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Stratonovich-Weyl Correspondence for the Generalized Poincaré Group

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Abstract. We construct a Stratonovich-Weyl correspondence for each unitary irreducible representation of the generalized Poincaré group $\mathbb{R}^{n+1} \rtimes SO_0(n,1)$ associated with an integral coadjoint orbit with little group $SO(n)$, generalizing some results of J. F. Cariñena, J. M. Gracia-Bondìa and J. C. Vàrilly [J. Phys. A: Math. Gen. 23 (1990) 901–933].

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1. Introduction

The notion of Stratonovich-Weyl correspondence was introduced in [32] as a generalization of the classical Weyl correspondence between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$ (see [1], [20]) to the setting of a Lie group acting on a homogeneous space. Stratonovich-Weyl correspondences were systematically studied by Gracia-Bondìa, Vàrilly and their co-workers, see in particular [18], [19], [22]. For a good review of Stratonovich-Weyl correspondences and related topics, we refer the reader to [21].

Definition 1.1. [21] Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and μ a (suitably normalized) G-invariant measure on M. Then a *Stratonovich-Weyl correspondence* for the triple (G,π,M) is an isomorphism W from a vector space of operators on $\mathcal{H}$ to a vector space of (generalized) functions on M (called *symbols*) satisfying the following properties:

1. Reality: the function $W(A^*)$ is the complex conjugate of $W(A)$;
2. Covariance: we have $W(\pi(g)\,A\,\pi(g)^{-1})(x) = W(A)(g^{-1}\cdot x)$;
3. Unitarity: we have $\displaystyle\int_M W(A)(x)W(B)(x)\,d\mu(x) = \mathrm{Tr}(AB)$.

*This paper is dedicated to the memory of my father, Alfred Cahen.

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Note that usually the space of operators on $\mathcal{H}$ is the Hilbert-Schmidt class and the symbols are square integrable functions on M.

Note also that the notion of Stratonovich-Weyl correspondence is very similar to that of invariant symbolic calculus which was studied in particular by Arazy and Upmeier for symmetric domains [2], [3].

In fact, in Definition 1.1, M is generally taken to be a coadjoint orbit of G which is associated with π by the Kirillov-Kostant method of orbits [25], [27]. A basic example is then the case when G is the $(2n+1)$-dimensional Heisenberg group. Each non-degenerate coadjoint orbit M of G is diffeomorphic to $\mathbb{R}^{2n}$ and is associated with a unitary irreducible representation π of G on $L^2(\mathbb{R}^n)$. In this case, it is well-known that the classical Weyl correspondence gives a Stratonovich-Weyl correspondence for the triple (G, π, M) [20], [21].

On the other hand, when G is a quasi-Hermitian Lie group and π a unitary irreducible representation of G which is holomorphically induced from a unitary character of a compactly embedded subgroup K of G (see [29]), we can obtain a Stratonovich-Weyl correspondence by taking the isometric part in the polar decomposition of the Berezin correspondence, see [19], [11], [12], [15]. In particular, this method works for the unitary irreducible representations of a semisimple compact Lie group, see [19] and [11].

The physical interest of Stratonovich-Weyl correspondences is explained in [18] and [22]. Let us also mention that, in the setting of Deformation Quantization, Stratonovich-Weyl correspondences can be used to construct invariant star products on coadjoint orbits, see [21].

In [18], Stratonovich-Weyl correspondences for the unitary irreducible representations of the Poincaré group $\mathbb{R}^4 \rtimes SO_0(3,1)$ corresponding to the massive particles with spin were obtained by using explicit coordinates on the coadjoint orbits associated with these representations.

In the present paper, we consider the case where $G = \mathbb{R}^{n+1} \rtimes SO_0(n,1)$ and π is a unitary irreducible representation of G whose associated coadjoint orbit has little group $SO(n)$. We call such representations (and such orbits) *massive* since they constitute the natural generalization of the above mentioned representations (and coadjoint orbits) of $\mathbb{R}^4 \rtimes SO_0(3,1)$.

By [8], each massive coadjoint orbit $\mathcal{O}$ of G is diffeomorphic to the product of $\mathbb{R}^{2n}$ by a coadjoint orbit o of $SO(n)$ (in fact, in [8] we exhibit a symplectic diffeomorphism from $\mathcal{O}$ onto $\mathbb{R}^{2n} \times o$ for the natural symplectic structures on these manifolds). This suggests us to combine the classical Weyl correspondence on $\mathbb{R}^{2n}$ with the Stratonovich-Weyl correspondence on o in order to get a Stratonovich-Weyl correspondence on $\mathcal{O}$. The first step is then the construction of the Stratonovich-Weyl quantizer as explained in Section 6. Our main result is the construction of the Stratonovich-Weyl correspondence which is accomplished in Section 7.

This paper is organized as follows. We begin by some generalities on the generalized Poincaré group (Section 2) and its massive representations (Section 3). Then we recall some well-known facts on the classical Weyl correspondence (Section 4) and on the Stratonovich-Weyl correspondence for the unitary irreducible representations of a compact Lie group (Section 5). In Section 6, we introduce the Stratonovich-Weyl quantizer for G and, in Section 7, we show that we obtain indeed a Stratonovich-Weyl correspondence. In Section 8, we consider the problem of extending the Stratonovich-Weyl correspondence to operators which are not Hilbert-Schmidt and we compute in particular the symbol of $d\pi(X)$ for X in the Lie algebra of G. We conclude the paper with some final remarks and perspectives (Section 9). Indeed, we can hope for further extensions of our results to semidirect products of the form $V \rtimes K$ where K is a connected semisimple Lie group acting linearly on a finite dimensional vector space V.

2. Preliminaries

Let $V = \mathbb{R}^{n+1}$ and let $K = SO_0(n,1)$ be the identity component of the group $SO(n,1)$ of all real $(n+1)\times(n+1)$ matrices of determinant 1 leaving invariant the bilinear form on V defined by

$$\langle v, v'\rangle = -\left(\sum_{k=1}^{n} v_i\, v_i'\right) + v_{n+1}\, v_{n+1}'.$$

Then K acts linearly on V and, for k in K and v in V, we denote by $k\cdot v$ the action of k on v (v being considered as a column vector). Thus we can form the semidirect product $G = V \rtimes K$. The multiplication of G is

$$(v,k)(v',k') = (v + k\cdot v', kk')$$

for each v, v' in V and k, k' in K. The group G is the called the *generalized Poincaré group*. The usual Poincaré group corresponds to the case $n = 3$. We denote by $(k,p) \to k\cdot p$ the action of K on V^* contragredient to the action of K on V.

Let $\mathfrak{k}$ be the Lie algebra of K. We denote by $(A,v) \to A\cdot v$ and $(A,p) \to A\cdot p$ the actions of $\mathfrak{k}$ on V and V^* which are obtained by differentiating the actions of K on V and V^*. For each $k \in K$ and $f \in \mathfrak{k}^*$, we denote by $k\cdot f$ the coadjoint action of k on f. For each v in V and p in V^* we define $v\wedge p \in \mathfrak{k}^*$ by $(v\wedge p)(A) = p(A\cdot v) = -(A\cdot p)(v)$ for $A \in \mathfrak{k}$. Then we have

$$k\cdot(v\wedge p) = k\cdot p \wedge k\cdot v$$

for each $k \in K$, $v \in V$ and $p \in V^*$. The Lie algebra $\mathfrak{g}$ of G is the vector space $V\times\mathfrak{k}$ equipped with the Lie bracket

$$[(a,A)\,,\,(a',A')] = (A\cdot a' - A'\cdot a\,,\,[A,A'])$$

for each a, a' in V and A, A' in $\mathfrak{k}$. We can identify $\mathfrak{g}^*$ with $V^*\times\mathfrak{k}^*$. The coadjoint action of G on $\mathfrak{g}^*$ is then given by

$$(v,k)\cdot(p,f) = (k\cdot p, k\cdot f + v\wedge k\cdot p)$$

for each $(v,k)\in G$ and $(p,f)\in\mathfrak{g}^*$. Also, we can identify K-equivariantly $\mathfrak{k}$ to its dual $\mathfrak{k}^*$ by using the bilinear form on $\mathfrak{k}$ defined by $\langle X,Y\rangle = \frac{1}{2}\,\mathrm{Tr}(XY)$ for each $X, Y\in\mathfrak{k}$ which is a multiple of the Killing form of $\mathfrak{k}$. Then $\mathfrak{g}^*$ can be identified to $V^*\times\mathfrak{k}$.

