the standard positive systems, so

$$\rho := \rho(\mathfrak{so}(a + b)) = \begin{cases} (-q, -q - 1, \dots, -p - q + 1 \mid q - 1, \dots, 0) & \text{if } a = 2p, \\ (-q, -q - 1, \dots, -p - q + 1 \mid q - 1, \dots, 1) & \text{if } a = 2p + 1. \end{cases}$$

For $a = 2p + 1$, this is ρ restricted to the compact part of the fundamental Cartan subalgebra; we drop a 0 from the second set of coordinates.

The infinitesimal character of $M(\mu)$ is as before,

$$\mu + \rho \simeq (p - 1/2, p - 3/2, \dots, 1/2 \mid q - 1, \dots, 1, 0).$$

The possible $\mu = (\mu_L \mid \mu_R)$ are given by the equations

$$\begin{aligned} (a_1, \dots, a_p \mid b_1, \dots, b_q) + (-q, -q - 1, \dots, -p - q + 1 \mid q - 1, \dots, 0) \\ = (\mp 1/2, -3/2, \dots, -p + 1/2 \mid q - 1, \dots, 0). \end{aligned}$$

When $2k + 2 < p + q$, and there are two more for $\widetilde{Spin}(2p, 2p)$:

$$\begin{aligned} &(a_1, \dots, a_p \mid b_1, \dots, b_p) + (-p, -p - 1, \dots, -2p + 1 \mid p - 1, \dots, 0) \\ &= (0, \dots, -p + 1 \mid p - 1/2, \dots, \pm 1/2). \end{aligned}$$

So there are four μ ,

- (i) $F(\mu_L \mid \mu_R) = F(q - 1/2, \dots, q - 1/2 \mid 0, \dots, 0)$.
- (ii) $F(\mu_L \mid \mu_R) = F(q + 1/2, q - 1/2, \dots, q - 1/2 \mid 0, \dots, 0)$.
- (iii) $F(\mu_L \mid \mu_R) = F(p, \dots, p \mid 1/2, \dots, 1/2)$.
- (iv) $F(\mu_L \mid \mu_R) = F(p, \dots, p \mid 1/2, \dots, 1/2, -1/2)$.

We call them Cases (i)–(iv). Again, Cases (iii) and (iv) only occur for $k + 1 = p = q$, i.e. $\widetilde{Spin}(2p, 2p)$. For $\widetilde{Spin}(2p + 1, 2q - 1)$, only Cases (i) and (ii) occur, and one 0 is dropped from the coordinates.

$M(\mu)$ is reducible, and the irreducible quotient $L(\mu)$ has associated variety $\mathcal{O}_C = [3 \ 2^{2k} \ 1^{2n-4k-3}]$. Its character is

$$L(\mu) = \sum_{w \in W(D_p^+)} (-1)^{\ell(w)} M(w \cdot \mu) \quad \text{where } w \cdot \mu = w(\mu + \rho) - \rho.$$

As a \mathfrak{k} -module, $M(\mu) = U(\mathfrak{k}) \otimes_{U(\mathfrak{q} \cap \mathfrak{k})} [S(\mathfrak{u}_1 \cap \mathfrak{s}) \otimes (F(\mu_L \mid \mu_R))]$. Embed $\mathfrak{so}(b) \subset \mathfrak{gl}(b)$ in case $b = 2q$, and $\mathfrak{so}(b - 1) \subset \mathfrak{gl}(b - 1)$ in case $b = 2q - 1$. Write $\mathfrak{gl}(b - \epsilon)$ with $\epsilon = 0$ if $b = 2q$ and $\epsilon = 1$ if $b = 2q - 1$.

Lemma 7.2. *As an $\mathfrak{l} \cap \mathfrak{k}$ -module,*

$$S^m(\mathfrak{u}_1 \cap \mathfrak{s}) = \sum F_{\mathfrak{gl}(p)}(m_1, \dots, m_p) \boxtimes F_{\mathfrak{gl}(b-\epsilon)}(m_1, \dots, m_p, 0, \dots, 0)$$

with $\sum m_i = m$. In all cases $p = k + 1 \leq b - \epsilon = k + 1 - \epsilon + r_-$.

Proof. $\mathfrak{u}_1 \cap \mathfrak{s}$ has highest weight $(1, 0, \dots, 0 \mid 1, 0, \dots, 0)$. The representation on the second factor is the standard one for $\mathfrak{so}(b - \epsilon)$. It is the restriction of the standard representation with highest weight $(1, 0, \dots, 0)$ of $\mathfrak{so}(b - \epsilon) \subset \mathfrak{gl}(b - \epsilon)$. The claim follows. ■

Definition 7.3. Let $W(\beta)$ and $F(\mu_R)$ be representations of $\mathfrak{so}(b - \epsilon)$. Then

$$(W_{\mathfrak{so}(b-\epsilon)}(\beta) \otimes F_{\mathfrak{so}(b-\epsilon)}(\mu_R))_{\mathfrak{gl}(p)}$$

means the sum (with multiplicity) of the composition factors of the tensor product whose highest weight has at most p nonzero coordinates replaced by the irreducible representations of $\mathfrak{gl}(p)$ with the same highest weight.

