

## Galois Groups and Group Actions on Lie Algebras

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**Abstract.** If  $\mathfrak{g} \subseteq \mathfrak{h}$  is an extension of Lie algebras over a field  $k$  such that  $\dim_k(\mathfrak{g}) = n$  and  $\dim_k(\mathfrak{h}) = n + m$ , then the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is explicitly described as a subgroup of the canonical semidirect product of groups  $\text{GL}(m, k) \rtimes \text{M}_{n \times m}(k)$ . An Artin type theorem for Lie algebras is proved: if a group  $G$  whose order is invertible in  $k$  acts as automorphisms on a Lie algebra  $\mathfrak{h}$ , then  $\mathfrak{h}$  is isomorphic to a skew crossed product  $\mathfrak{h}^G \#^\bullet V$ , where  $\mathfrak{h}^G$  is the subalgebra of invariants and  $V$  is the kernel of the Reynolds operator. The Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{h}^G)$  is also computed, highlighting the difference from the classical Galois theory of fields where the corresponding group is  $G$ . The counterpart for Lie algebras of Hilbert's Theorem 90 is proved and based on it the structure of Lie algebras  $\mathfrak{h}$  having a certain type of action of a finite cyclic group is described. Radical extensions of finite dimensional Lie algebras are introduced and it is shown that their Galois group is solvable. Several applications and examples are provided.

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### 1. Introduction

The complete description of the automorphism group  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$  of a given Lie algebra  $\mathfrak{h}$  is an old and notoriously difficult problem intimately related to the structure of Lie algebras. One of the most important results dealing with this problem shows that the automorphism group of a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero is generated, with few exceptions, by the invariant automorphisms [19, Theorem 4]. This allows for a full description of the automorphism group of any finite dimensional reductive Lie algebra. Beyond the theoretical interest in this problem, the description of the automorphism group of an arbitrary Lie algebra turns out to be of crucial importance for the construction of solutions to Einstein's field equations for Bianchi geometries, in the study of  $(4 + 1)$ -dimensional spacetimes with applications in cosmology [9, 14, 15, 10] or for discrete symmetries of differential equations. The classification of automorphism groups for indecomposable real Lie algebras is

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known only up to dimension six and it has been only recently finished [14, 15]. For other contributions to the subject see [3, 16] and the references therein. Perhaps the strongest motivation for studying  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$  comes from Hilbert's invariant theory, whose foundation was set at the level of Lie algebras in the classical papers [5, 6, 37, 38] - for more recent work on the subject we refer the reader to [7, 12, 33].

An action as automorphisms of a group  $G$  on a Lie algebra  $\mathfrak{h}$  is a morphism of groups  $\varphi: G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$ . Particular attention was given to the situation where  $G$  is a finite subgroup of  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$  with the canonical action on  $\mathfrak{h}$ ; in this case achieving the description of the subgroups of  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$  is the key step. If  $\varphi: G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$  is an action, then we can consider the subalgebra of invariants  $\mathfrak{h}^G$  and we obtain an extension  $\mathfrak{h}^G \subseteq \mathfrak{h}$  of Lie algebras. The fundamental problem of invariant theory [5, 17, 30, 23], in the setting of Lie algebras comes down to finding under which conditions on  $G$  and  $\mathfrak{h}$  the algebraic/geometric properties can be transferred between the two Lie algebras  $\mathfrak{h}^G$  and  $\mathfrak{h}$ . Turning to the problem we started with, since describing the automorphism group  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$  of a given Lie algebra  $\mathfrak{h}$  is an extremely complicated task, it is natural to start by considering only those automorphisms of  $\mathfrak{h}$  which fix a given subalgebra  $\mathfrak{g} \neq 0$  of  $\mathfrak{h}$ . Thus, we can define the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  of the extension  $\mathfrak{g} \subseteq \mathfrak{h}$  as the subgroup of all Lie algebra automorphisms  $\sigma: \mathfrak{h} \rightarrow \mathfrak{h}$  that fix  $\mathfrak{g}$ , i.e.  $\sigma(g) = g$ , for all  $g \in \mathfrak{g}$ . In an ideal situation, after computing  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  for as many subalgebras  $\mathfrak{g}$  of  $\mathfrak{h}$  as possible, we will have a complete picture on the entire group  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$  as well. Having defined the group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$ , the following question arises naturally:

*What is the counterpart in the context of Lie algebras of the classical Galois theory for fields?*

At first sight the chances of developing a promising Galois theory for Lie algebras are very low since even the basic concepts from field theory such as the algebraic/separable/normal extensions, the splitting fields of a polynomial, etc. are rather difficult to define in the context of Lie algebras. Moreover, it is unlikely to have a fundamental theorem establishing a bijective correspondence between the subgroups of  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  and the Lie subalgebras  $\mathfrak{g}'$  of  $\mathfrak{h}$  such that  $\mathfrak{g} \subseteq \mathfrak{g}' \subseteq \mathfrak{h}$  as in the case of the classical Galois theory (Example 3.1). On the other hand, several counterparts of the classical Galois theory for fields were proved in the context of associative algebras [11], differential Galois theory [27], Hopf algebras [31], von Neumann algebras [32], structured ring spectra [34] or stable homotopy theory [29]. In this context, invariant theory seems to provide a better approach for our problem: if  $G \leq \text{Aut}(K)$  is a finite group of automorphisms of a field  $K$  then the famous Artin's theorem states that  $k := K^G \subseteq K$  is a finite Galois extension of degree  $[K : k] = |G|$  and  $\text{Gal}(K/k) = G$  [24, Theorem 1.8]. Furthermore, the extension  $k \subseteq K$  has a normal basis as a consequence of being Galois; that is, there exists  $x \in K$  such that  $\{\sigma(x) \mid \sigma \in G\}$  is a  $k$ -basis of  $K$  [24, Theorem 1.8].

One of the many generalizations of Artin's theorem deals with arbitrary actions [31, Example 8.1.2]: if  $\varphi: G \rightarrow \text{Aut}(K)$  is an action as automorphisms of a finite group  $G$  on a field  $K$ , then  $K/K^G$  is a Galois extension in the classical sense with Galois group  $G$  if and only if  $G$  acts faithfully on  $K$ . A version of Artin's theorem for Hopf-Galois extensions (a concept which generalizes Galois extensions for fields [31, Example 8.1.2]) was obtained in [4, Theorem 1.18]. At this level, Hopf-Galois

extensions satisfying the normal basis property coincide with crossed products [31, Corollary 8.2.5]. This last observation allows us to restate Artin’s theorem in a more convenient but equivalent manner, as follows: if  $G \leq \text{Aut}(K)$  is a finite group of automorphisms of a field  $K$ , then  $K$  is isomorphic to a crossed product algebra  $k \#_{\sigma} k[G]^*$  between the field of invariants  $k = K^G$  and the dual algebra of the group algebra  $k[G]$ , associated to some cocycle  $\sigma: k[G]^* \otimes k[G]^* \rightarrow k$ . With this last conclusion in mind, we replace the category we work in: instead of fields we consider Lie algebras together with group actions as automorphisms. Now the question if an Artin type theorem holds for Lie algebras has a positive answer with a slight amending though: the role of the classical crossed product of Lie algebras (we use the terminology of [1, Section 4.1]), as it arises in the theory of Chevalley and Eilenberg [8] will be played by a new construction, called skew crossed product of Lie algebras, which is introduced in Section 2 as a generalization of the semidirect product of Lie algebras.

The paper is organized as follows: in Section 2 we recall the basic concepts used throughout. The key role in our approach is played by the recently introduced unified product of Lie algebras ([1]). As a special case of the unified product we will introduce in Example 2.2 the skew crossed product, denoted by  $\mathfrak{g} \#^{\bullet} V$ , whose construction requires a Lie algebra  $\mathfrak{g}$ , a right Lie  $\mathfrak{g}$ -module  $(V, \leftarrow)$  equipped with a twisted bracket operation and a cocycle  $\theta: V \times V \rightarrow \mathfrak{g}$  satisfying a set of axioms. If  $\mathfrak{g} \subseteq \mathfrak{h}$  is an extension of Lie algebras, Theorem 3.2 provides the explicit description of the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  as a subgroup of the canonical semidirect product  $\text{GL}_k(V) \rtimes \text{Hom}_k(V, \mathfrak{g})$  of groups, where  $V$  is a vector space that measures the codimension of  $\mathfrak{g}$  in  $\mathfrak{h}$ . i.e., the ‘degree’ of the extension  $\mathfrak{h}/\mathfrak{g}$ . We point out that the group  $\text{GL}_k(V) \rtimes \text{Hom}_k(V, \mathfrak{g})$  is a lot more complex than the classical general affine group  $\text{GL}_k(V) \rtimes V$  of an affine space  $V$ . Theorem 3.10 is the counterpart of Artin’s Theorem for Lie algebras: if  $G$  is a finite group of invertible order in  $k$  acting on a Lie algebra  $\mathfrak{h}$ , then the Lie algebra  $\mathfrak{h}$  is reconstructed as a skew crossed product  $\mathfrak{h} \cong \mathfrak{h}^G \#^{\bullet} V$  between the Lie subalgebra of invariants  $\mathfrak{h}^G$  and the kernel  $V$  of the Reynolds operator  $t: \mathfrak{h} \rightarrow \mathfrak{h}^G$ .

The Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{h}^G)$  is also described and Example 3.11 shows that even in the case of faithful actions, the group  $\text{Gal}(\mathfrak{h}/\mathfrak{h}^G)$  is different from  $G$ , as opposed to the classical Galois theory of fields where the two groups coincide. Theorem 3.12 is Hilbert’s 90 Theorem for Lie algebras: if  $G$  is a cyclic group then the kernel of the Reynolds operator  $t: \mathfrak{h} \rightarrow \mathfrak{h}^G$  is determined. The structure of Lie algebras  $\mathfrak{h}$  endowed with a certain type of action of a finite cyclic group is then described in Corollary 3.14:  $\mathfrak{h}$  is isomorphic to a semidirect product between  $\mathfrak{h}^G$  and an ideal of  $\mathfrak{h}$ . This is the Lie algebra counterpart of the structure theorem for cyclic Galois extensions of fields [24, Theorem 6.2]: if  $G \leq \text{Aut}(K)$  is a cyclic subgroup of order  $n$  of the automorphism group of a field  $K$  of characteristic zero and  $k := K^G$ , then  $K$  is isomorphic to the splitting field over  $k$  of a polynomial of the form  $X^n - a \in k[X]$ .

Section 4 is devoted to computing the Galois groups for several Lie algebra extensions. Corollary 4.1 shows that if  $\mathfrak{g} \subseteq \mathfrak{h}$  is a Lie subalgebra of codimension 1 in  $\mathfrak{h}$ , then the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is metabelian (in particular, solvable). Based on this, the Lie algebra counterpart of the concept of a radical extension of fields is

proposed in Definition 4.3. As in the classical Galois theory, Theorem 4.4 proves that the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  of a radical extension  $\mathfrak{g} \subseteq \mathfrak{h}$  of finite dimensional Lie algebras is a solvable group. Several other applications and concrete examples of Galois groups are presented. For instance, in Example 4.8 we present a Lie algebra extension  $\mathfrak{g} \subseteq \mathfrak{h}$  whose Galois group is trivial i.e.,  $\text{Gal}(\mathfrak{h}/\mathfrak{g}) = \{\text{Id}_{\mathfrak{h}}\}$ . However, we point out that in general the Galois group of a Lie algebra extension is far from being trivial.