Let K_0 be the subgroup of K consisting of all matrices of the form $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ where $k \in SO(n)$ and let $\mathfrak{k}_0$ be the Lie algebra of K_0. Then the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}_0$ in $\mathfrak{k}$ consists of all matrices of the form $\begin{pmatrix} 0 & u^t \\ u & 0 \end{pmatrix}$ where $u \in \mathbb{R}^n$ (the subscript t denotes transposition). Thus $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{k}$ and the map $(k, T) \to \exp(T)k$ is a diffeomorphism from $K_0 \times \mathfrak{p}$ onto K, see e.g. [23], p. 253.
Let $(e_1, e_2, \ldots, e_{n+1})$ be the standard basis of $V = \mathbb{R}^{n+1}$ and let

$$(e_1^*, e_2^*, \ldots, e_{n+1}^*)$$

be the dual basis. For each $p \in V^*$, we denote $p = (\tilde{p}, p_{n+1})$ where $\tilde{p} \in \mathbb{R}^n$ and $p_{n+1} \in \mathbb{R}$.
For each $1 \leq i,\ j \leq n+1$, we write E_{ij} for the matrix whose ij-th entry is 1 and all of the other entries are 0. The matrices $A_{ij} := E_{ji} - E_{ij}$ $(1 \leq i < j \leq n)$ form a basis for $\mathfrak{k}_0$ and the matrices $E_k := E_{kn+1} + E_{n+1k}$ $(1 \leq k \leq n)$ a basis for $\mathfrak{p}$.
Then, in the identification $\mathfrak{k}^* \simeq \mathfrak{k}$, the matrix A_{ij} $(1 \leq i < j \leq n)$ corresponds to the element $e_i \wedge e_j^*$ of $\mathfrak{k}^*$ and the matrix E_k $(1 \leq k \leq n)$ to $e_k \wedge e_{n+1}^*$. In particular, we have $\mathfrak{p} = \{v \wedge e_{n+1}^* : v \in V\}$.
Now, fix $m > 0$ and take $p_0 := m e_{n+1}^*$. Then we easily verify that K_0 is the stabilizer of p_0 for the action of K on V^* and, by using the decomposition $K = \exp(\mathfrak{p})K_0$, we see that the map $T \to \exp(T) \cdot p_0$ is a diffeomorphism from $\mathfrak{p}$ onto the orbit $Z(p_0)$ of p_0 in V^*. We also verify that $Z(p_0)$ is the sheet of the hyperboloid $-|\tilde{p}|^2 + p_{n+1}^2 = m^2$ defined by $p_{n+1} > 0$.
Given $p \in Z(p_0)$, we denote by $T(p)$ the unique element of $\mathfrak{p}$ such that $p = \exp(T(p)) \cdot p_0$ and we set $M(p) := \exp(T(p))$. Moreover, for each $p \in Z(p_0)$, we define the symmetric p^s of p with respect to p_0 by $p^s := M(p)^{-1} \cdot p_0$ and, for each $p, q \in Z(p_0)$, we define $s_q(p) := M(q) \cdot (M(q)^{-1} \cdot p)^s$.

In the following proposition, we collect some formulas which will be needed later.

Proposition 2.1. (1) *Let $u \in \mathbb{R}^n$ and $T = \begin{pmatrix} 0 & u^t \\ u & 0 \end{pmatrix}$. Then we have*

$$\exp(T) = \begin{pmatrix} I_n + \frac{\cosh|u| - 1}{|u|^2} u^t u & \frac{\sinh|u|}{|u|} u^t \\ \frac{\sinh|u|}{|u|} u & \cosh|u| \end{pmatrix} \tag{1}$$

where $|u|$ denotes the Euclidean norm of u.

(2) *Let $u \in \mathbb{R}^n$, $T = \begin{pmatrix} 0 & u^t \\ u & 0 \end{pmatrix}$ and $p = \exp(T) \cdot p_0$. Then we have $\tilde{p} = m\frac{\sinh|u|}{|u|} u$ and $p_{n+1} = m \cosh|u|$.*

(3) *Let $p = (\tilde{p}, p_{n+1}) \in Z(p_0)$. Then we have $p^s = (-\tilde{p}, p_{n+1})$.*

(4) *Let $p, q \in Z(p_0)$. Let $q' = s_q(p)$.*

Then we have
$$\tilde{q}' = -\tilde{p} + \frac{2}{m^2}\langle p, q\rangle \tilde{q} \tag{2}$$
and
$$q'_{n+1} = \frac{2}{m^2}\langle p, q\rangle q_{n+1} - p_{n+1}. \tag{3}$$

(5) *For each $k \in K_0$ and each $p \in Z(p_0)$ we have $M(k \cdot p) = kM(p)k^{-1}$ and $(k \cdot p)^s = k \cdot p^s$.*

Proof. (1) is proved by a direct computation. (2) and (3) follow immediately from (1). (4) results from a computation based on (1) and (2). To prove (5), note that for each $k \in K_0$ and each $p \in Z(p_0)$ we have

$$kM(p)k^{-1} = k\exp(T(p))k^{-1} = \exp(\mathrm{Ad}(k)T(p)) \in \exp(\mathfrak{p})$$

and $kM(p)k^{-1} \cdot p_0 = k \cdot p$. Hence $M(k \cdot p) = kM(p)k^{-1}$. The second assertion of (5) is an immediate consequence of (3). ■

3. Representations and orbits

In this section, we introduce the massive coadjoint orbits of G and the corresponding representations.

As in Section 2, let $p_0 := me^*_{n+1}$ where $m > 0$. Let $\varphi_0 \in \mathfrak{k}^* \simeq \mathfrak{k}$ and $\xi_0 = (p_0, \varphi_0) \in V^* \times \mathfrak{k}$. Consider the orbit $\mathcal{O}(\xi_0)$ of $\mathfrak{g}^* \simeq V^* \times \mathfrak{k}$ under the coadjoint action of G. Then the corresponding little group is $K_0 \simeq SO(n)$. Moreover since for each $v \in V$ we have

$$(v, I_{n+1}).(p_0, \varphi_0) = (p_0, \varphi_0 + v \wedge p_0),$$

we see that we may assume that $\varphi_0 \in \mathfrak{k}_0$ without loss of generality. We denote by $o(\varphi_0) \subset \mathfrak{k}_0$ the orbit of $\varphi_0 \in \mathfrak{k}_0 \simeq \mathfrak{k}_0^*$ under the (co)adjoint action of K_0.

Henceforth, we assume that $o(\varphi_0)$ is integral, that is, associated with a unitary irreducible representation (ρ, E) of K_0. Following [4], [10], [28], [35], we can briefly described the correspondence between $o(\varphi_0)$ and ρ as follows. Let T be a maximal torus of K_0 with Lie algebra $\mathfrak{t}$. Let $i\lambda \in i\mathfrak{t}^*$ be the highest weight of ρ. We defined $\varphi_0 \in \mathfrak{k}_0^*$ by taking $\varphi_0 = \lambda$ on $\mathfrak{t}$ and $\varphi_0 = 0$ on the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}_0$. The orbit of φ_0 under the (co)adjoint action of K_0 is then said to be associated with ρ.

Then $\mathcal{O}(\xi_0)$ is also integral [30]. More precisely, $\mathcal{O}(\xi_0)$ is associated with the unitarily induced representation $\pi := \mathrm{Ind}^G_{V \times K_0}(e^{i\langle p_0, \cdot\rangle} \otimes \rho)$. Moreover, by a result of Mackey, π is irreducible because ρ is irreducible [33]. The usual realization of π is obtained as follows. Let μ be the invariant measure of $Z(p_0)$. Then we can easily verify that $d\mu(p) = \frac{1}{p_{n+1}} d\tilde{p}$ where $d\tilde{p}$ denotes the Lebesgue measure on $\mathbb{R}^n$. Let us consider the Hilbert space $\mathcal{H} := L^2(Z(p_0), E)$ which is obtained by completion of the space of compactly supported smooth functions $\psi : Z(p_0) \to E$ with respect to the norm defined by

$$\|\psi\|^2 = \int_{Z(p_0)} \langle \psi(p), \psi(p)\rangle_E \, d\mu(p).$$

Then, for $(v, k) \in G$, the action of $\pi(v, k)$ is given by

$$(\pi\,(v, k)\,\psi)(p) = e^{i\langle p, v\rangle}\,\rho(M(p)^{-1}kM(k^{-1} \cdot p))\psi\,(k^{-1} \cdot p).$$

Here we will also use another realization of π, says π_0, which is obtained by means of the (unitary) intertwining operator $J : \mathcal{H}_0 := L^2(\mathbb{R}^n, E) \to \mathcal{H}$ defined by $(J\phi)(p) = p_{n+1}^{1/2}\phi(\tilde{p})$. Then we have

$$(\pi_0(v, k)\,\phi)(\tilde{p}) = (k^{-1} \cdot p)_{n+1}/p_{n+1})^{1/2} e^{i\langle p, v\rangle}\,\rho(M(p)^{-1}kM(k^{-1} \cdot p))\phi\,(\widetilde{k^{-1} \cdot p})$$

where $(v, k) \in G$ et $p = (\tilde{p}, p_{n+1}) \in Z(p_0)$.

