Journal of Lie Theory
Volume **28** (2018) 1063–1094

Polynomiality for the Poisson Centre of Truncated Maximal Parabolic Subalgebras

Florence Fauquant-Millet and Polyxeni Lamprou

Communicated by A. Pasquale

Abstract. We study the Poisson centre of truncated maximal parabolic subalgebras of a simple Lie algebra of type B, D or E_6. In particular we show that this centre is a polynomial algebra and compute the degrees of its generators. In roughly half of the cases the polynomiality of the Poisson centre was already known by a completely different method. For the rest of the cases, our approach is to construct an *algebraic slice* in the sense of Kostant given by an *adapted pair* and the computation of an improved upper bound for the Poisson centre.

Mathematics Subject Classification: 16W22, 17B22, 17B35.
Key Words and Phrases: Poisson centre, parabolic subalgebras, polynomiality, adapted pairs.

1. Introduction

The base field k is assumed to be algebraically closed of characteristic zero. For any finite dimensional Lie algebra $\mathfrak{a}$ over k, the symmetric algebra $S(\mathfrak{a})$ of $\mathfrak{a}$ is equipped with the Lie-Poisson bracket. We refer to the Poisson centres and semicentres with respect to this Poisson structure.

In this paper we continue our study on the Poisson semicentre of maximal parabolic subalgebras of a simple Lie algebra over k, that we initiated in [10].

Let $\mathfrak{q}$ be a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$ over k. Recall that the semicentre $Sy(\mathfrak{q})$ of the symmetric algebra $S(\mathfrak{q})$ of $\mathfrak{q}$ is the vector space generated by the semi-invariants of $S(\mathfrak{q})$ under the adjoint action of $\mathfrak{q}$. The semicentre $Sy(\mathfrak{q})$ coincides with the Poisson semicentre of $S(\mathfrak{q})$ (of $\mathfrak{q}$ for short).

The algebra of invariants $S(\mathfrak{q})^{\mathfrak{q}}$ of $S(\mathfrak{q})$ under the adjoint action of $\mathfrak{q}$ will be denoted by $Y(\mathfrak{q})$. Again the algebra $Y(\mathfrak{q})$ coincides with the Poisson centre of $S(\mathfrak{q})$ (of $\mathfrak{q}$ for short). By a result of [1, Satz 6.1], since $\mathfrak{q}$ is algebraic, there is a canonically defined algebraic subalgebra $\mathfrak{q}_\Lambda$ of $\mathfrak{q}$, called the canonical truncation of $\mathfrak{q}$, such that $Sy(\mathfrak{q}) = Y(\mathfrak{q}_\Lambda) := S(\mathfrak{q}_\Lambda)^{\mathfrak{q}_\Lambda}$. Actually $\mathfrak{q}_\Lambda$ is the largest subalgebra of $\mathfrak{q}$ which vanishes on the weights of $Sy(\mathfrak{q})$. One has also trivially that $Sy(\mathfrak{q}_\Lambda) = Y(\mathfrak{q}_\Lambda)$.

Recall that the Poisson centre $Y(\mathfrak{q})$ of $\mathfrak{q}$ is reduced to k, when $\mathfrak{q}$ is not equal to $\mathfrak{g}$ and $\mathfrak{g}$ is simple (see for example [13, 7.9] or [5, Chap. I, Sec. B, 8.2 (iv)]), whereas the Poisson semicentre $Sy(\mathfrak{q})$ of $\mathfrak{q}$ is never reduced to scalars by [3].

ISSN 0949–5932 / $2.50

By [7] – see also [13], [14] in the more general case of biparabolic (seaweed) subalgebras – we know that $Sy(\mathfrak{q})$ is lower and upper bounded, up to gradations, by polynomial algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, having the same number of generators. The weight of each generator of $\mathcal{A}$ may either be equal or be the double of the weight of the corresponding generator of $\mathcal{B}$. Moreover, it was shown that the coincidence of the formal characters $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ of these bounds is a sufficient condition for the polynomiality of $Sy(\mathfrak{q})$. The coincidence of $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ occurs often, for instance when $\mathfrak{g}$ is simple of type A or C and $\mathfrak{q}$ is any parabolic subalgebra of $\mathfrak{g}$.

However, the coincidence of $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ is not a necessary condition for the polynomiality of the Poisson semicentre and indeed there are examples where they do not coincide but the Poisson semicentre is polynomial, for example in the Borel case [12].

Let $\mathfrak{a}$ be any finite dimensional Lie algebra over k. By [3] or [5, Chap. I, Sec. B, 5.11, 5.12] we know that each semi-invariant of the field of fractions $\operatorname{Fract} S(\mathfrak{a})$ of $S(\mathfrak{a})$ is a quotient of two semi-invariants of $S(\mathfrak{a})$. Denote by $C(\mathfrak{a}) := (\operatorname{Fract} S(\mathfrak{a}))^{\mathfrak{a}}$ the field of invariant fractions of $S(\mathfrak{a})$. In [4, Problème 4] Dixmier asked whether the field $C(\mathfrak{a})$ is a purely transcendental extension of k (refered to as Dixmier's fourth problem).

Since $Sy(\mathfrak{q}_\Lambda) = Y(\mathfrak{q}_\Lambda)$, the field $C(\mathfrak{q}_\Lambda)$ is equal to the field of fractions $\operatorname{Fract}(Y(\mathfrak{q}_\Lambda))$ of $Y(\mathfrak{q}_\Lambda)$. Hence the polynomiality of $Sy(\mathfrak{q}) = Y(\mathfrak{q}_\Lambda)$ implies that the fields $C(\mathfrak{q}_\Lambda)$ and also $C(\mathfrak{q})$ are purely transcendental extensions of the base field k (the latter result follows by [21, Thm. 66], since then there exists a set of algebraically independent generators of $Sy(\mathfrak{q})$ formed by weight vectors - that is, by semi-invariants of $S(\mathfrak{q})$). This gives a positive answer to Dixmier's fourth problem for such parabolic subalgebras. However the polynomiality of the Poisson centre $Y(\mathfrak{q}_\Lambda)$ is a much stronger result.

Recently, several authors have been interested in the question of polynomiality of the Poisson centre of non-reductive algebraic Lie algebras; parabolic and biparabolic (seaweed) subalgebras of a simple Lie algebra $\mathfrak{g}$ over k were studied in [7], [8], [13], [14] and some particular semi-direct products were studied in [23], [24], [25], [29], [30], where polynomiality of the Poisson centre was shown. In [21] the author gives necessary and sufficient conditions for the Poisson centre or semicentre of certain finite dimensional Lie algebras to be polynomial.

So far, only one counterexample to the polynomiality of the Poisson semicentre of a biparabolic subalgebra $\mathfrak{q}$ is known, namely when $\mathfrak{g}$ is of type E_8 and $\mathfrak{q}$ is the maximal parabolic subalgebra of $\mathfrak{g}$, whose canonical truncation coincides with the centralizer of the highest root vector of $\mathfrak{g}$ [28].

In [10] we studied $Sy(\mathfrak{q})$ for $\mathfrak{q}$ a maximal parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$, when the lower and upper bounds $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ coincide (hence $Sy(\mathfrak{q})$ is polynomial) and we constructed slices for the coadjoint action, extending the Kostant Slice Theorem [20, Thm. 0.10].

In this paper we study the remaining cases for $\mathfrak{g}$ simple of type B or D and $\mathfrak{q}$ a maximal parabolic subalgebra of $\mathfrak{g}$ and we deduce the polynomiality of the Poisson semicentre $Sy(\mathfrak{q})$ by constructing slices for the coadjoint action and computing an "improved upper bound" (see below). The slices we constructed in [10] were given by *adapted pairs* (see Def 2.1) for the canonical truncations $\mathfrak{q}_\Lambda$ of the parabolic subalgebras $\mathfrak{q}$ that we studied. In this paper we construct adapted pairs for the remaining cases mentioned above.

Adapted pairs play the role of principal $\mathfrak{sl}_2$-triples in the non-reductive case and were introduced in [18]. They give an improved upper bound $\mathcal{B}'$ for the character of $Sy(\mathfrak{q}) = Y(\mathfrak{q}_\Lambda)$ [16]. When this bound is attained, in particular when it coincides with the character of the lower bound $\mathcal{A}$ mentioned above, polynomiality of $Sy(\mathfrak{q})$ follows and the adapted pair gives an algebraic slice (in the sense of [17, 7.6]) also called a Weierstrass section in [9], extending the Kostant Slice Theorem [20, Thm. 0.10] to non-reductive Lie algebras. By [9], this Weierstrass section is also an affine slice for the coadjoint action (in the sense of [17, 7.3]).

Some particular cases had already been studied by other authors and different methods. For example, it was shown in [22] that for all maximal parabolic subalgebras $\mathfrak{q}$ whose canonical truncation is the centralizer of the highest root vector of the simple Lie algebra (except in type E_8, where we have Yakimova's counterexample), the Poisson semicentre $Sy(\mathfrak{q})$ is a polynomial algebra over k. Furthermore, Heckenberger [11] showed by computer calculations that in type B_n, $2 \leq n \leq 4$, the Poisson semicentre $Sy(\mathfrak{q})$ is polynomial for all parabolic subalgebras $\mathfrak{q}$.

In [27] an affine slice for the coadjoint action of $\mathfrak{q}$ was constructed for some non truncated biparabolic subalgebras $\mathfrak{q}$ of a simple Lie algebra, which gave a positive answer to Dixmier's fourth problem for $C(\mathfrak{q})$. These biparabolic subalgebras $\mathfrak{q}$ do not coincide with the maximal parabolic subalgebras we are interested in.

Below, labeling of simple roots follows Bourbaki [2, Planches I-IX].

Adapted pairs need not exist for all truncated parabolic subalgebras and are very hard to construct in general. One may hope to construct such pairs if the truncated Cartan subalgebra $\mathfrak{h}_\Lambda = \mathfrak{h} \cap \mathfrak{q}_\Lambda$ (where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$) is large enough. The subalgebra $\mathfrak{h}_\Lambda$ is large if $\mathfrak{g}$ is of type A or if the parabolic subalgebra $\mathfrak{q}$ is maximal. In type A, adapted pairs were constructed for all truncated biparabolic subalgebras in [15]. However, we showed in [10, Sect. 10] (with $\mathfrak{g}$ simple of type F_4 and $\mathfrak{q}$ the maximal parabolic subalgebra corresponding to $\pi' = \{\alpha_1, \alpha_2, \alpha_4\}$) that adapted pairs do not exist for $\mathfrak{q}_\Lambda$, although $Y(\mathfrak{q}_\Lambda)$ is polynomial.

Assume that the parabolic subalgebra $\mathfrak{q}$ is maximal and that $\mathfrak{g}$ is simple of type B_n, $n \geq 2$, resp. D_n, $n \geq 4$. Let π be a set of simple roots α_i, $1 \leq i \leq n$, in $\mathfrak{g}$ and $\pi' = \pi \setminus \{\alpha_s\}$ be the subset of π corresponding to $\mathfrak{q}$. If $\mathfrak{g}$ is of type B_n then the bounds $\mathrm{ch}\,\mathcal{A}$ and $\mathrm{ch}\,\mathcal{B}$ for $\mathrm{ch}\,Sy(\mathfrak{q})$ coincide if and only if s is odd, or $n = s = 2$, or $n = s = 4$. If $\mathfrak{g}$ is of type D_n, then the bounds $\mathrm{ch}\,\mathcal{A}$ and $\mathrm{ch}\,\mathcal{B}$ for $\mathrm{ch}\,Sy(\mathfrak{q})$ coincide if and only if s is odd and $s \neq n-1$, or $s = n-1$ and s even, or $n = 4$ and $s \neq 2$.

In this paper we give an adapted pair for the rest of the truncated maximal parabolic subalgebras in type B and D. In particular, we prove a lemma of non-degeneracy (Lemma 6.1) which is a non-obvious generalization of [10, Lemma 5].

From the case D_6, $s = 6$, we also deduce in Section 10 an adapted pair for the truncated maximal parabolic subalgebra in $\mathfrak{g}$ simple of type E_7 corresponding to $\pi' = \pi \setminus \{\alpha_3\}$.

Assume now that $\mathfrak{g}$ is simple of type E_6 and that $\mathfrak{q}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ corresponding to the subset $\pi' = \pi \setminus \{\alpha_s\}$ of the set of simple roots π of $\mathfrak{g}$. Then the bounds $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ for $\operatorname{ch} Sy(\mathfrak{q})$ do not coincide if and only if $s = 1, 2, 6$. For $s = 2$ an adapted pair for $\mathfrak{q}_\Lambda$ was already constructed in [16]. So we construct in Section 11 an adapted pair for $s = 1$ or 6.

Finally we compute the improved upper bound $\mathcal{B}'$ mentioned above, show that it is attained and deduce that the Poisson centre $Y(\mathfrak{q}_\Lambda)$ of $\mathfrak{q}_\Lambda$ is polynomial, computing also the degrees of a set of homogeneous generators (Theorems 7.10, 9.7, 10.1, 11.1). We deduce that for all such maximal parabolic subalgebras $\mathfrak{q}$, Dixmier's fourth problem has a positive answer for $C(\mathfrak{q})$. Furthermore, as in [10] we obtain an algebraic and an affine slice for the dual of $\mathfrak{q}_\Lambda$.

Acknowledgements. We would like to thank A. Joseph for many fruitful discussions on adapted pairs and for his interest in our work. We are also grateful to A. Ooms for enlightening exchange of ideas on the polynomiality of the Poisson semicentre. Part of these results were presented by the first author in the Seminar at the Weizmann Institute of Science in Israel in April 2016 and in the Conference "Algebraic Modes of Representations and Nilpotent Orbits: the Canicular Days", celebrating A. Joseph's 75th birthday, in Israel in July 2017. The second author explained also part of these results in the Conference "Representation Theory in Samos" in Greece in July 2016.

2. Preliminaries

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over k and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, π a chosen set of simple roots, Δ^+ (resp. Δ^-) the set of positive (resp. negative) roots. We adopt the labeling of [2, Planches I-IX] for the simple roots in π.

For any $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ denote the corresponding root space of $\mathfrak{g}$ and fix a nonzero vector x_α in $\mathfrak{g}_\alpha$. Then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha$. For all $\alpha \in \pi$, denote by $\alpha^\vee$ the corresponding coroot. For any subset A of Δ, set $\mathfrak{g}_A = \bigoplus_{\alpha\in A} \mathfrak{g}_\alpha$.

For any subset π' of π, let $\Delta_{\pi'}$ be the subset of roots in Δ generated by π' and $\Delta^+_{\pi'}$, $\Delta^-_{\pi'}$ the sets of positive and negative roots in $\Delta_{\pi'}$ respectively. One defines the standard parabolic subalgebra $\mathfrak{p}_{\pi'}$ associated to π' to be the algebra $\mathfrak{p}_{\pi'} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-_{\pi'}$ where $\mathfrak{n}^-_{\pi'} = \bigoplus_{\alpha\in\Delta^-_{\pi'}} \mathfrak{g}_\alpha$. Its opposite algebra then is $\mathfrak{p}^-_{\pi'} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}$, with $\mathfrak{n}_{\pi'}$ defined similarly. The dual space $\mathfrak{p}^*_{\pi'}$ identifies with $\mathfrak{p}^-_{\pi'}$ via the Killing form K on $\mathfrak{g}$.

We denote by $W_{\pi'}$ the Weyl group associated to π' and by r_γ, for $\gamma \in \Delta_{\pi'}$ the reflection with respect to γ. Then $W_{\pi'}$ is the subgroup of the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$, generated by r_γ, for all $\gamma \in \Delta_{\pi'}$. The W-invariant non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ obtained from the Killing form on $\mathfrak{h}$ will be denoted by $(\ ,\ )$.

Let $\mathfrak{a}$ be a finite dimensional Lie algebra over k. The *semicentre* $Sy(\mathfrak{a})$ of its symmetric algebra $S(\mathfrak{a})$ (of $\mathfrak{a}$ for short) is defined to be the vector space spanned by the semi-invariants under the adjoint action of $\mathfrak{a}$. One has

$$Sy(\mathfrak{a}) = \bigoplus_{\lambda\in\mathfrak{a}^*} S(\mathfrak{a})_\lambda$$

where $S(\mathfrak{a})_\lambda = \{s \in S(\mathfrak{a}) \mid \forall x \in \mathfrak{a},\, (\mathrm{ad}\, x)s = \lambda(x)s\}$. It is a subalgebra of $S(\mathfrak{a})$. When $S(\mathfrak{a})_\lambda \neq \{0\}$, λ is called a weight of the semicentre $Sy(\mathfrak{a})$. Let $\Lambda(\mathfrak{a})$ denote the set of weights of $Sy(\mathfrak{a})$. When $\mathfrak{a} = \mathfrak{p}_{\pi'}$, the set $\Lambda(\mathfrak{p}_{\pi'})$ of weights of $Sy(\mathfrak{p}_{\pi'})$ may be identified with a subset of $\mathfrak{h}^*$ and we have also that $Sy(\mathfrak{p}_{\pi'})$ is equal to the algebra of invariants $S(\mathfrak{p}_{\pi'})^{\mathfrak{p}'_{\pi'}}$ of $S(\mathfrak{p}_{\pi'})$ under the adjoint action of the derived subalgebra $\mathfrak{p}'_{\pi'}$ of $\mathfrak{p}_{\pi'}$.

The *Poisson centre* $Y(\mathfrak{a})$ of $\mathfrak{a}$ is the centre of $S(\mathfrak{a})$ for its natural Poisson structure and it is also the set of the invariants in $S(\mathfrak{a})$ under the adjoint action of $\mathfrak{a}$, that is $Y(\mathfrak{a}) = S(\mathfrak{a})_0$. It is an algebra contained in the semicentre $Sy(\mathfrak{a})$ of $S(\mathfrak{a})$. Again $Sy(\mathfrak{a})$ is also the *Poisson semicentre* of $S(\mathfrak{a})$ for its natural Poisson structure.

If $\mathfrak{a}$ is algebraic, there is an algebraic subalgebra of $\mathfrak{a}$, called the *canonical truncation* of $\mathfrak{a}$, $\mathfrak{a}_\Lambda = \cap_{\lambda\in\Lambda(\mathfrak{a})}\ker\lambda$, such that $Sy(\mathfrak{a}) = Sy(\mathfrak{a}_\Lambda) = Y(\mathfrak{a}_\Lambda)$ [1, Satz 6.1]. The algebra $\mathfrak{a}_\Lambda$ is an ideal of $\mathfrak{a}$ containing the derived subalgebra of $\mathfrak{a}$.

The *index* of $\mathfrak{a}$, denoted by $\mathrm{ind}\,\mathfrak{a}$, is the minimal dimension of a stabilizer $\mathfrak{a}^f$ for $f \in \mathfrak{a}^*$. When $\mathfrak{a}$ is algebraic, the index of $\mathfrak{a}$ is also equal to the minimal codimension of a coadjoint orbit in $\mathfrak{a}^*$ [4, 1.11.3].
An element $y \in \mathfrak{a}^*$ is called *regular* in $\mathfrak{a}^*$ if its stabilizer $\mathfrak{a}^y$ is of minimal dimension (equal to $\mathrm{ind}\,\mathfrak{a}$).

Let $\pi' \subset \pi$. Then $\mathfrak{p}_{\pi'}$ is algebraic, and the canonical truncation $\mathfrak{p}_{\pi',\Lambda}$ of $\mathfrak{p}_{\pi'}$, defined to be the largest subalgebra of $\mathfrak{p}_{\pi'}$ that vanishes on the weights of $Sy(\mathfrak{p}_{\pi'})$, has the property that the Poisson centre $Y(\mathfrak{p}_{\pi',\Lambda})$ is equal to the Poisson semicentre $Sy(\mathfrak{p}_{\pi',\Lambda})$ and also equal to $Sy(\mathfrak{p}_{\pi'})$.

The canonical truncation of $\mathfrak{p}_{\pi'}$ was given explicitly in [8]. It is of the form $\mathfrak{p}_{\pi',\Lambda} = \mathfrak{n} \oplus \mathfrak{h}_\Lambda \oplus \mathfrak{n}^-_{\pi'}$ where $\mathfrak{h}_\Lambda$ is a subalgebra of $\mathfrak{h}$ called the truncated Cartan subalgebra (this is the largest subalgebra of $\mathfrak{h}$ which vanishes on the set of weights $\Lambda(\mathfrak{p}_{\pi'})$ of $Sy(\mathfrak{p}_{\pi'})$).

The Gelfand-Kirillov dimension of $Y(\mathfrak{p}_{\pi',\Lambda})$ is equal to the index of $\mathfrak{p}_{\pi',\Lambda}$. For more details, see [8, 2.4, 2.5, B.2].

Set $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{p}'_{\pi'}$. When $\pi' = \pi \setminus \{\alpha_s\}$, then $\mathfrak{h}_\Lambda = \mathfrak{h}'$ that is, $\mathfrak{h}_\Lambda$ is the vector space over k generated by all $\alpha^\vee$ with $\alpha \in \pi'$.

For convenience, we replace the truncated parabolic subalgebra $\mathfrak{p}_{\pi',\Lambda}$ by its opposite algebra $\mathfrak{p}^-_{\pi',\Lambda}$ (that is, the canonical truncation of the opposite algebra $\mathfrak{p}^-_{\pi'}$).