## 2. Preliminaries

**2.1. Notations and terminology.** All vector spaces, (bi)linear maps or Lie algebras are over an arbitrary field  $k$ . A map  $f: V \rightarrow W$  between two vector spaces is called trivial if  $f(v) = 0$ , for all  $v \in V$ . For two vector spaces  $V$  and  $W$  we denote by  $\text{Hom}_k(V, W)$  the abelian group of all linear maps from  $V$  to  $W$  and by  $\text{GL}_k(V) := \text{Aut}_k(V)$  the group of all linear automorphisms of  $V$ ; if  $V$  has dimension  $m$  over  $k$  then  $\text{GL}_k(V)$  is identified with the general linear group  $\text{GL}(m, k)$  of all  $m \times m$  invertible matrices over  $k$ . As usual,  $\text{SL}(m, k)$  stands for the special linear group of degree  $m$  over  $k$  which is the normal subgroup of  $\text{GL}(m, k)$  consisting of all  $m \times m$  matrices of determinant 1. Throughout this paper we use the right hand side convention for the *semidirect products* whose construction we briefly recall below. Let  $G$  and  $H$  be two groups and  $\triangleleft: H \times G \rightarrow H$  a right action as automorphisms of the group  $G$  on the group  $H$ , i.e. the following compatibility conditions hold for all  $g, g' \in G$  and  $h, h' \in H$ :

$$h \triangleleft 1 = h, \quad h \triangleleft (gg') = (h \triangleleft g) \triangleleft g' \quad (hh') \triangleleft g = (h \triangleleft g)(h' \triangleleft g)$$

The associated semidirect product  $G \rtimes H$  is the group structure on  $G \times H$  with multiplication given for any  $g, g' \in G$  and  $h, h' \in H$  by:

$$(g, h) \cdot (g', h') := (gg', (h \triangleleft g')h') \tag{1}$$

Let  $V$  and  $W$  be two vector spaces. Then there exists a *canonical right action* as automorphisms of the group  $\text{GL}_k(V)$  on the abelian group  $(\text{Hom}_k(V, W), +)$  given for any  $r \in \text{Hom}_k(V, W)$  and  $\sigma \in \text{GL}_k(V)$  by:

$$\triangleleft: \text{Hom}_k(V, W) \times \text{GL}_k(V) \rightarrow \text{Hom}_k(V, W), \quad r \triangleleft \sigma := r \circ \sigma$$

We shall denote by  $\mathbb{G}_W^V := \text{GL}_k(V) \rtimes \text{Hom}_k(V, W)$  the corresponding semidirect product, i.e.  $\mathbb{G}_W^V := \text{GL}_k(V) \times \text{Hom}_k(V, W)$ , with the multiplication given for any  $\sigma, \sigma' \in \text{GL}_k(V)$  and  $r, r' \in \text{Hom}_k(V, W)$  by:

$$(\sigma, r) \cdot (\sigma', r') := (\sigma \circ \sigma', r \circ \sigma' + r') \tag{2}$$

The unit of the group  $\mathbb{G}_W^V$  is  $(\text{Id}_V, 0)$ . Moreover,  $\text{GL}_k(V) \cong \text{GL}_k(V) \times \{0\}$  is a subgroup of  $\mathbb{G}_W^V$  and the abelian group  $\text{Hom}_k(V, W) \cong \{\text{Id}_V\} \times \text{Hom}_k(V, W)$  is a normal subgroup of  $\mathbb{G}_W^V$ . The relation  $(\sigma, r) = (\sigma, 0) \cdot (\text{Id}_V, r)$  gives an exact factorization  $\mathbb{G}_W^V = \text{GL}_k(V) \cdot \text{Hom}_k(V, W)$  of the group  $\mathbb{G}_W^V$  through the subgroup  $\text{GL}_k(V)$  and the abelian normal subgroup  $\text{Hom}_k(V, W)$ . Being a semidirect

product, the group  $\mathbb{G}_W^V$  is a split extension of  $\mathrm{GL}_k(V)$  by the abelian group  $\mathrm{Hom}_k(V, W)$ ; that is, it fits into an exact sequence of groups  $0 \rightarrow \mathrm{Hom}_k(V, W) \rightarrow \mathbb{G}_W^V \rightarrow \mathrm{GL}_k(V) \rightarrow 1$  and the canonical projection  $\mathbb{G}_W^V \rightarrow \mathrm{GL}_k(V) \rightarrow 1$  has a section that is a morphism of groups. The group  $\mathbb{G}_W^V$  constructed above will play a crucial role in the paper since the Galois group of an arbitrary extension of Lie algebras embeds in such a group. If  $V \cong k$  is a 1-dimensional vector space, then the group  $\mathbb{G}_W^k$  identifies with the semidirect product  $k^* \rtimes W$  of the multiplicative group of units  $(k^*, \cdot)$  with the abelian group  $(W, +)$  and will be denoted simply by  $\mathbb{G}_W$ . The multiplication on  $\mathbb{G}_W = k^* \rtimes W$  is given for any  $u, u' \in k^*$  and  $x, x' \in W$  by:

$$(u, x) \cdot (u', x') := (uu', u'x + x') \tag{3}$$

The non-abelian group  $\mathbb{G}_W$  is an extension of the abelian group  $k^*$  by the abelian group  $W = (W, +)$ ; hence,  $\mathbb{G}_W$  is a metabelian group (i.e., the derived subgroup  $[\mathbb{G}_W, \mathbb{G}_W]$  is abelian). In particular,  $\mathbb{G}_W$  is a 2-step solvable group. On the other hand, if  $W \cong k$  is a 1-dimensional vector space then  $\mathbb{G}_k^V = \mathrm{GL}_k(V) \rtimes V^*$ , and for finite dimensional vector spaces  $V$  the group can be identified with the general affine group  $\mathrm{Aff}(V) = \mathrm{GL}_k(V) \rtimes V$ .

**2.2. Groups acting on Lie algebras.** For all basic notions and undefined concepts pertaining to Lie algebra theory we refer the reader to [18, 19]. We denote by  $\mathfrak{gl}(m, k)$  (resp.  $\mathfrak{sl}(m, k)$ ) the general (resp. special) linear Lie algebra of all  $m \times m$  matrices (resp. all  $m \times m$  matrices of trace 0) having the bracket  $[A, B] := AB - BA$ . Representations of a Lie algebra  $\mathfrak{g}$  will be viewed as right Lie  $\mathfrak{g}$ -modules: a right Lie  $\mathfrak{g}$ -module is a vector space  $V$  together with a bilinear map  $\leftarrow: V \times \mathfrak{g} \rightarrow V$  such that  $x \leftarrow [a, b] = (x \leftarrow a) \leftarrow b - (x \leftarrow b) \leftarrow a$ , for all  $a, b \in \mathfrak{g}$  and  $x \in V$ . Let  $G$  be a group,  $\mathfrak{h}$  a Lie algebra and  $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{h})$  the group of all Lie algebra automorphisms of  $\mathfrak{h}$ . If  $\varphi: G \rightarrow \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{h})$  is a morphism of groups we will say that  $G$  acts as automorphisms on  $\mathfrak{h}$  and we shall denote  $\varphi(g)(x) = g \triangleright x$ , for all  $g \in G$  and  $x \in \mathfrak{h}$ . The action is called *faithful* if  $\varphi$  is injective. Since  $\varphi(g)$  is a Lie algebra map we have that  $g \triangleright [x, y] = [g \triangleright x, g \triangleright y]$ , for all  $g \in G$  and  $x, y \in \mathfrak{h}$ . The subalgebra of invariants  $\mathfrak{h}^G$  of the action  $\varphi$  of  $G$  on  $\mathfrak{h}$  is defined by:

$$\mathfrak{h}^G := \{x \in \mathfrak{h} \mid g \triangleright x = x, \forall g \in G\}$$

Then  $\mathfrak{h}^G \subseteq \mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{h}$ . If  $G$  is a finite group and  $|G|$  is invertible in the base field  $k$  then the *trace map* or *Reynolds operator* (we borrowed the terminology from the classical invariant theory of groups acting on associative algebras [17]) defined for any  $x \in \mathfrak{h}$  by:

$$t = t_\triangleright: \mathfrak{h} \rightarrow \mathfrak{h}^G, \quad t(x) := |G|^{-1} \sum_{g \in G} g \triangleright x \tag{4}$$

is a linear retraction of the canonical inclusion  $\mathfrak{h}^G \hookrightarrow \mathfrak{h}$ . Furthermore, for any  $a \in \mathfrak{h}^G$  and  $x \in \mathfrak{h}$  we have  $t([a, x]) = [a, t(x)]$ .

**Example 2.1.** (1) The basic example of a group acting on a Lie algebra is provided by any subgroup  $G$  of  $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{h})$  with the canonical action given by  $\sigma \triangleright x := \sigma(x)$ , for all  $\sigma \in G$  and  $x \in \mathfrak{h}$ . Automorphic Lie algebras [25] introduced

in the context of integrable systems are examples of Lie algebras of invariants - for further details we refer to [21].

(2) The group  $GL(n, k)$  acts on  $\mathfrak{gl}(n, k)$  (resp.  $\mathfrak{sl}(n, k)$ ) by conjugation, i.e.  $U \triangleright X := UXU^{-1}$ , for all  $U \in GL(n, k)$  and  $X \in \mathfrak{gl}(n, k)$  (resp.  $X \in \mathfrak{sl}(n, k)$ ). Thus, any subgroup of  $GL(n, k)$  (such as  $SL(n, k)$ , the permutation group  $S_n$  on  $n$  letters, the cyclic group  $C_n$  or more generally any finite group of order  $n$ ) acts on the Lie algebras  $\mathfrak{gl}(n, k)$  and  $\mathfrak{sl}(n, k)$  by the same action. The subalgebras of invariants for these actions are exactly the centralizers in  $\mathfrak{gl}(n, k)$  (resp.  $\mathfrak{sl}(n, k)$ ) of  $GL(n, k)$  (or its subgroups). For example, the symmetric group  $S_n$  acts as automorphisms on  $\mathfrak{gl}(n, k)$  via the action:

$$S_n \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{gl}(n, k)), \quad \tau \triangleright e_{ij} := (e_{1\tau(1)} + \dots + e_{n\tau(n)}) e_{ij} (e_{1\tau(1)} + \dots + e_{n\tau(n)})^{-1} \quad (5)$$

for all  $\tau \in S_n$  and  $i, j = 1, \dots, n$ , where  $e_{ij}$  is  $n \times n$  matrix which has 1 in the  $(i, j)^{\text{th}}$ -position and zeros elsewhere. We will describe the subalgebras of invariants  $\mathfrak{gl}(n, k)^{C_n}$  and respectively  $\mathfrak{gl}(n, k)^{S_n}$  of the action defined by 5. We start by looking at  $\mathfrak{gl}(n, k)^{C_n}$ , where the cyclic group  $C_n$  is considered to be the subgroup of  $S_n$  generated by the cycle  $(12\dots n)$ . An easy computation gives:

$$(12\dots n) \triangleright A = (e_{12} + e_{23} + e_{34} + \dots + e_{n1})A(e_{21} + e_{32} + e_{43} + \dots + e_{1n}). \quad (6)$$

Hence  $A = \sum_{i,j=1}^n a_{ij} e_{ij} \in \mathfrak{gl}(n, k)^{C_n}$  if and only if  $(12\dots n) \triangleright A = A$ . This yields

$$a_{11} = a_{nn}, \quad a_{1j} = a_{n,j-1}, \quad a_{j1} = a_{j-1,n}, \quad a_{ij} = a_{i-1,j-1}, \quad \text{for all } i, j \geq 2. \quad (7)$$

Therefore, by a careful analysis of the above compatibilities, we obtain that  $\mathfrak{gl}(n, k)^{C_n}$  is the  $n$ -dimensional subalgebra of  $\mathfrak{gl}(n, k)$  consisting of all  $n \times n$ -matrices of the form:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_n & \dots & a_{n-2} \\ \cdot & \cdot & \dots & \cdot \\ a_2 & a_3 & \dots & a_1 \end{pmatrix}$$

for all  $a_1, \dots, a_n \in k$ . Next in line is  $\mathfrak{gl}(n, k)^{S_n}$ . As  $S_n$  is generated by the transposition  $(12)$  and the cycle  $(12\dots n)$  it follows that a matrix  $A = \sum_{i,j=1}^n a_{ij} e_{ij} \in \mathfrak{gl}(n, k)^{S_n}$  if and only if  $\tau \triangleright A = A$ , for  $\tau = (12)$  and  $\tau = (12\dots n)$ . Using again the formula given in (5) we obtain:

$$(12) \triangleright A = (e_{12} + e_{21} + e_{33} + \dots + e_{nn})A(e_{12} + e_{21} + e_{33} + \dots + e_{nn}), \quad (8)$$

which yields  $a_{11} = a_{22}, \quad a_{12} = a_{21}, \quad a_{1j} = a_{2j}, \quad a_{j1} = a_{j2},$  for all  $j \geq 3$ .

