

On Annihilators of Bounded $(\mathfrak{g}, \mathfrak{k})$ -Modules

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Abstract. Let \mathfrak{g} be a semisimple Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ be a reductive subalgebra. We say that a \mathfrak{g} -module M is a *bounded $(\mathfrak{g}, \mathfrak{k})$ -module* if M is a direct sum of simple finite-dimensional \mathfrak{k} -modules and the multiplicities of all simple \mathfrak{k} -modules in this direct sum are universally bounded. The goal of this article is to show that the “boundedness” property for a simple $(\mathfrak{g}, \mathfrak{k})$ -module M is equivalent to a property of the associated variety of the annihilator of M (this is the closure of a nilpotent coadjoint orbit inside \mathfrak{g}^*) under the assumption that the main field is algebraically closed and of characteristic 0. In particular this implies that if M_1, M_2 are simple $(\mathfrak{g}, \mathfrak{k})$ -modules such that M_1 is bounded and the associated varieties of the annihilators of M_1 and M_2 coincide then M_2 is also bounded. This statement is a geometric analogue of a purely algebraic fact due to I. Penkov and V. Serganova and it was posed as a conjecture in my Ph.D. thesis.

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1. Introduction

A notion of admissible $(\mathfrak{g}, \mathfrak{k})$ -module for a generic pair of Lie algebras $(\mathfrak{g}, \mathfrak{k})$ is a quite straightforward generalization of both Harish-Chandra modules and \mathfrak{g} -modules of category \mathcal{O} . Bounded $(\mathfrak{g}, \mathfrak{k})$ -modules [14] are a very specific subclass of the admissible $(\mathfrak{g}, \mathfrak{k})$ -modules and it turns out that this class is related to the notion of spherical variety [23]. The structure of this subcategory is still very mysterious in general and few examples are evaluated explicitly, see [4], [16], [21].

The set of works on both Harish-Chandra modules and category \mathcal{O} seems to be infinite and one can find some introduction to these subjects in [5, 7]. We would like to mention explicitly the works [2, 4, 10, 11, 15, 17, 18, 19, 25].

The goal of this article is to show that this “boundedness” property for a simple $(\mathfrak{g}, \mathfrak{k})$ -module M is equivalent to a property of the associated variety of the annihilator of M (this is the closure of a nilpotent coadjoint orbit inside \mathfrak{g}^*) under the assumption that the main field is algebraically closed and of characteristic 0. Roughly speaking, a simple $(\mathfrak{g}, \mathfrak{k})$ -module M is bounded if and only if the associated variety of the annihilator of M is K -coisotropic with respect to the action of an algebraic group K attached to the pair $(\mathfrak{g}, \mathfrak{k})$.

In particular this implies that if M_1, M_2 are simple $(\mathfrak{g}, \mathfrak{k})$ -modules such that M_1 is bounded and the associated varieties of the annihilators of M_1 and M_2

coincide then M_2 is also bounded. This statement is a geometric analogue of [13, Theorem 4.3]. It was posed as a conjecture in my Ph. D. thesis [20, Conjecture 2.1].

2. Preliminary definitions and notions

Throughout this paper we fix an algebraically closed field \mathbb{F} of characteristic 0. All vector spaces, Lie algebras and varieties are defined over \mathbb{F} .

2.1. $(\mathfrak{g}, \mathfrak{k})$ -modules

Let \mathfrak{g} be a Lie algebra and \mathfrak{k} be a subalgebra of \mathfrak{g} . We say that a \mathfrak{g} -module M is a $(\mathfrak{g}, \mathfrak{k})$ -module if $\dim(U(\mathfrak{k}) \cdot m) < \infty$ for all $m \in M$, where $U(\mathfrak{k})$ is the universal enveloping algebra of \mathfrak{k} .

It is clear that any $(\mathfrak{g}, \mathfrak{k})$ -module is isomorphic to the direct limit of a directed set of finite-dimensional \mathfrak{k} -modules. Thus, for any given simple \mathfrak{k} -module V , we can define the Jordan-Hölder multiplicity $[M : V]$. If such multiplicities $[M : V]$ are finite for all simple \mathfrak{k} -modules V then M is called *admissible*. If there exists $C_M \in \mathbb{N}$ such that all multiplicities $[M : V]$ are universally bounded by C_M then M is called *bounded*.