This is related to the Littlewood rule for restriction from $\mathfrak{gl}(b - \epsilon)$ to $\mathfrak{so}(b - \epsilon)$.

Proposition 7.4. *Let $S(\mathfrak{p}^+) = \sum V(2m_1, \dots, 2m_p)$. Then*

$$\begin{aligned}
 & [V(\delta) \boxtimes W(\beta) : S(\mathfrak{u}_1 \cap \mathfrak{s}) \otimes F(\mu_L \mid \mu_R)]_{\mathfrak{gl}(p) \times \mathfrak{so}(b-\epsilon)} \\
 &= [(W_{\mathfrak{so}(b-\epsilon)}(\beta) \otimes F_{\mathfrak{so}(b-\epsilon)}(\mu_R)^*)_{\mathfrak{gl}(p)} \otimes S(\mathfrak{p}^+) \otimes F_{\mathfrak{gl}(p)}(\mu_L) : V(\delta)]_{\mathfrak{gl}(p)}. \tag{54}
 \end{aligned}$$

In particular the multiplicity is 0 unless $\beta_{p+1} = \dots = 0$.

Proof. We abbreviate $(a_1, \dots, a_p, 0, \dots, 0)$ as $(a, 0)$.

$$\begin{aligned}
 & [V(\delta) \boxtimes W(\beta) : S(\mathfrak{u}_1 \cap \mathfrak{s}) \otimes F(\mu_L \mid \mu_R)] \tag{55} \\
 &= \sum_a [V(\delta) \boxtimes W(\beta) : (F_{\mathfrak{gl}(p)}(a) \boxtimes F_{\mathfrak{gl}(b-\epsilon)}(a, 0) \mid_{\mathfrak{so}(b-\epsilon)}) \otimes F(\mu_L \mid \mu_R)] \\
 &= \sum_a [V(\delta) \boxtimes W(\beta) : (F_{\mathfrak{gl}(p)}(a) \otimes F_{\mathfrak{gl}(p)}(\mu_L)) \boxtimes (F_{\mathfrak{gl}(b-\epsilon)}(a, 0) \mid_{\mathfrak{so}(b-\epsilon)} \otimes F_{\mathfrak{so}(b-\epsilon)}(\mu_R))] \\
 &= \sum_a [V(\delta) : F_{\mathfrak{gl}(p)}(a) \otimes F_{\mathfrak{gl}(p)}(\mu_L)] \cdot [W(\beta) : F_{\mathfrak{gl}(b-\epsilon)}(a, 0) \mid_{\mathfrak{so}(b-\epsilon)} \otimes F_{\mathfrak{so}(b-\epsilon)}(\mu_R)] \\
 &= \sum_a [V(\delta) \otimes F_{\mathfrak{gl}(p)}(\mu_L)^* : F_{\mathfrak{gl}(p)}(a)] \cdot [W(\beta) \otimes F_{\mathfrak{so}(b-\epsilon)}(\mu_R)^* : F_{\mathfrak{gl}(b-\epsilon)}(a, 0) \mid_{\mathfrak{so}(b-\epsilon)}].
 \end{aligned}$$

Let
$$F_{\mathfrak{gl}(p)}(a) \otimes F_{\mathfrak{gl}(p)}(\mu_L) = \sum_{\gamma} c_{a, \mu_L}^{\gamma} F_{\mathfrak{gl}(p)}(\gamma). \tag{56}$$

Note that

$$c_{a, \mu_L}^{\gamma} = [F_{\mathfrak{gl}(p)}(a) \otimes F_{\mathfrak{gl}(p)}(\mu_L) : F_{\mathfrak{gl}(p)}(\gamma)] = [F_{\mathfrak{gl}(p)}(a) : F_{\mathfrak{gl}(p)}(\gamma) \otimes F_{\mathfrak{gl}(p)}(\mu_L)^*]. \tag{57}$$

So
$$F_{\mathfrak{gl}(p)}(\gamma) \otimes F_{\mathfrak{gl}(p)}(\mu_L)^* = \sum_a c_{a, \mu_L}^{\gamma} F_{\mathfrak{gl}(p)}(a). \tag{58}$$