From now on, we denote $\mathfrak{p}^-_{\pi',\Lambda}$ simply by $\mathfrak{p}$. Recall that $\mathfrak{p}^*$ may be identified with $\mathfrak{p}_{\pi',\Lambda}$ via the Killing form on $\mathfrak{g}$.

For any $\mathfrak{h}$-module $M = \bigoplus_{\nu\in\mathfrak{h}^*} M_\nu$ with finite dimensional weight spaces $M_\nu := \{m \in M \mid \forall\, h \in \mathfrak{h},\, h.m = \nu(h)m\}$, we may define its formal character by

$$\mathrm{ch}\, M = \sum_{\nu\in\mathfrak{h}^*} \dim M_\nu e^\nu.$$

Given two such $\mathfrak{h}$-modules M and M' write $\operatorname{ch} M \leq \operatorname{ch} M'$ if $\dim M_\nu \leq \dim M'_\nu$ for all $\nu \in \mathfrak{h}^*$ [14, 2.8].

Here we recall the formal characters $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ of the lower and the upper bounds mentioned in the introduction for $\operatorname{ch} Y(\mathfrak{p})$ given in [14, Thm. 6.7].

Let $E(\pi')$ be the set of $\langle \mathbf{ij}\rangle$-orbits of π, where $\mathbf{i}$ – which depends on π' – and $\mathbf{j}$ are the involutions of π defined for example in [10, 2.2]. For the reader's convenience, we give below their definition.

Let w_0 be the longest element of the Weyl group W of $(\mathfrak{g},\,\mathfrak{h})$ and w_0' the longest element of the Weyl group $W_{\pi'}$. For all $\alpha \in \pi$, we set $\mathbf{j}(\alpha) = -w_0(\alpha)$. Let $\alpha \in \pi'$, then we set $\mathbf{i}(\alpha) = -w_0'(\alpha)$. Let $\alpha \in \pi \setminus \pi'$. If $\mathbf{j}(\alpha) \notin \pi'$, then we set $\mathbf{i}(\alpha) = \mathbf{j}(\alpha)$. Otherwise let $r \in \mathbb{N}$ be the smallest integer such that $\mathbf{j}(w_0'w_0)^r(\alpha) \notin \pi'$. Then we set $\mathbf{i}(\alpha) = \mathbf{j}(w_0'w_0)^r(\alpha)$.

By [6, 3.2] we have that $\operatorname{GKdim} Y(\mathfrak{p}) = \operatorname{ind}\mathfrak{p} = |E(\pi')|$.

Denote by $\{\varpi_\alpha\}_{\alpha\in\pi}$ (resp. $\{\varpi'_\alpha\}_{\alpha\in\pi'}$) the set of fundamental weights associated to π (resp. to π'); the same sets sometimes are denoted by $\{\varpi_i\}_{\alpha_i\in\pi}$ and $\{\varpi'_i\}_{\alpha_i\in\pi'}$ respectively. Recall that the Poisson semicentre $Sy(\mathfrak{n}\oplus\mathfrak{h})$ of the Borel subalgebra $\mathfrak{n}\oplus\mathfrak{h}$ of $\mathfrak{g}$ is a polynomial algebra by [12]. Let $\mathcal{B}_\pi := \Lambda(\mathfrak{n}\oplus\mathfrak{h})$ (resp. $\mathcal{B}_{\pi'} := \Lambda(\mathfrak{n}_{\pi'}\oplus\mathfrak{h}')$) be the set of weights of the Poisson semicentre $Sy(\mathfrak{n}\oplus\mathfrak{h})$ (resp. $Sy(\mathfrak{n}_{\pi'}\oplus\mathfrak{h}')$): the weights of the generators of the Poisson semicentre of a Borel are listed in [12, Tables I and II] and [7, Table] for an erratum.

For all $\Gamma \in E(\pi')$, set

$$\delta_\Gamma = -\sum_{\gamma\in\Gamma}\varpi_\gamma - \sum_{\gamma\in\mathbf{j}(\Gamma)}\varpi_\gamma + \sum_{\gamma\in\Gamma\cap\pi'}\varpi'_\gamma + \sum_{\gamma\in\mathbf{i}(\Gamma\cap\pi')}\varpi'_\gamma$$

and $\varepsilon_\Gamma = \begin{cases} 1/2 & \text{if } \Gamma = \mathbf{j}(\Gamma), \text{ and } \sum_{\gamma\in\Gamma}\varpi_\gamma \in \mathcal{B}_\pi, \text{ and } \sum_{\gamma\in\Gamma\cap\pi'}\varpi'_\gamma \in \mathcal{B}_{\pi'} \\ 1 & \text{otherwise.}\end{cases}$. It is shown in [14, Thm. 6.7] that

$$\operatorname{ch}\mathcal{A} = \prod_{\Gamma\in E(\pi')}(1-e^{\delta_\Gamma})^{-1} \leq \operatorname{ch} Y(\mathfrak{p}) \leq \prod_{\Gamma\in E(\pi')}(1-e^{\varepsilon_\Gamma\delta_\Gamma})^{-1} = \operatorname{ch}\mathcal{B}.$$

In particular, if for all $\Gamma \in E(\pi')$, $\varepsilon_\Gamma = 1$, then the above inequalities are equalities and $Y(\mathfrak{p})$ is a polynomial algebra over k by [7].

Definition 2.1. An *adapted pair* for $\mathfrak{p}$ is a pair $(h,\,y) \in \mathfrak{h}_\Lambda \times \mathfrak{p}^*$ such that y is regular in $\mathfrak{p}^*$, and $(\operatorname{ad} h)\,y = -y$, where ad denotes the coadjoint action of $\mathfrak{p}$ on $\mathfrak{p}^*$.

Assume that there exists an adapted pair $(h,\,y) \in \mathfrak{h}_\Lambda \times \mathfrak{p}^*$ for $\mathfrak{p}$. By [9, 2.2] one may choose subsets $S,\,T \subset \Delta^+ \sqcup \Delta^-_{\pi'}$ such that $y = \sum_{\gamma\in S} a_\gamma x_\gamma$, with $a_\gamma \in k\setminus\{0\}$ for all $\gamma \in S$, and $\mathfrak{p}^* = (\operatorname{ad}\mathfrak{p})\,y \oplus \mathfrak{g}_T$. Moreover the subset T is such that $|T| = \dim\mathfrak{g}_T = \operatorname{ind}\mathfrak{p}$ since y is regular in $\mathfrak{p}^*$. Assume further that $S_{|\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}^*_\Lambda$. Then for each $\gamma \in T$ there exists a unique $t(\gamma) \in \mathbb{Q}S$ such that $\gamma + t(\gamma)$ vanishes on $\mathfrak{h}_\Lambda$. By [16, Lem. 6.11]

$$\operatorname{ch} Y(\mathfrak{p}) \leq \prod_{\gamma\in T}(1-e^{-(\gamma+t(\gamma))})^{-1} = \mathcal{B}'.$$

We call the right hand side $\mathcal{B}'$ an *improved upper bound* for $\operatorname{ch} Y(\mathfrak{p})$; in this work it is indeed always an improvement of the upper bound $\operatorname{ch}\mathcal{B}$ mentioned above.

Moreover by [16, Lem. 6.11] if the above lower bound $\operatorname{ch}\mathcal{A}$ and this improved upper bound $\mathcal{B}'$ coincide then the restriction map gives an isomorphism of algebras $Y(\mathfrak{p}) \simeq R[y+\mathfrak{g}_T]$, where $R[y+\mathfrak{g}_T]$ is the ring of polynomial functions on $y+\mathfrak{g}_T$, isomorphic to $S(\mathfrak{g}_T^*)$. Hence $Y(\mathfrak{p})$ is a polynomial algebra over k and $y+\mathfrak{g}_T$ is an algebraic slice in the sense of [17, 7.6], also called a Weierstrass section in [9]. By [9] it is also an affine slice in the sense of [17, 7.3] for the coadjoint action of the adjoint group of $\mathfrak{p}$ on $\mathfrak{p}^*$.

Assume that there exists an adapted pair (h, y) for $\mathfrak{p}$ and denote by V an h-stable complement of $(ad\,\mathfrak{p})\,y$ in $\mathfrak{p}^*$. Assume further that $Y(\mathfrak{p})$ is a polynomial algebra and let $f_1, \dots, f_l$ be homogeneous generators for $Y(\mathfrak{p})$ ($l = \operatorname{ind}\mathfrak{p}$). Then by [19, Cor. 2.3] if $m_1, \dots, m_l$ are the eigenvalues of $\operatorname{ad} h$ on an h-stable basis of V, one has that $\deg f_i = m_i + 1$ for all $1 \le i \le l$, up to a permutation of indices.

3. A lemma of regularity

Keep the notation of the previous section and recall ([10, Def. 2]) the definition of a *Heisenberg set* with centre $\gamma \in \Delta$. It is a subset Γ_γ of Δ such that $\gamma \in \Gamma_\gamma$ and for all $\alpha \in \Gamma_\gamma \setminus \{\gamma\}$, there exists a (unique) $\alpha' \in \Gamma_\gamma \setminus \{\gamma\}$ such that $\alpha + \alpha' = \gamma$.

Example 3.1. Set $\Delta = \sqcup \Delta_i$ where Δ_i is an irreducible root system and let β_i be the unique highest root of Δ_i. Take $(\Delta_i)_{\beta_i} := \{\alpha \in \Delta_i \mid (\alpha, \beta_i) = 0\}$ and decompose it into irreducible root systems Δ_{ij} with highest roots β_{ij}. Continuing we obtain a set of strongly orthogonal positive roots β_K, called the *Kostant cascade*, and irreducible root systems Δ_K, indexed by elements $K \in \mathbb{N} \cup \mathbb{N}^2 \cup \cdots$. The set $H_{\beta_K} := \{\alpha \in \Delta_K \mid (\alpha, \beta_K) > 0\}$ is a Heisenberg set with centre β_K and actually it is the maximal Heisenberg set with centre β_K, which is included in Δ^+.

Consider $\mathfrak{p}_{\pi',\Lambda} = \mathfrak{n} \oplus \mathfrak{h}_\Lambda \oplus \mathfrak{n}^-_{\pi'} \simeq (\mathfrak{p}^-_{\pi',\Lambda})^*$ the truncated parabolic subalgebra of $\mathfrak{g}$ associated to $\pi' \subset \pi$. Let S be a subset of $\Delta^+ \sqcup \Delta^-_{\pi'}$ and for all $\gamma \in S$ choose a Heisenberg set Γ_γ with centre γ in $\Delta^+ \sqcup \Delta^-_{\pi'}$. Assume that the sets Γ_γ are disjoint and set $\Gamma = \bigsqcup_{\gamma\in S} \Gamma_\gamma$ and $y = \sum_{\gamma \in S} a_\gamma x_\gamma \in \mathfrak{p}_{\pi',\Lambda}$, with $a_\gamma \in k \setminus \{0\}$ for all $\gamma \in S$. Set $O = \bigsqcup_{\gamma\in S} \Gamma^0_\gamma$, with $\Gamma^0_\gamma = \Gamma_\gamma \setminus \{\gamma\}$, and $\mathfrak{o} = \mathfrak{g}_{-O}$.

Denote by Φ_y the skew-symmetric bilinear form on $\mathfrak{g}$ such that, for all $x, x' \in \mathfrak{g}$, $\Phi_y(x, x') = K(y, [x, x'])$, where recall K is the Killing form on $\mathfrak{g}$.

The lemma below is a generalization of [10, Lem. 6] (see also [15, Thm. 8.6]).

Lemma 3.2. *Assume further that*

(i) *There exist disjoint subsets T^* and T of $\Delta^+ \sqcup \Delta^-_{\pi'}$, also disjoint from Γ, such that $\Delta^+ \sqcup \Delta^-_{\pi'} = \Gamma \sqcup T^* \sqcup T$.*

(ii) *The restriction of Φ_y to $\mathfrak{o} \times \mathfrak{o}$ is non-degenerate.*

(iii) *$S_{|\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}^*_\Lambda$.*

(iv) *For all $\beta \in T^*$, $x_\beta \in (\operatorname{ad}\mathfrak{p}^-_{\pi',\Lambda})\,y + \mathfrak{g}_T$.*

(v) $|T| = \operatorname{ind}\mathfrak{p}^-_{\pi',\Lambda}$.

Then $\mathfrak{p}_{\pi',\Lambda} = (\mathrm{ad}\,\mathfrak{p}^-_{\pi',\Lambda})\,y \oplus \mathfrak{g}_T$, *where* ad *denotes the coadjoint action. In particular,* y *is regular in* $\mathfrak{p}_{\pi',\Lambda}$. *Moreover, if we uniquely define* $h \in \mathfrak{h}_\Lambda$ *by the relations* $\gamma(h) = -1$ *for all* $\gamma \in S$, *then* (h, y) *is an adapted pair for* $\mathfrak{p}^-_{\pi',\Lambda}$.

Proof. Condition (i) implies that $\mathfrak{p}^-_{\pi',\Lambda} = \mathfrak{h}_\Lambda \oplus \mathfrak{o} \oplus \mathfrak{g}_{-S} \oplus \mathfrak{g}_{-T^*} \oplus \mathfrak{g}_{-T}$ and that $\mathfrak{p}_{\pi',\Lambda} = \mathfrak{h}_\Lambda \oplus \mathfrak{g}_O \oplus \mathfrak{g}_S \oplus \mathfrak{g}_{T^*} \oplus \mathfrak{g}_T$. Condition (ii) implies that

$$\mathfrak{g}_O \subset (\mathrm{ad}\,\mathfrak{o})y + \mathfrak{g}_S + \mathfrak{g}_T + \mathfrak{g}_{T^*}$$

since $O \cap S = \emptyset$. Condition (iii) implies that $\mathfrak{g}_S = (\mathrm{ad}\,\mathfrak{h}_\Lambda)y$ and that

$$\mathfrak{h}_\Lambda \subset (\mathrm{ad}\,\mathfrak{g}_{-S})y + \mathfrak{g}_O + \mathfrak{g}_S + \mathfrak{g}_T + \mathfrak{g}_{T^*}.$$

Condition (iv) implies that $\mathfrak{g}_{T^*} \subset (\mathrm{ad}\,\mathfrak{p}^-_{\pi',\Lambda})\,y + \mathfrak{g}_T$. Hence

$$\mathfrak{p}_{\pi',\Lambda} = \mathfrak{h}_\Lambda \oplus \mathfrak{g}_O \oplus \mathfrak{g}_S \oplus \mathfrak{g}_{T^*} \oplus \mathfrak{g}_T \subset (\mathrm{ad}\,\mathfrak{p}^-_{\pi',\Lambda})\,y + \mathfrak{g}_T.$$

Finally condition (v) and the fact that $\dim \mathfrak{g}_T = \mathrm{ind}\,\mathfrak{p}^-_{\pi',\Lambda} \leq \mathrm{codim}\,(\mathrm{ad}\,\mathfrak{p}^-_{\pi',\Lambda})\,y$ imply that the latter sum is direct. ■

Remark 3.3. Let us explain how adapted pairs for maximal parabolic subalgebras when both bounds $\mathrm{ch}\,\mathcal{A}$ and $\mathrm{ch}\,\mathcal{B}$ coincide were constructed in [10]. In these cases, we took $T^* = \emptyset$ and for the sets S and T we took elements of the Kostant cascade for $\mathfrak{g}$. Moreover for all $\gamma \in S \cap \Delta^+$, resp. $\gamma \in S \cap \Delta^-_{\pi'}$, the corresponding maximal Heisenberg set (3.1) H_γ in Δ^+, resp. $H_{-\gamma}$ in $\Delta^+_{\pi'}$, was taken to be the Heisenberg set Γ_γ, resp. $-\Gamma_{-\gamma}$. Unfortunately in the case of maximal parabolic subalgebras when the bounds $\mathrm{ch}\,\mathcal{A}$ and $\mathrm{ch}\,\mathcal{B}$ do not coincide, such a strategy does no more work.

4. Stationary roots

Keep the notation and hypotheses of Sections 2 and 3. Given $\gamma \in S$, for all $\alpha \in \Gamma^0_\gamma$ denote by α' the unique root in Γ^0_γ such that $\alpha + \alpha' = \gamma$ and let θ_γ be the involution in Γ^0_γ mapping $\alpha \in \Gamma^0_\gamma$ to α'. Denote by θ the involution in O induced by all θ_γ, $\gamma \in S$.

Clearly, the non-degeneracy of the restriction of Φ_y to $\mathfrak{o} \times \mathfrak{o}$ is immediate if, for all $\alpha \in O$, the only root β in O such that $\alpha + \beta \in S$ is $\beta = \theta(\alpha)$. Unfortunately this will not be the case in general, but Lemma 6.1 below will give sufficient conditions for the non-degeneracy of the restriction of Φ_y to $\mathfrak{o} \times \mathfrak{o}$. To state this lemma, we need further notation. In particular for each root $\alpha \in O$, we set $S_\alpha = \{\beta \in O \mid \alpha + \beta \in S\}$ and for all $n \geq 1$, $O_n = \{\alpha \in O \mid |S_\alpha| = n\}$. Note that $O_1 = \{\alpha \in O \mid \forall\,\beta \in O,\, \alpha + \beta \in S \Longrightarrow \beta = \theta(\alpha)\}$.

Let $\alpha \in O$. Set $\alpha^0 = \alpha$ and for all $i \in \mathbb{N}$ define $\alpha^i \in O$ inductively as follows. If $\theta(\alpha^i) \in O_1$, set $\alpha^{i+1} = \alpha^i$. Otherwise, let $\alpha^{i+1} \neq \alpha^i$ be a root in O such that $\alpha^{i+1} + \theta(\alpha^i) \in S$. For all $i \in \mathbb{N}$, set $\alpha^{(i)} = \theta(\alpha)^i$.

Note that, if $\alpha \in O_2$, then $\alpha^{(1)}$ is the only root in O distinct from $\theta(\alpha)$ such that $\alpha^{(1)} + \alpha \in S$. Similarly if $\theta(\alpha) \in O_2$, then α^1 is the only root in O distinct from α such that $\alpha^1 + \theta(\alpha) \in S$. Observe that $\theta(\alpha)^{(i)} = \alpha^i$.

We will say that $(\alpha^i)_{i\in\mathbb{N}}$ is a sequence of roots in O constructed from α; such a sequence always exists but in general is not unique. If for all $i\in\mathbb{N}$, $\theta(\alpha^i)\in O_1\sqcup O_2$, then $(\alpha^i)_{i\in\mathbb{N}}$ will be called *the* sequence of roots in O constructed from α, since in this case, α^i is uniquely defined, for all $i\in\mathbb{N}$.

Note that if $\theta(\alpha^i)\in O_1$ for some $i\in\mathbb{N}$, then $\alpha^j=\alpha^i$ for all $j\geq i$. Conversely, if $\alpha^i=\alpha^{i+1}$, then $\theta(\alpha^i)\in O_1$ and $\alpha^j=\alpha^i$, for all $j\geq i$. We call a minimal such i the rank of the sequence $(\alpha^j)_{j\in\mathbb{N}}$ and we say that the sequence is *stationary at rank* i. Note that if $\theta(\alpha^i)\notin O_1$ then $\alpha^{i+1}\notin O_1$.

Let $\alpha\in O$ and set $A_\alpha=\{\alpha^i,\,\theta(\alpha^i)\mid i\in\mathbb{N}\}$ for a sequence $(\alpha^i)_{i\in\mathbb{N}}$ of roots in O constructed from α.

Remark 4.1. Let $\alpha\in O$ and assume that $A_\alpha\cup A_{\theta(\alpha)}\subset O_1\sqcup O_2$. Let $i,\ j\in\mathbb{N}$.

(1) One has that $(\alpha^i)^j=\alpha^{i+j}$ and $(\alpha^{(i)})^j=\alpha^{(i+j)}$.

(2) Assume that $i\geq 1$ and that $\theta(\alpha^{i-1})\in O_2$. If $j\leq i$, then $(\alpha^i)^{(j)}=\theta(\alpha^{i-j})$ and if $j\geq i$, then $(\alpha^i)^{(j)}=\alpha^{(j-i)}$.

(3) Assume that $i\geq 1$ and that $\theta(\alpha^{(i-1)})\in O_2$. If $j\leq i$, then $(\alpha^{(i)})^{(j)}=\theta(\alpha^{(i-j)})$ and if $j\geq i$, then $(\alpha^{(i)})^{(j)}=\alpha^{j-i}$.

Proof. The definition of the roots α^i and $\alpha^{(i)}$ and an induction on j, noting that $\alpha^{i+1}=(\alpha^i)^1$ and that $\alpha^{(i+1)}=(\alpha^{(i)})^1$, give the assertions. ■

Definition 4.2. Let $\alpha\in O$. We say that α is a *stationary root* if $A_\alpha\cup A_{\theta(\alpha)}\subset O_1\sqcup O_2$ and if the sequences $(\alpha^i)_{i\in\mathbb{N}}$ and $(\alpha^{(i)})_{i\in\mathbb{N}}$ are stationary.
The set of stationary roots in O will be denoted by O_{st}.