2.2. Associated varieties

Let M be a finitely generated \mathfrak{g} -module M (for example M can be a simple \mathfrak{g} -module). We consider a finite-dimensional generating subspace M_0 of M and define spaces M_i ($i \geq 0$) inductively using the formula

$$M_{i+1} := M_i + \mathfrak{g} \cdot M_i.$$

The *associated graded object*

$$\text{gr } M := \bigoplus_{i \geq 0} M_i / M_{i-1}, \quad M_{-1} := 0,$$

is a module of the associated graded algebra $S(\mathfrak{g})$ of $U(\mathfrak{g})$. Let J be the annihilator of M in $S(\mathfrak{g})$. We denote by $\text{Var}(M)$ the set of points $\chi \in \mathfrak{g}^*$ such that $f(\chi) = 0$ for all $f \in J$.

It is well known that $\text{Var}(M)$ is the same for all choices of M_0 , see for example [20, p. 19], and therefore $\text{Var}(M)$ is a proper invariant of M . Next, we consider the annihilator $\text{Ann}_{U(\mathfrak{g})} M$ of M in $U(\mathfrak{g})$ as a left \mathfrak{g} -module and put

$$\text{GVar}(M) := \text{Var}(\text{Ann}_{U(\mathfrak{g})} M).$$

If M is simple then $\text{GVar}(M)$ is irreducible and equals the closure of a nilpotent coadjoint orbit [6]. We denote this orbit $\text{GVar}_0(M)$.

Define $\overline{G \cdot \text{Var}(M)}$ to be the closure of the G -translation of $\text{Var}(M)$. Note that in many interesting cases

$$\text{GVar}(M) = \overline{G \cdot \text{Var}(M)}.$$

Put $\sqrt{J} := \{f \in S(\mathfrak{g}) \mid \exists n \in \mathbb{Z}_{>0} : (f^n \in J)\}$. For any \mathfrak{g} -module M the ideal \sqrt{J} is known to be *involutive*, i.e. $\{\sqrt{J}, \sqrt{J}\} \subset \sqrt{J}$ where $\{\cdot, \cdot\}$ is the standard Poisson bracket on $S(\mathfrak{g})$, see [3].

Next, if M is a $(\mathfrak{g}, \mathfrak{k})$ -module then one can choose filtration $\{M_i\}$ to be \mathfrak{k} -stable and hence $\mathfrak{k} \subset J$. This condition immediately implies $\text{Var}(M) \subset \mathfrak{k}^\perp$, where $\mathfrak{k}^\perp := \{\chi \in \mathfrak{g}^* \mid \chi|_{\mathfrak{k}} = 0\}$.

2.3. Equivariant actions and symplectic varieties

We will use notions and terminology on symplectic geometry and algebraic groups actions of [24].

Let (W, ω) be a symplectic vector space. A subspace $V \subset W$ is called *isotropic* if $\omega|_V = 0$. A subspace $V \subset W$ is called *coisotropic* if the orthogonal complement to V in W with respect to ω is isotropic. A subspace $V \subset W$ is called *Lagrangian* if it is isotropic and coisotropic in the same time.

Let K be a reductive algebraic group and let X be a smooth irreducible K -variety. Assume further that X has a nondegenerate K -equivariant symplectic form ω , i.e. that X is a K -symplectic variety. Then the tangent spaces $T_x X$ to all the points of x are symplectic vector spaces with respect to the restrictions ω_x of ω to $T_x X$.

Definition 2.1. An irreducible subvariety $Y \subset X$ is called *isotropic* (respectively *coisotropic*, *Lagrangian*) if $T_y Y$ are isotropic (respectively *coisotropic*, *Lagrangian*) subspaces of $T_y X$ for all y from an open subset of Y . A symplectic action of K on X is called *coisotropic* if all K -orbits from an open subset of X are coisotropic.