The above compatibilities together with those in equation (7) come down to:  $a_{ii} = \alpha \in k$  and  $a_{ij} = \beta \in k$ , for all  $i, j = 1, \dots, n, i \neq j$ . Thus,

$$\mathfrak{gl}(n, k)^{S_n} = \left\{ \alpha I_n + \beta \sum_{\substack{i,j=1 \\ i \neq j}}^n e_{ij} \mid \alpha, \beta \in k \right\}.$$

Both subalgebras of invariants  $\mathfrak{gl}(n, k)^{C_n}$  and respectively  $\mathfrak{gl}(n, k)^{S_n}$  are abelian.

(3) The actions as automorphism of abelian groups on Lie algebras can be seen as the dual concept of the well studied gradings on Lie algebras: for an overview and the importance of the problem introduced by Kac [20] we refer to [13, 22, 36] and the references therein.

Let  $G = (G, +)$  be an abelian group and  $\hat{G}$  be the group of characters on  $G$ , i.e. all morphisms of groups  $G \rightarrow k^*$ . A  $G$ -graded Lie algebra is a Lie algebra  $\mathfrak{h}$  such that  $\mathfrak{h} = \bigoplus_{g \in G} \mathfrak{h}_g$ , where any  $\mathfrak{h}_g$  is a subspace of  $\mathfrak{h}$  such that  $[\mathfrak{h}_g, \mathfrak{h}_{g'}] \subseteq \mathfrak{h}_{g+g'}$ , for all  $g, g' \in G$ . If  $\mathfrak{h} = \bigoplus_{g \in G} \mathfrak{h}_g$  is a  $G$ -graded Lie algebra then the map

$$\varphi: \hat{G} \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h}), \quad \varphi(\chi)(x_g) := \chi(g) x_g$$

for all  $\chi \in \hat{G}$ ,  $g \in G$  and  $x_g \in \mathfrak{h}_g$  is a faithful action of  $\hat{G}$  on  $\mathfrak{h}$ . Conversely, if  $\varphi: \hat{G} \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$  is an injective morphism of groups, then  $\mathfrak{h} = \bigoplus_{g \in G} \mathfrak{h}_g$  is a  $G$ -graded Lie algebra where  $\mathfrak{h}_g := \{y \in \mathfrak{h} \mid \chi \triangleright y = \chi(g)y, \forall \chi \in \hat{G}\}$ . In some special cases we can say more. For instance, if  $k$  is an algebraically closed field of characteristic zero and  $G$  is a finitely generated abelian group, then there exists a one-to-one correspondence between the set of all  $G$ -gradings on a given Lie algebra  $\mathfrak{h}$  and the set of all faithful actions  $\hat{G} \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$  of  $\hat{G}$  on  $\mathfrak{h}$  [22, Proposition 4.1]. Working with actions instead of gradings comes with the advantage of not assuming the group  $G$  to be abelian nor the actions to be faithful.

(4) As a special case of (3) let us take  $\mathfrak{h} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{h}_i$  to be a  $\mathbb{Z}$ -graded Lie algebra. Then the multiplicative group of units  $k^*$  acts on  $\mathfrak{h}$  via the following morphism of groups:

$$\varphi: k^* \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h}), \quad \varphi(u)(y_i) := u^i y_i \tag{9}$$

for all  $u \in k^*$ ,  $i \in \mathbb{Z}$  and  $y_i \in \mathfrak{h}_i$  a homogeneous element of degree  $i$ . Moreover, the subalgebra of invariants  $\mathfrak{h}^{k^*} = \mathfrak{h}_0$ , the Lie subalgebra of all elements of degree zero. The typical example of a  $\mathbb{Z}$ -graded Lie algebra is the Witt algebra  $\mathfrak{W}$  which is the vector space having  $\{e_i \mid i \in \mathbb{Z}\}$  as a basis and the bracket  $[e_i, e_j] := (i - j) e_{i+j}$ , for all  $i, j \in \mathbb{Z}$ . Another example is given by  $\mathfrak{h} := \mathfrak{sl}(2, k)$ , the Lie algebra with basis  $\{e_1, e_2, e_3\}$  and the usual bracket  $[e_1, e_2] = e_3$ ,  $[e_1, e_3] = -2e_1$  and  $[e_2, e_3] = 2e_2$  viewed with the standard grading: namely  $e_1$  has degree  $-1$ ,  $e_2$  has degree  $1$  and  $e_3$  has degree  $0$ . We obtain that the group  $k^*$  acts on  $\mathfrak{sl}(2, k)$  via:

$$\varphi: k^* \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{sl}(2, k)), \quad u \triangleright (\alpha e_1 + \beta e_2 + \gamma e_3) := u^{-1} \alpha e_1 + u \beta e_2 + \gamma e_3 \tag{10}$$

for all  $u \in k^*$  and  $\alpha, \beta, \gamma \in k$ . The algebra of invariants  $\mathfrak{sl}(2, k)^{k^*}$  is the abelian Lie algebra having  $e_3$  as a basis.

### 2.3. Unified products and skew crossed product for Lie algebras.

We recall from [1] some concepts that will play a key role in the paper. Let  $\mathfrak{g} = (\mathfrak{g}, [-, -])$  be a Lie algebra and  $V$  a vector space. A *Lie extending system* of  $\mathfrak{g}$  through  $V$  is a system  $\Lambda(\mathfrak{g}, V) = (\leftarrow, \rightarrow, \theta, \{-, -\})$  consisting of four bilinear maps  $\leftarrow: V \times \mathfrak{g} \rightarrow V$ ,  $\rightarrow: V \times \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $\theta: V \times V \rightarrow \mathfrak{g}$ ,  $\{-, -\}: V \times V \rightarrow V$  satisfying the following compatibility conditions for any  $a, b \in \mathfrak{g}$ ,  $x, y, z \in V$ :

(L1)  $(V, \leftarrow)$  is a right Lie  $\mathfrak{g}$ -module,  $\theta(x, x) = 0$  and  $\{x, x\} = 0$

$$(L2) \quad x \rightharpoonup [a, b] = [x \rightharpoonup a, b] + [a, x \rightharpoonup b] + (x \leftarrow a) \rightharpoonup b - (x \leftarrow b) \rightharpoonup a$$

$$(L3) \quad \{x, y\} \leftarrow a = \{x, y \leftarrow a\} + \{x \leftarrow a, y\} + x \leftarrow (y \rightharpoonup a) - y \leftarrow (x \rightharpoonup a)$$

$$(L4) \quad \{x, y\} \rightharpoonup a = x \rightharpoonup (y \rightharpoonup a) - y \rightharpoonup (x \rightharpoonup a) + [a, \theta(x, y)] + \theta(x, y \leftarrow a) + \theta(x \leftarrow a, y)$$

$$(L5) \quad \sum_{(c)} \theta(x, \{y, z\}) + \sum_{(c)} x \rightharpoonup \theta(y, z) = 0$$

$$(L6) \quad \sum_{(c)} \{x, \{y, z\}\} + \sum_{(c)} x \leftarrow \theta(y, z) = 0$$

where  $\sum_{(c)}$  denotes the circular sum. The bilinear maps  $\leftarrow$  and  $\rightharpoonup$  are called the *actions* of  $\Lambda(\mathfrak{g}, V)$  and  $\theta$  is called the *cocycle* of  $\Lambda(\mathfrak{g}, V)$ .

Let  $\Lambda(\mathfrak{g}, V) = (\leftarrow, \rightharpoonup, \theta, \{-, -\})$  be an extending system of  $\mathfrak{g}$  through  $V$  and let  $\mathfrak{g} \natural_{\Lambda(\mathfrak{g}, V)} V = \mathfrak{g} \natural_{\Lambda(\mathfrak{g}, V)} V$  be the vector space  $\mathfrak{g} \times V$  with the bracket  $[-, -]$  defined for any  $a, b \in \mathfrak{g}$  and  $x, y \in V$  by:

$$[(a, x), (b, y)] := ([a, b] + x \rightharpoonup b - y \rightharpoonup a + \theta(x, y), \{x, y\} + x \leftarrow b - y \leftarrow a). \quad (11)$$

Then  $\mathfrak{g} \natural V$  is a Lie algebra [1, Theorem 2.2] called the *unified product* of  $\mathfrak{g}$  and  $\Lambda(\mathfrak{g}, V)$ , and contains  $\mathfrak{g} \cong \mathfrak{g} \times \{0\}$  as a Lie subalgebra. Conversely, let  $\mathfrak{g}$  be a Lie algebra,  $E$  a vector space such that  $\mathfrak{g}$  is a subspace of  $E$ . Then, any Lie algebra structure  $[-, -]_E$  on  $E$  containing  $\mathfrak{g}$  as a Lie subalgebra is isomorphic to a unified product: i.e.,  $(E, [-, -]_E) \cong \mathfrak{g} \natural V$ , for some extending system  $\Lambda(\mathfrak{g}, V) = (\leftarrow, \rightharpoonup, \theta, \{-, -\})$  of  $\mathfrak{g}$  through  $V$  ([1, Theorem 2.4]). As explained in [1], the well known bicrossed product as well as the crossed product of Lie algebras are special cases of unified products. First of all, we observe that the extending system of a Lie algebra  $\mathfrak{g}$  through  $V$  is a cocycle deformation of the concept of matched pair between two Lie algebras, as introduced in [26, 28]. Indeed, if  $\theta$  is the trivial map, then  $\Lambda(\mathfrak{g}, V) = (\leftarrow, \rightharpoonup, \theta := 0, \{-, -\})$  is a Lie extending system of  $\mathfrak{g}$  through  $V$  if and only if  $(V, \{-, -\})$  is a Lie algebra and  $(\mathfrak{g}, V, \leftarrow, \rightharpoonup)$  is a matched pair of Lie algebras. In this case, the associated unified product  $\mathfrak{g} \natural V = \mathfrak{g} \bowtie V$  is precisely the bicrossed product (also called bicrossproduct in [28, Theorem 4.1] and double Lie algebra in [26, Definition 3.3]) associated to the matched pair  $(\mathfrak{g}, V, \leftarrow, \rightharpoonup)$ . Secondly, if  $\leftarrow$  is the trivial map, then  $\Lambda(\mathfrak{g}, V) = (\leftarrow := 0, \rightharpoonup, \theta, \{-, -\})$  is a Lie extending system of  $\mathfrak{g}$  through  $V$  if and only if  $(V, \{-, -\})$  is a Lie algebra and the following four compatibilities hold for any  $g, h \in \mathfrak{g}$  and  $x, y, z \in V$ :

- (1)  $f(x, x) = 0$ ,
- (2)  $\{x, y\} \rightharpoonup g = x \rightharpoonup (y \rightharpoonup g) - y \rightharpoonup (x \rightharpoonup g) + [g, \theta(x, y)]$ ,
- (3)  $x \rightharpoonup [g, h] = [x \rightharpoonup g, h] + [g, x \rightharpoonup h]$ ,
- (4)  $\sum_{(c)} \theta(x, \{y, z\}) + \sum_{(c)} x \rightharpoonup \theta(y, z) = 0$ .