Assume that μ_L is such that its coordinates are all nonnegative; this is the case for $w(\mu + \rho) - \rho$ because the first p coordinates of $w(\mu + \rho)$ are of the form

$$(r_1 + 1/2, \dots, r_{\ell} + 1/2, -s_1 - 1/2, \dots, -s_{p-\ell} - 1/2)$$

with $0 \leq r_i, s_j \leq p$, and $-\rho = (p + q - 1, \dots, q \mid -q + 1, -q + 2, \dots, -1, 0)$. So the factors in the tensor product $F_{\mathfrak{gl}(p)}(a) \otimes F_{\mathfrak{gl}(p)}(\mu)$ have highest weights γ with nonnegative entries as well. Thus by (55) and (56),

$$\begin{aligned}
 & [V(\delta) \boxtimes W(\beta) : S(\mathfrak{u}_1 \cap \mathfrak{k}) \otimes F(\mu_L \mid \mu_R)] \tag{59} \\
 &= \sum_a \sum_{\gamma} c_{a, \mu_L}^{\gamma} [V(\delta) : F_{\mathfrak{gl}(p)}(\gamma)] \cdot [W(\beta) \otimes F_{\mathfrak{so}(b-\epsilon)}(\mu_R)^* : F_{\mathfrak{gl}(b-\epsilon)}(a, 0) \mid_{\mathfrak{so}(b-\epsilon)}].
 \end{aligned}$$

Proposition 7.1 implies

$$[F_{\mathfrak{so}(b-\epsilon)}(\tau) : F_{\mathfrak{gl}(b-\epsilon)}(a, 0) \mid_{\mathfrak{so}(b-\epsilon)}] = [F_{\mathfrak{gl}(p)}(\tau) \otimes S(\mathfrak{p}^+) : F_{\mathfrak{gl}(p)}(a)], \tag{60}$$

where the right hand side is a $\mathfrak{gl}(p)$ -multiplicity, and $S(\mathfrak{p}^+) = \sum V(2m_1, \dots, 2m_p)$ with $m_i \in \mathbb{N}$ as a $\mathfrak{gl}(p)$ -module. In particular τ can have at most p nonzero coordinates, and the right hand side in (60) is a $\mathfrak{gl}(p)$ -multiplicity.

The sum is over the a such that $\gamma = \delta$ occurs in $V(a)$. By (58),

$$\sum c_{a,\mu_L}^\delta F_{\mathfrak{gl}(p)}(a) = V(\delta) \otimes F_{\mathfrak{gl}(p)}(\mu_L)^*. \tag{61}$$

So the multiplicity is

$$\begin{aligned} & [V(\delta) \boxtimes W(\beta) : S(\mathfrak{u}_1 \cap \mathfrak{s}) \otimes F(\mu_L \mid \mu_R)] \\ &= [(W(\beta) \otimes F_{\mathfrak{so}(b-c)}(\mu_R)^*)_{\mathfrak{gl}(p)} \otimes S(\mathfrak{p}^+) : \sum_a c_{a,\mu_L,\delta} F_{\mathfrak{gl}(p)}(a)] \\ &= [(W(\beta) \otimes F_{\mathfrak{so}(b-c)}(\mu_R)^*)_{\mathfrak{gl}(p)} \otimes S(\mathfrak{p}^+) : V(\delta) \otimes F_{\mathfrak{gl}(p)}(\mu_L)^*] \\ &= [(W(\beta) \otimes F_{\mathfrak{so}(b-c)}(\mu_R)^*)_{\mathfrak{gl}(p)} \otimes S(\mathfrak{p}^+) \otimes F_{\mathfrak{gl}(p)}(\mu_L) : V(\delta)]. \quad \blacksquare \end{aligned} \tag{62}$$

We compute the multiplicity

$$[V(\delta) \boxtimes W(\beta) : S(\mathfrak{u}_1 \cap \mathfrak{s}) \otimes \sum_{w \in W(D_p^+)} \epsilon(w) F(w \cdot \mu)]. \tag{63}$$

There are four cases. Write $\mathcal{X}(\mu) := S(\mathfrak{u}_1 \cap \mathfrak{s}) \otimes \sum_{w \in W(D_p^+)} \epsilon(w) F(w \cdot \mu)$.

Cases (i) and (ii). $\mu_R = Triv$ and

$$\mu_L = \begin{cases} (q - 1/2, q - 1/2, \dots, q - 1/2) & \text{in Case (i),} \\ (q + 1/2, q - 1/2, \dots, q - 1/2) & \text{in Case (ii).} \end{cases}$$

Proposition 7.5.

$$V(\delta) \boxtimes W(\beta) = (\alpha_1 + q - 1/2, \dots, \alpha_p + q - 1/2 \mid \beta_1, \dots, \beta_p, 0, \dots, 0)$$

occurs in $\mathcal{X}(\mu)$, and with multiplicity 1 if and only if

$$\begin{cases} \alpha_1 \geq \beta_1 \geq \alpha_2 \geq \dots \geq \alpha_p \geq |\beta_p| & \text{when } a = b = 2p, \\ \alpha_1 \geq \beta_1 \geq \alpha_2 \geq \dots \geq \alpha_p \geq \beta_p \geq 0 & \text{otherwise,} \end{cases}$$

with $\alpha_i, \beta_j \in \mathbb{Z}$. In Case (i), $(\alpha) - (\beta)$ is in the root lattice, in Case (ii), $(\alpha) - (\beta) - (1, 0, \dots, 0)$ is in the root lattice.