Remark 4.3. Let $\alpha\in O$. If $\alpha\in O_{st}$ then $A_\alpha\cup A_{\theta(\alpha)}\subset O_{st}$ and conversely if $A_\alpha\cup A_{\theta(\alpha)}\subset O_1\sqcup O_2$ and if there exists $i_0\in\mathbb{N}$ such that α^{i_0} or $\alpha^{(i_0)}$ belongs to O_{st}, then $\alpha\in O_{st}$.

Proof. This easily follows from (1), (2) and (3) of Remark 4.1. ■

Lemma 4.4. *Let $\alpha\in O_{st}$. Let $\vartheta:O\to O$ be a permutation such that, for all $\gamma\in O$, $\gamma+\vartheta(\gamma)\in S$. Then the restriction of ϑ to $A_\alpha\cup A_{\theta(\alpha)}$ coincides with the involution θ.*

Proof. Denote by n_0 (resp. n_1) the rank of the stationary sequence $(\alpha^i)_{i\in\mathbb{N}}$ (resp. $(\alpha^{(i)})_{i\in\mathbb{N}}$). Since $\theta(\alpha^{n_0})\in O_1$ (resp. $\theta(\alpha^{(n_1)})\in O_1$) the map ϑ necessarily sends $\theta(\alpha^{n_0})$ (resp. $\theta(\alpha^{(n_1)})$) to α^{n_0} (resp. $\alpha^{(n_1)}$). Then a decreasing induction on i gives that, for all $0\leq i\leq n_0$, we have $\vartheta(\theta(\alpha^i))=\alpha^i$. Similarly we obtain that, for all $0\leq i\leq n_1$, $\vartheta(\theta(\alpha^{(i)}))=\alpha^{(i)}$. An increasing induction on i proves then that $\vartheta(\alpha^i)=\theta(\alpha^i)$ for all $0\leq i\leq n_0$ and that $\vartheta(\alpha^{(i)})=\theta(\alpha^{(i)})$ for all $0\leq i\leq n_1$. ■

5. Cyclic roots.

We also need to define what we call a *cyclic root*. We use the notation and hypotheses of Section 4.

Definition 5.1. Let $\alpha \in O$. We say that α is *a cyclic root* if there exist $\beta, \gamma \in O$ such that the following conditions are satisfied:

(i) $\theta(\alpha) + \gamma = \beta + \theta(\beta)$.

(ii) $\theta(\gamma) + \beta = \alpha + \theta(\alpha)$.

(iii) $\theta(\beta) + \alpha = \gamma + \theta(\gamma)$.

(iv) $\{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\} \subset O_2 \sqcup O_3$.

(v) $|\{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\}| = 6$.

(vi) If $\delta \in \{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\} \cap O_3$, then there exists $\tilde{\delta} \in S_\delta$ such that $\tilde{\delta} \in O_2$ and $\theta(\tilde{\delta}) \in O_1$.

The set of cyclic roots in O is denoted by O_{cyc}.

For $\alpha \in O_{cyc}$, set $C_\alpha = \{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\}$. Note that for $\delta \in C_\alpha \cap O_3$ then, with the above notation, $\tilde{\delta}$ is unique and $S_\delta \setminus S_\delta \cap C_\alpha = \{\tilde{\delta}\}$.

Remark 5.2. (1) If $\alpha \in O_{cyc}$ then all roots in C_α are cyclic roots.

(2) Suppose that $\alpha \in O_{cyc}$ and that $C_\alpha \subset O_2$. Then $\alpha \notin O_{st}$. Indeed the cyclic relations (i), (ii) and (iii) imply that $\alpha^1 = \gamma$, $\alpha^2 = \beta$ and $\alpha^3 = \alpha$, hence the sequence $(\alpha^i)_{i\in\mathbb{N}}$ is not stationary.

(3) Only two of conditions (i), (ii), (iii) above are necessary. Indeed any two of them imply the third one.

(4) With the above notation, let $\alpha \in O$ such that $\alpha = \tilde{\beta}$, with $\beta \in O_{cyc} \cap O_3$. Then $\alpha \notin O_{st}$ and $\alpha \notin O_{cyc}$. Indeed $\theta(\alpha) = \theta(\tilde{\beta}) \in O_1$, hence $\alpha \notin O_{cyc}$ and $\alpha^1 = \alpha$, but since $\tilde{\beta} \in S_\beta$, one has that $\alpha^{(1)} = \beta$, hence $\alpha \notin O_{st}$.

Lemma 5.3. *Let $\vartheta : O \to O$ be a permutation such that for all $\gamma \in O$, $\gamma + \vartheta(\gamma) \in S$. Then ϑ exchanges $\tilde{\delta}$ and $\theta(\tilde{\delta})$, where $\tilde{\delta}$ is the unique root in S_δ given by condition (vi) of Definition* 5.1*, for any $\delta \in O_{cyc} \cap O_3$.*

Proof. Let $\alpha \in O_{cyc}$ and consider $\delta \in C_\alpha$. Assume that $\delta \in O_3$. Since $\theta(\tilde{\delta}) \in O_1$, we have necessarily that $\vartheta(\theta(\tilde{\delta})) = \tilde{\delta}$. Now since $\tilde{\delta} \in O_2$, we have that $\vartheta(\tilde{\delta}) = \theta(\tilde{\delta})$ or δ, since moreover $\tilde{\delta} \in S_\delta$.
Assume that $\vartheta(\tilde{\delta}) = \delta$ and to simplify that $\delta = \alpha$. Then necessarily $\vartheta(\theta(\delta)) = \gamma$ by condition (i) of Definition 5.1. Then $\vartheta(\theta(\gamma)) = \beta$ by condition (ii) of Definition 5.1. But condition (iii) of Definition 5.1 implies then that $\vartheta(\theta(\beta)) = \alpha = \delta$ which is not possible, since $\delta = \alpha$ has already a preimage by ϑ. ∎

6. A lemma of non-degeneracy

The subset $S \subset \Delta^+ \sqcup \Delta^-_{\pi'}$ is divided into three disjoint subsets S^+, S^- and S^m defined as follows. Let S^+ (resp. S^-) be the subset of S consisting of those $\gamma \in S$ for which $\Gamma_\gamma \subset \Delta^+$ (resp. $\Gamma_\gamma \subset \Delta^-_{\pi'}$).

Let S^m be the subset of S consisting of those $\gamma \in S$, for which the Heisenberg set Γ_γ contains *both* positive and negative roots in $\Delta^+ \sqcup \Delta^-_{\pi'}$. We have $S = S^+ \sqcup S^- \sqcup S^m$ and we set $\Gamma^\pm = \bigsqcup_{\gamma\in S^\pm} \Gamma_\gamma$, $\Gamma^m = \bigsqcup_{\gamma\in S^m} \Gamma_\gamma$; then $\Gamma = \Gamma^+ \sqcup \Gamma^- \sqcup \Gamma^m$. For all $\gamma \in S$, recall that $\Gamma^0_\gamma = \Gamma_\gamma \setminus \{\gamma\}$, and set $O^\pm = \bigsqcup_{\gamma\in S^\pm} \Gamma^0_\gamma$, $O^m = \bigsqcup_{\gamma\in S^m} \Gamma^0_\gamma$; we have $O = O^+ \sqcup O^- \sqcup O^m$. Set also $\mathfrak{o}^\pm = \mathfrak{g}_{-O^\pm}$ and $\mathfrak{o}^m = \mathfrak{g}_{-O^m}$ so that $\mathfrak{o} = \mathfrak{g}_{-O} = \mathfrak{o}^+ \oplus \mathfrak{o}^- \oplus \mathfrak{o}^m$.

Lemma 6.1. *Assume that:*

(1) *$S_{|\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}^*_\Lambda$.*

(2) *If $\alpha \in \Gamma^0_\gamma$, with $\gamma \in S^+$, then $S_\alpha \cap O^+ = \{\theta(\alpha)\}$.*

(3) *If $\alpha \in \Gamma^0_\gamma$, with $\gamma \in S^-$, then $S_\alpha \cap O^- = \{\theta(\alpha)\}$.*

(4) *If $\alpha \in O$, with $S_\alpha \cap O^m \neq \emptyset$, then $\alpha \in O_{st}$ or $\alpha \in O_{cyc}$ or there exists $\beta \in O_{cyc} \cap O_3$ such that $\alpha = \tilde\beta$ or $\theta(\alpha) = \tilde\beta$, where $\tilde\beta$ is the unique root in S_β such that $\tilde\beta \in O_2$ and $\theta(\tilde\beta) \in O_1$.*

Let $y = \sum_{\gamma\in S} a_\gamma x_\gamma$, with $a_\gamma \in k \setminus \{0\}$ for all $\gamma \in S$. Then the restriction of the bilinear form Φ_y to $\mathfrak{o} \times \mathfrak{o}$ is non-degenerate.

Proof. Let ρ be the linear form on $\mathfrak{h}^*$ defined by $\rho(\alpha) = 1$ for all $\alpha \in \pi$ and define $z(t) = \sum_{\gamma\in S} t^{|\rho(\gamma)|} a_\gamma x_\gamma$ for $t \in k$. Set $d(t) = \det(\Phi_{z(t)|\mathfrak{o}\times\mathfrak{o}})$ which is a polynomial in t and let H_Λ denote the adjoint group of $\mathfrak{h}_\Lambda$.

Since by hypothesis (1) the elements of $S_{|\mathfrak{h}_\Lambda}$ are linearly independent it follows that, for $t_0 \in k$, $z(ct_0)$ and $z(t_0)$ are in the same H_Λ-coadjoint orbit for all $c \in k \setminus \{0\}$. Moreover $\mathfrak{o} \times \mathfrak{o}$ is stable under the adjoint action of H_Λ. Then the degeneracy of the restriction of the bilinear form $\Phi_{z(t_0)}$ to $\mathfrak{o}\times\mathfrak{o}$ is equivalent to the degeneracy of the restriction of the bilinear form $\Phi_{z(ct_0)}$ to $\mathfrak{o}\times\mathfrak{o}$ for all $c \in k\setminus\{0\}$, that is, $d(t_0) = 0$ is equivalent to $d(ct_0) = 0$ for all $c \in k \setminus \{0\}$. It follows that either $d(t)$ is identically zero or it vanishes only at $t = 0$. Hence $d(t)$ is a multiple of a single power of t (see also [15, Rem. 8.4]).

Let $\alpha \in O$ be such that $S_\alpha \cap O^m \neq \emptyset$.

Assume first that $\alpha \in O_{st}$. Then by Lemma 4.4, the only factor involving α in $\det(\Phi_{z(t)|\mathfrak{o}\times\mathfrak{o}})$ is $t^{2|\rho(\alpha+\theta(\alpha))|}$.

Assume now that $\alpha = \tilde\beta$, or $\theta(\alpha) = \tilde\beta$ with $\tilde\beta \in S_\beta$, $\beta \in O_{cyc} \cap O_3$, satisfying condition (vi) of Definition 5.1. Then, by Lemma 5.3, in the expansion of the determinant of $\Phi_{z(t)|\mathfrak{o}\times\mathfrak{o}}$ the only factor involving α is $t^{2|\rho(\alpha+\theta(\alpha))|}$.

Assume that $\alpha \in O_{cyc}$ and consider $C_\alpha = \{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\}$ verifying conditions (i)–(vi) of Definition 5.1. Then the matrix of $\Phi_{z(t)|\mathfrak{g}_{-C_\alpha}\times\mathfrak{g}_{-C_\alpha}}$ is, up to a nonzero scalar, of the form

$$\begin{pmatrix} 0 & 0 & 0 & t^{|\rho(s_1)|} & t^{|\rho(s_3)|} & 0 \\ 0 & 0 & 0 & 0 & t^{|\rho(s_2)|} & t^{|\rho(s_1)|} \\ 0 & 0 & 0 & t^{|\rho(s_2)|} & 0 & t^{|\rho(s_3)|} \\ -t^{|\rho(s_1)|} & 0 & -t^{|\rho(s_2)|} & 0 & 0 & 0 \\ -t^{|\rho(s_3)|} & -t^{|\rho(s_2)|} & 0 & 0 & 0 & 0 \\ 0 & -t^{|\rho(s_1)|} & -t^{|\rho(s_3)|} & 0 & 0 & 0 \end{pmatrix}$$

where $s_1 = \alpha + \theta(\alpha)$, $s_2 = \beta + \theta(\beta)$ and $s_3 = \gamma + \theta(\gamma)$.

Hence up to a nonzero scalar,

$$\det(\Phi_{z(t)_{|\mathfrak{g}-C_\alpha\times\mathfrak{g}-C_\alpha}}) = t^{2(|\rho(s_1)|+|\rho(s_2)|+|\rho(s_3)|)} = t^{2(|\rho(\alpha+\theta(\alpha))|+|\rho(\beta+\theta(\beta))|+|\rho(\gamma+\theta(\gamma))|)}$$

and by Lemma 5.3, it follows that the only factor involving α in $\det(\Phi_{z(t)_{|\mathfrak{o}\times\mathfrak{o}}})$ is $t^{2|\rho(\alpha+\theta(\alpha))|}$.

Let now $\alpha\in O^{\pm}$ be such that there exists $\beta\in S_\alpha\cap O^{\mp}$. By the above, if there is a factor in $\det(\Phi_{z(t)_{|\mathfrak{o}\times\mathfrak{o}}})$ involving α and β, that is, if $t^{|\rho(\alpha+\beta)|}$ appears as a factor in $\det(\Phi_{z(t)_{|\mathfrak{o}\times\mathfrak{o}}})$, then necessarily $S_\alpha\cap O^m=\emptyset$ and $S_\beta\cap O^m=\emptyset$. Then observe that $|\rho(\alpha+\beta)|<|\rho(\alpha)|+|\rho(\beta)|$, whilst

$$|\rho(\alpha+\theta(\alpha))|=|\rho(\alpha)|+|\rho(\theta(\alpha))| \text{ and } |\rho(\beta+\theta(\beta))|=|\rho(\beta)|+|\rho(\theta(\beta))|.$$

Since $d(t)$ is a multiple of a single power of t, the above observations and conditions (2) and (3) imply that $t^{|\rho(\alpha+\beta)|}$ cannot appear as a factor in $\det(\Phi_{z(t)_{|\mathfrak{o}\times\mathfrak{o}}})$.

Denote by $\widetilde{O}$ a choice of representatives in O modulo the involution θ. Then, up to a nonzero scalar, $d(t)=\det(\Phi_{z(t)_{|\mathfrak{o}\times\mathfrak{o}}})=\prod_{\alpha\in\widetilde{O}}t^{2|\rho(\alpha+\theta(\alpha))|}$.

Thus $\det(\Phi_{z(t)_{|\mathfrak{o}\times\mathfrak{o}}})\neq 0$ for $t\neq 0$ and the assertion of the lemma follows. ■

Remark 6.2. If $S^m=\emptyset$ then condition (4) is empty and the above lemma is [10, Lemma 5] or [15, Lemma 8.5].

By Lemma 3.2 (taking $T^*=\emptyset$) and Lemma 6.1 we obtain the following corollary.

Corollary 6.3. *Assume that the hypotheses of the previous lemma hold and that* $|T|=\operatorname{ind}\mathfrak{p}^-_{\pi',\Lambda}$, *where* $T=(\Delta^+\sqcup\Delta^-_{\pi'})\setminus\Gamma$. *Let* $y=\sum_{\gamma\in S}a_\gamma x_\gamma$, *with* $a_\gamma\in k\setminus\{0\}$ *for all* $\gamma\in S$, *and define* $h\in\mathfrak{h}_\Lambda$ *by* $\gamma(h)=-1$ *for all* $\gamma\in S$. *Then* $(h,\,y)$ *is an adapted pair for* $\mathfrak{p}^-_{\pi',\Lambda}$.

In what follows, we construct adapted pairs for the truncated maximal parabolic subalgebras $\mathfrak{p}$ in type B or D where the lower and upper bounds $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ of Section 2 do not coincide; $\mathfrak{p}$ is associated to the subsystem π' of π obtained by suppressing a root of even index. The construction of an adapted pair in these cases is much more involved than in [10].

7. Types B and non-extremal D

Types B and D (non-extremal) are very similar, so we will treat them together. In this section, $\mathfrak{g}$ is a simple Lie algebra of type B_n, $n\geq 2$ (resp. of type D_n, $n\geq 4$) and $\mathfrak{p}=\mathfrak{p}^-_{\pi',\Lambda}$ is the truncated maximal parabolic subalgebra associated to the subset $\pi'=\pi\setminus\{\alpha_s\}$ of π with s even, $2\leq s\leq n$ (resp. $2\leq s\leq n-2$). Then the lower and the upper bounds $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ of Section 2 for $\operatorname{ch}Y(\mathfrak{p})$ do not coincide in general (see Section 1). We will construct an adapted pair $(h,\,y)$ for $\mathfrak{p}$, a slice for its coadjoint action and show that $Y(\mathfrak{p})$ is polynomial in $\operatorname{ind}\mathfrak{p}$ generators. It will follow by the discussion in the introduction that the field $C(\mathfrak{p}^-_{\pi'})$ of invariant fractions is a purely transcendental extension of k.

As we said above, it is enough to find sets S,T that satisfy the conditions of Lemma 6.1 and of Corollary 6.3.

Recall that the truncated Cartan subalgebra in $\mathfrak{p}$ is $\mathfrak{h}_\Lambda = \mathfrak{h}' = \bigoplus_{1\le i\le n, i\neq s} k\,\alpha_i{}^\vee$. Denote by $\{\varepsilon_i \,|\, 1\le i\le n\}$ an orthonormal basis of $\mathbb{R}^n$ according to which the simple roots α_i $(1\le i\le n)$ of $\mathfrak{g}$ are expanded as in [2, Planche II] for type B_n, and in [2, Planche IV] for type D_n.

Recall the Kostant cascade for each type B_n, resp. D_n. For type B_n, the Kostant cascade is formed by the strongly orthogonal positive roots β_i and $\beta_{i'}$ given in [10, Table I] or in [12, Table II]. We have that

$$\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$$

for all $1\le i\le [n/2]$, and for n odd, $\beta_{(n+1)/2} = \varepsilon_n = \alpha_n$, and

$$\beta_{i'} = \alpha_{2i-1} = \varepsilon_{2i-1} - \varepsilon_{2i}$$

for all $1\le i\le [n/2]$. For type D_n, the Kostant cascade is formed by the strongly orthogonal positive roots β_i, $\beta_{i'}$, $\beta_{i''}$ given in [10, Table I] or in [12, Table II] (note that in [10, Table I], we had forgotten $\beta_{(n+1)/2}$ for n odd). We have that

$$\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$$

for all $1\le i\le [n/2]$, and for n odd, $\beta_{(n+1)/2} = \alpha_{n-2} = \varepsilon_{n-2} - \varepsilon_{n-1}$, and

$$\beta_{i'} = \varepsilon_{2i-1} - \varepsilon_{2i}$$

for all $1\le i\le [n/2]-1$. Finally if n is even, $\beta_{(\frac{n-2}{2})''} = \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$.

Set $S = S^+ \sqcup S^- \sqcup S^m$ with the following subsets $S^\pm$ and S^m:

For type B_n, with $2\le s\le n$, s even:

$S^- = \{\varepsilon_{s-i} - \varepsilon_i,\ -\beta_j = -\varepsilon_{2j-1} - \varepsilon_{2j} \mid 1\le i\le s/2-1,\ s/2+1\le j\le [n/2]\}$,

$S^m = \{\varepsilon_s\}$;

if $n = s$: $S^+ = \{\varepsilon_{2i-1} + \varepsilon_{2i} \mid 1\le i\le s/2-1\}$;

$$\text{if } n>s:\ S^+ = \left\{\varepsilon_{2i-1}+\varepsilon_{2i},\ \varepsilon_{s-1}+\varepsilon_{s+1},\ \varepsilon_{2j}+\varepsilon_{2j+1} \,\middle|\, \begin{array}{l} 1\le i\le s/2-1, \\ s/2+1\le j\le [(n-1)/2] \end{array}\right\}.$$

For type D_n, with $2\le s\le n-2$, s even:

$S^- = \{\varepsilon_{s-i} - \varepsilon_i,\ -\varepsilon_{2j-1} - \varepsilon_{2j} \mid 1\le i\le s/2-1,\ s/2+1\le j\le [(n-1)/2]\}$,

$S^m = \{\varepsilon_s - \varepsilon_n,\ \varepsilon_s + \varepsilon_n\}$,

$$S^+ = \left\{\varepsilon_{2i-1}+\varepsilon_{2i},\ \varepsilon_{s-1}+\varepsilon_{s+1},\ \varepsilon_{2j}+\varepsilon_{2j+1} \,\middle|\, \begin{array}{l} 1\le i\le s/2-1, \\ s/2+1\le j\le [(n-2)/2] \end{array}\right\}.$$

Clearly, $S\subset \Delta^+\sqcup\Delta^-_{\pi'}$ and $|S| = n-1 = \dim\mathfrak{h}_\Lambda$. We first show below that condition (1) of Lemma 6.1 holds.