3. Main theorem

Fix a semisimple Lie algebra \mathfrak{g} together with the adjoint group G of \mathfrak{g} , and also fix a subalgebra $\mathfrak{k} \subset \mathfrak{g}$. Assume that \mathfrak{k} is *algebraically reductive* in \mathfrak{g} , that is there exists a connected reductive algebraic subgroup K of G such that $\mathfrak{k} \subset \mathfrak{g}$ is the Lie algebra of $K \subset G$. If such a subgroup K exists then it is unique.

Theorem 3.1. *Assume that \mathfrak{g} is a semisimple Lie algebra and \mathfrak{k} an algebraically reductive subalgebra of \mathfrak{g} . Then, for a simple $(\mathfrak{g}, \mathfrak{k})$ -module M , the following conditions are equivalent:*

- (i) M is a bounded $(\mathfrak{g}, \mathfrak{k})$ -module,
- (ii) $G\text{Var}_0(M)$ is K -coisotropic.

It turns out that it is possible to reduce the case of any subalgebra \mathfrak{k} of a semisimple Lie algebra \mathfrak{g} to the case of algebraically reductive subalgebra, see Propositions 4.8, 4.9.

4. Algebraic subalgebras

Let \mathfrak{k} be a subalgebra of a semisimple Lie algebra \mathfrak{g} , G be the adjoint group of \mathfrak{g} , and let \hat{K} be the least algebraic subgroup of G such that the Lie algebra of \hat{K} contains \mathfrak{k} . Denote by $\hat{\mathfrak{k}}$ the Lie algebra of \hat{K} . Then, see [12, Subsection 3.3.3],

- (i) \hat{K} is connected and \hat{K} normalizes \mathfrak{k} ,
- (ii) $\hat{\mathfrak{k}}/\mathfrak{k}$ is abelian and $[\hat{\mathfrak{k}}, \hat{\mathfrak{k}}] = [\mathfrak{k}, \mathfrak{k}]$.

Definition 4.1. A subalgebra \mathfrak{k} is called *algebraic in \mathfrak{g}* if $\mathfrak{k} = \hat{\mathfrak{k}}$. A subalgebra \mathfrak{k} is called *reductive in \mathfrak{g}* if \hat{K} is reductive.

It is clear that if \mathfrak{k} is algebraically reductive in \mathfrak{g} then \mathfrak{k} is reductive in \mathfrak{g} . The inverse statement is false.

Example 4.2. Let $\mathfrak{g} \cong \mathfrak{sl}(3)$ be the Lie algebra of traceless 3×3 matrices and let \mathfrak{k} be the one-dimensional subalgebra spanned by

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -(\alpha + \beta) \end{pmatrix},$$

$\alpha, \beta \in \mathbb{F}$. If $\alpha, \beta \in \mathbb{F} \setminus \mathbb{Q}$ and $\frac{\alpha}{\beta} \in \mathbb{F} \setminus \mathbb{Q}$ then \hat{K} is an algebraic torus consisting of all diagonal 3×3 matrices with determinant 1, and therefore $\hat{\mathfrak{k}}$ is the space of traceless diagonal 3×3 matrices. In particular, this implies that \mathfrak{k} is reductive in $\mathfrak{sl}(3)$, but \mathfrak{k} is **not** algebraically reductive in $\mathfrak{sl}(3)$.

The following lemmas show that \hat{K} -modules, $\hat{\mathfrak{k}}$ -modules and \mathfrak{k} -modules are close to each other. Note that \mathfrak{k} is not assumed to be neither reductive nor algebraic in these lemmas.

Lemma 4.3. *Let V_1, V_2 be \hat{K} -modules. Then the following conditions are equivalent.*

$$(i) V_1 \cong V_2, \quad (ii) (V_1)|_{\hat{\mathfrak{k}}} \cong (V_2)|_{\hat{\mathfrak{k}}}, \quad (iii) (V_1)|_{\mathfrak{k}} \cong (V_2)|_{\mathfrak{k}}.$$

Proof. Conditions (i) and (ii) are equivalent for a connected group in characteristic 0. It is clear that (ii) implies (iii). It remains to show that (iii) implies (i).