In this case, the associated unified product  $\mathfrak{g} \natural V = \mathfrak{g} \# V$  is the classical crossed product of the Lie algebras  $\mathfrak{g}$  and  $V$  introduced in [8] in connection to the extension problem. The Lie algebra  $\mathfrak{g} \# V$  is an extension of  $V$  by  $\mathfrak{g}$ , which is an ideal of it. Although it is completely different from both the crossed and the bicrossed product, the following construction is also a special case of the unified product:

**Example 2.2.** Consider the bilinear map  $\dashv: V \times \mathfrak{g} \rightarrow \mathfrak{g}$  to be trivial, i.e.  $x \dashv g = 0$ , for all  $x \in V$  and  $g \in \mathfrak{g}$ . Then  $\Lambda(\mathfrak{g}, V) = (\dashv, \dashv := 0, \theta, \{-, -\})$  is a Lie extending system of  $\mathfrak{g}$  through  $V$  if and only if the following compatibility conditions hold for any  $a \in \mathfrak{g}$ ,  $x, y, z \in V$ :

- (T1)  $(V, \dashv)$  is a right Lie  $\mathfrak{g}$ -module,  $\theta(x, x) = 0$  and  $\{x, x\} = 0$
- (T2)  $\{x, y\} \dashv a = \{x, y \dashv a\} + \{x \dashv a, y\}$
- (T3)  $[\theta(x, y), a] = \theta(x, y \dashv a) + \theta(x \dashv a, y)$
- (T4)  $\sum_{(c)} \theta(x, \{y, z\}) = 0$
- (T5)  $\sum_{(c)} \{x, \{y, z\}\} + \sum_{(c)} x \dashv \theta(y, z) = 0$

In this case the trivial map  $\dashv$  will be omitted when writing down the Lie extending system  $\Lambda(\mathfrak{g}, V)$ . The associated unified product  $\mathfrak{g} \ltimes V$  will be denoted by  $\mathfrak{g} \#^\bullet V$  and we will call it the skew crossed product associated to the system  $\Lambda(\mathfrak{g}, V) = (\dashv, \theta, \{-, -\})$  satisfying (T1)–(T5). Thus,  $\mathfrak{g} \#^\bullet V$  is the vector space  $\mathfrak{g} \times V$  with the Lie bracket  $[-, -]$  defined for any  $a, b \in \mathfrak{g}$  and  $x, y \in V$  by:

$$[(a, x), (b, y)] := ([a, b] + \theta(x, y), \{x, y\} + x \dashv b - y \dashv a) \tag{12}$$

As already mentioned, the skew crossed product  $\mathfrak{g} \#^\bullet V$  is completely different from both the crossed as well as the bicrossed product of Lie algebras: in the construction of  $\mathfrak{g} \#^\bullet V$  the bilinear map  $\{-, -\}$  on  $V$  is not a Lie bracket (axiom (T5) is a deformation of the Jacobi identity) and moreover  $\mathfrak{g} \cong \mathfrak{g} \times \{0\}$  is only a subalgebra in  $\mathfrak{g} \#^\bullet V$ , not an ideal. An explicit example of a skew crossed product is given in Example 3.4 where we write  $\mathfrak{sl}(2, k)$  as a skew crossed product  $k \#^\bullet k^2$  between the abelian Lie algebras of dimensions one and two, associated to a certain right action  $\dashv$  and a cocycle  $\theta$ .

Moreover, if the cocycle  $\theta$  of a Lie extending structure  $\Lambda(\mathfrak{g}, V) = (\dashv, \theta, \{-, -\})$  is also the trivial map, then the skew crossed product  $\mathfrak{g} \#^\bullet V$  is just the usual semidirect product  $\mathfrak{g} \rtimes V$  of two Lie algebras written in the right side convention. We point out that in our notational convention the Lie algebra  $\mathfrak{g} \rtimes V$  contains  $V \cong \{0\} \times V$  as an ideal.

### 3. The Galois group of Lie algebra extensions

Let  $\mathfrak{g} \subseteq \mathfrak{h}$  be an extension of Lie algebras. We define the *Galois group*  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  as the subgroup of  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$  consisting of all Lie algebra automorphisms of  $\mathfrak{h}$  that fix  $\mathfrak{g}$ , i.e.:

$$\text{Gal}(\mathfrak{h}/\mathfrak{g}) := \{\sigma \in \text{Aut}_{\text{Lie}}(\mathfrak{h}) \mid \sigma(g) = g, \forall g \in \mathfrak{g}\}.$$

Since  $\text{Gal}(\mathfrak{h}/\mathfrak{g}) \leq \text{Aut}_{\text{Lie}}(\mathfrak{h})$  we can consider the subalgebra of invariants  $\mathfrak{h}^{\text{Gal}(\mathfrak{h}/\mathfrak{g})}$ . Of course, we have that  $\mathfrak{g} \subseteq \mathfrak{h}^{\text{Gal}(\mathfrak{h}/\mathfrak{g})}$ . As it can be seen from the example below, a fundamental theorem establishing a bijective correspondence between the subgroups of  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  and the Lie subalgebras  $\mathfrak{g}'$  of  $\mathfrak{h}$  such that  $\mathfrak{g} \subseteq \mathfrak{g}' \subseteq \mathfrak{h}$  does not hold in the context of Lie algebras.

**Example 3.1.** Let  $\mathfrak{h} := \text{aff}(2, k)$  be the 2-dimensional affine Lie algebra with basis  $\{e_1, e_2\}$  and bracket  $[e_1, e_2] = e_2$  and  $\mathfrak{g} := ke_1$  the abelian Lie subalgebra. Then  $\text{Gal}(\text{aff}(2, k)/\mathfrak{g})$  is isomorphic to  $k^*$  the multiplicative group of units of  $k$  while the subalgebra of invariants  $\text{aff}(2, k)^{k^*} = \mathfrak{g}$ . Of course, between  $\mathfrak{g}$  and  $\text{aff}(2, k)$  there are no proper intermediary subalgebras while  $k^*$  has many subgroups (such as the cyclic groups  $U_n(k)$  of  $n$ -roots of unity) whose subalgebras of invariants coincide with  $\mathfrak{g}$ . ■

In what follows we will describe the group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$ . First we fix a linear map  $p: \mathfrak{h} \rightarrow \mathfrak{g}$  such that  $p(g) = g$ , for all  $g \in \mathfrak{g}$  - such a map always exists as  $k$  is a field. Then  $V := \text{Ker}(p)$  is a subspace of  $\mathfrak{h}$  and a complement of  $\mathfrak{g}$  in  $\mathfrak{h}$ , that is  $\mathfrak{h} = \mathfrak{g} + V$  and  $\mathfrak{g} \cap V = \{0\}$ . Using  $p$  we define a Lie extending system of  $\mathfrak{g}$  through  $V$ , called the canonical extending system associated to  $p$ , where the bilinear maps  $\rightarrow: V \times \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $\leftarrow: V \times \mathfrak{g} \rightarrow V$ ,  $\theta: V \times V \rightarrow \mathfrak{g}$  and  $\{, \}: V \times V \rightarrow V$  are given by the following formulas [1, Theorem 2.4] for any  $g \in \mathfrak{g}$  and  $x, y \in V$ :

$$x \rightarrow g := p([x, g]), \quad x \leftarrow g := [x, g] - p([x, g]) \tag{13}$$

$$\theta(x, y) := p([x, y]), \quad \{x, y\} := [x, y] - p([x, y]) \tag{14}$$

Thus we can construct the unified product  $\mathfrak{g} \natural V$  associated to the canonical extending structure, which is a Lie algebra with the bracket given by the formula (11). The map  $\varphi: \mathfrak{g} \natural V \rightarrow \mathfrak{h}$ , defined by  $\varphi(g, x) := g + x$ , is an isomorphism of Lie algebras with the inverse given by  $\varphi^{-1}(y) := (p(y), y - p(y))$ , for all  $y \in \mathfrak{h}$ . Since  $\varphi$  fixes  $\mathfrak{g} \cong \mathfrak{g} \times \{0\}$  we obtain that the map

$$\text{Gal}(\mathfrak{h}/\mathfrak{g}) \rightarrow \text{Gal}(\mathfrak{g} \natural V/\mathfrak{g}), \quad \sigma \mapsto \varphi^{-1} \circ \sigma \circ \varphi \tag{15}$$

is an isomorphism of groups with the inverse given by  $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$ . It follows from [1, Lemma 2.5] that there exists a bijection between the set of all elements  $\psi \in \text{Gal}(\mathfrak{g} \natural V/\mathfrak{g})$  and the set of all pairs  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{g})$ , satisfying the following four compatibility conditions for any  $g \in \mathfrak{g}$ ,  $x, y \in V$ :

- (G1)  $\sigma(x \leftarrow g) = \sigma(x) \leftarrow g$ , that is  $\sigma: V \rightarrow V$  is a right Lie  $\mathfrak{g}$ -module map;
- (G2)  $r(x \leftarrow g) = [r(x), g] + (\sigma(x) - x) \rightarrow g$ ;
- (G3)  $\sigma(\{x, y\}) = \{\sigma(x), \sigma(y)\} + \sigma(x) \leftarrow r(y) - \sigma(y) \leftarrow r(x)$ ;
- (G4)  $r(\{x, y\}) = [r(x), r(y)] + \sigma(x) \rightarrow r(y) - \sigma(y) \rightarrow r(x) + \theta(\sigma(x), \sigma(y)) - \theta(x, y)$ .

The bijection is such that  $\psi = \psi_{(\sigma, r)} \in \text{Gal}(\mathfrak{g} \natural V/\mathfrak{g})$  corresponding to  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{g})$  is given by  $\psi(g, x) := (g + r(x), \sigma(x))$ , for all  $g \in \mathfrak{g}$  and  $x \in V$ . We point out that  $\psi_{(\sigma, r)}$  is indeed an element of  $\text{Gal}(\mathfrak{g} \natural V/\mathfrak{g})$  with the inverse given by  $\psi_{(\sigma, r)}^{-1}(g, x) = (g - r(\sigma^{-1}(x)), \sigma^{-1}(x))$ , for all  $g \in \mathfrak{g}$  and  $x \in V$ . We denote by  $\mathbb{G}_{\mathfrak{g}}^V(\leftarrow, \rightarrow, \theta, \{-, -\})$  the set of all pairs  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{g})$  satisfying the compatibility conditions (G1)–(G4). It is straightforward to see that  $\mathbb{G}_{\mathfrak{g}}^V(\leftarrow, \rightarrow, \theta, \{-, -\})$  is a subgroup of the semidirect product of groups  $\mathbb{G}_{\mathfrak{g}}^V := \text{GL}_k(V) \rtimes \text{Hom}_k(V, \mathfrak{g})$  with the group structure given by formula (2).