4. Classical Weyl correspondence

In this section, we review some well-known facts on the classical Weyl correspondence, see in particular [20].

The Weyl correspondence on $\mathbb{R}^{2n}$ can be defined as follows. For each f in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, we define the operator $\mathcal{W}(f)$ acting on the Hilbert space $L^2(\mathbb{R}^n)$ by

$$(\mathcal{W}(f)\phi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\langle y,z\rangle} f(x + \tfrac{1}{2}y, z)\, \phi(x+y)\, dy\, dz.$$

Let us consider the Fourier transform $\mathcal{F}_2 f$ of $f \in \mathcal{S}(\mathbb{R}^{2n})$ with respect to the second variable

$$(\mathcal{F}_2 f)(x,y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle y,z\rangle} f(x,z) dz.$$

Then the formula for $\mathcal{W}(f)$ can be written as

$$(\mathcal{W}(f)\phi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (\mathcal{F}_2 f)(\tfrac{1}{2}(x+y), x-y)\phi(y) dy$$

and the kernel of the operator $A := \mathcal{W}(f)$ is

$$k_A(x,y) = (2\pi)^{-n/2} (\mathcal{F}_2 f)(\tfrac{1}{2}(x+y), x-y).$$

Thus, by the Fourier inversion theorem, we get

$$f(x,y) = \int_{\mathbb{R}^n} e^{i\langle y,z\rangle} k_A(x + \tfrac{1}{2}z, x - \tfrac{1}{2}z)\, dz. \tag{4}$$

Let R be the operator on $\mathbb{R}^n$ defined by $R\phi(x) = 2^n \phi(-x)$. The following lemma will be needed later.

Lemma 4.1. (1) *For each trace class operator A on $L^2(\mathbb{R}^n)$ with kernel k_A, we have*

$$\operatorname{Tr}(AR) = \int_{\mathbb{R}^n} k_A(\tfrac{1}{2}x, -\tfrac{1}{2}x) dx.$$

(2) *For each $f \in \mathcal{S}(\mathbb{R}^{2n})$, we have* $\operatorname{Tr}(\mathcal{W}(f)R) = f(0,0)$.

Proof. If A is of trace class, then AR is also of trace class and its kernel is $k_{AR}(x,y) = 2^n k_A(x,-y)$. Then, by Mercer's theorem, we get

$$\operatorname{Tr}(AR) = \int_{\mathbb{R}^n} k_{AR}(x,x) dx = \int_{\mathbb{R}^n} k_A(\tfrac{1}{2}x, -\tfrac{1}{2}x) dx.$$

Thus the first assertion is proved. The second assertion immediately follows from equation (4). ∎

In the rest of this section, we interpret the Weyl correspondence as a Stratonovich-Weyl correspondence, see [21] for a different presentation.
Let G_0 be the Heisenberg group of dimension $2n+1$. We write the elements of G_0 as $[a,b,c]$ with $a,b\in\mathbb{R}^n$ and $c\in\mathbb{R}$. The multiplication of G_0 is given by

$$[a,b,c]\cdot[a',b',c']=[a+a',b+b',c+c'+\tfrac{1}{2}(ab'-a'b)].$$

Then G_0 acts on $\mathbb{R}^{2n}$ by

$$g\cdot(p,q)=(p+a,q+b),\quad g=[a,b,c]\in G_0.$$

Consider the unitary representation σ of G_0 on $L^2(\mathbb{R}^n)$ defined by

$$(\sigma(g)\phi)(x)=\exp\big(i(c-\langle b,x\rangle+\tfrac{1}{2}\langle a,b\rangle)\big)\,\phi(x-a),\quad g=[a,b,c]\in G_0.$$

For each $(a,b)\in\mathbb{R}^{2n}$, let $R(a,b)=\sigma([a,b,0])R\sigma([a,b,0])^{-1}$. Then, by an elementary computation, we see that for each $(a,b)\in\mathbb{R}^{2n}$, $\phi\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, we have

$$(R(a,b)\phi)(x)=2^n\exp(2i\langle b,a-x\rangle)\phi(2a-x).$$

If f is a function on $\mathbb{R}^{2n}$ and $g\in G_0$, we denote by L_gf the function on $\mathbb{R}^{2n}$ defined by $(L_gf)(\xi)=f(g^{-1}\cdot\xi)$. We have the following proposition.

Proposition 4.2. (1) *For each $g\in G_0$ and $f\in\mathcal{S}(\mathbb{R}^{2n})$, we have*

$$\sigma(g)\mathcal{W}(f)\sigma(g)^{-1}=\mathcal{W}(L_gf).$$

(2) *For each $f\in\mathcal{S}(\mathbb{R}^{2n})$ and $(a,b)\in\mathbb{R}^{2n}$, we have* $\operatorname{Tr}(\mathcal{W}(f)R(a,b))=f(a,b)$.

Proof. Let $g\in G_0$ and $f\in\mathcal{S}(\mathbb{R}^{2n})$. By making a simple change of variables, we can establish that we have $\sigma(g)\mathcal{W}(f)=\mathcal{W}(L_gf)\sigma(g)$ hence we get the first assertion. Now, let $(a,b)\in\mathbb{R}^{2n}$ and $g=[a,b,0]$. Then, by the first assertion and Lemma 4.1, we can write

$$\begin{aligned}\operatorname{Tr}(\mathcal{W}(f)R(a,b))&=\operatorname{Tr}(\mathcal{W}(f)\sigma(g)R\sigma(g)^{-1})=\operatorname{Tr}(\sigma(g)^{-1}\mathcal{W}(f)\sigma(g)R)\\&=\operatorname{Tr}(\mathcal{W}(L_{g^{-1}}f)R)=(L_{g^{-1}}f)(0,0)=f(a,b)\end{aligned}$$

and the second assertion is proved. ∎

So, we can see that $\mathcal{W}^{-1}$ appears as a Stratonovich-Weyl correspondence for $(G_0,\sigma,\mathbb{R}^{2n})$ and that $\mathcal{W}^{-1}$ can be recovered from the operators $R(a,b)$, $a,b\in\mathbb{R}^n$, via the formula $\mathcal{W}^{-1}(A)(a,b)=\operatorname{Tr}(AR(a,b))$ for each trace class operator A on $L^2(\mathbb{R}^n)$ (see also [13]).
Also, it is well-known that the Weyl calculus can be extended to much larger classes of symbols (see for instance [24]). Here we only consider a class of C^∞-functions $f(x,y)$ which are polynomials in the variable y. If $f(x,y)=u(x)y^\alpha$ where $u\in C^\infty(\mathbb{R}^n)$ then we have

$$(\mathcal{W}(f)\phi)(x)=\left(i\frac{\partial}{\partial y}\right)^{\alpha}\big(u(x+\tfrac{1}{2}y)\,\phi(x+y)\big)\Big|_{y=0},\tag{5}$$

see [34]. In particular, if $f(x,y)=u(x)$ then $(\mathcal{W}(f)\phi)(x)=u(x)\,\phi(x)$ and if $f(x,y)=u(x)y_k$ then

$$(\mathcal{W}(f)\phi)(x)=i\big(\tfrac{1}{2}\partial_ku(x)\,\phi(x)+u(x)\partial_k\phi(x)\big).\tag{6}$$

5. Stratonovich-Weyl correspondence for $SO(n)$

For a detailed study of the Stratonovich-Weyl correspondences for compact Lie groups, we refer the reader to [19], [7] and in the particular case of $SU(2)$ to [22] and [31].

Here we first introduce the Berezin correspondence associated with ρ, see [5], [6], [4], [35] and [10].