Assume (iii). Consider the vector space $\text{Hom}_{\mathbb{F}}(V_1, V_2)$. We have $\text{Hom}_{\mathfrak{k}}(V_1, V_2) \neq 0$.

Fix
$$0 \neq \phi \in \text{Hom}_{\mathfrak{k}}(V_1, V_2) \tag{1}$$

such that ϕ defines the isomorphism between V_1 and V_2 . Let H be the stabilizer of ϕ in \hat{K} ; it is clear that H is an algebraic subgroup of \hat{K} . Condition (1) implies that \mathfrak{k} is a subalgebra of the Lie algebra of H . The definition of \hat{K} implies that $H = \hat{K}$. Therefore ϕ is an isomorphism of \hat{K} -modules. ■

Lemma 4.4. *Let V be a \hat{K} -module and let $W \subset V$ be an \mathbb{F} -subspace of V . The following conditions are equivalent.*

$$(i) W \text{ is } \hat{K}\text{-stable}, \quad (ii) W \text{ is } \hat{\mathfrak{k}}\text{-stable}, \quad (iii) W \text{ is } \mathfrak{k}\text{-stable}.$$

Proof. Conditions (i) and (ii) are equivalent for a connected group in characteristic 0. It is clear that (ii) implies (iii). It remains to show that (iii) implies (i).

Assume (iii). Let d be the dimension of W . Consider variety $\text{Gr}(d; V)$ of d -dimensional subspaces of V and denote by $[W]$ the point of $\text{Gr}(d; V)$ defined by W . Let H be the stabilizer of $[W]$ in \hat{K} ; it is clear that H is an algebraic subgroup of \hat{K} . Condition (iii) implies that \mathfrak{k} is a subalgebra of the Lie algebra

of H . The definition of \hat{K} implies that $H = \hat{K}$. Therefore W is a \hat{K} -submodule of V . ■

Corollary 4.5. *Let V be a finite-dimensional \hat{K} -module. Then the lattices of \hat{K} -submodules, $\hat{\mathfrak{k}}$ -submodules, and \mathfrak{k} -submodules of V coincide.*

Corollary 4.6. *Let M be a direct limit of finite-dimensional \hat{K} -modules. Then*

- (i) M is a bounded $(\hat{\mathfrak{k}}, \hat{\mathfrak{k}})$ -module if and only if M is a bounded $(\mathfrak{k}, \mathfrak{k})$ -module,
- (ii) M is an admissible $(\hat{\mathfrak{k}}, \hat{\mathfrak{k}})$ -module if and only if M is an admissible $(\mathfrak{k}, \mathfrak{k})$ -module.

Proof. This statement is implied by Lemma 4.3 and Corollary 4.5. ■

We wish to establish a connection between the admissible $(\mathfrak{g}, \hat{\mathfrak{k}})$ -modules and the admissible $(\mathfrak{g}, \mathfrak{k})$ -modules. The first step is as follows.

Lemma 4.7. *Let M be an admissible $(\mathfrak{g}, \mathfrak{k})$ -module. Then M is a $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module.*

Proof. Let M_0 be a \mathfrak{k} -stable subspace of M . Consider the $\hat{\mathfrak{k}}$ -submodule \hat{M}_0 of M generated by M_0 . It is enough to show that $\dim \hat{M}_0 < \infty$. Set

$$M_{i+1} := \hat{\mathfrak{k}}M_i + M_i, \quad i \geq 0.$$

It is clear that each M_i is finite dimensional and $\bigcup_i M_i = \hat{M}_0$. The definition of M_i implies that there exists a surjective \mathfrak{k} -morphism

$$\phi : \hat{\mathfrak{k}}/\mathfrak{k} \otimes M_i \twoheadrightarrow M_{i+1}/M_i$$

of \mathfrak{k} -modules. Theorem 3 of [12, Subsection 3.3.3] implies that

$$[\hat{\mathfrak{k}}, \hat{\mathfrak{k}}] = [\mathfrak{k}, \mathfrak{k}] = [\hat{\mathfrak{k}}, \mathfrak{k}]$$

and thus that $(\hat{\mathfrak{k}}/\mathfrak{k})$ is a trivial \mathfrak{k} -module. Therefore the list of simple \mathfrak{k} -subquotients of M_{i+1} equals the list of simple \mathfrak{k} -subquotients of M_i for all $i \geq 0$. Hence these lists equal to the list of simple \mathfrak{k} -subquotients of M_0 . If $\lim_{i \rightarrow \infty} \dim M_i = \infty$ then the multiplicity of at least one such a subquotient would tend to infinity with $i \rightarrow \infty$. That is incompatible with the assumption that M is an admissible $(\mathfrak{g}, \mathfrak{k})$ -module. ■