Lemma 7.1. *$S_{|\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}_\Lambda^*$.*

Proof. Set $S=\{s_i\}_{1\leq i\leq n-1}$ and choose $\{h_i\}_{1\leq i\leq n-1}=\{\alpha_i^\vee\}_{1\leq i\leq n,\, i\neq s}$ as a basis of $\mathfrak{h}_\Lambda$. Assume first that $2\leq s\leq n-2$.
For $\mathfrak{g}$ of type D_n, set $s_{n-2}=\varepsilon_s-\varepsilon_n$ and $s_{n-1}=\varepsilon_s+\varepsilon_n$. For $\mathfrak{g}$ of type B_n, set $s_{n-2}=\varepsilon_s$ and $s_{n-1}=\varepsilon_{n-1}+\varepsilon_n$ if n is odd, resp. $s_{n-1}=-\varepsilon_{n-1}-\varepsilon_n$ if n is even.
For $\mathfrak{g}$ of type D_n, set $s'_{n-2}=\varepsilon_s$ and $s'_{n-1}=\varepsilon_n$.
For $\mathfrak{g}$ of type B_n, set $s'_{n-2}=s_{n-2}=\varepsilon_s$ and $s'_{n-1}=\varepsilon_n-\varepsilon_{s+1}$ if n is odd, resp. $s'_{n-1}=\varepsilon_n+\varepsilon_{s+1}$ if n is even.
Finally set $s'_i=s_i$ for all $1\leq i\leq n-3$ and $S'=\{s'_i\}_{1\leq i\leq n-1}$. It is sufficient to prove that $\det(s'_i(h_j))_{1\leq i,j\leq n-1}\neq 0$. Order the basis of $\mathfrak{h}_\Lambda$ as

$$\{\alpha_{2i}^\vee,\, \alpha_{s-1}^\vee,\, \alpha_{2j-1}^\vee,\, \alpha_{s-2j-1}^\vee,\, \alpha_k^\vee \mid 1\leq i\leq s/2-1,\, 1\leq j\leq [s/4],\ s+1\leq k\leq n\}$$

without repetitions and the elements of S' as

$$\{\beta_i, \varepsilon_s,\, \varepsilon_{s-i}-\varepsilon_i, \varepsilon_{s-1}+\varepsilon_{s+1},\, -\varepsilon_{s+2j-1}-\varepsilon_{s+2j},\\ \varepsilon_{s+2j}+\varepsilon_{s+2j+1},\, s'_{n-1} \mid 1\leq i\leq s/2-1,\ 1\leq j\leq (n-s-2)/2\}$$

if n is even, and, if n is odd as

$$\{\beta_i,\, \varepsilon_s,\, \varepsilon_{s-i}-\varepsilon_i,\, \varepsilon_{s-1}+\varepsilon_{s+1},\, -\varepsilon_{s+2j-1}-\varepsilon_{s+2j},\\ \varepsilon_{s+2j}+\varepsilon_{s+2j+1},\, -\varepsilon_{n-2}-\varepsilon_{n-1},\, s'_{n-1} \mid 1\leq i\leq s/2-1,\ 1\leq j\leq (n-s-3)/2\}.$$

Then one checks that $\quad (s'_i(h_j))_{1\leq i,j\leq n-1}=\begin{pmatrix} A & 0 & 0\\ * & B & 0\\ * & * & C\end{pmatrix},$

where A (resp. B) is a $(s/2-1)\times(s/2-1)$ (resp. $(s/2)\times(s/2)$) lower triangular matrix with 1 (resp. -1) on the diagonal. Moreover $C=\left(\begin{smallmatrix} C' & 0\\ * & C''\end{smallmatrix}\right)$ with C' an $(n-s-2)\times(n-s-2)$ lower triangular matrix with alternating 1 and -1 on the diagonal and C'' a 2×2 invertible matrix. We conclude that $\det(s'_i(h_j))_{1\leq i,j\leq n-1}\neq 0$.

Now assume $n=s$, resp. $n=s+1$, and $\mathfrak{g}$ is of type B_n. Order the basis of $\mathfrak{h}_\Lambda$ as

$$\{\alpha_{2i}^\vee,\, \alpha_{s-1}^\vee,\, \alpha_{2j-1}^\vee,\, \alpha_{s-2j-1}^\vee \mid 1\leq i\leq s/2-1,\, 1\leq j\leq [s/4]\},$$

resp. $\quad \{\alpha_{2i}^\vee,\, \alpha_{s-1}^\vee,\, \alpha_{2j-1}^\vee,\, \alpha_{s-2j-1}^\vee,\, \alpha_{s+1}^\vee \mid 1\leq i\leq s/2-1,\, 1\leq j\leq [s/4]\},$

without repetitions and the elements of S as

$$\{\beta_i, \varepsilon_s,\, \varepsilon_{s-i}-\varepsilon_i \mid 1\leq i\leq s/2-1\},$$

resp. $\quad \{\beta_i, \varepsilon_s,\, \varepsilon_{s-i}-\varepsilon_i,\, \varepsilon_{s-1}+\varepsilon_{s+1} \mid 1\leq i\leq s/2-1\}.$

Then we obtain that

$$(s_i(h_j))_{1\leq i,j\leq n-1}=\begin{pmatrix} A & 0\\ * & B\end{pmatrix},\ \text{resp.}\ (s_i(h_j))_{1\leq i,j\leq n-1}=\begin{pmatrix} A & 0 & 0\\ * & B & 0\\ * & * & 2\end{pmatrix},$$

with the same matrices A and B as above. Hence $\det(s_i(h_j))_{1\leq i,j\leq n-1}\neq 0$. This completes the proof of the lemma. ∎

Recall the (maximal in Δ^+) Heisenberg set H_{β_i} with centre β_i defined in Example 3.1 for every positive root β_i of the Kostant cascade.
For type B_n, with $2\le s\le n$, s even, we set:
For all $i\in\mathbb{N}$, $1\le i\le s/2-1$, $\Gamma_{\beta_i}=H_{\beta_i}\setminus\{\varepsilon_{2i-1},\,\varepsilon_{2i}\}\subset\Delta^+$.

$$\Gamma_{\varepsilon_{s-1}+\varepsilon_{s+1}}=\{\varepsilon_{s-1}+\varepsilon_{s+1},\,\varepsilon_{s-1}\pm\varepsilon_i,\,\varepsilon_{s+1}\mp\varepsilon_i\mid s+2\le i\le n\}\subset\Delta^+.$$

For all $i\in\mathbb{N}$, $s/2+1\le i\le[(n-1)/2]$,

$$\Gamma_{\varepsilon_{2i}+\varepsilon_{2i+1}}=\{\varepsilon_{2i}+\varepsilon_{2i+1},\,\varepsilon_{2i}\pm\varepsilon_j,\,\varepsilon_{2i+1}\mp\varepsilon_j\mid 2i+2\le j\le n\}\subset\Delta^+.$$

For all $i\in\mathbb{N}$, $1\le i\le s/2-1$,

$$\Gamma_{\varepsilon_{s-i}-\varepsilon_i}=\{\varepsilon_{s-i}-\varepsilon_i,\,\varepsilon_j-\varepsilon_i,\,\varepsilon_{s-i}-\varepsilon_j\mid i+1\le j\le s-i-1\}\subset\Delta^-_{\pi'}.$$

For all $i\in\mathbb{N}$, $s/2+1\le i\le[n/2]$,

$$\Gamma_{-\varepsilon_{2i-1}-\varepsilon_{2i}}=\{-\varepsilon_{2i-1}-\varepsilon_{2i},\,-\varepsilon_{2i-1}\pm\varepsilon_j,\,-\varepsilon_{2i}\mp\varepsilon_j\mid 2i+1\le j\le n\}\subset\Delta^-_{\pi'}.$$

$$\Gamma_{\varepsilon_s}=\left\{\varepsilon_s,\,\varepsilon_i,\,\varepsilon_s-\varepsilon_i,\varepsilon_s+\varepsilon_j,\,-\varepsilon_j\;\middle|\;\begin{matrix}1\le i\le n,\ i\ne s,\\ s+1\le j\le n\end{matrix}\right\}\subset\Delta^+\sqcup\Delta^-_{\pi'}.$$

For type D_n, with $2\le s\le n-2$, s even, we set:
For all $i\in\mathbb{N}$, $1\le i\le s/2-1$,

$$\Gamma_{\beta_i}=H_{\beta_i}\setminus\{\varepsilon_{2i-1}-\varepsilon_n,\,\varepsilon_{2i}+\varepsilon_n\}\subset\Delta^+.$$

$$\Gamma_{\varepsilon_{s-1}+\varepsilon_{s+1}}=\{\varepsilon_{s-1}+\varepsilon_{s+1},\,\varepsilon_{s-1}+\varepsilon_i,\,\varepsilon_{s+1}-\varepsilon_i,\,\varepsilon_{s-1}-\varepsilon_j,\,\varepsilon_{s+1}+\varepsilon_j\mid s+2\le i\le n,\,s+2\le j\le n-1\}\subset\Delta^+.$$

For all $i\in\mathbb{N}$, $s/2+1\le i\le[(n-2)/2]$,

$$\Gamma_{\varepsilon_{2i}+\varepsilon_{2i+1}}=\{\varepsilon_{2i}+\varepsilon_{2i+1},\,\varepsilon_{2i}-\varepsilon_j,\,\varepsilon_j+\varepsilon_{2i+1},\,\varepsilon_{2i}+\varepsilon_k,\,\varepsilon_{2i+1}-\varepsilon_k\mid 2i+2\le j\le n,\,2i+2\le k\le n-1\}\subset\Delta^+.$$

For all $i\in\mathbb{N}$, $1\le i\le s/2-1$,

$$\Gamma_{\varepsilon_{s-i}-\varepsilon_i}=\{\varepsilon_{s-i}-\varepsilon_i,\,\varepsilon_j-\varepsilon_i,\,\varepsilon_{s-i}-\varepsilon_j\mid i+1\le j\le s-i-1\}\subset\Delta^-_{\pi'}.$$

For all $i\in\mathbb{N}$, $s/2+1\le i\le[(n-1)/2]$,

$$\Gamma_{-\varepsilon_{2i-1}-\varepsilon_{2i}}=\{-\varepsilon_{2i-1}-\varepsilon_{2i},\,-\varepsilon_{2i-1}-\varepsilon_j\,,\,\varepsilon_j-\varepsilon_{2i},\,-\varepsilon_{2i-1}+\varepsilon_k,\,-\varepsilon_k-\varepsilon_{2i}\mid 2i+1\le j\le n-1,\,2i+1\le k\le n\}\subset\Delta^-_{\pi'}.$$

$$\Gamma_{\varepsilon_s-\varepsilon_n}=\{\varepsilon_s-\varepsilon_n,\,\varepsilon_s-\varepsilon_{2i-1},\,\varepsilon_{2i-1}-\varepsilon_n,\,\varepsilon_s+\varepsilon_{2j+1},\,-\varepsilon_{2j+1}-\varepsilon_n\mid 1\le i\le[n/2],\,i\ne s/2+1,\,s/2\le j\le[(n-2)/2]\}\subset\Delta^+\sqcup\Delta^-_{\pi'}.$$

$$\Gamma_{\varepsilon_s+\varepsilon_n}=\{\varepsilon_s+\varepsilon_n,\,\varepsilon_s-\varepsilon_{2i},\,\varepsilon_{2i}+\varepsilon_n,\,\varepsilon_s-\varepsilon_{s+1},\,\varepsilon_{s+1}+\varepsilon_n,\,\varepsilon_s+\varepsilon_{2j},\,-\varepsilon_{2j}+\varepsilon_n\mid 1\le i\le[(n-1)/2],\,i\ne s/2,\,s/2+1\le j\le[(n-1)/2]\}\subset\Delta^+\sqcup\Delta^-_{\pi'}.$$

By construction, the sets Γ_γ, $\gamma\in S$, are disjoint Heisenberg sets with centre γ, included in $\Delta^+\sqcup\Delta^-_{\pi'}$.

For $\mathfrak{g}$ of type B_n, resp. D_n, denote by π'_1 the connected component of π' of type A_{s-1} and π'_2 the connected component of π' of type B_{n-s}, resp. of type D_{n-s} for $n \geq s+3$ or of type $\mathrm{A}_1 \times \mathrm{A}_1$ for $n = s+2$. Observe that for all $i \in \mathbb{N}$, $1 \leq i \leq s/2-1$, $\Gamma_{\varepsilon_{s-i}-\varepsilon_i} \subset \Delta^-_{\pi'_1}$ and for all $i \in \mathbb{N}$, $s/2+1 \leq i \leq [n/2]$, resp. $s/2+1 \leq i \leq [(n-1)/2]$, $\Gamma_{-\varepsilon_{2i-1}-\varepsilon_{2i}} \subset \Delta^-_{\pi'_2}$.

Remark 7.2. Assume that $\mathfrak{g}$ is of type B_n.
(1) If $\alpha \in O \setminus O_1 \sqcup O_2$ then $\alpha = \varepsilon_i - \varepsilon_j \in O^-$ with $1 \leq j < s/2 < i \leq s-1-j$. Moreover in this case, $S_{\varepsilon_i-\varepsilon_j} \cap O^- = \{\theta(\varepsilon_i - \varepsilon_j)\}$ and $S_{\varepsilon_i-\varepsilon_j} \cap O^m = \emptyset$. Hence conditions (3) and (4) of Lemma 6.1 are satisfied for such a root.
(2) For $i, j \neq s+1$, $\varepsilon_i + \varepsilon_j \in O_1$ unless $\varepsilon_i + \varepsilon_j = \varepsilon_{s-1} + \varepsilon_s \in T$ (where T is the complement of $\Gamma = \sqcup_{\gamma \in S}\Gamma_\gamma$ in $\Delta^+ \sqcup \Delta^-_{\pi'}$). Moreover $\varepsilon_s + \varepsilon_{s+1} \in O_1$ and $\varepsilon_s - \varepsilon_i \in O_1$ for all $i \geq s/2$, $i \neq s$.

We show below that conditions (2) and (4) for $\alpha \in O^+$, resp. conditions (3) and (4) for $\alpha \in O^-$, of Lemma 6.1, are satisfied.

Lemma 7.3. *Let $\alpha \in O^\pm$. Then $S_\alpha \cap O^\pm = \{\theta(\alpha)\}$ and if $S_\alpha \cap O^m \neq \emptyset$ then condition* (4) *of Lemma* 6.1 *is satisfied.*

Proof. Let $\alpha \in O^\pm$. By direct computation, one verifies that $S_\alpha \cap O^\pm = \{\theta(\alpha)\}$.

Assume that $\mathfrak{g}$ is of type B_n and that $S_\alpha \cap O^m \neq \emptyset$. Then necessarily $\alpha = \varepsilon_j - \varepsilon_s$ and $S_\alpha \cap O^m = \{\beta = \varepsilon_s - \varepsilon_{s-j}\}$ with $s/2 < j \leq s-2$.

If j is odd, $\theta(\alpha) = \varepsilon_{j+1} + \varepsilon_s \in O_1$, resp. if j is even, $\theta(\alpha) = \varepsilon_{j-1} + \varepsilon_s \in O_1$, by Remark 7.2(2). We will assume that j is odd; the other case is very similar.

Recall the sequences of roots in O constructed from a root in O in Section 4. Since $\theta(\alpha) \in O_1$, we have that the sequence $(\alpha^k)_{k\in\mathbb{N}}$ constructed from α is stationary at rank 0. We will determine the sequence $(\alpha^{(k)})_{k\in\mathbb{N}}$ constructed from $\theta(\alpha)$. Recall that $\alpha^{(0)} = \theta(\alpha)$, then since $\alpha = \theta(\alpha^{(0)}) \in O_2$, we necessarily have $\alpha^{(1)} = \beta = \varepsilon_s - \varepsilon_{s-j}$. Hence, $\theta(\beta) = \varepsilon_{s-j} \in O_2$ with $S_{\varepsilon_{s-j}} = \{\beta, \varepsilon_{s-j+1}\}$. Then $\alpha^{(2)} = \varepsilon_{s-j+1} \in O^m$ and $\theta(\alpha^{(2)}) = \varepsilon_s - \varepsilon_{s-j+1}$. Then $\alpha^{(3)} = \varepsilon_{j-1} - \varepsilon_s$ and $\theta(\alpha^{(3)}) = \varepsilon_{j-2} + \varepsilon_s \in O_1$ by Remark 7.2(2) (unless $j = s/2+1$ in which case already $\theta(\alpha^{(2)}) \in O_1$). We conclude that the sequence $(\alpha^{(k)})_{k\in\mathbb{N}}$ is stationary at rank at most 3. Since $A_\alpha \cup A_{\theta(\alpha)} \subset O_1 \sqcup O_2$ by Remark 7.2(1), we get $\alpha \in O_{st}$.

Now assume that $\mathfrak{g}$ is of type D_n and that there exists $\beta \in O^m$ such that $\alpha+\beta \in S$. Consider the case when $\alpha \in \Gamma^0_{\beta_i} \subset O^+$ with $1 \leq i \leq s/2-1$. Then four possibilities occur: $\alpha = \varepsilon_{2i} - \varepsilon_n$ and $\beta = \varepsilon_s - \varepsilon_{2i} \in \Gamma^0_{\varepsilon_s+\varepsilon_n}$, $\alpha = \varepsilon_{2i-1} + \varepsilon_n$ and $\beta = \varepsilon_s - \varepsilon_{2i-1} \in \Gamma^0_{\varepsilon_s-\varepsilon_n}$, $\alpha = \varepsilon_{2i-1} - \varepsilon_s$ and $\beta = \varepsilon_s - \varepsilon_{s-2i+1} \in \Gamma^0_{\varepsilon_s-\varepsilon_n}$ with $s-2i+1 < 2i-1$, $\alpha = \varepsilon_{2i} - \varepsilon_s$ and $\beta = \varepsilon_s - \varepsilon_{s-2i} \in \Gamma^0_{\varepsilon_s+\varepsilon_n}$ with $s-2i < 2i$.

We consider just one of the two first cases, that is when $\alpha = \varepsilon_{2i-1} + \varepsilon_n$ and $\beta = \varepsilon_s - \varepsilon_{2i-1} \in \Gamma^0_{\varepsilon_s-\varepsilon_n}$. Then $\alpha+\beta = \varepsilon_s+\varepsilon_n$, $\theta(\alpha) = \varepsilon_{2i}-\varepsilon_n$ and $\theta(\beta) = \varepsilon_{2i-1}-\varepsilon_n$. One verifies that there exists $\gamma = \varepsilon_s - \varepsilon_{2i} \in \Gamma^0_{\varepsilon_s+\varepsilon_n}$ such that $\theta(\alpha) + \gamma = \varepsilon_s - \varepsilon_n$, $\theta(\beta) + \theta(\gamma) = \varepsilon_{2i-1} + \varepsilon_{2i}$ and that $\alpha, \theta(\alpha), \theta(\beta), \theta(\gamma) \in O_2$. If $i = 1$ or $s-2i+1 \leq 2i-1$ (resp. $s-2i \leq 2i$) then $\beta \in O_2$ (resp. $\gamma \in O_2$). Otherwise $\beta \in O_3$, $\tilde{\beta} = \varepsilon_{s-2i+1} - \varepsilon_s \in O_2 \cap S_\beta$, and $\theta(\tilde{\beta}) = \varepsilon_{s-2i+2} + \varepsilon_s \in O_1$ (resp. $\gamma \in O_3$,

$\tilde{\gamma} = \varepsilon_{s-2i} - \varepsilon_s \in O_2 \cap S_\gamma$ and $\theta(\tilde{\gamma}) = \varepsilon_{s-2i-1} + \varepsilon_s \in O_1$). Hence $\alpha \in O_{cyc}$ and by Remark 5.2(1), the roots β, γ, $\theta(\alpha)$, $\theta(\beta)$, $\theta(\gamma)$ are also cyclic roots.
We consider only one of the two last cases. Suppose that $\alpha = \varepsilon_{2i-1} - \varepsilon_s$ and $\beta = \varepsilon_s - \varepsilon_{s-2i+1}$ with $s-2i+1 < 2i-1$. By the above, $\beta \in O_{cyc} \cap O_3$ and $\tilde{\beta} = \alpha$. Similar computations may be done in the other cases. ■

We show below that condition (4) of Lemma 6.1 is satisfied for $\alpha \in O^m$.