Without loss of generality, we can assume that E is a space of holomorphic sections of a complex line bundle over $o(\varphi_0)$. Let $\varphi \in o(\varphi_0)$. For each $\hat{\varphi} \neq 0$ in the fiber over φ, there exists a unique section $e_{\hat{\varphi}} \in E$ (a coherent state) such that $a(\varphi) = \langle a, e_{\hat{\varphi}} \rangle_E e_{\hat{\varphi}}$ for each $a \in E$.

The Berezin calculus on $o(\varphi_0)$ associates with each operator B on E the complex-valued function $s(B)$ on $o(\varphi_0)$ defined by

$$s(B)(\varphi) = \frac{\langle B e_{\hat{\varphi}}, e_{\hat{\varphi}} \rangle_E}{\langle e_{\hat{\varphi}}, e_{\hat{\varphi}} \rangle_E}$$

which is called the symbol of B. We denote by $Sy(o(\varphi_0))$ the space of all such symbols. Then we have the following proposition, see [17], [4] and [10].

Proposition 5.1. (1) *The map* $B \to s(B)$ *from* $\mathrm{End}(E)$ *onto* $Sy(o(\varphi_0))$ *is a linear isomorphism.*

(2) *For each operator* B *on* E*, we have* $s(B^*) = \overline{s(B)}$.

(3) *For each* $\varphi \in o(\varphi_0)$*,* $h \in K_0$ *and* $B \in \mathrm{End}(E)$*, we have*

$$s(B)(h \cdot \varphi) = s(\rho(h) B \rho(h)^{-1})(\varphi).$$

(4) *For each* $U \in \mathfrak{k}_0$ *and* $\varphi \in o(\varphi_0)$*, we have* $s(d\rho(U))(\varphi) = i\langle \varphi, U \rangle$.

We equip $\mathrm{End}(E)$ with the Hilbert-Schmidt norm. We consider $Sy(o(\varphi_0))$ as a (finite dimensional) subspace of $L^2(o(\varphi_0), \nu)$ where ν denotes the invariant measure on $o(\varphi_0)$. Then we can introduce the unitary part w in the polar decomposition of $s : \mathrm{End}(E) \to Sy(o(\varphi_0))$. We immediately see that w inherits some properties from s and that w is a Stratonovich-Weyl correspondence for $(K_0, \rho, o(\varphi_0))$ [19], [12]. Moreover, for each $\varphi \in o(\varphi_0)$, there exists a unique $\omega(\varphi) \in \mathrm{End}(E)$ such that, for each $B \in \mathrm{End}(E)$,

$$w(B)(\varphi) = \mathrm{Tr}(B\omega(\varphi)). \tag{7}$$

The map $\varphi \to \omega(\varphi)$ is called a *Stratonovich-Weyl quantizer* and the properties of w are reflected by similar properties of ω, see for instance [21]. In particular, the invariance property of w is equivalent to the fact that for each $h \in K_0$ and $\varphi \in o(\varphi_0)$, we have

$$\omega(h \cdot \varphi) = \rho(h)\omega(\varphi)\rho(h)^{-1}. \tag{8}$$

In the rest of this paper, we fix a section (defined on a dense open subset of $o(\varphi_0)$) $\varphi \to h_\varphi$ for the action of K_0 on $o(\varphi_0)$. Such sections always exist, see [14].

6. Stratonovich-Weyl quantizer for G

The aim of this section is to introduce a good candidate for the Stratonovich-Weyl quantizer associated with π. The first step is to parametrize $\mathcal{O}(\xi_0)$. We denote by V_0 the subspace of V generated by $e_1, e_2, \ldots, e_n$.

Proposition 6.1. *Let Ψ be the map from $Z(p_0)\times V_0\times o(\varphi_0)$ to $\mathfrak{g}^*$ defined by*

$$\Psi(q,v,\varphi)=(q,M(q)\cdot(\varphi+v\wedge p_0)).$$

Then we have

(1) *Ψ is a diffeomorphism from $Z(p_0)\times V_0\times o(\varphi_0)$ onto $\mathcal{O}(\xi_0)$;*

(2) *The image by Ψ of the measure $d\mu(p)dv\,d\nu(\varphi)$ on $Z(p_0)\times V_0\times o(\varphi_0)$ is an invariant measure on $\mathcal{O}(\xi_0)$ which will be denoted by μ_0;*

(3) *The map $\xi=\Psi(q,v,\varphi)\to g_\xi:=(M(q)\cdot v, M(q)\cdot h_\varphi)$ is a section for the action of G on $\mathcal{O}(\xi_0)$, that is we have $g_\xi\cdot\xi_0=\xi$ for each $\xi\in\mathcal{O}(\xi_0)$.*

Proof. (1) Let (q,v,φ) and (q',v',φ') in $Z(p_0)\times V_0\times o(\varphi_0)$ be such that $\Psi(q,v,\varphi)=\Psi(q',v',\varphi')$. Then we have $q=q'$ and $\varphi-\varphi'=(v'-v)\wedge p_0$. Since $\varphi-\varphi'\in\mathfrak{p}$ and $(v'-v)\wedge p_0\in\mathfrak{k}_0$, we get $\varphi=\varphi'$ and $(v'-v)\wedge p_0=0$ hence $v=v'$ by recalling that the elements $e_1\wedge e^*_{n+1}, e_2\wedge e^*_{n+1},\ldots,e_n\wedge e^*_{n+1}$ form a basis of $\mathfrak{p}$, see Section 2. Consequently, Ψ is one-to-one.
Now, let $(q,v,\varphi)\in Z(p_0)\times V_0\times o(\varphi_0)$. Let $h\in K_0$ such that $h\cdot\varphi_0=\varphi$. Then we have

$$(M(q)\cdot v, M(q)\cdot h)\cdot(p_0,\varphi_0)=(q,M(q)\cdot\varphi+(M(q)\cdot v)\wedge q)=\Psi(q,v,\varphi).$$

This shows that Ψ takes values in $\mathcal{O}(\xi_0)$. On the other hand, for each $\xi\in\mathcal{O}(\xi_0)$, there exists an element (w,k) in G such that $(w,k)\cdot\xi_0=\xi$. Then, putting $q=k\cdot p_0$ and writing $k=M(q)h$ for the Cartan decomposition of $k\in K$ where $q=k\cdot p_0$, we get

$$\xi=(q,M(q)\cdot(h\cdot\varphi_0)+w\wedge q)=\big(q,M(q)\cdot(h\cdot\varphi_0+(M(q)^{-1}\cdot w)\wedge p_0)\big).$$

Thus, denoting by v the projection of $M(q)^{-1}\cdot w$ on V_0 in the decomposition $V\simeq V_0\times\mathbb{R}$, we find $\xi=\Psi(q,v,h\cdot\varphi_0)$. Hence Ψ is surjective.

(2) First, we write the action of G on $\mathcal{O}(\xi_0)$ in the coordinates (q,v,φ). Let $g=(w,k)\in G$ and $(q,v,\varphi)\in Z(p_0)\times V_0\times o(\varphi_0)$. Let $h:=M(k\cdot q)^{-1}kM(q)\in K_0$ and let w' be the projection of $M(k\cdot q)^{-1}\cdot w$ on V_0 with respect to $V\simeq V_0\times\mathbb{R}$. Then we have

$$\begin{aligned}g\cdot\Psi(q,v,\varphi)=&(k\cdot q, M(k\cdot q)\cdot(h\cdot\varphi+(h\cdot v+w')\wedge p_0))\\=&\Psi(k\cdot q,h\cdot v+w',h\cdot\varphi).\end{aligned}$$

From this, it is clear that $d\mu(p)dv\,d\nu(\varphi)$ is let invariant by the action of G on $Z(p_0)\times V_0\times o(\varphi_0)$ which corresponds to the action of G on $\mathcal{O}(\xi_0)$, hence the result. This also implies that Ψ is regular hence a diffeomorphism and therefore we have completed the proof of (1). Finally, (3) is easy to verify. ∎

The following lemma is useful for starting the construction of a Stratonovich-Weyl quantizer. Recall that $\mathcal{H} := L^2(Z(p_0), E)$.

Lemma 6.2. *Let $\Omega(\xi_0)$ be an operator on $\mathcal{H}$. For each $\xi \in \mathcal{O}(\xi_0)$, let*

$$\Omega(\xi) := \pi(g_\xi)\Omega(\xi_0)\pi(g_\xi)^{-1}. \tag{9}$$

Then we have that $\Omega(g' \cdot \xi) := \pi(g')\Omega(\xi_0)\pi(g')^{-1}$ for each $g' \in G$ and $\xi \in \mathcal{O}(\xi_0)$, if and only if $\Omega(\xi_0)$ commute to the operator $\pi(g)$ for each g in the stabilizer $G(\xi_0)$ of ξ_0 in G.