Proof. By Proposition 7.4, equation (63) becomes

$$\begin{aligned} & [V(\delta) \boxtimes W(\beta) : S(\mathfrak{u}_1 \cap \mathfrak{s}) \otimes \sum_{w \in W(D_p^+)} \epsilon(w) F(w \cdot \mu_L \mid 0)] \\ &= [W(\beta) \otimes S(\mathfrak{p}^+) \otimes \sum_{w \in W(D_p^+)} \epsilon(w) F(w \cdot \mu_L) : V(\delta)] \end{aligned} \tag{64}$$

In all cases $W(\beta)$ is a representation of $\mathfrak{gl}(p)$ where the highest weight has been padded by 0's to make a highest weight of $\mathfrak{gl}(p)$.

We can do the computation in a different Lie algebra, $\mathfrak{g}' = \mathfrak{sp}(2p)$. Consider the parabolic subalgebra $\mathfrak{p}' = \mathfrak{l}' + \mathfrak{u}'$ corresponding to $\xi' = (1, \dots, 1)$, where $\mathfrak{l}' \cong \mathfrak{gl}(p)$.

Let $M(\lambda') = U(\mathfrak{g}') \otimes_{U(\overline{\mathfrak{p}'})} F(\lambda')$ be the generalized Verma module with λ' such that

$$\lambda' + \rho(C_p) = \begin{cases} (-1/2, -3/2, \dots, -p + 1/2) & \text{in case (i),} \\ (1/2, -3/2, \dots, -p + 1/2) & \text{in case (ii).} \end{cases} \tag{65}$$

Then $\rho(C_p) = (-1, \dots, -p)$ and

$$\lambda' = \begin{cases} (1/2, \dots, 1/2) & \text{in case (i),} \\ (3/2, 1/2, \dots, 1/2) & \text{in case (ii).} \end{cases} \tag{66}$$

The quotient $L(\lambda')$ is one of the metaplectic representations with $\mathfrak{l}' \cap \mathfrak{k}'$ -structure

$$\begin{cases} \sum_{m \in \mathbb{N}} F_{\mathfrak{gl}(p)}(2m + 1/2, 1/2, \dots, 1/2) & \text{in case (i),} \\ \sum_{m \in \mathbb{N}} F_{\mathfrak{gl}(p)}(2m + 3/2, 1/2, \dots, 1/2) & \text{in case (ii)} \end{cases} \tag{67}$$

and character analogous to $L(\lambda)$:

$$L(\lambda') = \sum_{w \in W(D_p^+)} \epsilon(w) M(w \cdot \lambda') = \sum_{w \in W(D_p^+)} \epsilon(w) S(\mathfrak{u}') \otimes F(w \cdot \lambda'). \tag{68}$$

The infinitesimal character $\lambda' + \rho(C_p) = (\mp 1/2, -3/2, \dots, -p + 1/2)$ is the same as the first p coordinates of the infinitesimal character of $M(\lambda)$. Furthermore, on the first p coordinates,

$$\begin{aligned} w \cdot \mu_L &= w(\mu_L + \rho(D_n)) - \rho(D_n) = w(\lambda' + \rho(C_p)) - \rho(C_p) + (q - 1, \dots, q - 1) \\ &= w \cdot \lambda' + (q - 1, \dots, q - 1). \end{aligned} \tag{69}$$

The multiplicity in (64) is therefore

$$[W(\beta) \otimes S(\mathfrak{u}') \otimes \sum_{w \in W(D_p^+)} \epsilon(w) F(w \cdot \lambda') : V(\delta + (-q + 1, \dots, -q + 1))]_{\mathfrak{gl}(p)}. \tag{70}$$

The multiplicity in (70) becomes

$$\begin{cases} [W(\beta) \otimes \sum_m F_{\mathfrak{gl}(p)}(2m + 1/2, 1/2, \dots, 1/2) : V(\delta + (-q + 1))] \\ \quad = [W(\beta) \otimes F_{\mathfrak{gl}(p)}(2m, 0, \dots, 0) : V(\delta + (-q + 1/2))] & \text{in case (i),} \\ [W(\beta) \otimes \sum_m F_{\mathfrak{gl}(p)}(2m + 3/2, 1/2, \dots, 1/2) : V(\delta)] \\ \quad = [W(\beta) \otimes F_{\mathfrak{gl}(p)}(2m + 1, 0, \dots, 0) : V(\delta + (-q + 1/2))] & \text{in case (ii).} \end{cases} \tag{71}$$