For any  $(\sigma, r)$  and  $(\sigma', r') \in \mathbb{G}_{\mathfrak{g}}^V(\leftarrow, \rightarrow, \theta, \{-, -\})$ ,  $g \in \mathfrak{g}$  and  $x \in V$  we have:

$$\psi_{(\sigma, r)} \circ \psi_{(\sigma', r')}(g, x) = (g + r'(x) + r(\sigma'(x)), \sigma(\sigma'(x))) = \psi_{(\sigma \circ \sigma', r \circ r' + r')}(g, x)$$

i.e.  $\psi_{(\sigma, r)} \circ \psi_{(\sigma', r')} = \psi_{(\sigma \circ \sigma', r \circ r' + r')}$ . Finally, we recall that  $\mathfrak{h} = \mathfrak{g} + V$  and  $\mathfrak{g} \cap V = \{0\}$ , i.e. any element  $y \in \mathfrak{h}$  has a unique decomposition as  $y = g + x$ , for  $g \in \mathfrak{g}$  and  $x \in V = \text{Ker}(p)$ . Putting all together we proved the following:

**Theorem 3.2.** *Let  $\mathfrak{g} \subseteq \mathfrak{h}$  be an extension of Lie algebras,  $p: \mathfrak{h} \rightarrow \mathfrak{g}$  a linear retraction of the inclusion  $\mathfrak{g} \subseteq \mathfrak{h}$ ,  $V = \text{Ker}(p)$ . We consider the canonical Lie extending system  $\Lambda(\mathfrak{g}, V) = (\leftarrow, \rightarrow, \theta, \{-, -\})$  associated to  $p$ . Then there exists an isomorphism of groups defined for any  $(\sigma, r) \in \mathbb{G}_{\mathfrak{g}}^V(\leftarrow, \rightarrow, \theta, \{-, -\})$ ,  $g \in \mathfrak{g}$  and  $x \in V$  by:*

$$\Omega: \mathbb{G}_{\mathfrak{g}}^V(\leftarrow, \rightarrow, \theta, \{-, -\}) \rightarrow \text{Gal}(\mathfrak{h}/\mathfrak{g}), \quad \Omega(\sigma, r)(g + x) := g + r(x) + \sigma(x). \tag{16}$$

*In particular, there exists an embedding  $\text{Gal}(\mathfrak{h}/\mathfrak{g}) \hookrightarrow \text{GL}_k(V) \rtimes \text{Hom}_k(V, \mathfrak{g})$ , where the right hand side is the semidirect product associated to the canonical right action of  $\text{GL}_k(V)$  on  $\text{Hom}_k(V, \mathfrak{g})$ .*

In the finite dimensional case we obtain the Lie algebra counterpart of the fact that the Galois group of a Galois extension of fields of degree  $m$  embeds in the symmetric group  $S_m$ .

**Corollary 3.3.** *Let  $\mathfrak{g} \subseteq \mathfrak{h}$  be an extension of Lie algebras such that  $\dim_k(\mathfrak{g}) = n$  and  $\dim_k(\mathfrak{h}) = n + m$ . Then the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  embeds in the canonical semidirect product of groups  $\text{GL}(m, k) \rtimes M_{n \times m}(k)$ .*

The first examples based on Theorem 3.2 are given below. More examples and applications will be presented in Section 4.

**Example 3.4.** Consider the extension of Lie algebras  $ke_3 \subseteq \mathfrak{sl}(2, k)$  with the notations of Example 2.1 and take  $p: \mathfrak{sl}(2, k) \rightarrow ke_3$  given by  $p(e_1) = p(e_2) := 0$  and  $p(e_3) := e_3$ . Then  $V = \text{Ker}(p) = ke_1 + ke_2$  and the canonical Lie extending system associated to  $p$  given by (13)–(14) takes the following form:  $\rightarrow: V \times ke_3 \rightarrow V$  and  $\{-, -\}: V \times V \rightarrow V$  are both trivial maps, while the action  $\leftarrow: V \times ke_3 \rightarrow V$  and the cocycle  $\theta: V \times V \rightarrow ke_3$  are given by  $e_1 \leftarrow e_3 = -2e_1$ ,  $e_2 \leftarrow e_3 = 2e_2$  and  $\theta(e_1, e_1) = \theta(e_2, e_2) = 0$ ,  $\theta(e_1, e_2) = -\theta(e_2, e_1) = e_3$ . In particular, this shows that  $\mathfrak{sl}(2, k)$  is isomorphic as a Lie algebra to the skew crossed product  $ke_3 \#^{\bullet} V$  of the abelian Lie algebras of dimensions one and two.

Now, an element  $\sigma \in \text{GL}_k(V)$  will be written in the matrix form  $\sigma = (\sigma_{ij}) \in M_2(k)$  and a linear map  $r \in \text{Hom}_k(V, ke_3)$  as a family of two scalars  $r = (r_1, r_2) \in k^2$  given by  $r(e_1) = r_1e_3$  and  $r(e_2) = r_2e_3$ . A straightforward computation proves that the pair  $(\sigma = (\sigma_{ij}), r = (r_1, r_2))$  satisfies (G1)–(G4) if and only if  $r_1 = r_2 = 0$ ,  $\sigma_{12} = \sigma_{21} = 0$  and  $\sigma_{11}\sigma_{22} = 1$ . This proves that the group  $\mathbb{G}_{ke_3}^V(\leftarrow, \theta)$  identifies with the group of units  $k^*$  and hence  $\text{Gal}(\mathfrak{sl}(2, k)/ke_3) \cong k^*$ . More precisely,  $\tau \in \text{Gal}(\mathfrak{sl}(2, k)/ke_3)$  if and only if there exists  $u \in k^*$  such that  $\tau(ae_1 + be_2 + ce_3) = uae_1 + u^{-1}be_2 + ce_3$ , for all  $a, b, c \in k$ . ■

Our next example proves that the Galois group of the extension of two consecutive Lie Heisenberg algebras is the 2-dimensional special affine group  $SL_2(k) \rtimes k^2$ .

**Example 3.5.** Let  $n \in \mathbb{N}^*$  and consider  $\mathfrak{h}^{2n+1}$  to be the  $(2n + 1)$ -dimensional Heisenberg Lie algebra having  $\{x_1, \dots, x_n, y_1, \dots, y_n, w\}$  as a basis and the bracket given by  $[x_i, y_i] = w$ , for all  $i = 1, \dots, n$ . If we consider the canonical Lie algebra extension  $\mathfrak{h}^{2n+1} \subset \mathfrak{h}^{2n+3}$ , then there exists an isomorphism of groups:

$$\text{Gal}(\mathfrak{h}^{2n+3}/\mathfrak{h}^{2n+1}) \cong SL_2(k) \rtimes k^2$$

where  $SL_2(k) \rtimes k^2$  is the semidirect product of groups corresponding to the canonical right action  $\triangleleft: k^2 \times SL_2(k) \rightarrow k^2$  given by  $(a, b) \triangleleft B = (a, b)B$ , for all  $(a, b) \in k^2$ ,  $B \in SL_2(k)$ . First, we point out that  $\mathfrak{h}^{2n+3}$  can be realized as a unified product between  $\mathfrak{h}^{2n+1}$  and the vector space  $V$  with  $k$  basis  $\{x_{n+1}, y_{n+1}\}$  corresponding to the Lie extending system with one non-trivial map, namely  $\theta: V \times V \rightarrow \mathfrak{h}^{2n+1}$  given by  $\theta(x_{n+1}, y_{n+1}) = w$ . The conclusion now follows by applying Theorem 3.2. First notice that any pair  $(\sigma, r) \in GL_k(V) \times \text{Hom}_k(V, \mathfrak{g})$  fulfills trivially the compatibility conditions (G1) and (G3). Furthermore, the compatibility condition (G2) yields  $r(x_{n+1}) = \alpha w$  and  $r(y_{n+1}) = \beta w$  for some  $\alpha, \beta \in k$ .

Finally, if we denote  $\sigma(x_{n+1}) = ax_{n+1} + bx_{n+1}$  and resp.  $\sigma(y_{n+1}) = cx_{n+1} + dx_{n+1}$  for some  $a, b, c, d \in k$ , the compatibility condition (G4) gives  $ad - bc = 1$ . Therefore, the set of pairs  $(\sigma, r) \in GL_k(V) \times \text{Hom}_k(V, \mathfrak{g})$  satisfying (G1) – (G4) is in fact equal to  $SL_2(k) \times k^2$ . The proof is now finished by identifying the maps  $\sigma \in GL_k(V)$  and  $r \in \text{Hom}_k(V, \mathfrak{g})$  with their corresponding matrices in  $SL_2(k)$  respectively  $k^2$  and noticing that the multiplication on  $\mathbb{G}_{\mathfrak{h}^{2n+1}}^V(\theta)$  comes down to that corresponding to the semidirect product induced by the action defined above. ■

Now follows an example of a Lie algebra extension with a metabelian Galois group.

**Example 3.6.** For a positive integer  $n$ , let  $\mathfrak{l}(2n + 1)$  be the metabelian Lie algebra with basis  $\{E_i, F_i, G \mid i = 1, \dots, n\}$  and bracket given by  $[E_i, G] = E_i$ ,  $[G, F_i] = F_i$ , for all  $i = 1, \dots, n$ . Then there exists an isomorphism of groups:

$$\text{Gal}(\mathfrak{l}(2n + 3)/\mathfrak{l}(2n + 1)) \cong (k^* \times k^*) \rtimes \mathcal{M}_{n \times 2}(k),$$

where  $(k^* \times k^*) \rtimes \mathcal{M}_{n \times 2}(k)$  is the semidirect product corresponding to the right action  $\triangleleft: \mathcal{M}_{n \times 2}(k) \times (k^* \times k^*) \rightarrow \mathcal{M}_{n \times 2}(k)$  given by

$$B \triangleleft (a \ b) = B \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

for all  $a, b \in k^*$  and  $B \in \mathcal{M}_{n \times 2}(k)$ . Indeed,  $\mathfrak{l}(2n + 3)$  can be written as a unified product between  $\mathfrak{l}(2n + 1)$  and the vector space  $V$  with basis  $\{E_{n+1}, F_{n+1}\}$  corresponding to the extending structure with only one non-trivial map, namely  $\dashv: V \times \mathfrak{l}(2n + 1) \rightarrow V$  given by  $E_{n+1} \dashv G = E_{n+1}$  and  $F_{n+1} \dashv G = -F_{n+1}$ . Now if  $(\sigma, r) \in GL_k(V) \times \text{Hom}_k(V, \mathfrak{l}(2n + 1))$  a careful analysis of the compatibility conditions (G1) – (G4) yields:

$$\begin{aligned} \sigma(E_{n+1}) &= aE_{n+1}, & \sigma(F_{n+1}) &= bF_{n+1}, & a, b \in k, & ab \neq 0 \\ r(E_{n+1}) &= \sum_{i=1}^n \alpha_i E_i, & r(F_{n+1}) &= \sum_{i=1}^n \beta_i F_i, & \alpha_i, \beta_i \in k, & i = 1, 2, \dots, n. \end{aligned}$$

Thus, the pairs  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{l}(2n + 1))$  are parameterized by  $(k^* \times k^*) \times \mathcal{M}_{n \times 2}(k)$  and the conclusion follows by Theorem 3.2. ■

In the sequel we consider three general examples.