Lemma 4.7 implies that if M is an admissible $(\mathfrak{g}, \mathfrak{k})$ -module then

- (i) $\text{Var}(M) \subset (\hat{\mathfrak{k}})^\perp$, and
- (ii) $\text{Var}(M)$ is stable under the action of \hat{K} .

Proposition 4.8. *Let \mathfrak{g} be a semisimple Lie algebra, let M be a simple \mathfrak{g} -module, and let $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra. Then*

- (i) M is a bounded $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module if and only if M is a bounded $(\mathfrak{g}, \mathfrak{k})$ -module,
- (ii) M is an admissible $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module if and only if M is an admissible $(\mathfrak{g}, \mathfrak{k})$ -module.

Proof. It is straightforward to argue that M is a bounded $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module (resp. bounded $(\mathfrak{g}, \mathfrak{k})$ -module) if and only if $\mathbb{F}[\text{Var}(M)]$ is a bounded $(\hat{\mathfrak{k}}, \hat{\mathfrak{k}})$ -module (resp. bounded $(\mathfrak{k}, \mathfrak{k})$ -module), see [18, Proof of Proposition 2.6]. In the same way we have that M is an admissible $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module (resp. admissible $(\mathfrak{g}, \mathfrak{k})$ -module) if and only if $\mathbb{F}[\text{Var}(M)]$ is an admissible $(\hat{\mathfrak{k}}, \hat{\mathfrak{k}})$ -module (resp. admissible $(\mathfrak{k}, \mathfrak{k})$ -module). Therefore it is enough to verify the following facts:

- (a') $\mathbb{F}[\text{Var}(M)]$ is a bounded $(\hat{\mathfrak{k}}, \hat{\mathfrak{k}})$ -module if and only if $\mathbb{F}[\text{Var}(M)]$ is a bounded $(\mathfrak{k}, \mathfrak{k})$ -module.
- (b') $\mathbb{F}[\text{Var}(M)]$ is an admissible $(\hat{\mathfrak{k}}, \hat{\mathfrak{k}})$ -module if and only if $\mathbb{F}[\text{Var}(M)]$ is an admissible $(\mathfrak{k}, \mathfrak{k})$ -module.

These statements are implied by Corollary 4.6 and Lemma 4.7. ■

Let L be a maximal connected reductive subgroup of \hat{K} . Such a subgroup is unique up to conjugacy. Denote by $\mathfrak{l} \subset \mathfrak{g}$ the Lie algebra of L . It is clear that \mathfrak{l} is algebraically reductive in \mathfrak{g} . We conclude this section with the following statement.

Proposition 4.9. *Let M be a simple $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module. Then*

- (a) *M is an admissible $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module if and only if M is an admissible $(\mathfrak{g}, \mathfrak{l})$ -module,*
- (b) *M is a bounded $(\mathfrak{g}, \hat{\mathfrak{k}})$ -module if and only if M is a bounded $(\mathfrak{g}, \mathfrak{l})$ -module.*

Proof. This is implied by two facts: (1) the restriction of a simple finite-dimensional \hat{K} -module to L is simple, and (2) the restrictions of two nonisomorphic simple finite-dimensional \hat{K} -modules to L are nonisomorphic. ■

5. A geometric approach to $(\mathfrak{g}, \mathfrak{k})$ -modules

5.1. Notation for symplectic varieties

Let (W, ω) be a symplectic vector space and $V \subset W$ be a subspace. Denote by $\text{rk}(V)$ the *rank* of $\omega|_V$ and set

$$\text{def}(V) := \dim V - \text{rk}(V), \quad \text{cork}(V) := \dim W - 2 \dim V + \text{rk}(V).$$

The rank of the orthogonal complement to V in W equals $\text{cork}(V)$. It is easy to verify that V is coisotropic if and only if $\text{cork}(V) = 0$.