By the Littlewood-Richardson rule,

$$W(\beta) \otimes \sum_{k \in \mathbb{N}} F_{\mathfrak{gl}(p)}(k, 0, \dots, 0) = \sum V(\beta_1 + m_1, \dots, \beta_p + m_p), \tag{72}$$

where the sum is taken over the set $\{m_i \in \mathbb{N} \mid \sum m_i = k, m_{i+1} \leq \beta_i - \beta_{i+1}, 1 \leq i \leq p - 1\}$. Therefore, in case (i),

$$\begin{aligned} & [W(\beta) \otimes \sum_{m \in \mathbb{N}} F_{\mathfrak{gl}(p)}(2m, 0, \dots, 0) : V(\delta - q + 1/2)] \\ &= \begin{cases} 1 & \text{if } \delta_1 - q + 1/2 \geq \beta_1 \geq \dots \geq \delta_p - q + 1/2 \geq \beta_p \geq 0, p < q \\ 1 & \text{if } \delta_1 - q + 1/2 \geq \beta_1 \geq \dots \geq \delta_p - q + 1/2 \geq |\beta_p|, p = q \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{73}$$

When the multiplicity is 1, there is an additional restriction that $(\delta) - (q - 1/2) - (\beta)$ be in the root lattice.

The case $[W(\beta) \otimes \sum_{m \in \mathbb{N}} F_{\mathfrak{gl}(p)}(2m + 1, 0, \dots, 0) : V(\delta - q + 1/2)]$ is the same, but the multiplicity is 1 only when $(\delta) - (q - 1/2) - (\beta) - (1, 0, \dots, 0)$ is in the root lattice. Recalling that the notation is $(\alpha) = (\delta) - (p - 1/2)$, the assertions in the Proposition follow. ■

Cases (iii) and (iv). In these cases $\mu_R = (1/2, \dots, 1/2, \pm 1/2) = Spin_{\pm}$, and $\mu_L = (p, \dots, p)$. The multiplicity of $V(\delta) \boxtimes W(\beta)$ in Proposition (7.4) is

$$\sum_a c_{a, \mu_L, \delta} [W(\beta) \otimes F_{\mathfrak{so}_{2p}}(\mu_R)^* : F_{\mathfrak{gl}_{2p}}(a, 0) \mid_{\mathfrak{so}_{2p}}].$$

Since the coordinates of the factors of $F_{\mathfrak{gl}_{2p}}(a, 0)$ are all nonnegative integers, the same has to hold for $W(\beta) \otimes F_{\mathfrak{so}_{2p}}(\mu_R)^*$. It follows that all the coordinates of β are half-integers, so strictly greater than 0 except for possibly the last one.

Proposition 7.6. (1) *In Case (iii),*

$$[V(\delta) \otimes W(\beta) : \mathcal{X}(\mu)] = \begin{cases} 1 & \text{if } \delta_1 - p + 1/2 \geq \beta_1 \geq \dots \geq \delta_p - p + 1/2 \geq |\beta_p|, \\ & (\delta) - (p - 1/2) - (\beta) \text{ in the root lattice,} \\ 0 & \text{otherwise.} \end{cases}$$

(2) *In Case (iv),*

$$[V(\delta) \otimes W(\beta) : \mathcal{X}(\mu)] = \begin{cases} 1 & \text{if } \delta_1 - p + 1/2 \geq \beta_1 \geq \dots \geq \delta_p - p + 1/2 \geq |\beta_p|, \\ & (\delta) - (p - 1/2) - (\beta) - (1, 0, \dots, 0) \text{ in the root lattice,} \\ 0 & \text{otherwise.} \end{cases}$$

In these cases, the coordinates of δ are in \mathbb{Z} , the coordinates of β in $(\mathbb{Z} + 1/2)$.

Proof. We will use the following two lemmas and the Littlewood rule.

Lemma 7.7. *Assume $F_{\mathfrak{so}_{2p}}(\beta)$ is a module for $\mathfrak{so}(2p)$. Then*

$$F_{\mathfrak{so}_{2p}}(\beta) \otimes Spin = \sum F_{\mathfrak{so}_{2p}}(\beta_i + \epsilon_i/2), \quad \epsilon_i = \pm 1$$

and the sum is over the ϵ_i such that $(\beta_i + \epsilon_i/2)$ is a highest weight.

Proof. Omitted. ■

Lemma 7.8. *Assume $F_{\mathfrak{gl}(p)}(\beta)$ is a module for $\mathfrak{gl}(p)$. Then*

$$F_{\mathfrak{gl}(p)}(\beta) \otimes F_{\mathfrak{gl}(p)}(1/2, \dots, 1/2, \underbrace{-1/2, \dots, -1/2}_k) = \sum F_{\mathfrak{gl}(p)}(\beta_i + \epsilon_i/2), \quad \epsilon_i = \pm 1,$$

with exactly k ϵ_i 's equal to -1 , and the sum is over the ϵ_i such that $(\beta_i + \epsilon_i/2)$ is a highest weight.