**Example 3.7.** Let  $\mathfrak{g} \rtimes \mathfrak{h}$  be a semidirect product of two Lie algebras  $\mathfrak{g}$  and  $\mathfrak{h}$  written in the right hand side convention as indicated in Example 2.2: thus,  $\mathfrak{g} \rtimes \mathfrak{h}$  is associated to a right action of  $\mathfrak{g}$  on  $\mathfrak{h}$  denoted by  $\leftarrow : \mathfrak{h} \times \mathfrak{g} \rightarrow \mathfrak{h}$ . If  $\mathfrak{g}$  is abelian and  $\mathfrak{h}$  is a perfect Lie algebra (i.e.  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$ ), then there exists an isomorphism of groups:

$$\text{Gal}(\mathfrak{g} \rtimes \mathfrak{h}/\mathfrak{g}) \cong \text{Aut}_{\text{Lie}}^{\leftarrow}(\mathfrak{h})$$

where  $\text{Aut}_{\text{Lie}}^{\leftarrow}(\mathfrak{h})$  denotes the set of Lie algebra automorphisms of  $\mathfrak{h}$  which are also right Lie  $\mathfrak{g}$ -module maps, i.e.

$$\text{Aut}_{\text{Lie}}^{\leftarrow}(\mathfrak{h}) = \{u \in \text{Aut}_{\text{Lie}}(\mathfrak{h}) \mid u(x \leftarrow g) = u(x) \leftarrow g \text{ for all } g \in \mathfrak{g}, x \in \mathfrak{h}\}.$$

We just apply Theorem 3.2. Indeed, let  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{g})$  satisfying the compatibility conditions (G1)–(G4). As  $\mathfrak{g}$  is abelian (G2) comes down to  $r(\{x, y\}) = 0$  for all  $x, y \in \mathfrak{h}$ . Now since  $\mathfrak{h}$  is a perfect Lie algebra we obtain  $r = 0$ . Then (G2) is trivially fulfilled while (G1) and (G3) imply that  $\sigma$  is a right Lie  $\mathfrak{g}$ -module map respectively a Lie algebra map. ■

The next example computes the Galois group of the extension  $\mathfrak{g}' \subset \mathfrak{g}$  for a special class of Lie algebras  $\mathfrak{g}$ , namely the non-perfect ones with  $C_{\mathfrak{g}}(\mathfrak{g}') = \{0\}$ , where  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$  is the derived algebra of  $\mathfrak{g}$  and  $C_{\mathfrak{g}}(\mathfrak{g}')$  denotes the centralizer of  $\mathfrak{g}'$  in  $\mathfrak{g}$ . A generic example of such a Lie algebra is for instance  $\mathfrak{g} := \mathfrak{gl}(n, k) \rtimes k^n$ , the semidirect product of Lie algebras corresponding to the canonical action of  $\mathfrak{gl}(n, k)$  on  $k^n$ .

**Example 3.8.** Let  $\mathfrak{g}$  be a non-perfect Lie algebra such that  $C_{\mathfrak{g}}(\mathfrak{g}') = \{0\}$ . Then there exists an isomorphism of groups

$$\text{Gal}(\mathfrak{g}/\mathfrak{g}') \cong \text{GL}_k(V)$$

where  $V$  is a complement as vector spaces of  $\mathfrak{g}'$  in  $\mathfrak{g}$ . First we write  $\mathfrak{g}$  as the unified product between  $\mathfrak{g}'$  and  $V$  associated to the Lie extending system whose non-trivial maps are given as follows:  $x \rightarrow g = [x, g]$  and  $\theta(x, y) = [x, y]$ , for all  $g \in \mathfrak{g}'$ ,  $x, y \in V$ . Now let  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{g})$  satisfying the compatibility conditions (G1)–(G4). One can easily see that (G1) and (G3) are trivially fulfilled while (G2) and (G4) come down to the following compatibilities:

$$[r(x) + \sigma(x) - x, g] = 0, \quad [r(x) + \sigma(x), r(y) + \sigma(y)] = [x, y]$$

for all  $g \in \mathfrak{g}'$ ,  $x, y \in V$ . We obtain that  $r(x) + \sigma(x) - x \in C_{\mathfrak{g}}(\mathfrak{g}') = \{0\}$ , for all  $x \in V$ . Thus  $r = \text{Id}_V - \sigma$  and hence the second equation is now trivially fulfilled.

Therefore, the pairs  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{g})$  satisfying the compatibility conditions (G1)–(G4) are of the form  $(\sigma, \text{Id}_V - \sigma)$  with  $\sigma \in \text{GL}_k(V)$ . In this case the multiplication given by equation (2) becomes  $(\sigma, \text{Id}_V - \sigma) \cdot (\sigma', \text{Id}_V - \sigma') = (\sigma \circ \sigma', \text{Id}_V - \sigma \circ \sigma')$  and thus  $\mathbb{G}_{\mathfrak{g}'}^V(\dashv, \theta)$  is isomorphic to  $\text{GL}_k(V)$ . The conclusion now follows from Theorem 3.2. ■

Let  $\mathfrak{H}(\mathfrak{g})$  be the *holomorph* [35] Lie algebra of a Lie algebra  $\mathfrak{g}$ , so that we can write  $\mathfrak{H}(\mathfrak{g}) = \mathfrak{g} \times \text{Der}(\mathfrak{g})$ , endowed with the Lie bracket given by the formula:  $[(g, \varphi), (h, \psi)] = \{[g, h] + \varphi(h) - \psi(g), [\varphi, \psi]\}$ , for all  $g, h \in \mathfrak{g}$  and  $\varphi, \psi \in \text{Der}(\mathfrak{g})$ .

**Example 3.9.** Let  $\mathfrak{g}$  be a complete Lie algebra (i.e.  $\mathfrak{g}$  has trivial center and only inner derivations; see [19] for further details). Then there exists an isomorphism of groups:

$$\text{Gal}(\mathfrak{H}(\mathfrak{g})/\mathfrak{g}) \cong \text{Aut}_{\text{Lie}}(\mathfrak{g})$$

To start with we point out that since  $\mathfrak{g}$  is complete all derivations are inner, i.e.  $\text{Der}(\mathfrak{g}) = \{\text{ad}_x \mid x \in \mathfrak{g}\}$ . It can be easily seen that  $\mathfrak{h}(\mathfrak{g})$  is a unified product between  $\mathfrak{g}$  and  $\text{Der}(\mathfrak{g})$  corresponding to the extending system whose non-trivial maps are given as follows:

$$\text{ad}_x \dashv g = [x, g], \quad \{\text{ad}_x, \text{ad}_y\} = \text{ad}_{[x, y]}$$

for all  $g, x, y \in \mathfrak{g}$ . Consider now  $(\sigma, r) \in \text{GL}_k(\text{Der}(\mathfrak{g})) \times \text{Hom}_k(\text{Der}(\mathfrak{g}), \mathfrak{g})$  satisfying the compatibility conditions (G1)–(G4). As for any  $x \in \mathfrak{g}$  we have  $\sigma(\text{ad}_x) \in \text{Der}(\mathfrak{g})$  it follows that  $\sigma(\text{ad}_x) = \text{ad}_{\tau(x)}$  for some bijective linear map  $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$ . One can easily see that (G1) is trivially fulfilled, (G3) comes down to  $\tau$  being a Lie algebra map while (G2) yields:

$$[r(\text{ad}_x) + \tau(x) - x, g] = 0, \text{ for all } x, g \in \mathfrak{g}$$

Hence  $r(\text{ad}_x) + \tau(x) - x \in Z(\mathfrak{g})$  for all  $x \in \mathfrak{g}$ , where  $Z(\mathfrak{g})$  denotes the center of the Lie algebra  $\mathfrak{g}$ ; as  $\mathfrak{g}$  is complete we obtain  $r(\text{ad}_x) = x - \tau(x)$  for all  $x \in \mathfrak{g}$ . Under these assumptions (G4) is also trivially fulfilled. To summarize, we proved that any pair  $(\sigma, r) \in \text{GL}_k(\text{Der}(\mathfrak{g})) \times \text{Hom}_k(\text{Der}(\mathfrak{g}), \mathfrak{g})$  satisfying the compatibility conditions (G1)–(G4) is implemented by a Lie algebra automorphism  $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$  as follows:

$$\sigma(\text{ad}_x) = \text{ad}_{\tau(x)}, \quad r(\text{ad}_x) = x - \tau(x)$$

for all  $x \in \mathfrak{g}$ . An easy computation shows that the map which sends each pair  $(\sigma, r) \in \text{GL}_k(\text{Der}(\mathfrak{g})) \times \text{Hom}_k(\text{Der}(\mathfrak{g}), \mathfrak{g})$  to the corresponding Lie algebra automorphism  $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$  is a group automorphism between  $\mathbb{G}_{\mathfrak{g}}^{\text{Der}(\mathfrak{g})}(\dashv, \theta)$  and  $\text{Aut}_{\text{Lie}}(\mathfrak{g})$ . Now Theorem 3.2 is the last step in obtaining the desired conclusion. ■

We now specialize the discussion to extensions of the form  $\mathfrak{h}^G \subseteq \mathfrak{h}$ , where  $G$  is a group acting on a Lie algebra  $\mathfrak{h}$ . Our approach has Artin’s theorem [24, Theorem 1.8] as source of inspiration: if  $G \leq \text{Aut}(K)$  is a finite group of automorphisms of a field  $K$ , then  $K$  is isomorphic to a crossed product algebra  $k \#_{\sigma} k[G]^*$  between the field of invariants  $k = K^G$  and the dual algebra of the group algebra  $k[G]$ , associated to some cocycle  $\sigma: k[G]^* \otimes k[G]^* \rightarrow k$ . In what follows we will prove the Lie algebra counterpart of this very important result.