Let (X, ω) be a symplectic variety with Y being an irreducible subvariety of X . We have that $\text{def}(T_x Y), \text{cork}(T_x Y)$ are constant for all y from an open subset of Y . Define $\text{def}(Y), \text{cork}(Y)$ to be the respective constants.

Assume further that (X, ω) is endowed with a symplectic action of K . We have that $\text{def}(K \cdot x), \text{cork}(K \cdot x)$ are constant for all x from an open subset of X . Define $\text{def}_K(X), \text{cork}_K(X)$ to be the respective constants.

Definition 5.1. A coisotropic subvariety $Y \subset X$ is called *special* if, for all $y \in Y$,

$$\omega(T_y Y, T_y(K \cdot y)) = 0.$$

5.2. Notation for K -varieties

Let K be a reductive algebraic group and let X be a K -variety. We denote by $m_K(X)$ the *maximal dimension* of a K -orbit on X . Further choose a Borel subgroup B_K of K and set

$$c_K(X) := \dim X - m_{B_K}(X).$$

We say that X is K -spherical if X is irreducible and $c_K(X) = 0$. It is easy to check that $c_K(X)$ is the same for all choices of B_K .

5.3. $(\mathfrak{g}, \mathfrak{k})$ -modules and actions of algebraic groups

Assume that \mathfrak{k} is algebraically reductive in \mathfrak{g} . Denote by K the respective subgroup of G . The following propositions provide a connection between algebraic properties of $(\mathfrak{g}, \mathfrak{k})$ -modules and geometric properties of the action of K on $\text{Var}(M)$.

Proposition 5.2 ([18, Proposition 2.6]). *Let M be a $(\mathfrak{g}, \mathfrak{k})$ -module. Then M is bounded if and only if all irreducible components of $\text{Var}(M)$ are K -spherical varieties.*

Lemma 5.3 ([19, Lemma 2]). *Let M be a $(\mathfrak{g}, \mathfrak{k})$ -module. Then M is admissible if and only if the only closed orbit of $\text{Var}(M)$ is $0 \in \mathfrak{g}^*$.*

Proposition 5.4 ([19, Theorem 8]). *Let M be a simple admissible $(\mathfrak{g}, \mathfrak{k})$ -module. Let V_1, \dots, V_n be the irreducible components of $V(M)$. Then*

- (a) *the closure of $G \cdot V_i$ contains the unique closed G -orbit, denoted by \mathcal{O}_i ,*
- (b) *$V_i \cap \mathcal{O}_i$ is a Lagrangian subvariety of \mathcal{O}_i .*

6. Proof of Theorem 3.1

6.1. Preliminary results

We will need the following statement due to D. Timashev and V. Zhgun [24].

Proposition 6.1. *Let X be a K -symplectic variety and Y be a K -special subvariety of X . Then*

$$2c_K(Y) = \text{cork}_K X + 2 \dim Y - \dim X, \tag{2}$$

Proof. It is a straightforward corollary of [24, Theorem 7]. ■

We will also need a statement due to I. Losev [9]. It is quite technical and we require more notation to deal with it: Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit and $\overline{\mathcal{O}} \subset \mathfrak{g}^*$ be the closure of \mathcal{O} . The inclusion $\mathfrak{k} \hookrightarrow \mathfrak{g}$ induces the dual map $\psi : \mathfrak{g}^* \rightarrow \mathfrak{k}^*$. Put $\mu := \psi|_{\overline{\mathcal{O}}}$. Denote by $\overline{\mathcal{O}}//K$ the categorical quotient of $\overline{\mathcal{O}}$ by K , denote by $\mathfrak{k}^*//K$ the categorical quotient of \mathfrak{k}^* by K . There is the induced map

$$\mu//K : \overline{\mathcal{O}}//K \rightarrow \mathfrak{k}^*//K. \tag{3}$$

Proposition 6.2. (a) *The codimension of any fiber of $\mu//K$ is at least $\text{def}_K(\mathcal{O})$.*
 (b) *The dimension of any fiber of $\mu//K$ is at most $\text{cork}_K(\mathcal{O})$.*

Proof. This statement is a straightforward corollary of [9, Theorem 1.2.1]. To apply this theorem we need to check that $\overline{\mathcal{O}}$ is an affine K -irreducible Hamiltonian K -variety, see the definitions in [9]. This is true thanks to a minor modification of [9, Example 3.2.1].