Lemma 7.4. *Let $\alpha \in O^m$. Then condition* (4) *of Lemma* 6.1 *is satisfied.*

Proof. Assume first that $\mathfrak{g}$ is of type B_n. Let $\alpha \in O^m$ $(= \Gamma^0_{\varepsilon_s})$. Note that $S_\alpha \cap O^m \neq \emptyset$, since it contains $\theta(\alpha)$. Recall that

$$\Gamma^0_{\varepsilon_s} = \{\varepsilon_i,\, \varepsilon_s - \varepsilon_i,\, -\varepsilon_j,\, \varepsilon_s + \varepsilon_j \mid 1 \leq i \leq n,\ i \neq s,\ s+1 \leq j \leq n\}.$$

We will show that the sequences of roots in O constructed from the roots in $\Gamma^0_{\varepsilon_s}$ are stationary and that all the elements of these sequences and their image by θ lie in $O_1 \sqcup O_2$. Note that this will prove that $\alpha \in O_{st}$.
For $\alpha = -\varepsilon_j$, with $s+1 \leq j \leq n$, we have that $\theta(\alpha) \in O_1$ by Remark 7.2(2), hence the sequence (α^i) is stationary at rank 0.
For $\alpha = \varepsilon_s + \varepsilon_j$, with $s+1 \leq j \leq n$, we have $\theta(\alpha) \in O_2$ and $\alpha^1 = -\varepsilon_{j+1}$ if j is odd, resp. $\alpha^1 = -\varepsilon_{j-1}$ if j is even. Then $\theta(\alpha^1) \in O_1$, hence (α^i) is stationary at rank 1.
For $\alpha = \varepsilon_i$ with $i \geq s+2$ then $\theta(\alpha) \in O_1$. Also for $\alpha = \varepsilon_s - \varepsilon_i$ and $i \geq s+2$, $\theta(\alpha) \in O_2$ and $\alpha^1 = \varepsilon_{i+1}$ if i is even, $\alpha^1 = \varepsilon_{i-1}$ if i is odd and $\theta(\alpha^1) \in O_1$. We conclude as above.
For $\alpha = \varepsilon_{s\pm1}$, then $\theta(\alpha) \in O_1$ and we are done. For $\alpha = \varepsilon_s - \varepsilon_{s\pm1}$ then $\theta(\alpha) = \varepsilon_{s\pm1} \in O_2$ and $\alpha^1 = \varepsilon_{s\mp1}$ is such that $\theta(\alpha^1) = \varepsilon_s - \varepsilon_{s\mp1} \in O_1$.
For $\alpha = \varepsilon_i$ with $1 \leq i \leq s-2$, $\theta(\alpha) \in O_1$ if $i \geq s/2$, otherwise $\theta(\alpha) \in O_2$. In the latter case, by the proof of Lemma 7.3, we obtain that the sequence of roots in O constructed from α is stationary.
It remains to consider $\alpha = \varepsilon_s - \varepsilon_i$, with $1 \leq i \leq s-2$. Then $\theta(\alpha) = \varepsilon_i \in O_2$ and $\alpha^1 = \varepsilon_{i+1}$ if i is odd, $\alpha^1 = \varepsilon_{i-1}$ if i is even. By the above, $\theta(\alpha^1) \in O_1$ if $i > s/2$, or $i = s/2$ and $s/2$ odd, or $i = s/2 - 1$ and $s/2$ even, and we are done. In the other cases, $\theta(\alpha^1) \in O_2$ by the above, which also gives that the sequence of roots in O constructed from α^1 and then from α is stationary.
Finally we observe that all roots of the sequences and their image by θ lie in $O_1 \sqcup O_2$.
If now $\mathfrak{g}$ is of type D_n then similar computations as above or as in the proof of Lemma 7.3 may be done to prove that condition (4) is satisfied. ■

Now denote by T the complement of $\Gamma = \Gamma^+ \sqcup \Gamma^- \sqcup \Gamma^m = \bigsqcup_{\gamma \in S} \Gamma_\gamma$ in $\Delta^+ \sqcup \Delta^-_{\pi'}$.

Lemma 7.5. $|T| = \operatorname{ind} \mathfrak{p}$.

Proof. Assume first that $\mathfrak{g}$ is of type B_n. One checks that:

For $n = s$, $T = \{\varepsilon_{s-1} + \varepsilon_s,\, \varepsilon_{2i-1} - \varepsilon_{2i} \mid 1 \leq i \leq s/2\}$,

for $n > s$, $T = \{\varepsilon_{s-1} + \varepsilon_s,\, \varepsilon_{s-1} - \varepsilon_{s+1},\, \varepsilon_{2i-1} - \varepsilon_{2i},\, -\varepsilon_{s+2j-1} + \varepsilon_{s+2j},\, \varepsilon_{s+2k} - \varepsilon_{s+2k+1} \mid 1 \leq i \leq s/2,\, 1 \leq j \leq [(n-s)/2],\, 1 \leq k \leq [(n-s-1)/2]\}$.

If now $\mathfrak{g}$ is of type D_n, with $s \leq n-2$, then we obtain the same set T. It follows that $|T| = n - s/2 + 1$. On the other hand, recall that the index of $\mathfrak{p}$ equals the number of $\langle \mathbf{ij} \rangle$-orbits in π where $\mathbf{i}$ and $\mathbf{j}$ are the involutions of π described in Section 2. Here the $\langle \mathbf{ij} \rangle$-orbits in π are $\Gamma_t = \{\alpha_t, \alpha_{s-t}\}$ for $1 \leq t \leq s/2 - 1$, $\Gamma_{s/2} = \{\alpha_{s/2}\}$ and $\Gamma_t = \{\alpha_t\}$ for $s \leq t \leq n$. They are $n - s/2 + 1$ in number hence $\operatorname{ind} \mathfrak{p} = n - s/2 + 1$. ■

Remark 7.6. All conditions of Lemma 3.2 are satisfied (by setting $T^* = \emptyset$). Hence by defining $h \in \mathfrak{h}_\Lambda$ by $\gamma(h) = -1$, for all $\gamma \in S$, and by setting $y = \sum_{\gamma \in S} x_\gamma$ we obtain an adapted pair (h, y) for $\mathfrak{p}^-_{\pi', \Lambda}$.

The semisimple element h of the adapted pair is uniquely defined by the relations $\gamma(h) = -1$ for all $\gamma \in S$. Below we compute the values of h on the elements of T, that is the $\operatorname{ad} h$ eigenvalues on the complement $\mathfrak{g}_T$ of the $\operatorname{ad} \mathfrak{p}^-_{\pi', \Lambda}$-orbit of y.

Lemma 7.7. *The eigenvalues of* $\operatorname{ad} h$ *on* $\mathfrak{g}_T$ *are:*
$s + 4i - 1 = (\varepsilon_{2i-1} - \varepsilon_{2i})(h)$ *for all* $i \in \mathbb{N}$, $1 \leq i \leq [s/4]$;
$3s - 4i + 1 = (\varepsilon_{2i-1} - \varepsilon_{2i})(h)$ *for all* $i \in \mathbb{N}$, $[s/4] + 1 \leq i \leq s/2 - 1$;
$s/2 + 1 = (\varepsilon_{s-1} - \varepsilon_s)(h)$;
$s/2 - 1 = (\varepsilon_{s-1} + \varepsilon_s)(h)$;
$s + 1 = (\varepsilon_{s-1} - \varepsilon_{s+1})(h)$;
$s + 4j - 1 = (-\varepsilon_{s+2j-1} + \varepsilon_{s+2j})(h)$, *for all* $j \in \mathbb{N}$, $1 \leq j \leq [(n-s-1)/2]$;
$s + 4j + 1 = (\varepsilon_{s+2j} - \varepsilon_{s+2j+1})(h)$, *for all* $j \in \mathbb{N}$, $1 \leq j \leq [(n-s-2)/2]$;

$$2n - s - 1 = \begin{cases} (-\varepsilon_{n-1} + \varepsilon_n)(h) & \text{if } n \text{ even} \\ (\varepsilon_{n-1} - \varepsilon_n)(h) & \text{if } n \text{ odd} \end{cases} \quad \textit{if } \mathfrak{g} \textit{ is of type } \mathrm{B}_n;$$

$$n - s/2 - 1 = \begin{cases} (-\varepsilon_{n-1} + \varepsilon_n)(h) & \text{if } n \text{ even} \\ (\varepsilon_{n-1} - \varepsilon_n)(h) & \text{if } n \text{ odd} \end{cases} \quad \textit{if } \mathfrak{g} \textit{ is of type } \mathrm{D}_n.$$

Then we have that $s + 2k - 1$ *is an eigenvalue of* $\operatorname{ad} h$ *on* $\mathfrak{g}_T$, *for all* $k \in \mathbb{N}$, $1 \leq k \leq n - s$, *if* $\mathfrak{g}$ *is of type* B_n, *resp. for all* $k \in \mathbb{N}$, $1 \leq k \leq n - s - 1$, *if* $\mathfrak{g}$ *is of type* D_n.

Proof. Follows by direct computation, since the semisimple element h of the above adapted pair (h, y) for $\mathfrak{p}^-_{\pi', \Lambda}$ is

$$h = \sum_{k=1}^{[s/4]} (\frac{s}{2} + 2k - 1)\varepsilon_{2k-1} + \sum_{k=[s/4]+1}^{s/2-1} (\frac{3s}{2} - 2k)\varepsilon_{2k-1} - \sum_{k=1}^{[s/4]} (\frac{s}{2} + 2k)\varepsilon_{2k}$$
$$- \sum_{k=[s/4]+1}^{s/2-1} (\frac{3s}{2} + 1 - 2k)\varepsilon_{2k} + \frac{s}{2}\varepsilon_{s-1} - \varepsilon_s + \sum_{k=1}^{[(n-s+u)/2]} (-2k + 1 - \frac{s}{2})\varepsilon_{s+2k-1} + \sum_{k=1}^{[(n-s-1+u)/2]} (2k + \frac{s}{2})\varepsilon_{s+2k}$$

with $u = 0$, resp. $u = 1$, if $\mathfrak{g}$ is of type D_n, resp. B_n. ■

Recall the bounds $\operatorname{ch} \mathcal{A}$ and $\operatorname{ch} \mathcal{B}$ for $\operatorname{ch} Y(\mathfrak{p})$ as well as the improved upper bound $\mathcal{B}'$ of Section 2. We will show that the lower bound $\operatorname{ch} \mathcal{A}$ and the improved upper bound $\mathcal{B}'$ coincide, hence $Y(\mathfrak{p})$ is a polynomial algebra over k.

Lemma 7.8. *For $\mathfrak{g}$ of type* B_n*, one has:*

If $n = s$: $$\mathrm{ch}\,\mathcal{A} = (1 - e^{-2\varpi_n})^{-2}(1 - e^{-4\varpi_n})^{-(n/2-1)}, \quad (1)$$

if $n > s$: $$\mathrm{ch}\,\mathcal{A} = (1 - e^{-\varpi_s})^{-2}(1 - e^{-2\varpi_s})^{-(n-1-s/2)}. \quad (2)$$

For $\mathfrak{g}$ of type D_n*, one has:*

$$\mathrm{ch}\,\mathcal{A} = (1 - e^{-\varpi_s})^{-3}(1 - e^{-2\varpi_s})^{-(n-2-s/2)}. \quad (3)$$

Proof. Assume first that $\mathfrak{g}$ is of type B_n. The lower bound for $\mathrm{ch}\,Y(\mathfrak{p})$ is $\mathrm{ch}\,\mathcal{A} = \prod_{\Gamma\in E(\pi')}(1 - e^{\delta_\Gamma})^{-1} \leq \mathrm{ch}\,Y(\mathfrak{p})$. We will compute it explicitly. As we already said in the proof of Lemma 7.5, the set of $\langle\mathbf{ij}\rangle$-orbits in π is $E(\pi') = \{\Gamma_{s/2} := \{\alpha_{s/2}\},\, \Gamma_t := \{\alpha_t,\, \alpha_{s-t}\},\, \Gamma_u := \{\alpha_u\} \mid 1 \leq t \leq s/2 - 1,\, s \leq u \leq n\}$. It remains to compute δ_Γ for each $\Gamma \in E(\pi')$.
Let $\Gamma \in E(\pi')$. Since $\mathbf{j} = \mathrm{id}_\pi$ and $\mathbf{i}(\Gamma \cap \pi') = \mathbf{j}(\Gamma) \cap \pi'$, one has

$$\delta_\Gamma = -2(\sum_{\gamma\in\Gamma}\varpi_\gamma - \sum_{\gamma\in\Gamma\cap\pi'}\varpi'_\gamma).$$

Assume first that $n = s$. Then the Levi factor of $\mathfrak{p}$ is of type A_{n-1} and one may check that for all $1 \leq t \leq n-1$, $\varpi_t - \varpi'_t = 2(t/n)\varpi_n$. Then for all $1 \leq t \leq n/2-1$, $\delta_{\Gamma_t} = -2(\varpi_t - \varpi'_t + \varpi_{n-t} - \varpi'_{n-t}) = -4\varpi_n$ and $\delta_{\Gamma_n} = \delta_{\Gamma_{n/2}} = -2\varpi_n$. Hence for $n = s$, one has the equality (1).
Assume now that $n > s$. Then the Levi factor of $\mathfrak{p}$ is the product of a simple Lie algebra of type A_{s-1} and a simple Lie algebra of type B_{n-s}.
For all $1 \leq t \leq s - 1$, one checks that $\varpi_t - \varpi'_t = (t/s)\varpi_s$. Then, for all $1 \leq t \leq s/2 - 1$, one has $\delta_{\Gamma_t} = -2\varpi_s$ and $\delta_{\Gamma_{s/2}} = -\varpi_s$. On the other hand, for all $s+1 \leq t \leq n-1$, one has that $\varpi_t - \varpi'_t = \varpi_s$, hence $\delta_{\Gamma_t} = -2\varpi_s$. Finally $\varpi_n - \varpi'_n = (1/2)\varpi_s$ and $\delta_{\Gamma_n} = -\varpi_s$, whereas $\delta_{\Gamma_s} = -2\varpi_s$, since $\Gamma_s \cap \pi' = \emptyset$. We conclude that for $n > s$, one has the equality (2).
Now if $\mathfrak{g}$ is of type D_n, the computation of the δ_Γ, $\Gamma \in E(\pi')$, is exactly as above, except for the $\langle\mathbf{ij}\rangle$-orbit $\Gamma_{n-1} = \{\alpha_{n-1}\}$, for which $\delta_{\Gamma_{n-1}} = -2(\varpi_{n-1} - \varpi'_{n-1}) = -\varpi_s$. Hence equality (3). ■

Lemma 7.9. *If $\mathfrak{g}$ is of type* B_n*, then the improved upper bound $\mathcal{B}'$ is given by the right hand side of* (1) *if $n = s$, resp. of* (2) *if $n > s$. If $\mathfrak{g}$ is of type* D_n*, then $\mathcal{B}'$ is given by the right hand side of* (3)*. Hence* $\mathrm{ch}\,\mathcal{A} = \mathcal{B}'$ *and then $Y(\mathfrak{p})$ is a polynomial algebra over k.*

Proof. Recall from Section 2 that the improved upper bound for $\mathrm{ch}\,Y(\mathfrak{p})$ is $\mathcal{B}' = \prod_{\gamma\in T}(1 - e^{-(\gamma+t(\gamma))})^{-1}$, where for all $\gamma \in T$, $t(\gamma)$ is the unique element in $\mathbb{Q}S$ such that $\gamma + t(\gamma)$ is a multiple of ϖ_s. We will compute $t(\gamma)$, for all $\gamma \in T$.
Assume first that $\mathfrak{g}$ is of type B_n and that $n = s$ and recall that

$$T = \{\varepsilon_{s-1} + \varepsilon_s,\, \varepsilon_{2i-1} - \varepsilon_{2i} \mid 1 \leq i \leq s/2\}.$$

Recall also that $S = \{\varepsilon_s,\, \varepsilon_{s-i} - \varepsilon_i,\, \varepsilon_{2j-1} + \varepsilon_{2j} \mid 1 \leq i,\, j \leq s/2 - 1\}$ and that $\varpi_s = \varpi_n = 1/2(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$. By direct calculation, one may verify that:

$$t(\varepsilon_{s-1} + \varepsilon_s) = (\varepsilon_1 + \varepsilon_2) + (\varepsilon_3 + \varepsilon_4) + \ldots + (\varepsilon_{n-3} + \varepsilon_{n-2})$$

and $$\varepsilon_{s-1} + \varepsilon_s + t(\varepsilon_{s-1} + \varepsilon_s) = 2\varpi_n.$$

Moreover $\quad t(\varepsilon_{s-1}-\varepsilon_s)=(\varepsilon_1+\varepsilon_2)+\ldots+(\varepsilon_{n-3}+\varepsilon_{n-2})+2\varepsilon_n$

and $\quad \varepsilon_{s-1}-\varepsilon_s+t(\varepsilon_{s-1}-\varepsilon_s)=2\varpi_n.$

Let $1\leq i\leq s/2-1$. If $n\leq 4i-2$,

$$t(\varepsilon_{2i-1}-\varepsilon_{2i})=2\sum_{j=1}^{n-2i}(\varepsilon_{n-j}-\varepsilon_j)+4\sum_{j=1}^{n/2-i}(\varepsilon_{2j-1}+\varepsilon_{2j})$$
$$+2\sum_{j=n/2-i+1}^{i-1}(\varepsilon_{2j-1}+\varepsilon_{2j})+(\varepsilon_{2i-1}+\varepsilon_{2i})+2\varepsilon_n$$

and $(\varepsilon_{2i-1}-\varepsilon_{2i})+t(\varepsilon_{2i-1}-\varepsilon_{2i})=4\varpi_n$. If $n>4i-2$,

$$t(\varepsilon_{2i-1}-\varepsilon_{2i})=2\sum_{j=1}^{2i-1}(\varepsilon_{n-j}-\varepsilon_j)+4\sum_{j=1}^{i-1}(\varepsilon_{2j-1}+\varepsilon_{2j})$$
$$+2\sum_{j=i+1}^{n/2-i}(\varepsilon_{2j-1}+\varepsilon_{2j})+3(\varepsilon_{2i-1}+\varepsilon_{2i})+2\varepsilon_n$$

and $(\varepsilon_{2i-1}-\varepsilon_{2i})+t(\varepsilon_{2i-1}-\varepsilon_{2i})=4\varpi_n$. Hence for all $1\leq i\leq s/2-1$,

$$\varepsilon_{2i-1}-\varepsilon_{2i}+t(\varepsilon_{2i-1}-\varepsilon_{2i})=4\varpi_n.$$

We conclude that, when $n=s$, the product $\prod_{\gamma\in T}(1-e^{-(\gamma+t(\gamma))})^{-1}$ is given by the right hand side of equality (1) of Lemma 7.8 and hence coincides with the lower bound for $\operatorname{ch}Y(\mathfrak{p})$.

Now assume that $n>s$. The previous computations hold if we replace n by s and $2\varpi_n$ by ϖ_s (and so $4\varpi_n$ by $2\varpi_s$). Then we may recover $t(\gamma)$ and $\gamma+t(\gamma)$ for $\gamma=\varepsilon_{s-1}+\varepsilon_s$, $\gamma=\varepsilon_{s-1}-\varepsilon_s$ or $\gamma=\varepsilon_{2i-1}-\varepsilon_{2i}$, $1\leq i\leq s/2-1$, by the above. It remains to compute $t(\gamma)$, $\gamma+t(\gamma)$ for the rest of the elements in T. One has

$$t(\varepsilon_{s-1}-\varepsilon_{s+1})=2((\varepsilon_1+\varepsilon_2)+\ldots+(\varepsilon_{s-3}+\varepsilon_{s-2}))+(\varepsilon_{s-1}+\varepsilon_{s+1})+2\varepsilon_s$$

and $(\varepsilon_{s-1}-\varepsilon_{s+1})+t(\varepsilon_{s-1}-\varepsilon_{s+1})=2\varpi_s$. For $1\leq j\leq[(n-s)/2]$, one has

$$t(-\varepsilon_{s+2j-1}+\varepsilon_{s+2j})=2((\varepsilon_1+\varepsilon_2)+\ldots+(\varepsilon_{s-3}+\varepsilon_{s-2}))+2(\varepsilon_{s-1}+\varepsilon_{s+1})$$
$$-2\sum_{k=1}^{j-1}(\varepsilon_{s+2k-1}+\varepsilon_{s+2k})+2\sum_{k=1}^{j-1}(\varepsilon_{s+2k}+\varepsilon_{s+2k+1})-(\varepsilon_{s+2j-1}+\varepsilon_{s+2j})+2\varepsilon_s$$

and $(-\varepsilon_{s+2j-1}+\varepsilon_{s+2j})+t(-\varepsilon_{s+2j-1}+\varepsilon_{s+2j})=2\varpi_s$. For $1\leq j\leq[(n-s-1)/2]$, one has

$$t(\varepsilon_{s+2j}-\varepsilon_{s+2j+1})=2((\varepsilon_1+\varepsilon_2)+\ldots+(\varepsilon_{s-3}+\varepsilon_{s-2}))+2(\varepsilon_{s-1}+\varepsilon_{s+1})$$
$$-2\sum_{k=1}^{j}(\varepsilon_{s+2k-1}+\varepsilon_{s+2k})+2\sum_{k=1}^{j-1}(\varepsilon_{s+2k}+\varepsilon_{s+2k+1})+(\varepsilon_{s+2j}+\varepsilon_{s+2j+1})+2\varepsilon_s$$

and $\quad (\varepsilon_{s+2j}-\varepsilon_{s+2j+1})+t(\varepsilon_{s+2j}-\varepsilon_{s+2j+1})=2\varpi_s.$

We conclude that for $n > s$ the product $\prod_{\gamma\in T}(1-e^{-(\gamma+t(\gamma))})^{-1}$ is given by the right hand side of equality (2) of Lemma 7.8 and hence coincides with the lower bound for $\operatorname{ch} Y(\mathfrak{p})$.

For $\mathfrak{g}$ of type D_n, the computations are very similar to type B_n above. Comparing the sets S in both types, we observe that the sets $S^{\pm}$ differ only by one element. Recall that the set T is the same for both types. More precisely, if n is odd, then for $\gamma = \varepsilon_{n-1}-\varepsilon_n \in T$, one obtains that $t(\varepsilon_{n-1}-\varepsilon_n)+(\varepsilon_{n-1}-\varepsilon_n)=\varpi_s$ instead of $2\varpi_s$ in type B_n.