Proof. Omitted. ■

Consider
$$\sum_a c_{a,\mu_L,\delta} [W(\beta) \otimes Spin_{\pm}^* \mid_{\mathfrak{gl}} \otimes S(\mathfrak{p}^+) : F_{\mathfrak{gl}}(a)].$$

As before, this is $[W(\beta) \otimes (Spin_{\pm})^* \otimes S(\mathfrak{p}^+) \otimes F_{\mathfrak{gl}(p)}(\mu_L) : V(\delta)]_{\mathfrak{gl}_p}$.

Replace $F_{\mathfrak{gl}(p)}(\mu_L)$ by $\sum_{w \in W(D_p^+)} \epsilon(w) F_{\mathfrak{gl}(p)}(w(\mu_L + \rho(D_{2p})) - \rho(D_{2p})) :$

$$[W(\beta) \otimes (Spin_{\pm})^* \otimes S(\mathfrak{p}^+) \otimes \sum_{w \in W(D_p^+)} \epsilon(w) F_{\mathfrak{gl}(p)}(w \cdot \mu_L) : V(\delta)]. \tag{74}$$

Write $Ch(W)$ for the character of W .

Lemma 7.9.

$$Ch \left[(Spin_{\pm})^* \otimes S(\mathfrak{p}^+) \otimes \sum_{w \in W(D_p^+)} \epsilon(w) F_{\mathfrak{gl}(p)}(w \cdot \mu_L) \right] = Ch(L(\lambda')) \otimes e^{(p-1, \dots, p-1)},$$

where $L(\lambda')$ is the metaplectic representation introduced in the last section.

Proof.
$$\begin{aligned} Ch(S(\mathfrak{p}^+)) &= \frac{1}{\prod_{\alpha \in \Delta(\mathfrak{p}^+)} (1 - e^\alpha)} = (-1)^q \frac{e^{-\rho(\Delta(\mathfrak{p}^+))}}{\prod_{\alpha \in \Delta(\mathfrak{p}^+)} (e^{\alpha/2} - e^{-\alpha/2})} \\ &= (-1)^q \frac{e^{-\rho(\Delta(\mathfrak{p}^+))}}{\Delta_{\mathfrak{p}^+}} = (-1)^q \frac{e^{(\frac{-p+1}{2}, \dots, \frac{-p+1}{2})}}{\Delta_{\mathfrak{p}^+}}, \end{aligned} \tag{75}$$

where $q = |\Delta(\mathfrak{p}^+)|$. The right hand side can be rewritten as

$$\begin{aligned} RHS &= e^{(p-1, \dots, p-1)} \cdot Ch[L(\lambda')] \\ &= e^{(p-1, \dots, p-1)} \cdot Ch \left[\sum_{w \in W(D_p^+)} \epsilon(w) M(w \cdot \lambda') \right] \\ &= e^{(p-1, \dots, p-1)} \cdot Ch[S(\mathfrak{p}^+)] \cdot Ch \left[\sum_{w \in W(D_p^+)} \epsilon(w) F_{\mathfrak{gl}(p)}(w \cdot \lambda') \right] \\ &= \frac{(-1)^q}{\Delta_{\mathfrak{p}^+}} e^{(\frac{p-1}{2}, \dots, \frac{p-1}{2})} \sum_{w \in W(D_p^+)} \epsilon(w) \frac{\sum_{x \in W(A_{p-1})} \epsilon(x) e^{x[w \cdot \lambda' + \rho(\mathfrak{gl}_p)]}}{\Delta_{\mathfrak{gl}(p)}} \\ &= \frac{(-1)^q}{\Delta_{\mathfrak{p}^+}} e^{(\frac{p-1}{2}, \dots, \frac{p-1}{2})} e^{(\frac{p+1}{2}, \dots, \frac{p+1}{2})} \sum_{w \in W(D_p^+)} \epsilon(w) \frac{\sum_{x \in W(A_{p-1})} \epsilon(x) e^{xw(\mp 1/2, -\frac{3}{2}, \dots, -p+1/2)}}{\Delta_{\mathfrak{gl}(p)}} \\ &= (-1)^q e^{(p, \dots, p)} \sum_{v \in W(D_p)} \frac{\epsilon(v) e^{v(\mp 1/2, -\frac{3}{2}, \dots, -p+1/2)}}{\Delta_{D_p}} = (-1)^q e^{(p, \dots, p)} Ch[Spin_{\pm}^*]. \end{aligned}$$