As a last step, we note that $\underline{\text{def}}_K(X)$ (used in [9, Theorem 1.2.1]) equals $\text{def}_K(X)$ thanks to [9, Remark 3.1.5]. This completes the proof of (a).

Formula above [24, Theorem 2] states that

$$\text{cork}_K(\overline{\mathcal{O}}) = \dim(\overline{\mathcal{O}}/K) - \text{def}_K(\overline{\mathcal{O}}) \tag{4}$$

where $\dim(\overline{\mathcal{O}}/K) := \dim \overline{\mathcal{O}} - m_K(\overline{\mathcal{O}})$. Thanks to [9, Theorem 1.2.9(1)] we have

$$\dim(\overline{\mathcal{O}}//K) = \dim(\overline{\mathcal{O}}/K). \tag{5}$$

It is clear that (a), (4), and (5) imply (b). ■

Finally, we need the following statement.

Proposition 6.3 ([1, Proposition 2.7]). *Let $\mathcal{O}_1, \mathcal{O}_2 \subset \mathfrak{g}^*$ be coadjoint orbits of G such that $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$. Let K be a reductive subgroup of G . If \mathcal{O}_2 is K -coisotropic then \mathcal{O}_1 is K -coisotropic.*

6.2. The proof

We use notation of Section 3 and of Theorem 3.1. Fix a simple $(\mathfrak{g}, \mathfrak{k})$ -module M . Then $\text{GVar}(M)$ is irreducible [6]. Theorem 3.1 is implied by the following propositions.

Proposition 6.4. *If M is a bounded $(\mathfrak{g}, \mathfrak{k})$ -module then $\text{GVar}_0(M)$ is K -coisotropic.*

Proposition 6.5. *If $\text{GVar}_0(M)$ is K -coisotropic then M is an admissible $(\mathfrak{g}, \mathfrak{k})$ -module.*

Proposition 6.6. *If M is an admissible $(\mathfrak{g}, \mathfrak{k})$ -module and $\text{GVar}_0(M)$ is K -coisotropic then M is a bounded $(\mathfrak{g}, \mathfrak{k})$ -module.*

Proof of Proposition 6.4. Proposition 5.2 implies that if M is a bounded $(\mathfrak{g}, \mathfrak{k})$ -module then all irreducible components of $\text{Var}(M)$ are K -spherical varieties. The Gelfand-Kirillov dimension of $U(\mathfrak{g})/\text{Ann}_{U(\mathfrak{g})} M$ equals $\dim \text{GVar}(M)$. Theorem 9.11 of [8] implies that the Gelfand-Kirillov dimension of M is at least $\frac{1}{2} \dim \text{GVar}(M)$. Hence there exists an irreducible component V of $\text{Var}(M)$ such that

$$\dim V \geq \frac{1}{2} \dim \text{GVar}(M).$$

Let \mathcal{O} be the nilpotent coadjoint which is dense in $G \cdot V$. Proposition 5.4 implies that $V \cap \mathcal{O}$ is Lagrangian in \mathcal{O} and hence that

$$\dim V = \frac{1}{2} \dim \mathcal{O} \geq \frac{1}{2} \dim \text{GVar}(M).$$

On the other hand we have that $\mathcal{O} \subset \text{GVar}(M)$. The variety $\text{GVar}(M)$ is irreducible [6] and thus

$$\text{GVar}(M) = \overline{\mathcal{O}}.$$

Recall that $V \cap \mathcal{O}$ is Lagrangian in \mathcal{O} , and therefore $V \cap \mathcal{O} \subset \mathcal{O}$ is special in the sense of [24, Definition 4]. Next, Proposition 6.1 implies that

$$2c(V) = \text{cork}M + 2 \dim V - \dim \mathcal{O} \tag{6}$$

(notation of [24]), and thus $2c(V) = \text{cork}M$. The variety V is K -spherical and therefore $c(V) = 0$. Hence $\text{cork}M = 0$ and M is K -coisotropic. ■