Proof. The proof is easy and left as an exercice. ∎

Now, in the spirit of [18], we combine the operator R of Section 4 with $\omega(\varphi_0)$ (see Section 5) and $\Omega(\xi_0)$ is then taken to be of the form

$$(\Omega(\xi_0)\psi)(p) := a(p_{n+1})\omega(\varphi_0)\psi(p^s). \tag{10}$$

Here the function $a : \mathbb{R} \to \mathbb{R}$ is a normalization factor which is necessary to ensure unitarity of the Stratonovich-Weyl correspondence. Note also that we have $G(\xi_0) = \mathbb{R}e_{n+1} \times K_0(\varphi_0)$ where $K_0(\varphi_0)$ denotes the stabilizer of φ_0 in K_0. From this we see immediately that $\Omega(\xi_0)$ commutes to $\pi(g)$ for each $g \in G(\xi_0)$ hence we can apply Lemma 6.2.
In the rest of the section, we give an explicit expression for $\Omega(\xi)$, $\xi \in \mathcal{O}(\xi_0)$. For convenience, we denote $\alpha(p) := p_{n+1}$ for $p \in Z(p_0)$.

Proposition 6.3. *Let $\xi = \Psi(q, v, \varphi)$ where $(q, v, \varphi) \in Z(p_0) \times V_0 \times o(\varphi_0)$. Then, for each $\psi \in \mathcal{H}$ and $p \in Z(p_0)$, we have*

$$\begin{aligned}(\Omega(\xi)\psi)(p) = {} & e^{i(\langle M(q)^{-1}\cdot p-(M(q)^{-1}\cdot p)^s, v\rangle)}\, a(\alpha(M(q)^{-1} \cdot p)) \\ & \times \rho\left(M(p)^{-1}M(q)M(M(q)^{-1} \cdot p)\right)\omega(\varphi) \\ & \times \rho\left(M(M(q)^{-1} \cdot p)M(q)^{-1}M(s_q(p))\right)\, \psi(s_q(p)).\end{aligned}$$

Proof. This is an easy but tedious computation based on equation (9), in which we use (5) of Proposition 2.1 and equation (8). ∎

7. Stratonovich-Weyl correspondence for G

In this section we show that the Stratonovich-Weyl quantizer Ω introduced in Section 6 leads actually to a Stratonovich-Weyl correspondence.
For simplicity, for each $p, q \in Z(p_0)$, we set

$$\begin{aligned} m_1(p,q) :={} & M(p)^{-1}M(q)M(M(q)^{-1} \cdot p); \\ m_2(p,q) :={} & M(M(q)^{-1} \cdot p)M(q)^{-1}M(s_q(p)). \end{aligned}$$

Also, for each $(q, v, \varphi) \in Z(p_0) \times V_0 \times o(\varphi_0)$ and $p \in Z(p_0)$, we set

$$\begin{aligned}\beta(\xi, p) := {} & e^{i(\langle M(q)^{-1}\cdot p-(M(q)^{-1}\cdot p)^s, v\rangle)} a(\alpha(M(q)^{-1} \cdot p)) \\ & \cdot \rho(m_1(p,q))\omega(\varphi)\rho(m_2(p,q)) \end{aligned} \tag{11}$$

where $\xi = \Psi(q, v, \varphi)$. Then we have, for each $\psi \in \mathcal{H}$,

$$(\Omega(\xi)\psi)(p) = \beta(\xi, p)\psi(s_q(p)). \tag{12}$$

For each trace class operator A on $\mathcal{H}$, we define

$$W(A)(\xi) := \mathrm{Tr}(A\Omega(\xi))$$

and we will show that we can choose the function a in equation (10) so that W is a Stratonovich-Weyl correspondence.
For each trace class operator A on $\mathcal{H}$, we denote by K_A the *kernel of* A, that is, for each $\psi \in \mathcal{H}$ and $p \in Z(p_0)$, we have

$$(A\psi)(p) = \int_{Z(p_0)} K(p, q)\psi(q)\, d\mu(q).$$

Now, we give an integral formula for W.

Proposition 7.1. *Let A be a trace class operator on $\mathcal{H}$. Let $\xi = \Psi(q, v, \varphi)$ where $(q, v, \varphi) \in Z(p_0) \times V_0 \times o(\varphi_0)$. Then we have*

$$\begin{aligned} W(A)(\xi) = \int_{\mathbb{R}^n} & e^{-2i\langle \tilde{p}, v\rangle} w(\rho(m_2(p, M(q)\cdot p^s)) K_A(M(q)\cdot p, M(q)\cdot p^s) \\ & \times \rho(m_1(p, M(q)\cdot p^s)))(\varphi) \tfrac{a(\alpha(p))}{\alpha(p)}\, d\tilde{p}. \end{aligned}$$

Proof. Let A be a trace class operator on $\mathcal{H}$ and $\xi = \Psi(q, v, \varphi)$. Clearly, the kernel of $A\Omega(\xi)$ is given by

$$K_{A\Omega(\xi)}(p, q') = K_A(p, s_q(q'))\beta(\xi, s_q(q')).$$

Now, we apply the Mercer's theorem and we make the change of variables $p \to M(q)\cdot p$. Then we get

$$\begin{aligned} W(A)(\xi) &= \int_{Z(p_0)} \mathrm{Tr}(K_{A\Omega(\xi)}(p, p))\, d\mu(p) \\ &= \int_{Z(p_0)} \mathrm{Tr}(K_A(M(q)\cdot p, M(q)\cdot p^s)\beta(\xi, M(q)\cdot p^s)\, d\mu(p). \end{aligned}$$

But equation (11) gives

$$\beta(\xi, M(q)\cdot p^s) = e^{-2i\langle \tilde{p}, v\rangle} a(\alpha(p))\rho(m_1(p, M(q)\cdot p^s))\omega(\varphi)\rho(m_2(p, M(q)\cdot p^s)).$$

Hence, taking into account that $d\mu(p) = (1/\alpha(p))d\tilde{p}$, the desired result is obtained by applying equation (7). ■

We need the following technical lemma.

Lemma 7.2. *Let $p \in Z(p_0)$. Then the Jacobian of the map*

$$\tilde{q} \to \tilde{q}' = -\tilde{p} + \tfrac{2}{m^2}\langle p, q\rangle \tilde{q}$$

is $m^2 \left(\frac{2}{m^2}\right)^n \langle p, q\rangle^{n-1} \frac{q'_{n+1}}{q_{n+1}}$.

Proof. We can compute the differential of the function f defined on $\mathbb{R}^n$ by

$$f(x) = (-\langle \tilde{p}, x\rangle + p_{n+1}\sqrt{m^2+|x|^2})x.$$

We easily find that for each $x, y \in \mathbb{R}^n$ we have

$$df(x)(y) = \langle -\tilde{p} + \tfrac{p_{n+1}}{x_{n+1}}, y\rangle x + \langle p, \bar{x}\rangle y$$

where $x_{n+1} := \sqrt{m^2+|x|^2}$ and $\bar{x} = (x, x_{n+1})$.
Now, let $w_1, w_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. This is a simple exercice to verify that the determinant of the linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$y \to \langle w_1, y\rangle w_2 + \lambda y$$

is $\lambda^{n-1}(\langle w_1, w_2\rangle + \lambda)$. Applying this, we find that the Jacobian of f at point $\tilde{q}$ is

$$\langle p, q\rangle^{n-1}(\langle p, q\rangle + \langle -\tilde{p} + \tfrac{p_{n+1}}{q_{n+1}}\tilde{q}, \tilde{q}\rangle).$$

Moreover, we have

$$\begin{aligned}\langle p, q\rangle - \langle \tilde{p} + \tfrac{p_{n+1}}{q_{n+1}}\tilde{q}, \tilde{q}\rangle =& 2\langle p, q\rangle - p_{n+1}q_{n+1} + \tfrac{p_{n+1}}{q_{n+1}}|\tilde{q}|^2\\ =& 2\langle p, q\rangle + \tfrac{p_{n+1}}{q_{n+1}}(|\tilde{q}|^2 - q_{n+1}^2)\\ =& 2\langle p, q\rangle - m^2\tfrac{p_{n+1}}{q_{n+1}} = m^2\tfrac{q'_{n+1}}{q_{n+1}}\end{aligned}$$

by equation (2). The result follows. ∎

For each Hilbert-Schmidt operator A on $\mathcal{H}$, we denote by $\|A\|_2$ the Hilbert-Schmidt norm of A. Moreover, for each function $F : \mathcal{O}(\xi_0) \to \mathbb{C}$ which is square integrable with respect to μ_0, we set

$$\|F\|_2^2 := \int_{\mathcal{O}(\xi_0)} |F(\xi)|^2\, d\mu_0(\xi).$$

Proposition 7.3. *If we take $a(x) = m^{(1-n)/2}(2/\pi)^{n/2}x^{n/2}$ then, for each trace class operator A on $\mathcal{H}$, we have $\|W(A)\|_2 = \|A\|_2$.*