On the other hand,

$$\begin{aligned} LHS &= Ch[Spin_{\pm}^*] \cdot (-1)^q \frac{e^{(\frac{-p+1}{2}, \dots, \frac{-p+1}{2})}}{\Delta_{\mathfrak{p}^+}} \cdot \sum_{w \in W(D_p^+)} \epsilon(w) \frac{\sum_{x \in W(A_{p-1})} \epsilon(x) e^{x[w \cdot \mu_L + \rho(\mathfrak{gl}_p)]}}{\Delta_{\mathfrak{gl}_p}} \\ &= Ch[Spin_{\pm}^*] \cdot (-1)^q \frac{e^{(\frac{-p+1}{2}, \dots, \frac{-p+1}{2})}}{\Delta_{\mathfrak{p}^+}} \cdot e^{(\frac{3p-1}{2}, \dots, \frac{3p-1}{2})} \frac{\sum_{v \in W(D_p)} \epsilon(v) e^{v(0, -1, \dots, -p+1)}}{\Delta_{\mathfrak{gl}_p}} \\ &= (-1)^q \cdot Ch[Spin_{\pm}^*] \cdot e^{(p, \dots, p)} \cdot \frac{\Delta_{D_p}}{\Delta_{\mathfrak{p}^+} \Delta_{\mathfrak{gl}(p)}} = (-1)^q \cdot Ch[Spin_{\pm}^*] \cdot e^{(p, \dots, p)}. \end{aligned}$$

The lemma follows. ■

Thus (74) becomes $[W(\beta) \otimes L(\lambda') : V(\delta + (-p + 1))].$ (76)

By the same argument as in cases (i) and (ii), we get in case (iii),

$$\begin{aligned} &[W(\beta) \otimes \sum_{k \in \mathbb{N}} F_{\mathfrak{gl}(p)}(2k, 0, \dots, 0) : V(\delta - p + 1/2)] \\ &= \begin{cases} 1 & \text{if } \delta_1 - p + 1/2 \geq \beta_1 \geq \dots \geq \delta_p - p + 1/2 \geq |\beta_p|, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (77)$$

When the multiplicity is 1, there is an additional restriction that $(\delta) - (p - 1/2) - (\beta)$ be in the root lattice. In case (iv),

$$\begin{aligned} &[W(\beta) \otimes \sum_{k \in \mathbb{N}} F_{\mathfrak{gl}(p)}(2k + 1, 0, \dots, 0) : V(\delta - p + 1/2)] \\ &= \begin{cases} 1 & \text{if } \delta_1 - p + 1/2 \geq \beta_1 \geq \dots \geq \delta_p - p + 1/2 \geq |\beta_p|, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (78)$$

The multiplicity is 1 only when $(\delta) - (p - 1/2) - (\beta) - (1, 0, \dots, 0)$ is in the root lattice. Note that $\delta \in \mathbb{Z}^p$ and $\beta \in (\mathbb{Z} + 1/2)^p$ in these cases. ■

We now apply the cohomological induction functor Π from [14].

Theorem 7.10. *The \tilde{K} -spectra of the representations constructed are as follows. We again use $S := \sum(a_i + b_j)$.*

Case 1. $a = b = 2p$

- (i) $(a_p + 1/2, \dots, a_p + 1/2 \mid b_1, \dots, b_p)$ with $S \in 2\mathbb{Z}$,
 - (ii) $(a_p + 1/2, \dots, a_p + 1/2 \mid b_1, \dots, b_p)$ with $S \in 2\mathbb{Z} + 1$,
 - (iii) $(a_p + 1, \dots, a_p + 1 \mid b_1 + 1/2, \dots, b_p + 1/2)$ with $S \in 2\mathbb{Z}$,
 - (iv) $(a_p + 1, \dots, a_p + 1 \mid b_1 + 1/2, \dots, b_p + 1/2)$ with $S \in 2\mathbb{Z} + 1$,
- satisfying $a_1 \geq b_1 \geq \dots \geq a_p \geq |b_p|$, $a_i, b_j \in \mathbb{Z}$.

Case 3. $a = 2p = 2k + 2$, $b = 2q = 2k + 2 + 2r_-$ with $r_- > 0$

- (i) $(a_p + r_- + 1/2, \dots, a_p + r_- + 1/2 \mid b_1, \dots, b_p, 0, \dots, 0)$ with $S \in 2\mathbb{Z}$,
 - (ii) $(a_p + r_- + 1/2, \dots, a_p + r_- + 1/2 \mid b_1, \dots, b_p, 0, \dots, 0)$ with $S \in 2\mathbb{Z} + 1$,
- satisfying $a_1 \geq b_1 \geq \dots \geq a_p \geq |b_p|$, $a_i, b_j \in \mathbb{Z}$.

Case 4. $a = 2p + 1 = 2q - 1 = b$

- (i) $(a_p + 1/2, \dots, a_p + 1/2 \mid b_1, \dots, b_p)$ with $S \in 2\mathbb{Z}$,
 - (ii) $(a_p + 1/2, \dots, a_p + 1/2 \mid b_1, \dots, b_p)$ with $S \in 2\mathbb{Z} + 1$,
- satisfying $a_1 \geq b_1 \geq \dots \geq a_p \geq b_p \geq 0$, $a_i, b_j \in \mathbb{Z}$.