Proof of Proposition 6.5. It is enough to show that M is an admissible $(\mathfrak{g}, \mathfrak{k})$ -module, or equivalently that $\mathbb{F}[\text{Var}(M)]$ is an admissible $(\mathfrak{k}, \mathfrak{k})$ -module, see [18, Proof of Proposition 2.6]. The last statement is equivalent to the condition

$$\dim(\mathbb{F}[\text{Var}(M)]^K) < \infty,$$

see [22, Theorem 3.24]. The embedding $\mathfrak{k} \hookrightarrow \mathfrak{g}$ defines the sequence of maps

$$S(\mathfrak{k}) \hookrightarrow S(\mathfrak{g}) \twoheadrightarrow \mathbb{F}[\text{GVar}(M)] \twoheadrightarrow \mathbb{F}[\text{Var}(M)].$$

Further we have

$$S(\mathfrak{k})^K \hookrightarrow S(\mathfrak{g})^K \twoheadrightarrow \mathbb{F}[\text{GVar}(M)]^K \twoheadrightarrow \mathbb{F}[\text{Var}(M)]^K.$$

Next, we need the following lemma.

Lemma 6.7. *For any maximal ideal $m \in S(\mathfrak{k})^K$ the algebra*

$$\mathbb{F}[\text{GVar}(M)]^K / m\mathbb{F}[\text{GVar}(M)]^K \tag{7}$$

is finite dimensional.

Proof. Set $\mathcal{O} := \text{GVar}_0(M)$. It is easy to verify that (7) is equivalent to the statement that all the fibers of (3) are at most 0-dimensional. The last statement is implied by Proposition 6.2. ■

Put $m_0 := S(\mathfrak{k})^K \cap (\mathfrak{k}S(\mathfrak{k}))$. It is clear that m_0 is a maximal ideal of $S(\mathfrak{k})^K$. Next,

$$\text{Var}(M) \subset \mathfrak{k}^\perp$$

implies that $\mathbb{F}[\text{Var}(M)]$ is annihilated by $\mathfrak{k}S(\mathfrak{g})$. Therefore $\mathbb{F}[\text{Var}(M)]^K$ is also annihilated by m_0 . Hence $\mathbb{F}[\text{Var}(M)]^K$ is a quotient of

$$\mathbb{F}[\text{GVar}(M)]^K / m_0\mathbb{F}[\text{GVar}(M)]^K.$$

By Lemma 6.7 we have

$$\dim \mathbb{F}[\text{GVar}(M)]^K / m_0\mathbb{F}[\text{GVar}(M)]^K < \infty,$$

and therefore $\dim \mathbb{F}[\text{Var}(M)]^K < \infty$. ■

Proof of Proposition 6.6. Let V_1, \dots, V_n be the irreducible components of $\text{Var}(M) \subset \text{GVar}(M)$. Proposition 2.6 of [18] implies that it is enough to show that all V_i are K -spherical varieties.

Fix i . The variety $G \cdot V_i$ contains the unique dense orbit \mathcal{O}_i . Moreover, $V_i \cap \mathcal{O}_i$ is a coisotropic subvariety of \mathcal{O}_i [3]. On the other hand, V_i is K -isotropic [19, Theorem 2]. This implies that V_i is K -special in a sense of [24, Definition 4]. Applying formula (6) we have

$$2c(V_i) = \text{cork}(\mathcal{O}_i).$$

Next, $c(V_i) = 0$ if and only if V_i is K -spherical and thus we need to check that

$$\text{cork } \mathcal{O}_i = 0,$$

i.e. that the action of K on \mathcal{O}_i is coisotropic. This is a consequence of Proposition 6.3 and the facts that $\mathcal{O}_i \subset \text{GVar}(M)$ and $\text{GVar}_0(M)$ is K -coisotropic, see also [9, Theorem 1.2.4]. \blacksquare

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