Let  $G$  be a finite group whose order  $|G|$  is invertible in  $k$  and suppose  $G$  is acting on the Lie algebra  $\mathfrak{h}$  via the group morphism  $\varphi: G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$ ,  $\varphi(g)(x) = g \triangleright x$ , for all  $g \in G$  and  $x \in \mathfrak{h}$ . Our goal is to describe the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{h}^G)$  and to rebuild  $\mathfrak{h}$  from the subalgebra of invariants  $\mathfrak{h}^G$  and an extra set of data. We mention that if  $G \leq \text{Aut}_{\text{Lie}}(\mathfrak{h})$  is a subgroup of the Lie algebra automorphism of  $\mathfrak{h}$  acting on  $\mathfrak{h}$  via the canonical action  $\sigma \triangleright x := \sigma(x)$ , for all  $\sigma \in G \leq \text{Aut}_{\text{Lie}}(\mathfrak{h})$ , then we have  $G \subseteq \text{Gal}(\mathfrak{h}/\mathfrak{h}^G)$  - as opposed to the classical Artin's theorem, we will see that for Lie algebras we are far from having equality in the inclusion  $G \subseteq \text{Gal}(\mathfrak{h}/\mathfrak{h}^G)$ . Since  $|G|$  is invertible in  $k$ , we can choose the trace map  $t: \mathfrak{h} \rightarrow \mathfrak{h}^G$  defined by  $t(x) := |G|^{-1} \sum_{\gamma \in G} \gamma \triangleright x$ , for all  $x \in \mathfrak{h}$  as a linear retraction of the inclusion  $\mathfrak{h}^G \hookrightarrow \mathfrak{h}$ . We shall compute the canonical extending system of  $\mathfrak{h}^G$  through  $V := \text{Ker}(t)$  associated to the trace map  $t$  using the formulas (13)–(14). For any  $x \in V$  and  $g \in \mathfrak{h}^G$  we have:

$$\begin{aligned} x \rightarrow g &= t([x, g]) = |G|^{-1} \sum_{\gamma \in G} \gamma \triangleright [x, g] = |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright x, \gamma \triangleright g] \\ &= |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright x, g] = [t(x), g] = 0 \end{aligned}$$

where the equalities in the last line follow from  $g \in \mathfrak{h}^G$  and  $x \in V = \text{Ker}(t)$ , respectively. Moreover, we can easily see that the action  $\leftarrow$ , the cocycle  $\theta$  and the quasi-bracket  $\{-, -\}$  on  $V$  take the form:

$$x \leftarrow g = [x, g], \quad \theta(x, y) = |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright x, \gamma \triangleright y] \tag{17}$$

$$\{x, y\} = [x, y] - |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright x, \gamma \triangleright y] \tag{18}$$

for all  $x, y \in V$  and  $g \in \mathfrak{h}^G$ . The left action  $\rightarrow$  being the trivial map has an important consequence: using Example 2.2 it follows that the unified product  $\mathfrak{h}^G \ltimes V$  associated to this canonical extending system of  $\mathfrak{h}^G$  by  $V$  reduces to a skew crossed product  $\mathfrak{h}^G \#^\bullet V$  and the map defined for any  $g \in \mathfrak{h}^G$  and  $x \in V$  by:

$$\varphi: \mathfrak{h}^G \#^\bullet V \rightarrow \mathfrak{h}, \quad \varphi(g, x) := g + x \tag{19}$$

is an isomorphism of Lie algebras. The Lie bracket on  $\mathfrak{h}^G \#^\bullet V$  given by equation (12) takes the following form:

$$\begin{aligned} [(g, x), (g', x')] &:= \left( [g, g'] + |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright x, \gamma \triangleright x'], \right. \\ &\quad \left. [x, x'] - |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright x, \gamma \triangleright x'] + [x, g'] - [x', g] \right) \end{aligned} \tag{20}$$

for all  $g, g' \in \mathfrak{h}^G$  and  $x, x' \in V$ . Given a group  $G$  acting on a Lie algebra  $\mathfrak{h}$ , the isomorphism given in equation (19) provides the reconstruction of  $\mathfrak{h}$  from the subalgebra of invariants  $\mathfrak{h}^G$ . We continue our investigation in order to describe the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{h}^G)$ . Since the components of the canonical extending

system given by equations (17)–(18) are implemented only by the action  $\varphi$  of  $G$  on  $\mathfrak{h}$  we shall denote the group  $\mathbb{G}_{\mathfrak{h}^G}^V (\leftarrow, \rightarrow, \theta, \{-, -\})$  constructed in Theorem 3.2 by  $\mathbb{G}_{\mathfrak{h}^G}^V (\varphi)$ . Thus  $\mathbb{G}_{\mathfrak{h}^G}^V (\varphi)$  consist of the set of all pairs  $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, \mathfrak{h}^G)$  satisfying the following compatibility conditions for all  $g \in \mathfrak{h}^G$  and  $x, y \in V$ :

$$\begin{aligned} \sigma([x, g]) &= [\sigma(x), g], & r([x, g]) &= [r(x), g] \\ \sigma([x, y]) - [\sigma(x), \sigma(y)] &= [\sigma(x), r(y)] + [r(x), \sigma(y)] + \\ &+ |G|^{-1} \sum_{\gamma \in G} \sigma([\gamma \triangleright x, \gamma \triangleright y]) - |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright \sigma(x), \gamma \triangleright \sigma(y)] \\ r([x, y]) - [r(x), r(y)] &= |G|^{-1} \sum_{\gamma \in G} r([\gamma \triangleright x, \gamma \triangleright y]) + \\ &+ |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright \sigma(x), \gamma \triangleright \sigma(y)] - |G|^{-1} \sum_{\gamma \in G} [\gamma \triangleright x, \gamma \triangleright y] \end{aligned}$$

which is exactly what is left from the axioms (G1)–(G4) after using equations (17)–(18) and the fact that  $\rightarrow$  is the trivial action. We note that the first two compatibilities above show that  $\sigma$  and  $r$  are morphisms of right Lie  $\mathfrak{h}^G$ -modules while the last two compatibilities measures how far they are from being Lie algebra maps.  $\mathbb{G}_{\mathfrak{h}^G}^V (\varphi)$  is a group with multiplication given by equation (2). We record all this facts in the following:

**Theorem 3.10.** (Artin’s Theorem for Lie algebras) *Let  $G$  be a finite group of invertible order in  $k$  acting on a Lie algebra  $\mathfrak{h}$  via  $\varphi: G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$ . Let  $\mathfrak{h}^G \subseteq \mathfrak{h}$  be the subalgebra of invariants and  $V = \text{Ker}(t)$ , where  $t: \mathfrak{h} \rightarrow \mathfrak{h}^G$  is the trace map. Then:*

(1) *The map defined for any  $g \in \mathfrak{h}^G$  and  $x \in V$  by:*

$$\varphi: \mathfrak{h}^G \#^\bullet V \rightarrow \mathfrak{h}, \quad \varphi(g, x) := g + x \tag{21}$$

*is an isomorphism of Lie algebras, where  $\mathfrak{h}^G \#^\bullet V$  is the skew crossed product of Lie algebras having the bracket given by (20).*

(2) *The map defined for any  $(\sigma, r) \in \mathbb{G}_{\mathfrak{h}^G}^V (\varphi)$ ,  $g \in \mathfrak{h}^G$  and  $x \in V$  by:*

$$\Omega: \mathbb{G}_{\mathfrak{h}^G}^V (\varphi) \rightarrow \text{Gal}(\mathfrak{h}/\mathfrak{h}^G), \quad \Omega(\sigma, r)(g + x) := g + r(x) + \sigma(x) \tag{22}$$

*is an isomorphism of groups.*

Even if  $G$  is a finite subgroup of  $\text{Aut}_{\text{Lie}}(\mathfrak{h})$ , Theorem 3.10 (2) shows that we are far away from having  $\text{Gal}(\mathfrak{h}/\mathfrak{h}^G) \cong G$  as in the case of fields. We present a relevant example below:

**Example 3.11.** Let  $k$  be a field of characteristic  $\neq 2$  and  $\varphi: k^* \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{sl}(2, k))$  the action of  $k^*$  on  $\mathfrak{sl}(2, k)$  given by equation (10). The subalgebra of invariants  $\mathfrak{sl}(2, k)^{k^*}$  of this action is just  $ke_3$  and Example 3.4 shows that the Galois group

$\text{Gal}(\mathfrak{sl}(2, k)/\mathfrak{sl}(2, k)^{k^*})$  is isomorphic to  $k^*$ . This situation occurs rarely. Indeed, if we consider  $G := U_2(k) = \{\pm 1\}$  the cyclic subgroup of  $k^*$  of roots of unity of order two and the same action of  $U_2(k) \leq k^*$  on  $\mathfrak{sl}(2, k)$  as above we obtain the same subalgebra of invariants, namely  $\mathfrak{sl}(2, k)^{U_2(k)} = ke_3$ . Hence we have that  $\text{Gal}(\mathfrak{sl}(2, k)/\mathfrak{sl}(2, k)^{U_2(k)}) \cong k^* \neq U_2(k)$ . ■

A crucial step in applying Theorem 3.10 is the description of the kernel of the trace map  $t: \mathfrak{h} \rightarrow \mathfrak{h}^G$ , which heavily depends of the group  $G$  and on the action  $\varphi$ . In the case of cyclic groups acting on fields this kernel is described by Hilbert’s theorem [24, Theorem 6.3]. As a nice surprise its counterpart for Lie algebras is also true but the proof uses completely different techniques.

**Theorem 3.12.** (Hilbert’s Theorem 90 for Lie algebras) *Let  $G$  be a finite cyclic group generated by an element  $\gamma$  whose order  $n$  is invertible in  $k$ . Let  $\varphi: G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$  be a morphism of groups and  $t: \mathfrak{h} \rightarrow \mathfrak{h}^G$  the trace map. Then  $\text{Ker}(t) = \{y - \gamma \triangleright y \mid y \in \mathfrak{h}\}$ .*

**Proof.** It is straightforward to see that  $t(y - \gamma \triangleright y) = 0$ , for all  $y \in \mathfrak{h}$ . Conversely, let  $a \in \text{Ker}(t)$ . We define recursively the sequence of elements  $(d_i)_{i \geq 0}$  of  $\text{Hom}_k(\mathfrak{h}, \mathfrak{h})$  by the formulas:

$$d_0(y) := a + y, \quad d_{i+1}(y) := a + \gamma \triangleright d_i(y),$$

for all  $i \geq 0$  and  $y \in \mathfrak{h}$ . Thus, we have

$$d_1(y) = a + \gamma \triangleright a + \gamma \triangleright y, \dots, d_{n-2}(y) = a + \gamma \triangleright a + \dots + \gamma^{n-2} \triangleright a + \gamma^{n-2} \triangleright y$$

and using  $t(a) = 0$  we obtain  $d_{n-1}(y) = \gamma^{n-1} \triangleright y$  and hence

$$d_n(y) = a + \gamma \triangleright (\gamma^{n-1} \triangleright y) = a + y = d_0(y).$$

Therefore  $d_n = d_0$  i.e., the sequence  $(d_i)_{i \geq 0}$  is periodic. Now, in the abelian group  $\text{Hom}_k(\mathfrak{h}, \mathfrak{h})$  we add all the equalities listed below:

$$d_1 = a + \gamma \triangleright d_0, \quad d_2 = a + \gamma \triangleright d_1, \quad \dots, \quad d_{n-1} = a + \gamma \triangleright d_{n-2}, \quad d_n = a + \gamma \triangleright d_{n-1}$$

and using  $d_n = d_0$ , we obtain

$$\sum_{i=0}^{n-1} d_i = n a + \gamma \triangleright \left( \sum_{i=0}^{n-1} d_i \right).$$

If  $d_0 + d_1 + \dots + d_{n-1} = 0$  in the abelian group  $\text{Hom}_k(\mathfrak{h}, \mathfrak{h})$ , we obtain using the invertibility of  $n$  in  $k$ , that  $a = 0 = 0 - \gamma \triangleright 0$  and we are done. On the other hand, if  $d_0 + d_1 + \dots + d_{n-1} \neq 0$  we can pick some  $z \in \mathfrak{h}$  such that  $y := \sum_{i=0}^{n-1} d_i(z) \neq 0$ . Then, using  $d_n(z) = d_0(z)$ , we have:

$$n a + \gamma \triangleright y = n a + \sum_{i=0}^{n-1} \gamma \triangleright d_i(z) = \sum_{i=0}^{n-1} (a + \gamma \triangleright d_i(z)) = \sum_{i=0}^{n-1} d_{i+1}(z) = y$$

This shows that  $a = n^{-1} (y - \gamma \triangleright y)$  and the proof is finished. ■