Proof. By parametrizing $\mathcal{O}(\xi_0)$ by $Z(p_0) \times V_0 \times o(\varphi_0)$ as in Proposition 6.1, we can write

$$\|W(A)\|_2^2 = \int_{Z(p_0)\times V_0\times o(\varphi_0)} |W(A)(\Psi(q, v, \varphi))|^2\, d\mu(q)\, dv\, d\nu(\varphi).$$

Now, by interpreting the integral expression for $W(A)(\xi)$ given in Proposition 7.1 as a Fourier transform and by using the Fourier inversion theorem, we get

$$\begin{aligned}\|W(A)\|_2^2 = \pi^n \int_{Z(p_0)\times V_0\times o(\varphi_0)} &|w(\rho(m_2(p, M(q)\cdot p^s))K_A(M(q)\cdot p, M(q)\cdot p^s)\\ &\times \rho(m_1(p, M(q)\cdot p^s)))(\varphi)|^2 \left(\tfrac{a(\alpha(p))}{\alpha(p)}\right)^2 d\mu(q)\, dv\, d\nu(\varphi).\end{aligned}$$

Recall that w is unitary in the sense that for each operator B on E we have

$$\|B\|_2^2 = \int_{o(\varphi_0)} |w(B)(\varphi)|^2 d\nu(\varphi).$$

Then we have $\|W(A)\|_2^2 =$

$$= \pi^n \int_{Z(p_0)\times\mathbb{R}^n} \|\rho(m_2(p, M(q)\cdot p^s))K_A(M(q)\cdot p, M(q)\cdot p^s)\rho(m_1(p, M(q)\cdot p^s))\|_2^2 \times \left(\tfrac{a(\alpha(p))}{\alpha(p)}\right)^2 d\mu(q)\, d\tilde{p}$$

$$= \pi^n \int_{Z(p_0)\times\mathbb{R}^n} \|K_A(M(q)\cdot p, M(q)\cdot p^s)\|_2^2 \left(\tfrac{a(\alpha(p))}{\alpha(p)}\right)^2 d\mu(q)\, d\tilde{p}$$

$$= \pi^n \int_{Z(p_0)\times\mathbb{R}^n} \|K_A(p, M(q)\cdot (M(q)^{-1}\cdot p)^s)\|_2^2 \tfrac{a(\alpha(M(q)^{-1}\cdot p))^2}{\alpha(p)\alpha(M(q)^{-1}\cdot p)}\, d\mu(q)\, d\tilde{p}.$$

But by Proposition 2.1, if $q' = s_q(p)$ then we have

$$\tilde{q}' = -\tilde{p} + \tfrac{2}{m^2}\langle p, q\rangle \tilde{q}$$

so that, by taking Lemma 7.2 into account, we obtain

$$\|W(A)\|_2^2 = 2^{-n}\pi^n m^{n-1} \int_{Z(p_0)\times Z(p_0)} \|K_A(p, q')\|_2^2 \frac{a(\frac{1}{m}\langle p, q\rangle)^2}{\frac{1}{m^n}\langle p, q\rangle^n} d\mu(q')\, d\mu(p).$$

Hence, by taking a as above, we finally get

$$\|W(A)\|_2^2 = \int_{Z(p_0)\times Z(p_0)} \|K_A(p, q')\|_2^2 d\mu(q')\, d\mu(p) = \|A\|_2^2.$$

This completes the proof. ∎

Since the trace class operators on $\mathcal{H}$ are dense in the Hilbert-Schmidt class, an immediate consequence of Proposition 7.3 is that W can be extended to a unique unitary map (also denoted by W) from the space of all Hilbert-Schmidt operators on $\mathcal{H}$ to $L^2(M, \mu)$.

Proposition 7.4. *The map W is a Stratonovich-Weyl correspondence for the triple $(G, \pi, \mathcal{O}(\xi_0))$, namely, for each Hilbert-Schmidt operator A on $\mathcal{H}$ we have*

(1) $W(\pi(g)\, A\, \pi(g)^{-1})(\xi) = W(A)(g^{-1}\cdot\xi)$ *for each* $g \in G$ *and* $\xi \in \mathcal{O}(\xi_0)$*;*

(2) $\|W(A)\|_2 = \|A\|_2$*;*

(3) $W(A^*) = \overline{W(A)}$.

Proof. It is sufficient to prove the assertions for trace class operators. Assertion (1) immediately follows from the invariance property of Ω and Assertion (2) from Proposition 7.3 . Moreover, to prove Assertion (3), first note that $\Omega(\xi_0)$ is self-adjoint since $\omega(\varphi_0)$ is. This implies that, for each $\xi \in \mathcal{O}(\xi_0)$, $\Omega(\xi)$ is also self-adjoint. Then, for each trace class operator A on $\mathcal{H}$ and each $\xi \in \mathcal{O}(\xi_0)$, we have

$$W(A^*)(\xi) = \mathrm{Tr}(A^*\Omega(\xi)) = \overline{\mathrm{Tr}(\Omega(\xi)A)} = \overline{W(A)(\xi)}.$$

Hence the proposition is proved. ∎

8. Extension of the Stratonovich-Weyl correspondence

The classical Weyl correspondence can be extended to symbols which are not necessarily square integrable functions and to operators which are not necessarily Hilbert-Schmidt, see Section 4. Here we aim to extend similarly W to operators which are not Hilbert-Schmidt. Since our method is based on the properties of the classical Weyl correspondence, it is convenient to work with π_0 instead of π (see Section 3).

Clearly, we can define a Stratonovich-Weyl correspondence W_0 for the triple $(G, \pi_0, \mathcal{O}(\xi_0))$ by the equation

$$W_0(A_0) := W(JA_0J^{-1})$$

for each trace class operator A_0 on $\mathcal{H}_0 = L^2(\mathbb{R}^n, E)$, with the notation of Section 3. The Stratonovich-Weyl quantizer Ω_0 corresponding to W_0 is then

$$\Omega_0(\xi) := J\Omega(\xi)J^{-1}, \qquad \xi \in \mathcal{O}(\xi_0).$$

In particular, from equation (10) we get immediately

$$(\Omega_0(\xi_0)\phi)(\tilde{p}) = c_{m,n}p_{n+1}^{n/2}\omega(\varphi_0)\phi(-\tilde{p}) \tag{13}$$

for each $\phi \in \mathcal{H}_0$, $c_{m,n}$ denoting the constant $m^{(1-n)/2}(2/\pi)^{n/2}$.

Let us introduce some additional notation. Clearly, one has $\mathcal{H}_0 = L^2(\mathbb{R}^n) \otimes E$. For each $\phi_0 \in L^2(\mathbb{R}^n)$ and each $v \in E$ we denote by $\phi_0 \otimes v$ the function $x \to \phi_0(x)v$. Moreover, if A_1 is an operator on $\mathcal{H}_0$ and A_2 is an operator on E then we denote by $A_1 \otimes A_2$ the operator on $\mathcal{H}$ defined by $(A_1 \otimes A_2)(\phi_0 \otimes v) = A_1\phi_0 \otimes A_2 v$.

Also, if f_1 is a complex valued function on $\mathbb{R}^{2n}$ and f_2 is a complex valued function on $o(\varphi_0)$, we denote by $f_1 \otimes f_2$ the function on $\mathbb{R}^{2n} \times o(\varphi_0)$ defined by $(f_1 \otimes f_2)(x, y, \varphi) = f_1(x, y) f_2(\varphi)$ for $x, y \in \mathbb{R}^n$ and $\varphi \in o(\varphi_0)$.