Similarly, if n is even, then for $\gamma = -\varepsilon_{n-1}+\varepsilon_n \in T$ one has the equality $t(-\varepsilon_{n-1}+\varepsilon_n)+(-\varepsilon_{n-1}+\varepsilon_n)=\varpi_s$ instead of $2\varpi_s$ in type B_n.

Hence the improved upper bound for $\operatorname{ch} Y(\mathfrak{p})$ in type D_n differs from the improved upper bound in type B_n only by this factor. We conclude that the improved upper bound for $\operatorname{ch} Y(\mathfrak{p})$ is equal to the lower bound also in type D_n. ∎

Theorem 7.10. *Let $\mathfrak{g}$ be a simple Lie algebra of type B_n, $n\geq 2$, resp. of type D_n, $n\geq 4$, and let $\mathfrak{p}=\mathfrak{p}^-_{\pi',\Lambda}$ be a truncated maximal parabolic subalgebra of $\mathfrak{g}$ associated to $\pi'=\pi\setminus\{\alpha_s\}$, where s is an even integer, $s\leq n$, resp. $s\leq n-2$. There exists an adapted pair $(h,\, y)$ for $\mathfrak{p}$ and an affine slice $y+\mathfrak{g}_T$ in $\mathfrak{p}^*$ such that restriction of functions gives an isomorphism of algebras between $Y(\mathfrak{p})$ and the ring $R[y+\mathfrak{g}_T]$ of polynomial functions on $y+\mathfrak{g}_T$.*

In particular $Y(\mathfrak{p})$ is a polynomial algebra over k and the field $C(\mathfrak{p}^-_{\pi'})$ of invariant fractions is a purely transcendental extension of k. The degrees of a set of homogeneous generators of the polynomial algebra $Y(\mathfrak{p})$ are given by the eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}_T$ computed in Lemma 7.7, each augmented by one.

Proof. Follows by the previous lemma and by what we said at the end of Section 2. ∎

Remark 7.11. In the particular case $s=2$ polynomiality was known by [22] and an adapted pair was constructed in [16]. Our adapted pair is equivalent to the adapted pair of Joseph $(h',\, y'=\sum_{s\in S'} x_s)$, in the sense of [9, 2.1.1]. Indeed one verifies that for $\mathfrak{g}$ of type B_n, $w=\prod_{k=1}^{[(n-1)/2]} r_{\varepsilon_{2k+1}}\circ r_{\alpha_1}\in W_{\pi'}$ and sends bijectively S to S'.

For $\mathfrak{g}$ of type D_n, setting $r_{i,\,j}=r_{\varepsilon_i-\varepsilon_j}\circ r_{\varepsilon_i+\varepsilon_j}$, one verifies $w=\prod_{k=1}^{2m-1} r_{2k+1,\,2k+3}\circ r_{\alpha_1}$ (resp. $w=\prod_{k=1}^{2m-3} r_{2k+1,\,2k+3}\circ r_{\alpha_1}\circ r_{n-1,\,n}$) if $n=4m+u$ with $u\in\{1,\,2,\,3\}$ (resp. if $n=4m$) and $m\neq 0$ is such that $w\in W_{\pi'}$ and sends bijectively S to S'.

8. Another lemma of non-degeneracy

It remains to consider the case when the simple Lie algebra $\mathfrak{g}$ is of type D_n and when the truncated maximal parabolic subalgebra $\mathfrak{p}$ corresponds to $\pi'=\pi\setminus\{\alpha_n\}$ with n even (such a case will be called the extremal case). In this extremal case, we can no more use the set S constructed in the non-extremal case in type D_n. Moreover the set S and the Heisenberg sets Γ_γ, $\gamma\in S$ (see next section) that we have found in this extremal case will not verify Lemma 6.1 (since they produce more roots in O_3 than in the non-extremal case). However they verify a new lemma of non-degeneracy, where the notions of stationary roots and of cyclic roots that we have defined in Definitions 4.2 and 5.1 need to be slightly extended.

Recall the hypotheses and the notation of Section 4, especially the definition of a sequence $(\alpha^i)_{i\in\mathbb{N}}$ of roots in O constructed from the root $\alpha\in O$ and the set $A_\alpha=\{\alpha^i,\,\theta(\alpha^i)\mid i\in\mathbb{N}\}$. But first the condition $(*)$ below will be needed.

Condition $(*)$: If $\alpha\in O_3$, then there exists $\alpha'\in S_\alpha\setminus\{\theta(\alpha)\}\cap O_2$ such that $\theta(\alpha')\in O_1$.

Assume that condition $(*)$ is satisfied for $\alpha\in O_3$ and choose a root α' as above. Then we define $\alpha^{(1)}$ as the unique root in O, distinct from α' and from $\theta(\alpha)$, such that $\alpha^{(1)}+\alpha\in S$. If $\theta(\alpha^{(1)})\in O_3$ satisfies condition $(*)$, we define $\alpha^{(2)}$ similarly. If at each step i, condition $(*)$ is satisfied for the root $\theta(\alpha^{(i)})$ if it belongs to O_3 or if $\theta(\alpha^{(i)})\in O_1\sqcup O_2$, then the sequence $(\alpha^{(i)})_{i\in\mathbb{N}}$ of roots in O constructed from $\alpha^{(0)}=\theta(\alpha)$ is uniquely defined.

Remark 8.1. Let $\alpha\in O$ such that $A_\alpha\cup A_{\theta(\alpha)}\subset O_1\sqcup O_2\sqcup O_3$, with condition $(*)$ satisfied for all roots in $(A_\alpha\cup A_{\theta(\alpha)})\cap O_3$. In particular this implies that the sequences $(\alpha^i)_{i\in\mathbb{N}}$ and $(\alpha^{(i)})_{i\in\mathbb{N}}$ are uniquely defined and moreover Remark 4.1(1) still applies. Hence if there exists $i_0\in\mathbb{N}$ such that the sequence of roots in O constructed from α^{i_0}, resp. from $\alpha^{(i_0)}$, is stationary, then the sequence of roots in O constructed from α, resp. from $\theta(\alpha)$, is also stationary.

We can now give the definition of an extended stationary root.

Definition 8.2. Let $\alpha\in O$. We will say that α is an *extended stationary root* if $A_\alpha\cup A_{\theta(\alpha)}\subset O_1\sqcup O_2\sqcup O_3$, with condition $(*)$ satisfied for all roots in $(A_\alpha\cup A_{\theta(\alpha)})\cap O_3$ and if the sequences $(\alpha^{(i)})_{i\in\mathbb{N}}$ and $(\alpha^i)_{i\in\mathbb{N}}$ are stationary. The set of extended stationary roots will be denoted by O^e_{st}.

Remark 8.3. $O_{st}\subset O^e_{st}$.

Recall conditions (i)–(vi) of Definition 5.1 and its notation. We replace condition (vi) by condition (vie) below. Let $\alpha\in O$ be a root verifying conditions (i)–(v) of Definition 5.1.

Condition (vie): If $\delta\in C_\alpha\cap O_3$, then there exists $\tilde\delta\in S_\delta$ such that $A_{\tilde\delta}\subset O_1\sqcup O_2\sqcup O_3$ with condition $(*)$ satisfied for all roots in $A_{\tilde\delta}\cap O_3$, and the sequence $(\tilde\delta^i)_{i\in\mathbb{N}}$ is stationary.

Remark 8.4. (1) Using the above notation it is clear that the root $\tilde\delta$ is unique since $\tilde\delta\in S_\delta\setminus S_\delta\cap C_\delta$.

(2) Condition (vi) of Definition 5.1 implies condition (vie), since $\theta(\tilde\delta)\in O_1$ implies that $\tilde\delta^1=\tilde\delta$.

We can now give the definition of an extended cyclic root.

Definition 8.5. Let $\alpha\in O$. We say that α is an *extended cyclic root* if there exist $\beta,\gamma\in O$ satisfying conditions (i)–(v) of Definition 5.1 and condition (vie) above. The set of extended cyclic roots will be denoted by O^e_{cyc}.

Remark 8.6. $O_{cyc} \subset O^e_{cyc}$.

Similarly to Lemma 4.4, we obtain the following lemma.

Lemma 8.7. *Let $\alpha \in O^e_{st}$. Let $\vartheta : O \to O$ be a permutation such that, for all $\gamma \in O$, $\gamma + \vartheta(\gamma) \in S$. Then the restriction of ϑ to $A_\alpha \cup A_{\theta(\alpha)}$ coincides with the involution θ. Moreover the map ϑ exchanges β' and $\theta(\beta')$, where β' is the chosen root in $S_\beta \setminus \{\theta(\beta)\} \cap O_2$ such that $\theta(\beta') \in O_1$ for any $\beta \in (A_\alpha \cup A_{\theta(\alpha)}) \cap O_3$.*

Proof. It is similar to the proof of Lemma 4.4, noting that necessarily, with the above notation, the map ϑ sends $\theta(\beta')$ to β' since $\theta(\beta') \in O_1$. ■

Similarly to Lemma 5.3, we obtain the following lemma.

Lemma 8.8. *Let $\vartheta : O \to O$ be a permutation such that for all $\gamma \in O$, $\gamma + \vartheta(\gamma) \in S$. Then ϑ exchanges $\tilde{\delta}^i$ and $\theta(\tilde{\delta}^i)$, for all $i \in \mathbb{N}$, where $\tilde{\delta}$ is the root in S_δ given by condition (vie) of Definition 8.5, for any $\delta \in O^e_{cyc} \cap O_3$.*

Similarly to Lemma 6.1, we get the following new lemma of non-degeneracy.

Lemma 8.9. *Assume that:*
(1) *$S_{|\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}^*_\Lambda$.*
(2) *If $\alpha \in \Gamma^0_\gamma$, with $\gamma \in S^+$, then $S_\alpha \cap O^+ = \{\theta(\alpha)\}$.*
(3) *If $\alpha \in \Gamma^0_\gamma$, with $\gamma \in S^-$, then $S_\alpha \cap O^- = \{\theta(\alpha)\}$.*
(4) *If $\alpha \in O$, with $S_\alpha \cap O^m \neq \emptyset$, then $\alpha \in O^e_{st}$ or $\alpha \in O^e_{cyc}$ or there exists $\beta \in O^e_{cyc} \cap O_3$ and $i \in \mathbb{N}$ such that $\alpha = \tilde{\beta}^i$ or $\theta(\alpha) = \tilde{\beta}^i$, where $\tilde{\beta}$ is the unique root in S_β given by condition* (vie) *of Definition* 8.5.

Let $y = \sum_{\gamma\in S} a_\gamma x_\gamma$, with $a_\gamma \in k \setminus \{0\}$ for all $\gamma \in S$. Then the restriction of the bilinear form Φ_y to $\mathfrak{o} \times \mathfrak{o}$ is non-degenerate.

Proof. Works like that of Lemma 6.1, using Lemmas 8.7 and 8.8. ■

9. Type D, the extremal case.

In this section, we assume that the simple Lie algebra $\mathfrak{g}$ is of type D_n with $n \geq 6$ and n even and we consider $\mathfrak{p} = \mathfrak{p}^-_{\pi',\Lambda}$ the truncated maximal parabolic subalgebra of $\mathfrak{g}$ associated to $\pi' = \pi\setminus\{\alpha_n\}$. Then the lower and the upper bounds of Section 2 for $\mathrm{ch}\, Y(\mathfrak{p})$ do not coincide. We will construct an adapted pair for $\mathfrak{p}$ and then prove that $Y(\mathfrak{p})$ is a polynomial algebra over k. Since the case when $\pi' = \pi \setminus \{\alpha_{n-1}\}$ is symmetric, this will also prove that $Y(\mathfrak{p})$ is polynomial when $\pi' = \pi \setminus \{\alpha_{n-1}\}$.
Here $\mathfrak{p}$ may be viewed as the semi-direct product of its Levi factor $\mathfrak{g}' \simeq \mathfrak{sl}_n$ and its nilpotent radical $\mathfrak{m}$, which is in this case an abelian $\mathfrak{g}'$-module, isomorphic to $\Lambda^2 k^n$ as a $\mathfrak{sl}_n$-module. Then one may apply [25, Thm. 2.3] to conclude that $Y(\mathfrak{p})$ is a polynomial algebra. However [25, Thm. 2.3] does not give the degrees of a set of homogeneous generators. In our present work we will also compute their degrees (see Lemma 9.5 and Thm 9.7). We set

$$\begin{aligned} S = \; & \{\varepsilon_{2i-1} + \varepsilon_{2i},\, \varepsilon_{n-3} + \varepsilon_{n-1},\, \varepsilon_n - \varepsilon_{n-3},\, \varepsilon_{n-2} - \varepsilon_{n-4},\, \varepsilon_{n-4} - \varepsilon_{n-5}, \\ & \varepsilon_{n-3} - \varepsilon_{n-6},\, \varepsilon_{n-2j} - \varepsilon_{n-2j-2} \mid 1 \leq i \leq n/2-2,\, 3 \leq j \leq n/2-2\}. \end{aligned}$$

One checks that $S \subset \Delta^+ \sqcup \Delta^-_{\pi'}$ and that $|S| = n-1 = \dim \mathfrak{h}_\Lambda$. We first prove below that condition (1) of Lemma 8.9 holds.

Lemma 9.1. *$S_{|\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}^*_\Lambda$.*

Proof. Set $S = \{s_i\}_{1\le i\le n-1}$ with $s_i = \varepsilon_{2i-1}+\varepsilon_{2i}$ for all $i\in\mathbb{N}$, $1\le i\le n/2-2$, $s_{n/2-1} = \varepsilon_{n-3}+\varepsilon_{n-1}$, $s_{n/2} = \varepsilon_{n-4}-\varepsilon_{n-5}$, $s_{n/2+1} = \varepsilon_{n-2}-\varepsilon_{n-4}$, $s_{n/2+2} = \varepsilon_n-\varepsilon_{n-3}$, $s_{n/2+3} = \varepsilon_{n-3}-\varepsilon_{n-6}$, $s_{n/2+k} = \varepsilon_{n-2k+2}-\varepsilon_{n-2k}$ for all $k\in\mathbb{N}$, $4\le k\le n/2-1$. Then set $s'_i = s_i$ for all $i\in\mathbb{N}$, $1\le i\le n-1$, $i\ne n/2-1$, $s'_{n/2-1} = \varepsilon_{n-1}+\varepsilon_n = s_{n/2-1}+s_{n/2+2}$ and $S' = \{s'_i\}_{1\le i\le n-1}$.
If we choose $\{h_i\}_{1\le i\le n-1} = \{\alpha_i^\vee\}_{1\le i\le n-1}$ as a basis of $\mathfrak{h}_\Lambda$, it is sufficient to show that $\det(s'_i(h_j))_{1\le i,j\le n-1}\ne 0$.
By ordering S' as above and the basis of $\mathfrak{h}_\Lambda$ as

$$\{\alpha^\vee_{2i},\, \alpha^\vee_{n-5},\, \alpha^\vee_{n-3},\, \alpha^\vee_{n-1},\, \alpha^\vee_{n-2j-1} \mid 1\le i\le n/2-1,\ 3\le j\le n/2-1\},$$

one checks that the matrix $(s'_i(h_j))_{1\le i,,j\le n-1}$ is a lower triangular matrix with 1 on the first $n/2-2$ diagonal elements, then -1, -2, -1, -1 on the next diagonal elements and then 1 on the $n/2-3$ last diagonal elements. Hence $\det(s'_i(h_j))_{1\le i,,j\le n-1} = 2$ and the lemma. ∎

Now we define the Heisenberg sets Γ_γ with centre γ, for all $\gamma\in S$, by setting:
For all $k\in\mathbb{N}$, $2\le k\le n/2-3$,

$$\begin{aligned}
\Gamma_{\varepsilon_{2k}-\varepsilon_{2k-2}} &= \{\varepsilon_{2k}-\varepsilon_{2k-2},\, \varepsilon_{2k}-\varepsilon_i,\, \varepsilon_i-\varepsilon_{2k-2} \mid 1\le i\le 2k-3\},\\
\Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}} &= \{\varepsilon_{n-3}-\varepsilon_{n-6},\, \varepsilon_{n-3}-\varepsilon_i,\, \varepsilon_i-\varepsilon_{n-6} \mid 1\le i\le n-7\},\\
\Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}} &= \{\varepsilon_{n-4}-\varepsilon_{n-5},\, \varepsilon_{n-3}-\varepsilon_{n-5},\, \varepsilon_{n-4}-\varepsilon_{n-3},\\
&\qquad \varepsilon_{n-4}-\varepsilon_{2i},\, \varepsilon_{2i}-\varepsilon_{n-5} \mid 1\le i\le n/2-3\},\\
\Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}} &= \{\varepsilon_{n-2}-\varepsilon_{n-4},\, \varepsilon_{n-2}-\varepsilon_{n-1},\, \varepsilon_{n-1}-\varepsilon_{n-4},\\
&\qquad \varepsilon_{n-2}-\varepsilon_n,\, \varepsilon_n-\varepsilon_{n-4},\, \varepsilon_{n-2}-\varepsilon_i,\, \varepsilon_i-\varepsilon_{n-4} \mid 1\le i\le n-5\},\\
\Gamma_{\varepsilon_n-\varepsilon_{n-3}} &= \{\varepsilon_n-\varepsilon_{n-3},\, \varepsilon_n-\varepsilon_{n-2},\, \varepsilon_{n-2}-\varepsilon_{n-3},\\
&\qquad \varepsilon_n-\varepsilon_{n-1},\, \varepsilon_{n-1}-\varepsilon_{n-3},\, \varepsilon_n-\varepsilon_i,\, \varepsilon_i-\varepsilon_{n-3} \mid 1\le i\le n-6\},\\
\Gamma_{\varepsilon_{n-3}+\varepsilon_{n-1}} &= \{\varepsilon_{n-3}+\varepsilon_{n-1},\, \varepsilon_{n-3}+\varepsilon_n,\, \varepsilon_{n-1}-\varepsilon_n,\\
&\qquad \varepsilon_{n-3}-\varepsilon_n,\, \varepsilon_n+\varepsilon_{n-1},\, \varepsilon_{n-3}-\varepsilon_{n-2},\, \varepsilon_{n-2}+\varepsilon_{n-1},\\
&\qquad \varepsilon_{n-3}+\varepsilon_{n-2},\, -\varepsilon_{n-2}+\varepsilon_{n-1},\, \varepsilon_{n-1}-\varepsilon_i,\, \varepsilon_i+\varepsilon_{n-3} \mid 1\le i\le n-5\}.
\end{aligned}$$

It is easy to check that the $n/2+1$ sets defined above are Heisenberg sets, which we denote by Γ_{γ_j}, $1\le j\le n/2+1$, whose centre will be denoted by $\gamma_j\in S$.
Let $i\in\mathbb{N}$, $1\le i\le n/2-2$, and recall that $\beta_i = \varepsilon_{2i-1}+\varepsilon_{2i}$ is an element of the Kostant cascade of $\mathfrak{g}$ (see Section 7) and that we denote by H_{β_i} the maximal Heisenberg set in Δ^+ with centre β_i (see Example 3.1).
We define below every Heisenberg set Γ_{β_i} with centre β_i, $1\le i\le n/2-2$, by decreasing induction on i. First we set

$$\begin{aligned}
\Gamma_{\beta_{n/2-2}} = (H_{\beta_{n/2-2}} \setminus \bigsqcup_{1\le j\le n/2+1} \Gamma_{\gamma_j}\cap H_{\beta_{n/2-2}}) &\sqcup \{\varepsilon_i+\varepsilon_{n-4},\, \varepsilon_{n-5}-\varepsilon_i \mid 1\le i\le n-6\}\\
&\sqcup \{\varepsilon_{n-4}-\varepsilon_{2i-1},\, \varepsilon_{2i-1}+\varepsilon_{n-5} \mid 1\le i\le n/2-3\}.
\end{aligned}$$

One has $\beta_{n/2-2} = \varepsilon_{n-5} + \varepsilon_{n-4}$ and

$$\Gamma_{\beta_{n/2-2}} = \{\beta_{n/2-2},\ \varepsilon_{n-5} + \varepsilon_i,\ \varepsilon_{n-4} - \varepsilon_i,\ \varepsilon_{n-5} - \varepsilon_j,\ \varepsilon_j + \varepsilon_{n-4},\ \varepsilon_{n-5} + \varepsilon_{2k-1},\ \varepsilon_{n-4} - \varepsilon_{2k-1} \mid n-2 \le i \le n,\ 1 \le j \le n,\ j \notin \{n-5,\, n-4\},\ 1 \le k \le n/2-3\}.$$

Set $\gamma_j = \beta_{n-j}$ for all $j \in \mathbb{N}$, $n/2+2 \le j \le n-1$, and suppose, for $2 \le k \le n/2-2$, that we have defined the Heisenberg set Γ_{γ_j} with centre $\gamma_j \in S$, for all $j \in \mathbb{N}$, $1 \le j \le n/2+k$. Then we set

$$\Gamma_{\gamma_{n/2+k+1}} = \Gamma_{\beta_{n/2-k-1}} = (H_{\beta_{n/2-k-1}} \setminus \bigsqcup_{1 \le j \le n/2+k} \Gamma_{\gamma_j} \cap H_{\beta_{n/2-k-1}}) \sqcup \{\varepsilon_i + \varepsilon_{n-2k-2},\, \varepsilon_{n-2k-3} - \varepsilon_i \mid 1 \le i \le n-2k-4\}.$$

Observe that, for $1 \le i \le n/2-3$,

$$\Gamma_{\beta_i} = \{\beta_i,\, \varepsilon_{2i-1} + \varepsilon_{2j-1},\, \varepsilon_{2i} - \varepsilon_{2j-1},\, \varepsilon_{2i-1} - \varepsilon_{2k-1},\, \varepsilon_{2i} + \varepsilon_{2k-1},\, \varepsilon_{2i-1} \pm \varepsilon_u,\, \varepsilon_{2i} \mp \varepsilon_u \mid i+1 \le j \le n/2-3,\, i+1 \le k \le n/2-2,\, n-2 \le u \le n\} \sqcup \{\varepsilon_{2i-1} - \varepsilon_v,\, \varepsilon_{2i} + \varepsilon_v \mid 1 \le v \le 2i-2\}.$$

One checks that, for every $\gamma_j \in S$, $1 \le j \le n-1$, the set Γ_{γ_j} is a Heisenberg set with centre γ_j. Moreover by construction all these Heisenberg sets are disjoint and $\Gamma = \bigsqcup_{\gamma \in S} \Gamma_\gamma \subset \Delta^+ \sqcup \Delta^-_{\pi'}$.
Finally one has that $S^+ = \{\beta_1\}$. For $n \ge 8$, one has that $S^- = \emptyset$ and $S^m = S \setminus \{\beta_1\}$. For $n = 6$, one has that $S^- = \{\varepsilon_6 - \varepsilon_3\}$ and $S^m = S \setminus \{\beta_1,\, \varepsilon_6 - \varepsilon_3\}$.