Case 6. $a = 2p + 1 = 2k + 1$, $b = 2q - 1 = 2k + 3 + 2r_-$

- (i) $(a_p + r_- + 3/2, \dots, a_p + r_- + 3/2 \mid b_1, \dots, b_p, 0, \dots, 0)$ with $S \in 2\mathbb{Z}$,
 - (ii) $(a_p + r_- + 3/2, \dots, a_p + r_- + 3/2 \mid b_1, \dots, b_p, 0, \dots, 0)$ with $S \in 2\mathbb{Z} + 1$,
- satisfying $a_1 \geq b_1 \geq \dots \geq a_p \geq b_p \geq 0$, $a_i, b_j \in \mathbb{Z}$.

Case 8. $a = 2p + 1 = 2k + 3$, $b = 2q - 1 = 2k + 1 + 2r_-$

- (i) $(a_p + r_- - 1/2, \dots, a_p + r_- - 1/2 \mid b_1, \dots, b_p, 0, \dots, 0)$ with $S \in 2\mathbb{Z}$,
 - (ii) $(a_p + r_- - 1/2, \dots, a_p + r_- - 1/2 \mid b_1, \dots, b_p, 0, \dots, 0)$ with $S \in 2\mathbb{Z} + 1$,
- satisfying $a_1 \geq b_1 \geq \dots \geq a_p \geq b_p \geq 0$, $a_i, b_j \in \mathbb{Z}$.

Proof. The main result we use is Corollary 4.160 in [14], which computes Π^j of a generalized Verma module. The formulas in the previous sections write $L(\mu)$ as a sum of generalized Verma modules for \mathfrak{k} .

We treat Case 3 in detail. The term $W(\beta)$ is already $\mathfrak{so}(2p) \times \mathfrak{so}(2q)$ -finite, only $V(\delta)$ is affected. Then

$$\delta + \rho(D_p) = (\delta_1, \delta_2 - 1, \dots, \delta_p - p + 1) \tag{79}$$

is formed of all negative terms. We need a w that makes it dominant for our choice of positive system $\mathfrak{b} \subset \bar{\mathfrak{q}}$; w is the long Weyl group element. The highest weight is

$$(p - 1 - \delta_p, \dots, 1 - \delta_2, -\delta_1) - (0, \dots, -p + 1) = (p - 1 - \delta_p, \dots, p - 1 - \delta_1)$$

Going to the more standard positive system for type D_p , the highest weight is

$$(\delta_1 - p + 1, \dots, \delta_p - p + 1). \tag{80}$$

In Cases (i) and (ii), since $\delta \in (\mathbb{Z} + 1/2)^p$, $\beta \in \mathbb{Z}^p$, we get that the K -spectrum of $\Pi^j(L(\lambda))$ is $\sum(a_1 + r_- + 1/2, \dots, a_p + r_- + 1/2 \mid b_1, \dots, b_p, 0, \dots, 0)$ where a_i, b_j are integers and the multiplicity is 1 precisely when

$$a_1 \geq b_1 \geq \dots \geq a_p \geq |b_p|.$$

For case (i), the additional condition is that $(a \mid b)$ belongs to the root lattice of D_{2p} . For case (ii), the additional condition is that $(a \mid b) - (1, 0, \dots, 0 \mid 0, \dots, 0)$ belongs to the root lattice of D_{2p} . Therefore, the additional parity conditions hold.

In the case $a = b = 2p$ (i.e. Case 1), we get the \tilde{K} -spectra for case (i) and (ii) by replacing $r_- = 0$. In Cases (iii) and (iv), since $\delta \in \mathbb{Z}^p$, $\beta \in (\mathbb{Z} + 1/2)^p$, we get that the K -spectrum of $\Pi^j(L(\lambda))$ is $\sum(a_1 + 1, \dots, a_p + 1 \mid b_1 + 1/2, \dots, b_p + 1/2)$ where a_i, b_j are integers and the multiplicity is 1 precisely when

$$a_1 \geq b_1 \geq \dots \geq a_p \geq |b_p|.$$

For case (iii), the additional condition is that $(a \mid b)$ belongs to the root lattice of D_{2p} . For case (iv), the additional condition is that $(a \mid b) - (1, 0, \dots, 0 \mid 0, \dots, 0)$ belongs to the root lattice of D_{2p} . So we get the additional parity conditions.

The argument for $a = 2p = 1$, $b = 2q + 1$ is essentially the same. In equation (79), $\rho(D_p)$ is replaced by $\rho(B_p)$. The highest weight in (80) becomes

$$(\delta_1 - p, \dots, \delta_p - p).$$

The K -spectrum of $\Pi^j(L(\lambda))$ is therefore

$$\sum (a_1 + q - p - 1/2, \dots, a_p + q - p - 1/2 \mid b_1, \dots, b_p, 0, \dots, 0),$$

with $a_1 \geq b_1 \geq \dots \geq a_p \geq b_p \geq 0$ and $a_i, b_j \in \mathbb{Z}$.

In Case 4, $q - p = 1$, in Case 6, $q - p = r_- + 2$, and in Case 8, $q - p = r_-$. Consequently, the \tilde{K} -spectra are as in the statement of the theorem, with the additional parity conditions due to the same reason as above. ■

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