Recall that R denotes the operator on $L^2(\mathbb{R}^n)$ defined by $(R\phi_0)(\tilde{p}) = 2^n\phi_0(-\tilde{p})$. Let M be the operator on $L^2(\mathbb{R}^n)$ defined by

$$(M\phi_0)(\tilde{p}) = 2^{-n}c_{m,n}p_{n+1}^{n/2}\phi_0(\tilde{p}).$$

Now, note that if f is a function on $\mathbb{R}^{2n} \times o(\varphi_0)$ which is of the form $f = f_1 \otimes f_2$ as above with $f_2 \in Sy(o(\varphi_0))$ (see Section 5), then we can consider the operator $\mathcal{W}'(f)$ on $\mathcal{H}_0$ defined by

$$\mathcal{W}'(f) := \mathcal{W}(f_1) \otimes w^{-1}(f_2).$$

Of course, we can extend $\mathcal{W}'$ to finite sums of such functions f.

Let us make the following remark. Let A_1 be a trace class operator on $L^2(\mathbb{R}^n)$ and A_2 an operator on E. Let $A_0 = A_1 \otimes A_2$. Then we have, by Lemma 4.1,

$$\begin{aligned} W_0(A_0)(\xi_0) &= \operatorname{Tr}(A_0\Omega_0(\xi_0)) = \operatorname{Tr}(A_1MR)\operatorname{Tr}(A_2\omega(\varphi_0)) \\ &= \mathcal{W}^{-1}(A_1M)(0,0)w(A_2)(\varphi_0) = \mathcal{W}'^{-1}((M \otimes Id_E)A_0)(0,0,\varphi_0). \end{aligned}$$

This naturally suggests to extend W as follows. If A_0 is an operator on $\mathcal{H}_0$, we set

$$W_0(A_0)(\xi) = \mathcal{W}'^{-1}((M \otimes Id_E)\pi_0(g_\xi)A_0\pi_0(g_\xi)^{-1})(0, 0, \varphi_0)$$

provided that $\mathcal{W}'^{-1}((M \otimes Id_E)\pi_0(g_\xi)A_0\pi_0(g_\xi)^{-1})$ is well-defined. In the rest of this section, we give some simple examples of operators A_0 such that $W_0(A_0)$ can be defined by this way.
We need the following lemma. Let $p_{\mathfrak{k}_0}$ and $p_{\mathfrak{p}}$ be the projection operators of $\mathfrak{k}$ onto $\mathfrak{k}_0$ and $\mathfrak{p}$ associated with the decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{p}$.

Lemma 8.1. [8], [9]

(1) *For each $p \in Z(p_0)$ and $U \in \mathfrak{k}$ we have*

$$\begin{aligned} L(p,U) :=& \frac{d}{dt}\left(M(p)^{-1}\exp(tU)M(\exp(-tU)\cdot p)\right)\Big|_{t=0} \\ =& p_{\mathfrak{k}_0}(U) - \tanh\Big(\tfrac{1}{2}\operatorname{ad} T(p)\Big)p_{\mathfrak{p}}(U). \end{aligned}$$

(2) *For each $X = (u, U) \in \mathfrak{g}$, we have*
$(d\pi_0(X)\phi)(\tilde{p}) = i\langle p, u\rangle\phi(\tilde{p}) + d\rho(L(p,U))\phi(\tilde{p}) - \frac{(U\cdot p)_{n+1}}{2p_{n+1}}\phi(\tilde{p}) - d\phi(\tilde{p})(\widetilde{U\cdot p})$.
More precisely, we have

$$(d\pi_0(0, A_{ij})\phi)(\tilde{p}) = d\rho(A_{ij})\phi(\tilde{p}) + p_j\frac{\partial\phi}{\partial p_i}(\tilde{p}) - p_i\frac{\partial\phi}{\partial p_j}(\tilde{p}), \quad 1 \le i, j \le n,$$

$$(d\pi_0(0, T_k)\phi)(\tilde{p}) = d\rho\left(L(p, T_k)\right)\phi(\tilde{p}) - p_{n+1}\frac{\partial\phi}{\partial p_k}(\tilde{p}) - \frac{p_k}{2p_{n+1}}\phi(\tilde{p}), 1 \le k \le n,$$

$$(d\pi_0(e_k, 0)\phi)(\tilde{p}) = -ip_k\phi(\tilde{p}), \quad 1 \le k \le n,$$

$$(d\pi_0(e_{n+1}, 0)\phi)(\tilde{p}) = ip_{n+1}\phi(\tilde{p}).$$

From this lemma, we can deduce the following result.

Proposition 8.2. *For each $X_1, X_2, \ldots, X_p \in \mathfrak{g}$, $W_0(d\pi_0(X_1X_2\cdots X_p))$ is well-defined.*

Proof. Let $X_1, X_2, \ldots, X_p \in \mathfrak{g}$. Let $\xi \in \mathcal{O}(\xi_0)$. Define $Y_k := \operatorname{Ad}(g_\xi)^{-1}X_k$ for $k = 1, 2, \ldots, p$. Then one has

$$\pi_0(g_\xi)^{-1}d\pi_0(X_1X_2\cdots X_p)\pi_0(g_\xi) = d\pi_0(Y_1Y_2\cdots Y_p).$$

Now, by induction we see that $d\pi_0(Y_1Y_2\cdots Y_p)$ is a sum of operators of the form $A_1 \otimes A_2$ where A_1 is a differential operator on $\mathbb{R}^n$ with polynomial coefficients and A_2 an operator on E. Then $W_0(d\pi_0(Y_1Y_2\cdots Y_p))(\xi_0)$ is well-defined hence $W_0(d\pi_0(X_1X_2\cdots X_p))$ is. ■

Now, we compute $W_0(d\pi_0(X))$ for $X \in \mathfrak{g}$.

Proposition 8.3. (1) *We have*

$$W_0(d\pi_0(0, A_{ij}))(\xi_0) = 2^{-n} c_{m,n} m^{\frac{n}{2}} w(d\rho(A_{ij}))(\varphi_0), \quad 1 \leq i, j \leq n,$$
$$W_0(d\pi_0(0, T_k))(\xi_0) = 0, \quad W_0(d\pi_0(e_k, 0))(\xi_0) = 0, \quad 1 \leq k \leq n,$$
$$W_0(d\pi_0(e_{n+1}, 0))(\xi_0) = i2^{-n} c_{m,n} m^{1+\frac{n}{2}}.$$

(2) *For each* $X = (u, U) \in \mathfrak{g}$ *and each* $\xi = \Psi(q, v, \varphi) \in \mathcal{O}(\xi_0)$, *we have*

$$W_0(d\pi_0(X))(\xi) = 2^{-n} c_{m,n} m^{\frac{n}{2}} \Big(i\langle q, u\rangle + i\langle (\mathrm{Ad}(M(q))^{-1}U) \cdot v, p_0\rangle$$
$$+ w\big(p_{\mathfrak{k}_0}(\mathrm{Ad}(M(q))^{-1}U)\big)(\varphi)\Big).$$

Proof. (1) Let $X = (0, A_{ij})$ with $1 \leq i, j \leq n$. Then, by Lemma 8.1, we can write

$$(M \otimes \mathrm{id}_E) d\pi_0(X) = M \otimes d\rho(A_{ij}) + 2^{-n} c_{m,n} p_{n+1}^{\frac{n}{2}} \left(p_j \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial p_j} \right) \otimes \mathrm{id}_E$$

and, by equation (6), we find easily that

$$\mathcal{W}^{-1} \left(p_{n+1}^{\frac{n}{2}} \left(p_j \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial p_j} \right) \right) = p_{n+1}^{\frac{n}{2}} (p_j q_i - p_i q_j)$$

hence

$$\mathcal{W}^{-1} \left(p_{n+1}^{\frac{n}{2}} \left(p_j \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial p_j} \right) \right) (0,0) = 0.$$

Consequently, we get

$$\mathcal{W}'^{-1} \left((M \otimes \mathrm{id}_E) d\pi_0(X) \right) (0, 0, \varphi_0) = 2^{-n} m^{n/2} c_{m,n} w(d\rho(A_{ij}))(\varphi_0).$$

Now, let $X = (0, T_k)$ with $1 \leq k \leq n$. For each $p \in Z(p_0)$, we write $L(p, T_k) = \sum_{i,j} a_{ij}(p) A_{ij}$ for the decomposition of $L(p, T_k) \in \mathfrak{k}_0$ in the basis (A_{ij}). We get

$$(M \otimes \mathrm{id}_E) d\pi_0(X)$$
$$= 2^{-n} c_{m,n} \left(\sum_{i,j} a_{ij}(p) p_{n+1}^{\frac{n}{2}} \otimes d\rho(A_{ij}) - \left(p_{n+1}^{\frac{n}{2}+1} \frac{\partial}{\partial p_k} - \tfrac{1}{2} p_k p_{n+1}^{\frac{n}{2}-1} \right) \otimes \mathrm{id}_E \right).$$

But on the one hand we can verify that

$$\mathcal{W}^{-1} \left(p_{n+1}^{\frac{n}{2}+1} \frac{\partial}{\partial p_k} + \tfrac{1}{2} p_k p_{n+1}^{\frac{n}{2}-1} \right) (0,0) = 0$$

and on the other hand we have

$$\mathcal{W}'^{-1} \Big(\sum_{i,j} a_{ij}(p) p_{n+1}^{\frac{n}{2}} \otimes d\rho(A_{ij}) \Big) (0, 0, \varphi_0) = m^{\frac{n}{2}} \sum_{i,j} a_{ij}(p_0) w(d\rho(A_{ij}))(\varphi_0)$$
$$= m^{\frac{n}{2}}\, w\Big(\sum_{i,j} a_{ij}(p_0) A_{ij} \Big)(\varphi_0) = 0$$

since $\sum_{i,j} a_{ij}(p_0)A_{ij} = L(p_0, T_k) = 0$ by Lemma 8.1. Hence $W_0(d\pi_0(0, T_k))(\xi_0) = 0$ for $k = 1, 2, \dots, n$. The computations are similar (and more simple) for the remaining cases.