Lemma 9.2. *Conditions* (2), (3), *and* (4) *of Lemma* 8.9 *hold.*

Proof. **(a)** One checks that all roots in $\Gamma^0_{\varepsilon_{2k} - \varepsilon_{2k-2}}$, $2 \le k \le n/2-3$, belong to O^e_{st}. Let us explain the case when $\alpha = \varepsilon_{2k} - \varepsilon_i$, $1 \le i \le 2k-3$, with i even. If $2k = n-6$, then $\alpha \in O_2$, $\alpha^{(1)} = \varepsilon_{n-7} + \varepsilon_i$ and $\theta(\alpha^{(1)}) \in O_1$, hence $\alpha^{(2)} = \alpha^{(1)}$. If $i = 2$, then $\theta(\alpha) \in O_2$, $\alpha^1 = \varepsilon_1 + \varepsilon_{2k-2}$ and $\theta(\alpha^1) \in O_1$, hence $\alpha^2 = \alpha^1$. In the other cases, $\alpha \in O_3$ and $\theta(\alpha) \in O_3$ and they verify condition $(*)$ of Section 8. Indeed for $2k \le n-8$, $\alpha' = \varepsilon_{2k-1} + \varepsilon_i \in \Gamma^0_{\beta_{i/2}} \cap O_2$, $\theta(\alpha') = \varepsilon_{i-1} - \varepsilon_{2k-1} \in O_1$ and $\alpha^{(1)} = \varepsilon_{i+2} - \varepsilon_{2k} \in \Gamma^0_{\varepsilon_{2k+2} - \varepsilon_{2k}}$. Similarly for $i \ge 4$, $\theta(\alpha)' = \varepsilon_{i-1} + \varepsilon_{2k-2} \in \Gamma^0_{\beta_{k-1}} \cap O_2$, $\theta(\theta(\alpha)') = \varepsilon_{2k-3} - \varepsilon_{i-1} \in O_1$ and $\alpha^1 = \varepsilon_{2k-2} - \varepsilon_{i-2} \in \Gamma^0_{\varepsilon_{2k-2} - \varepsilon_{2k-4}}$. Then one deduces by induction that the sequences $(\alpha^{(i)})_{i \in \mathbb{N}}$ and $(\alpha^i)_{i \in \mathbb{N}}$ are stationary and that $A_\alpha \cup A_{\theta(\alpha)} \subset O_1 \sqcup O_2 \sqcup O_3$ with condition $(*)$ satisfied for all roots in $(A_\alpha \cup A_{\theta(\alpha)}) \cap O_3$.

(b) One checks that all roots in $\Gamma^0_{\varepsilon_{n-3} - \varepsilon_{n-6}}$ belong to O^e_{st}. Let us explain the case when $\alpha = \varepsilon_{n-3} - \varepsilon_i$, $1 \le i \le n-7$, with i even. One has $\theta(\alpha) = \varepsilon_i - \varepsilon_{n-6}$. If $i = 2$ then $\theta(\alpha) \in O_2$, $\alpha^1 = \varepsilon_1 + \varepsilon_{n-6} \in \Gamma^0_{\beta_{n/2-3}} \cap O_2$ and $\theta(\alpha^1) \in O_1$ hence $\alpha^2 = \alpha^1$. Moreover $\alpha \in O_3$ and for $i \ge 4$, $\theta(\alpha) \in O_3$ and both roots verify condition $(*)$ of Section 8. Indeed $\alpha' = \varepsilon_{n-1} + \varepsilon_i \in \Gamma^0_{\beta_{i/2}} \cap O_2$, $\theta(\alpha') = \varepsilon_{i-1} - \varepsilon_{n-1} \in O_1$, $\theta(\alpha)' = \varepsilon_{i-1} + \varepsilon_{n-6} \in \Gamma^0_{\beta_{n/2-3}} \cap O_2$ and $\theta(\theta(\alpha)') = \varepsilon_{n-7} - \varepsilon_{i-1} \in O_1$. Moreover for $i \ge 4$, $\alpha^1 = \varepsilon_{n-6} - \varepsilon_{i-2} \in \Gamma^0_{\varepsilon_{n-6} - \varepsilon_{n-8}}$, and paragraph (a) above gives that $\alpha^1 \in O^e_{st}$. Then, by Remark 8.1, the sequence $(\alpha^i)_{i \in \mathbb{N}}$ is stationary and $A_\alpha \subset O_1 \sqcup O_2 \sqcup O_3$ with condition $(*)$ satisfied for all roots in $A_\alpha \cap O_3$. On the other hand, one has that $\alpha^{(1)} = \varepsilon_{i+2} - \varepsilon_{n-3} \in \Gamma^0_{\varepsilon_n-\varepsilon_{n-3}} \cap O_2$, $\theta(\alpha^{(1)}) = \varepsilon_n - \varepsilon_{i+2} \in O_2$, (unless $i = n-8$, in which case $\theta(\alpha^{(1)}) \in O_1$), $\alpha^{(2)} = \varepsilon_{i+4} - \varepsilon_n \in \Gamma^0_{\beta_{(i+4)/2}} \cap O_2$ and $\theta(\alpha^{(2)}) = \varepsilon_{i+3} + \varepsilon_n \in O_1$. Hence $\alpha^{(3)} = \alpha^{(2)}$ and the sequence $(\alpha^{(i)})_{i\in\mathbb{N}}$ is stationary and $A_{\theta(\alpha)} \subset O_1 \sqcup O_2 \sqcup O_3$ with condition $(*)$ satisfied for all roots in $A_{\theta(\alpha)} \cap O_3$. This proves that $\alpha \in O^e_{st}$.

(c) One checks that all roots in $\Gamma^0_{\varepsilon_{n-4}-\varepsilon_{n-5}}$ belong to O^e_{cyc}. Let us explain the case when $\alpha = \varepsilon_{n-4} - \varepsilon_{2i}$, $1 \leq i \leq n/2 - 3$. Let $\beta = \varepsilon_{2i-1} - \varepsilon_{n-5} \in \Gamma^0_{\beta_i}$ and $\gamma = \varepsilon_{2i-1} + \varepsilon_{n-5} \in \Gamma^0_{\varepsilon_{n-4}+\varepsilon_{n-5}}$. Then α, β, γ verify the cyclic relations (i)-(iii) of Definition 8.5 and β, $\theta(\beta)$, γ, $\theta(\gamma)$ belong to O_2. Moreover α and $\theta(\alpha)$ belong to O_3 (unless $2i = n-6$, in which case $\alpha \in O_2$ or $i = 1$, in which case $\theta(\alpha) \in O_2$). If $2i \leq n-8$, $\tilde{\alpha} = \varepsilon_{2i+2} - \varepsilon_{n-4} \in \Gamma^0_{\varepsilon_{n-2}-\varepsilon_{n-4}} \cap O_3 \cap S_\alpha$ is such that $\theta(\tilde{\alpha}) = \varepsilon_{n-2} - \varepsilon_{2i+2} \in O_2$. If $2i \leq n-10$, $\tilde{\alpha}^1 = \varepsilon_{2i+4} - \varepsilon_{n-2} \in \Gamma^0_{\beta_{i+2}} \cap O_2$ and $\theta(\tilde{\alpha}^1) = \varepsilon_{2i+3} + \varepsilon_{n-2} \in O_1$ and if $2i = n-8$, then $\tilde{\alpha}^1 = \varepsilon_{n-3} - \varepsilon_{n-2} \in \Gamma^0_{\varepsilon_{n-3}+\varepsilon_{n-1}} \cap O_2$ and $\theta(\tilde{\alpha}^1) = \varepsilon_{n-2} + \varepsilon_{n-1} \in O_1$. Hence $\tilde{\alpha}^2 = \tilde{\alpha}^1$ and the sequence $(\tilde{\alpha}^i)_{i\in\mathbb{N}}$ is stationary.
Let $\tilde{\alpha}' = \varepsilon_{2i+1} + \varepsilon_{n-4} \in \Gamma^0_{\beta_{n/2-2}} \cap O_2 \cap S_{\tilde{\alpha}}$. Then $\theta(\tilde{\alpha}') = \varepsilon_{n-5} - \varepsilon_{2i+1} \in O_1$. Hence $\tilde{\alpha}$ satisfies condition $(*)$ and actually α verifies condition (vie) of Definition 8.5. Similarly for $i \geq 2$, one checks that $\theta(\alpha)$ verifies condition (vie) of Definition 8.5.

(d) One checks that, for a root α in $\Gamma^0_{\varepsilon_{n-2}-\varepsilon_{n-4}}$, there exists $\beta \in O_3 \cap O^e_{cyc}$ and $i \in \mathbb{N}$ such that $\alpha = \tilde{\beta}^i$ or $\theta(\alpha) = \tilde{\beta}^i$, unless some particular cases for which $\alpha \in O^e_{st}$. For instance assume that $\alpha = \varepsilon_{n-2} - \varepsilon_i$, $1 \leq i \leq n-7$ with i odd. Let $\beta = \varepsilon_{i+3} - \varepsilon_{n-5}$ if $i \leq n-9$, resp. $\beta = \varepsilon_{n-3} - \varepsilon_{n-5}$ if $i = n-7$. By paragraph (c) above one has that $\beta \in \Gamma^0_{\varepsilon_{n-4}-\varepsilon_{n-5}} \cap O_3 \cap O^e_{cyc}$.Then $\tilde{\beta} = \varepsilon_{n-5} - \varepsilon_{i+1} \in \Gamma^0_{\beta_{n/2-2}} \cap S_\beta \cap O_2$, $\theta(\tilde{\beta}) = \varepsilon_{n-4} + \varepsilon_{i+1} \in O_2$ and $\tilde{\beta}^1 = \varepsilon_i - \varepsilon_{n-4} \in \Gamma^0_{\varepsilon_{n-2}-\varepsilon_{n-4}} \cap O_2$ and $\alpha = \theta(\tilde{\beta}^1) \in O_1$. Hence $\theta(\alpha) = \tilde{\beta}^1$.

(e) One checks that all roots in $\Gamma^0_{\varepsilon_n-\varepsilon_{n-3}}$ belong to O^e_{st} and that, for $n = 6$, condition (3) of Lemma 8.9 is satisfied for all $\alpha \in \Gamma^0_{\varepsilon_6-\varepsilon_3}$.

(f) One checks that all roots in $\Gamma^0_{\varepsilon_{n-3}+\varepsilon_{n-1}}$ belong to O^e_{st}, except for the following cases. If $\alpha = \varepsilon_{n-3} - \varepsilon_{n-2}$, resp. $\alpha = \varepsilon_{n-2} + \varepsilon_{n-1}$, with $n \geq 10$, then $\alpha = \tilde{\beta}^1$, resp. $\theta(\alpha) = \tilde{\beta}^1$, with $\beta = \varepsilon_{n-4} - \varepsilon_{n-8} \in \Gamma^0_{\varepsilon_{n-4}-\varepsilon_{n-5}} \cap O_3 \cap O^e_{cyc}$ by paragraph (c) above. If $\alpha = \varepsilon_{n-1} - \varepsilon_{n-5}$, or if $\alpha = \varepsilon_{n-5} + \varepsilon_{n-3}$, then $\alpha \in O^e_{cyc}$ by paragraph (c) above.

(g) For a root α in $\Gamma^0_{\beta_{n/2-2}}$, one checks, using paragraphs (c) or (d) above, that $\alpha \in O^e_{st}$ or that $\alpha \in O^e_{cyc}$, except for the following cases. If $\alpha = \varepsilon_{n-5} - \varepsilon_j$, resp. $\alpha = \varepsilon_j + \varepsilon_{n-4}$, $1 \leq j \leq n-6$, j even, then $\alpha = \tilde{\beta}$, resp. $\theta(\alpha) = \tilde{\beta}$, with $\beta \in \Gamma^0_{\varepsilon_{n-4}-\varepsilon_{n-5}} \cap O^e_{cyc} \cap O_3$ by paragraph (c) above. One also checks that, for $n = 6$, condition (2) of Lemma 8.9 holds for all $\alpha \in \Gamma^0_{\beta_1}$.

(h) Let α be a root in $\Gamma^0_{\beta_i}$, with $1 \leq i \leq n/2 - 3$. (Observe that this implies that $n \geq 8$). Using the above paragraphs, one checks that $\alpha \in O^e_{st}$, except for the following cases. If $\alpha = \varepsilon_{2i-1} - \varepsilon_{n-5}$, or if $\alpha = \varepsilon_{2i} + \varepsilon_{n-5}$ then $\alpha \in O^e_{cyc}$. If $\alpha = \varepsilon_{2i-1} + \varepsilon_{n-2}$, resp. $\alpha = \varepsilon_{2i} - \varepsilon_{n-2}$, and $i \geq 3$, then $\theta(\alpha) = \tilde{\beta}^1$, resp. $\alpha = \tilde{\beta}^1$, with $\beta \in \Gamma^0_{\varepsilon_{n-4}-\varepsilon_{n-5}} \cap O^e_{cyc} \cap O_3$ by paragraph (c) above. One also checks that condition (2) of Lemma 8.9 holds for all $\alpha \in \Gamma^0_{\beta_1}$. ∎

We denote by T the complement of the set $\Gamma=\bigsqcup_{\gamma\in S}\Gamma_\gamma$ in $\Delta^+\sqcup\Delta^-_{\pi'}$.

Lemma 9.3. $|T|=\operatorname{ind}\mathfrak{p}$.

Proof. One checks that
$T=\{\varepsilon_{n-3}-\varepsilon_{n-1},\,\varepsilon_{n-2}+\varepsilon_n,\,\varepsilon_n-\varepsilon_{n-5},\,\varepsilon_{n-3}-\varepsilon_{n-4},\,\varepsilon_{n-2k}-\varepsilon_{n-2k-1}\mid 3\le k\le n/2-1\}$. Then $|T|=n/2+1$. Moreover the $\langle\mathbf{ij}\rangle$-orbits in π are $\Gamma_t=\{\alpha_t,\,\alpha_{n-t}\}$ for all $1\le t\le n/2-1$, $\Gamma_{n/2}=\{\alpha_{n/2}\}$ and $\Gamma_n=\{\alpha_n\}$. They are $n/2+1$ in number, hence the lemma. ∎

Remark 9.4. All conditions of Lemma 3.2 are satisfied (with $T^*=\emptyset$). Hence defining $h\in\mathfrak{h}_\Lambda$ by $\gamma(h)=-1$ for all $\gamma\in S$, and setting $y=\sum_{\gamma\in S}x_\gamma$ we obtain an adapted pair $(h,\,y)$ for $\mathfrak{p}^-_{\pi',\Lambda}$.

Lemma 9.5. *The eigenvalues of* $\operatorname{ad}h$ *on* $\mathfrak{g}_T$ *are:*
$2(n-i)+1=(\varepsilon_{2i}-\varepsilon_{2i-1})(h)$ *for all* $i\in\mathbb{N}$, $1\le i\le n/2-3$*;*
$n+5=(\varepsilon_{n-3}-\varepsilon_{n-1})(h)$*;*
$n/2-1=(\varepsilon_{n-2}+\varepsilon_n)(h)$*;*
$n/2+1=(\varepsilon_n-\varepsilon_{n-5})(h)$*;*
$n/2+3=(\varepsilon_{n-3}-\varepsilon_{n-4})(h)$*.*
From the first two equalities, we deduce that $n+3+2k$ *is an eigenvalue of* $\operatorname{ad}h$ *on* $\mathfrak{g}_T$*, for all* $k\in\mathbb{N}$, $1\le k\le n/2-2$*.*

Proof. Let us give the semisimple element h of the above adapted pair for $\mathfrak{p}^-_{\pi',\Lambda}$. If $n=6$ we have that $h=-\varepsilon_2+5\varepsilon_3-2\varepsilon_4-6\varepsilon_5+4\varepsilon_6$. If $n\ge 8$, we have

$$h=-n\varepsilon_1+\sum_{k=1}^{n/2-4}(k-n)\varepsilon_{2k+1}+\sum_{k=1}^{n/2-3}(n-k)\varepsilon_{2k}-\varepsilon_{n-4}$$
$$+(n/2+2)\varepsilon_{n-3}-2\varepsilon_{n-2}-(n/2+3)\varepsilon_{n-1}+(n/2+1)\varepsilon_n.$$

The lemma follows by direct calculation. ∎

Then we have the following lemma.

Lemma 9.6. *The lower bound* $\operatorname{ch}\mathcal{A}$ *for* $\operatorname{ch}Y(\mathfrak{p})$ *is equal to*

$$\prod_{\Gamma\in E(\pi')}(1-e^{\delta_\Gamma})^{-1}=(1-e^{-2\varpi_n})^{-3}(1-e^{-4\varpi_n})^{-(n/2-2)}$$

and this is also the improved upper bound $\mathcal{B}'$ *for* $\operatorname{ch}Y(\mathfrak{p})$*.*

Proof. Indeed one checks that, for all $t\in\mathbb{N}$, $1\le t\le n-2$, $\varpi_t-\varpi'_t=(2t/n)\varpi_n$ and that $\varpi_{n-1}-\varpi'_{n-1}=((n-2)/n)\varpi_n$ and the lemma follows by an easy computation. ∎

By what we said at the end of Section 2 one can now give the following theorem.

Theorem 9.7. *Let* $\mathfrak{g}$ *be a simple Lie algebra of type* D_n *with* n *an even integer,* $n\ge 6$*, and let* $\mathfrak{p}=\mathfrak{p}^-_{\pi',\Lambda}$ *be the truncated maximal parabolic subalgebra of* $\mathfrak{g}$ *associated to* $\pi'=\pi\setminus\{\alpha_n\}$*.*

There exists an adapted pair (h, y) for $\mathfrak{p}$ and an affine slice $y+\mathfrak{g}_T$ in $\mathfrak{p}^*$ such that restriction of functions gives an isomorphism of algebras between $Y(\mathfrak{p})$ and the ring $R[y+\mathfrak{g}_T]$ of polynomial functions on $y+\mathfrak{g}_T$.
In particular $Y(\mathfrak{p})$ is a polynomial algebra over k and the field $C(\mathfrak{p}^-_{\pi'})$ of invariant fractions is a purely transcendental extension of k. The degrees of a set of homogeneous generators are the eigenvalues plus one of $\operatorname{ad} h$ on $\mathfrak{g}_T$ (given in Lemma 9.5).

10. Type E_7.

Let $\mathfrak{g}$ be simple of type E_7 and let $\mathfrak{p}=\mathfrak{p}^-_{\pi',\Lambda}$ be the truncated maximal parabolic subalgebra corresponding to $\pi'=\pi\setminus\{\alpha_3\}$. Let β_1 be the unique highest root of $\mathfrak{g}$ and let $H_{\beta_1}=\{\beta\in\Delta^+\mid(\beta,\beta_1)>0\}$ be the maximal Heisenberg set with centre β_1 in Δ^+. Then notice that the set $\Delta\setminus(H_{\beta_1}\sqcup -H_{\beta_1})$ is a root system of type D_6 and removing α_3 corresponds to removing the extremal root from a system of type D_6. Write $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ for the root $\sum_{i=1}^7 a_i\alpha_i$ (with a_i some integers). The sets S and T given in Section 9 for type D_6 with $s=6$ lead us to taking for S the set

$$\begin{aligned} S=\ &\{\beta_1,(0,1,1,2,2,2,1),(0,1,1,1,1,0,0),(0,-1,0,-1,-1,0,0),\\ &(0,0,0,0,-1,-1,0),(0,0,0,0,0,0,-1)\}\end{aligned}$$

and for T the set

$$\begin{aligned} T=\ &\{(-1,0,0,0,0,0,0),(0,0,0,1,1,0,0),(0,0,1,1,0,0,0),\\ &(0,-1,0,-1,-1,-1,-1),(0,0,0,0,0,-1,0)\}.\end{aligned}$$

More explicitly, we have added to the set S in type D_6 with $s=6$ (rewritten with respect to the roots in type E_7) the highest root β_1, and to the set T in type D_6 with $s=6$ (rewritten with respect to the roots in type E_7) we have added the negative root $-\alpha_1$.
For every $\gamma\in S\setminus\{\beta_1\}$, we take the same Heisenberg set Γ_γ with centre γ (rewritten with respect to the roots in type E_7) as in type D_6 with $s=6$ and we add the maximal Heisenberg set H_{β_1}. Observe that if $\alpha\in H_{\beta_1}$ and $\beta\in\Gamma_\gamma$ with $\gamma\in S\setminus\{\beta_1\}$ then one has that $\alpha+\beta\notin S$.
Hence, by the extremal case in type D_6 (see Remark 9.4), it follows that all conditions of Lemma 3.2 (with $T^*=\emptyset$) hold for $y=\sum_{\gamma\in S}x_\gamma$. Then defining $h\in\mathfrak{h}_\Lambda$ by $\gamma(h)=-1$ for all $\gamma\in S$, one obtains that (h, y) is an adapted pair for $\mathfrak{p}$.
Finally we show that $Y(\mathfrak{p})$ is polynomial. For this purpose we need to calculate the $\langle\mathbf{ij}\rangle$-orbits in π and the lower and improved upper bounds for $\operatorname{ch}Y(\mathfrak{p})$. The orbits are: $\Gamma_1=\{\alpha_1\},\Gamma_2=\{\alpha_3\},\Gamma_3=\{\alpha_2,\alpha_7\},\Gamma_4=\{\alpha_4,\alpha_6\}$ and $\Gamma_5=\{\alpha_5\}$. For the lower bound, we need to compute δ_Γ for each orbit Γ.
Let $\{\varepsilon_i\}_{1\le i\le 8}$ be an orthonormal basis of $\mathbb{R}^8$ according to which the simple roots of $\mathfrak{g}$ are expanded as in [2, Planche VI].
Recall that the fundamental weights ϖ_i', $1\le i\le 7$, $i\ne 3$, are those for the Levi factor of $\mathfrak{p}$. A direct computation gives that $\delta_{\Gamma_1}=-2(\varpi_1-\varpi_1')=-\varpi_3$. Similarly one gets $\delta_{\Gamma_2}=\delta_{\Gamma_3}=\delta_{\Gamma_5}=-2\varpi_3$ and $\delta_{\Gamma_4}=-4\varpi_3$. Hence the lower bound is

$$\operatorname{ch}\mathcal{A}=(1-e^{-\varpi_3})^{-1}(1-e^{-2\varpi_3})^{-3}(1-e^{-4\varpi_3})^{-1}.$$

Now for the improved upper bound, for each $\gamma \in T$ we will find $t(\gamma) \in \mathbb{Q}S$ such that $\gamma + t(\gamma)$ is a multiple of ϖ_3. Denote by s_i the i-th element of S as it is written above.
For $\gamma = -\alpha_1$: $t(\gamma) = 2s_1$ and $\gamma + t(\gamma) = \varpi_3$.
For $\gamma = \alpha_4 + \alpha_5$: $t(\gamma) = 6s_1 + 3(s_2 + s_3) + 2(s_4 + s_5) + s_6$ and $\gamma + t(\gamma) = 4\varpi_3$.
For $\gamma = \alpha_3 + \alpha_4$: $t(\gamma) = 3s_1 + s_2 + s_3$ and $\gamma + t(\gamma) = 2\varpi_3$.
For $\gamma = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)$: $t(\gamma) = 3s_1 + 2s_2 + s_3 + s_5$ and $\gamma + t(\gamma) = 2\varpi_3$.
For $\gamma = -\alpha_6$: $t(\gamma) = 3s_1 + 2s_2 + s_3 + s_4 + s_5 + s_6$ and $\gamma + t(\gamma) = 2\varpi_3$.

We deduce that the lower bound coincides with the improved upper bound. Thus $Y(\mathfrak{p})$ is a polynomial algebra over k. Then one checks that

$$h = -\alpha_1^\vee - \frac{13}{2}\alpha_2^\vee + 3\alpha_4^\vee + \frac{11}{2}\alpha_5^\vee - 2\alpha_6^\vee - \frac{1}{2}\alpha_7^\vee.$$

The eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}_T$ are respectively 2, 5, 7, 9, 17, hence the degrees of a set of homogeneous generators of $Y(\mathfrak{p})$ are 3, 6, 8, 10, 18.
We can now give the following theorem.

Theorem 10.1. *Let $\mathfrak{p}^-_{\pi'}$ be the maximal parabolic subalgebra of the simple Lie algebra $\mathfrak{g}$ of type* E_7 *corresponding to $\pi' = \pi \setminus \{\alpha_3\}$. Then the Poisson semicentre $Sy(\mathfrak{p}^-_{\pi'})$ is a polynomial algebra over k in five homogeneous generators, having degrees* 3, 6, 8, 10, 18 *respectively. Moreover there exists an affine slice $y + \mathfrak{g}_T$ in $(\mathfrak{p}^-_{\pi',\Lambda})^*$, which is also a Weierstrass section for $Y(\mathfrak{p}^-_{\pi',\Lambda})$ and the field $C(\mathfrak{p}^-_{\pi'})$ of invariant fractions is a purely transcendental extension of k.*

11. Type E_6

Recall that the numbering of simple roots follows [2, Planche V]. In type E_6 we know that the Poisson centre of the truncated maximal parabolic subalgebra associated to $\pi' = \pi \setminus \{\alpha_s\}$ is polynomial for $s = 3, 4, 5$ by [7] (since both bounds $\operatorname{ch}\mathcal{A}$ and $\operatorname{ch}\mathcal{B}$ coincide), resp. for $s = 2$ by [22] and an adapted pair was constructed in [10], resp. in [16]. It remains to examine the cases $s = 1, 6$, and by symmetry we may just assume that $s = 6$. In the latter case, the truncated parabolic subalgebra $\mathfrak{p} = \mathfrak{p}^-_{\pi',\Lambda}$ may be viewed as the semi-direct product $\mathfrak{g}' \ltimes \mathfrak{m}$, where $\mathfrak{g}'$ is the Levi factor of $\mathfrak{p}$ (of type D_5), and $\mathfrak{m}$ is the nilpotent radical of $\mathfrak{p}$, which is an abelian $\mathfrak{g}'$-module, isomorphic to the half-spin representation k^{16} of $\mathfrak{so}_{10}$. Moreover the group Spin_{10} acts on $\mathfrak{m}$ with a dense open orbit, which has no divisors in the complement, and the stabiliser of an element in this orbit is $Q = \mathrm{Spin}_7 \ltimes \exp(k^8)$ (see [26, Summary Table]). By [29, Prop. 3.10] the algebra of invariants $S(\mathfrak{q})^Q$ (with $\mathfrak{q} = \operatorname{Lie} Q$) is a polynomial ring in three generators and the general theory of [29] asserts that $Y(\mathfrak{p}^-_{\pi',\Lambda})$ is also polynomial in the same number of generators (but it does not give the degrees of the generators).
Here we give an adapted pair for $\mathfrak{p}^-_{\pi',\Lambda}$ and show that for this pair, the improved upper bound $\mathcal{B}'$ coincides with the lower bound $\operatorname{ch}\mathcal{A}$. We also compute the degrees of the three generators of the polynomial algebra $Y(\mathfrak{p}^-_{\pi',\Lambda})$.
Recall the strongly orthogonal positive roots β_1, β_2, β_3, β_4 of the Kostant cascade for Δ^+ (see [12, Table I] or [10, Table I]) and β_1', β_2', β_3' and $\beta_{1'}'$ for $\Delta^+_{\pi'} = -\Delta^-_{\pi'}$ (see Section 7).

We choose for S the set $S=\{\beta_1,\,\beta_2,\,\beta_3,\,-\beta'_1,\,-\beta'_2+\alpha_2\}$. In terms of simple roots, by writing as $(a_1,\,a_2,\,a_3,\,a_4,\,a_5,\,a_6)$ the root $\sum_{i=1}^6 a_i\alpha_i$, our chosen set S is

$$S=\ \{(1,\,2,\,2,\,3,\,2,\,1),\,(1,\,0,\,1,\,1,\,1,\,1),\,(0,\,0,\,1,\,1,\,1,\,0),\\ (-1,\,-1,\,-2,\,-2,\,-1,\,0),\,(0,\,0,\,0,\,-1,\,-1,\,0)\}.$$

We easily check that $S_{|\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}_\Lambda^*$, hence condition (iii) of Lemma 3.2 is satisfied.

Set $\Gamma_{\beta_1}=H_{\beta_1}\setminus\{(0,\,1,\,1,\,1,\,0,\,0),\,(1,\,1,\,1,\,2,\,2,\,1)\}$, where H_{β_1} is the maximal Heisenberg set with centre β_1 in Δ^+ as defined in Example 3.1. Then set $\Gamma_{\beta_2}=H_{\beta_2}\setminus\{(1,\,0,\,1,\,1,\,1,\,0),\,(0,\,0,\,0,\,0,\,0,\,1)\}$ and $\Gamma_{\beta_3}=H_{\beta_3}$. Set $\Gamma_{-\beta'_1}=-H_{\beta'_1}$ and $\Gamma_{-\beta'_2+\alpha_2}=\{-\beta'_2+\alpha_2,\,-\alpha_4,\,-\alpha_5\}$. We easily check that all these sets are disjoint Heisenberg sets. Now we choose

$$T^*=\{(1,\,1,\,1,\,2,\,2,\,1),\,(1,\,0,\,1,\,1,\,1,\,0),\,-\alpha_1,\,-\alpha_2,\,-(\alpha_2+\alpha_4),-(\alpha_2+\alpha_4+\alpha_5)\}$$

and $$T=\{\alpha_4,\,\alpha_6,\,\alpha_2+\alpha_3+\alpha_4\}.$$

The $\langle\mathbf{ij}\rangle$-orbits in π are $\Gamma_1:=\{\alpha_1,\ \alpha_6\}$, $\Gamma_2:=\{\alpha_2,\,\alpha_3,\,\alpha_5\}$ and $\Gamma_3:=\{\alpha_4\}$, hence condition (v) of Lemma 3.2 is satisfied.

By [12, Lemma 2.2] or [10, Lemma 3 (2)], one has that $\Delta^+=\bigsqcup_{i=1}^4 H_{\beta_i}$. Moreover $H_{\beta_4}=\{\beta_4=\alpha_4\}$. Hence one has that $\Delta^+=\Gamma_{\beta_1}\sqcup\Gamma_{\beta_2}\sqcup\Gamma_{\beta_3}\sqcup(T^*\cap\Delta^+)\sqcup T$.

Similarly one has that $\Delta^+_{\pi'}=H_{\beta'_1}\sqcup H_{\beta'_2}\sqcup H_{\beta'_3}\sqcup H_{\beta'_{1'}}$.

Moreover $H_{\beta'_2}=\{\beta'_2,\,\alpha_2,\,\alpha_4+\alpha_5,\,\alpha_2+\alpha_4,\,\alpha_5\}$, $H_{\beta'_3}=\{\beta'_3=\alpha_4\}$ and $H_{\beta'_{1'}}=\{\beta'_{1'}=\alpha_1\}$. Hence one has that $\Delta^-_{\pi'}=\Gamma_{-\beta'_1}\sqcup\Gamma_{-\beta'_2+\alpha_2}\sqcup(T^*\cap\Delta^-_{\pi'})$. Thus condition (i) of Lemma 3.2 is satisfied.

To prove condition (ii) of Lemma 3.2, it suffices to prove Lemma 6.1, noting that $S^+=\{\beta_1,\,\beta_2,\,\beta_3\}$, $S^-=\{-\beta'_1,\,-\beta'_2+\alpha_2\}$ and $S^m=\emptyset$. Using [10, Lemma 3 (5)], conditions (2) and (3) of Lemma 6.1 follow directly and condition (4) is empty. Hence condition (ii).

Finally condition (iv) of Lemma 3.2 can be verified by direct computation.

All conditions of Lemma 3.2 are satisfied. Thus we obtain an adapted pair $(h,\,y)$ for $\mathfrak{p}^-_{\pi',\Lambda}$ by setting $y=\sum_{\gamma\in S}x_\gamma$, and $h\in\mathfrak{h}_\Lambda$ such that $\gamma(h)=-1$ for all $\gamma\in S$.

Now we compute the lower bound $\operatorname{ch}\mathcal{A}$ and the improved upper bound $\mathcal{B}'$ for $\operatorname{ch}Y(\mathfrak{p}^-_{\pi',\Lambda})$. Recall that the fundamental weights ϖ'_i, $i\in\{1,\ldots,5\}$, are those of the Levi factor of $\mathfrak{p}=\mathfrak{p}^-_{\pi',\Lambda}$. A direct computation gives that $\delta_{\Gamma_1}=-2(\varpi_1+\varpi_6-\varpi'_1)=-3\varpi_6$, $\delta_{\Gamma_2}=-2(\varpi_2+\varpi_3+\varpi_5-\varpi'_2-\varpi'_3-\varpi'_5)=-6\varpi_6$, and $\delta_{\Gamma_3}=-2(\varpi_4-\varpi'_4)=-3\varpi_6$. Hence the lower bound for $\operatorname{ch}Y(\mathfrak{p})$ is

$$\operatorname{ch}\mathcal{A}=(1-e^{-3\varpi_6})^{-2}(1-e^{-6\varpi_6})^{-1}.$$

We now compute the improved upper bound $\mathcal{B}'$. Recall that for every $\gamma\in T$ we need to compute the unique element $t(\gamma)\in\mathbb{Q}S$ such that $\gamma+t(\gamma)$ is a multiple of ϖ_6.

For $\gamma=\alpha_4$: $t(\gamma)=5\beta_1+3\beta_2+3\beta_3+2(-\beta'_2+\alpha_2)+4(-\beta'_1)$ and $\gamma+t(\gamma)=6\varpi_6$.

For $\gamma=\alpha_6$: $t(\gamma)=2\beta_1+\beta_2+\beta_3+(-\beta'_1)$ and $\gamma+t(\gamma)=3\varpi_6$.

For $\gamma=\alpha_2+\alpha_3+\alpha_4$: $t(\gamma)=2\beta_1+2\beta_2+\beta_3+2(-\beta'_1)$ and $\gamma+t(\gamma)=3\varpi_6$.

Hence the improved upper bound coincides with the lower bound and $Y(\mathfrak{p}^-_{\pi',\Lambda})$ is a polynomial algebra over k. Note that the element $h \in \mathfrak{h}_\Lambda$ such that $\gamma(h) = -1$ for all $\gamma \in S$ is $h = -2\alpha_1^\vee - \alpha_2^\vee + \alpha_3^\vee + 6\alpha_4^\vee - 5\alpha_5^\vee$. Then the eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}_T$ are 5, 7 and 17, hence the degrees of a set of homogeneous generators for $Y(\mathfrak{p})$ are 6, 8 and 18.

We can now give the following theorem.

Theorem 11.1. *Let $\mathfrak{p}^-_{\pi'}$ be the maximal parabolic subalgebra of the simple Lie algebra $\mathfrak{g}$ of type E_6 corresponding to $\pi' = \pi \setminus \{\alpha_6\}$. Then the Poisson semicentre $Sy(\mathfrak{p}^-_{\pi'})$ is a polynomial algebra over k in three homogeneous generators, having degrees 6, 8 and 18 respectively. Moreover there exists an affine slice $y + \mathfrak{g}_T$ in $(\mathfrak{p}^-_{\pi',\Lambda})^*$, which is also a Weierstrass section for $Y(\mathfrak{p}^-_{\pi',\Lambda})$ and the field $C(\mathfrak{p}^-_{\pi'})$ of invariant fractions is a purely transcendental extension of k.*

References

[1] W. Borho, P. Gabriel, R. Rentschler: *Primideale in Einhüllenden auflösbarer Lie-Algebren. Beschreibung durch Bahnenräume*, Lecture Notes Math. **357**, Springer, Berlin et al. (1973).

[2] N. Bourbaki: *Groupes et Algèbres de Lie*, Chapitres 4, 5 et 6, Masson, Paris et al. (1981).

[3] J. Dixmier: *Sur le centre de l'algèbre enveloppante d'une algèbre de Lie*, C. R. Acad. Sc. Paris **265** (1967) 408–410.

[4] J. Dixmier: *Algèbres Enveloppantes*, Les Grands Classiques de Gauthier-Villars, Editions Jacques Gabay, Paris et al. (1974).

[5] F. Fauquant-Millet: *Sur la polynomialité de certaines algèbres d'invariants d'algèbres de Lie*, Mémoire d'Habilitation à Diriger des Recherches, http://tel.archives-ouvertes.fr/tel-00994655 (2014).

[6] F. Fauquant-Millet, A. Joseph: *Sur les semi-invariants d'une sous-algèbre parabolique d'une algèbre enveloppante quantifiée*, Transform. Groups **6** (2001) 125–142.

[7] F. Fauquant-Millet, A. Joseph: *Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple*, Ann. Sci. Éc. Norm. Sup. **38** (2005) 155–191.

[8] F. Fauquant-Millet, A. Joseph: *La somme des faux degrés – un mystère en théorie des invariants*, Adv. Math. **217** (2008) 1476–1520.

[9] F. Fauquant-Millet, A. Joseph: *Adapted pairs and Weierstrass sections*, http://arxiv.org/abs/1503.02523 (2015).

[10] F. Fauquant-Millet, P. Lamprou: *Slices for maximal parabolic subalgebras of a semisimple Lie algebra*, Transform. Groups **22** (2017) 911–932.

[11] I. Heckenberger: *On the semi-centre of $U(\mathfrak{p})$ for parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{so}_7$ and $\mathfrak{so}_9$*, unpublished notes.

[12] A. Joseph: *A preparation theorem for the prime spectrum of a semisimple Lie algebra*, J. Algebra **48** (1977) 241–289.

[13] A. Joseph: *On semi-invariants and index for biparabolic (seaweed) algebras I*, J. Algebra **305** (2006) 487–515.

[14] A. Joseph: *On semi-invariants and index for biparabolic (seaweed) algebras II*, J. Algebra **312** (2007) 158–193.

[15] A. Joseph: *Slices for biparabolic coadjoint actions in type A*, J. Algebra **319** (2008) 5060–5100.

[16] A. Joseph: *Compatible adapted pairs and a common slice theorem for some centralizers*, Transform. Groups **13** (2008) 637–669.

[17] A. Joseph: *An algebraic slice in the coadjoint space of the Borel and the Coxeter element*, Adv. Math. **227** (2011) 522–585.

[18] A. Joseph, P. Lamprou: *Maximal Poisson commutative subalgebras for truncated parabolic subalgebras of maximal index in $\mathfrak{sl}_n$*, Transform. Groups **12** (2007) 549–571.

[19] A. Joseph, D. Shafrir: *Polynomiality of invariants, unimodularity and adapted pairs*, Transform. Groups **15** (2010) 851–882.

[20] B. Kostant: *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963) 327–404.

[21] A. Ooms: *The polynomiality of the Poisson center and semi-center of a Lie algebra and Dixmier's fourth problem*, J. Algebra **477** (2017) 95–146.

[22] D. Panyushev, A. Premet, O. Yakimova: *On symmetric invariants of centralisers in reductive Lie algebras*, J. Algebra **313** (2007) 343–391.

[23] D. Panyushev, O. Yakimova: *A remarkable contraction of semisimple Lie algebras*, Ann. Inst. Fourier (Grenoble) **62** (2012) 2053–2068.

[24] D. Panyushev, O. Yakimova: *Parabolic contractions of semisimple Lie algebras and their invariants*, Sel. Math. New Ser. **19** (2013) 699–717.

[25] D. Panyushev, O. Yakimova: *Symmetric invariants related to representations of exceptional simple groups*, Trans. Moscow Math. Soc. **78** (2017) 161–170.

[26] V. L. Popov, E. B. Vinberg: *Invariant Theory*, in: Algebraic Geometry IV, Encyclopedia Math. Sci. **55**, Springer, Berlin et al. (1994) 123–284.

[27] P. Tauvel, R. W. T. Yu: *Affine Slice for the Coadjoint Action of a Class of Biparabolic Subalgebras of a Semisimple Lie Algebra*, Algebr. Represent. Theor. **16** (2013) 859–872.

[28] O. Yakimova: *A counterexample to Premet's and Joseph's conjectures*, Bull. Lond. Math. Soc. **39** (2007) 749–754.

[29] O. Yakimova: *Symmetric invariants of $\mathbb{Z}_2$-contractions and other semi-direct products*, Int. Math. Res. Notices **2017** (2017) 1674–1716.

[30] O. Yakimova: *Some semi-direct products with free algebras of symmetric invariants*, in: Perspectives in Lie Theory, F. Callegaro, G. Carnovale, F. Caselli, C. De Concini, A. De Sole (eds.), INdAM Series **19**, Springer, Cham (2017) 267–279.

Florence Fauquant-Millet
Univ Lyon, UJM Saint-Etienne
CNRS UMR 5208
Institut Camille Jordan
F-42023 Saint-Etienne
France
florence.millet@univ-st-etienne.fr

Polyxeni Lamprou
polyxeni.lamprou@googlemail.com

Received November 18, 2017
and in final form June 28, 2018