(2) Let $X = (u, U) \in \mathfrak{g}$. By (1) we have

$$W(d\pi_0(X))(\xi_0) = 2^{-n} m^{n/2} c_{m,n} \left(i\langle p_0, u\rangle + w(p_{\mathfrak{k}_0}(U))(\varphi_0)\right).$$

Let $(q, v, \varphi) \in Z(p_0) \times V_0 \times o(\varphi_0)$ and $\xi = \Psi(q, v, \varphi)$. Then we have

$$W(d\pi_0(X))(\xi) = W(d\pi_0(\mathrm{Ad}(g_\xi)^{-1}X))(\xi_0).$$

Recall that $g_\xi = (M(q) \cdot v, M(q)u_\varphi)$ and note that

$$\mathrm{Ad}(g_\xi)^{-1}X = \big(u_\varphi^{-1}M(q)^{-1} \cdot u + u_\varphi^{-1} \cdot (\mathrm{Ad}(M(q))^{-1}U) \cdot v, (\mathrm{Ad}(M(q)u_\varphi)^{-1}U\big).$$

Thus we obtain

$$\begin{aligned} W(d\pi_0(X))(\xi) = 2^{-n} c_{m,n} m^{n/2} \Big(& i\langle q, u\rangle + i\langle p_0, (\mathrm{Ad}(M(q))^{-1}U) \cdot v\rangle \\ & + w\big(p_{\mathfrak{k}_0}(\mathrm{Ad}(u_\varphi^{-1}M(q))^{-1})U\big)(\varphi_0)\Big) \end{aligned}$$

hence the result since

$$\begin{aligned} w\big(p_{\mathfrak{k}_0}(\mathrm{Ad}(u_\varphi^{-1}M(q))^{-1})U)\big)(\varphi_0) &= w\big(\mathrm{Ad}(u_\varphi)^{-1}p_{\mathfrak{k}_0}(\mathrm{Ad}(M(q))^{-1})U)\big)(\varphi_0) \\ &= w\big(p_{\mathfrak{k}_0}(\mathrm{Ad}(M(q))^{-1})U)\big)(\varphi). \end{aligned}$$

This completes the proof. ∎

9. Final remarks and perspectives

9.1. In the case of the original Poincaré group $\mathbb{R}^4 \rtimes SO_0(3,1)$, we can recover the Stratonovich-Weyl correspondence constructed in [18]. This can be verified as follows. In this case ($n = 3$), equation (13) becomes

$$(\Omega_0(\xi_0)\phi)(\tilde{p}) := m^{-1}(2\pi)^{3/2} p_4^{3/2} \omega(\varphi_0)\phi(-\tilde{p}).$$

Up to normalization, this is precisely equation (36) in [18], evaluated at ξ_0.
Note also that in this case $o(\varphi_0)$ is a 2-sphere and see [19] and [31] for a detailed study of ω.

9.2. The Stratonovich-Weyl correspondence is not unique. Let $W_1, W_2 : \mathcal{L}_2(\mathcal{H}) \to L^2(\mathcal{O}(\xi_0), \mu_0)$ be two Stratonovich-Weyl correspondences. Then $T := W_2 \circ W_1^{-1}$ is a unitary operator on $L^2(\mathcal{O}(\xi_0), \mu_0)$ which commutes with complex conjugation and with the left regular representation τ of G defined on $L^2(\mathcal{O}(\xi_0), \mu_0)$ by $(\tau(g)f)(\xi_0) = f(g^{-1} \cdot \xi)$ for $g \in G$ and $\xi \in \mathcal{O}(\xi_0)$. So, as noticed in [19], the set of such operators T provides the measure of the nonuniqueness of the Stratonovich-Weyl correspondence.

For instance, suppose that τ is multiplicity-free, that is, τ can be decomposed as a direct integral $\tau \simeq \int_S \pi_s \, dm(s)$ where $S \subset \hat{G}$ and m denotes a Borel measure on $\hat{G}$. Then, writing $L^2(\mathcal{O}(\xi_0),\mu_0)\simeq\int_S \mathcal{H}_s\,dm(s)$ for the corresponding direct integral of Hilbert spaces, we see by Schur's lemma that each T can be diagonalized as $\int_S(\pm\,\mathrm{id}_{\mathcal{H}_s})\,dm(s)$.

However, we have not yet found in the literature references giving precise informations on the desintegration of τ. In contrast, there are many results on the decomposition of $L^2(G/H)$ for G semisimple Lie group with subgroup H, see for instance [26] and its references.

9.3. An alternative way to obtain a Stratonovich-Weyl correspondence is to take the Stratonovich-Weyl quantizer $\tilde{\Omega}$ as

$$(\tilde{\Omega}(\xi_0)\psi)(p)=\omega(\varphi_0)\phi(p^s).$$

Then we define $\tilde{\Omega}(\xi)$ $(\xi\in\mathcal{O}(\xi_0))$ and $\tilde{W}$ as above, that is, one has $\tilde{W}(A):=\mathrm{Tr}(A\tilde{\Omega}(\xi))$ for each trace class operator A on $\mathcal{H}$. Thus, by the proof of Proposition 7.3, we get

$$\|\tilde{W}(A)\|_2^2=\int_{Z(p_0)\times Z(p_0)}\|K_A(p,q')\|_2^2\tfrac{1}{\langle p,q\rangle^n}d\mu(q')\,d\mu(p)$$

for each trace class operator A. By using (4) of Proposition 2.1, we easily verify

$$\langle p,q'\rangle=-m^2+\tfrac{2}{m^2}\langle p,q\rangle^2,$$

hence
$$\langle p,q\rangle^2=\tfrac{m^2}{2}(m^2+\langle p,q'\rangle)\geq\tfrac{m^2}{4}.$$

Hence $\tilde{W}$ is an isomorphism from the trace class operators to $L^2(\mathcal{O}(\xi_0),\mu_0)$ which can be extended to an isomorphism from the Hilbert-Schmidt class to $L^2(\mathcal{O}(\xi_0),\mu_0)$ and then we have already constructed an invariant calculus in the sense of [3].

We can thus obtain a Stratonovich-Weyl correspondence by taking the unitary part in the polar decomposition of $\tilde{W}$, in the spirit of [19] and [12]. However, this Stratonovich-Weyl correspondence has some inconvenients: it is difficult to make explicit computations with it and to extend it to operators which are not trace class as it was done for W.

9.4. The natural generalization of the example studied here is the case of a semidirect product $G=V\rtimes K$ where K is a connected semisimple (non compact) Lie group acting linearly on a finite dimensional vector space V. Also, we assume that π is a unitary irreducible representation of G which is associated with a coadjoint orbit $\mathcal{O}$ of G and that the corresponding little group is a maximal compact K_0 of K. The formulas for π, Ψ and Ω can be easily extended to this context as well as the expression of W (Section 7) provided that $2\tilde{p}$ is replaced by $p-p^s$ in equation (7.1). But, in order to imitate the proof of Proposition 7.3, we have in particular to show that the map $p\to p-p^s$ is a diffeomorphism from $Z(p_0)$ onto a n-dimensional vector subspace of V^* and we have not yet succeeded in proving this conjecture (see, however, some partial results concerning the closed notion of invariant symbolic calculus in [16]). Moreover, the problem of extending W as in Section 8 seems to be delicate too.

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