**Definition 3.13.** Let  $\mathfrak{h}$  be a Lie algebra,  $G$  a group,  $\varphi: G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$  a morphism of groups,  $\gamma \in G$  and  $\mathfrak{h}_\gamma := \{y - \gamma \triangleright y \mid y \in \mathfrak{h}\}$ . The action  $\varphi$  is called  $\gamma$ -abelian if, for all  $g \neq g' \in G$  and  $z, z' \in \mathfrak{h}_\gamma$ ,

$$[g \triangleright z, g' \triangleright z'] = 0. \tag{23}$$

The structure theorem for cyclic Galois extensions of fields [24, Theorem 6.2] can be rephrased as follows: if  $G \leq \text{Aut}(K)$  is a cyclic subgroup of order  $n$  of the group of automorphisms of a field  $K$  of characteristic zero and  $k := K^G$ , then  $K$  is isomorphic to the splitting field over  $k$  of a polynomial of the form  $X^n - a \in k[X]$ . The Lie algebra counterpart of this result now follows by replacing the splitting field with the semidirect product of Lie algebras:

**Corollary 3.14.** Let  $\mathfrak{h}$  be a Lie algebra,  $G$  a finite cyclic group generated by an element  $\gamma$  whose order  $n$  is invertible in  $k$  and  $\varphi: G \rightarrow \text{Aut}_{\text{Lie}}(\mathfrak{h})$  a  $\gamma$ -abelian morphism of groups. Then, the map defined for any  $g \in \mathfrak{h}^G$  and  $x \in \mathfrak{h}_\gamma$  by:

$$\varphi: \mathfrak{h}^G \rtimes \mathfrak{h}_\gamma \rightarrow \mathfrak{h}, \quad \varphi(g, x) := g + x \tag{24}$$

is an isomorphism of Lie algebras, where  $\mathfrak{h}^G \rtimes \mathfrak{h}_\gamma$  is the semidirect product of Lie algebras associated to the right action  $\leftarrow: \mathfrak{h}_\gamma \times \mathfrak{h}^G \rightarrow \mathfrak{h}_\gamma$ , given by  $x \leftarrow g := [x, g]$ .

**Proof.** Using Theorem 3.12 together with Theorem 3.10 we only need to prove that the cocycle  $\theta: \mathfrak{h}_\gamma \times \mathfrak{h}_\gamma \rightarrow \mathfrak{h}^G$  given by (17) is the trivial map. Moreover, in this case it also follows that the bracket  $\{-, -\}$  on  $\mathfrak{h}_\gamma$  depicted in (18) coincides with the Lie bracket on  $\mathfrak{h}$ , i.e.  $\{x, y\} = [x, y]$ , for all  $x, y \in \mathfrak{h}_\gamma$  and  $\mathfrak{h}_\gamma$  is an ideal of  $\mathfrak{h}$ . Indeed, let  $y - \gamma \triangleright y$  and  $y' - \gamma \triangleright y'$  be two elements of  $\mathfrak{h}_\gamma$ , for some  $y, y' \in \mathfrak{h}$ . Then we have:

$$\begin{aligned} \theta(y - \gamma \triangleright y, y' - \gamma \triangleright y') &= n^{-1} \sum_{\delta \in G} [\delta \triangleright (y - \gamma \triangleright y), \delta \triangleright (y' - \gamma \triangleright y')] \\ &= n^{-1} \sum_{i=0}^{n-1} [\gamma^i \triangleright (y - \gamma \triangleright y), \gamma^i \triangleright (y' - \gamma \triangleright y')] \\ &= n^{-1} \left[ \sum_{i=0}^{n-1} \gamma^i \triangleright (y - \gamma \triangleright y), \sum_{i=0}^{n-1} \gamma^i \triangleright (y - \gamma \triangleright y) \right] = 0 \end{aligned}$$

where in the third equality we used the fact that  $\varphi$  is a  $\gamma$ -abelian action while the final equality holds due to the following trivial identity:  $\sum_{i=0}^{n-1} \gamma^i \triangleright (y - \gamma \triangleright y) = 0$ . ■

**Remark 3.15.** Under the assumptions of Corollary 3.14 we can provide an simpler description of the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{h}^G) \cong \mathbb{G}_{\mathfrak{h}^G}^{\mathfrak{h}_\gamma}(\varphi)$ . Indeed, in this case the group  $\mathbb{G}_{\mathfrak{h}^G}^{\mathfrak{h}_\gamma}(\varphi)$  as defined in Theorem 3.10 consists of the set of all pairs  $(\sigma, r) \in \text{GL}_k(\mathfrak{h}_\gamma) \times \text{Hom}_k(\mathfrak{h}_\gamma, \mathfrak{h}^G)$  satisfying the following compatibility conditions for any  $g \in \mathfrak{h}^G$  and  $x, y \in \mathfrak{h}_\gamma$ :

$$\begin{aligned} \sigma([x, g]) &= [\sigma(x), g], & r([x, g]) &= [r(x), g], & r([x, y]) &= [r(x), r(y)] \\ \sigma([x, y]) - [\sigma(x), \sigma(y)] &= [\sigma(x), r(y)] + [r(x), \sigma(y)] \end{aligned}$$

which is a subgroup in the semidirect product of groups  $\text{GL}_k(\mathfrak{h}_\gamma) \rtimes \text{Hom}_k(\mathfrak{h}_\gamma, \mathfrak{h}^G)$ .

### 4. Applications and examples

In this section we present some applications as well as explicit examples of Galois groups. The simplest case is that of extensions  $\mathfrak{g} \subseteq \mathfrak{h}$  for which the codimension of  $\mathfrak{g}$  in  $\mathfrak{h}$  is equal to 1. In this case we will show that the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is metabelian. To this end, consider  $\mathfrak{g} \subseteq \mathfrak{h}$  to be an extension of Lie algebras such that  $\mathfrak{g}$  has codimension 1 in  $\mathfrak{h}$ . Thus, we can write  $\mathfrak{h} = \mathfrak{g} + V$ , where  $V := kx$ , for a fixed element  $x \in \mathfrak{h} \setminus \mathfrak{g}$ . We choose the map  $p$  defined by  $p(x) := 0$  and  $p(g) = g$ , for all  $g \in \mathfrak{g}$ , as a retraction of the inclusion map  $\mathfrak{g} \hookrightarrow \mathfrak{h}$ . Now, the space of all Lie extending systems of  $\mathfrak{g}$  through  $V = kx$  is parameterized by the set  $\text{TwDer}(\mathfrak{g})$  of all twisted derivations of  $\mathfrak{g}$  [1, Proposition 4.4]. Recall that a twisted derivation of a Lie algebra  $\mathfrak{g}$  is a pair  $(\lambda, \Delta)$  consisting of two linear maps  $\lambda: \mathfrak{g} \rightarrow k$  and  $\Delta: \mathfrak{g} \rightarrow \mathfrak{g}$  such that for any  $g, h \in \mathfrak{g}$ :

$$\lambda([g, h]) = 0, \quad \Delta([g, h]) = [\Delta(g), h] + [g, \Delta(h)] + \lambda(g)\Delta(h) - \lambda(h)\Delta(g) \quad (25)$$

The bijection between the set of all Lie extending structures of  $\mathfrak{g}$  through  $V = kx$  and  $\text{TwDer}(\mathfrak{g})$  is given by the two-sided formula:

$$x \leftarrow g := \lambda(g)x, \quad x \rightarrow g := \Delta(g), \quad \theta := 0, \quad \{-, -\} := 0 \quad (26)$$

for all  $g \in \mathfrak{g}$ . Let  $(\lambda, \Delta) \in \text{TwDer}(\mathfrak{g})$  be the twisted derivation associated to the canonical Lie extending system of  $\mathfrak{g}$  through  $V$  arising from  $p$  via (26) and denote by  $\mathfrak{g}_{(\lambda, \Delta)} := \mathfrak{g} \natural kx$  the corresponding unified product. For future use, we mention that the Lie algebra  $\mathfrak{g}_{(\lambda, \Delta)}$  can be defined (see [2]) as the vector space  $\mathfrak{g} \times k$  with the bracket given as follows for all  $x, y \in \mathfrak{g}$  and  $a, b \in k$ :

$$\{(x, a), (y, b)\} := ([x, y] + b\Delta(x) - a\Delta(y), b\lambda(x) - a\lambda(y)). \quad (27)$$

Of course  $\mathfrak{g}_{(\lambda, \Delta)}$  contains  $\mathfrak{g} \cong \mathfrak{g} \times \{0\}$  as a subalgebra of codimension 1. We observe that  $(\lambda := 0, \Delta) \in \text{TwDer}(\mathfrak{g})$  if and only if  $\Delta \in \text{Der}(\mathfrak{g})$  is a classical derivation of  $\mathfrak{g}$ ; in this case we shall denote  $\mathfrak{g}_{(\Delta)} := \mathfrak{g}_{(0, \Delta)}$ , for any  $\Delta \in \text{Der}(\mathfrak{g})$ . As an example, we mention that if  $\mathfrak{g}$  is a perfect Lie algebra then the first compatibility condition in (25) yields  $\text{TwDer}(\mathfrak{g}) = \{0\} \times \text{Der}(\mathfrak{g})$ .

Continuing our investigation we get that  $\text{Gal}(\mathfrak{h}/\mathfrak{g}) \cong \text{Gal}(\mathfrak{g}_{(\lambda, \Delta)}/\mathfrak{g})$ , which is a special case of the isomorphism given by (15). We denote by  $\mathbb{G}_{\mathfrak{g}}(\lambda, \Delta)$  the set of all pairs  $(u, g_0) \in k^* \times \mathfrak{g}$  satisfying the following compatibility condition for all  $g \in \mathfrak{g}$ :

$$\lambda(g)g_0 = [g_0, g] + (u - 1)\Delta(g). \quad (28)$$

Then  $\mathbb{G}_{\mathfrak{g}}(\lambda, \Delta)$  is a subgroup in the metabelian group  $\mathbb{G}_{\mathfrak{g}} := k^* \rtimes \mathfrak{g}$  whose multiplication is given by equation (3), that is  $(u, g) \cdot (u', g') := (uu', u'g + g')$ , for all  $u, u' \in k^*$  and  $g, g' \in \mathfrak{g}$ . We can now prove the following:

**Corollary 4.1.** *Let  $\mathfrak{g} \subseteq \mathfrak{h}$  be a Lie subalgebra of codimension 1 in  $\mathfrak{h}$  and  $(\lambda, \Delta) \in \text{TwDer}(\mathfrak{g})$  the twisted derivation defined by equation (26) for a fixed  $x \in \mathfrak{h} \setminus \mathfrak{g}$ . Then there exists an isomorphism of groups given for any  $(u, g_0) \in \mathbb{G}_{\mathfrak{g}}(\lambda, \Delta)$ ,  $g \in \mathfrak{g}$  and  $\alpha \in k$  by:*

$$\Omega: \mathbb{G}_{\mathfrak{g}}(\lambda, \Delta) \rightarrow \text{Gal}(\mathfrak{h}/\mathfrak{g}), \quad \Omega(u, g_0)(g + \alpha x) := g + \alpha g_0 + u\alpha x. \quad (29)$$

*In particular, the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is metabelian and hence solvable.*

**Proof.** We apply Theorem 3.2. Since  $V = kx$ , any linear automorphism  $\sigma: V \rightarrow V$  is uniquely determined by an invertible element  $u \in k^*$  via  $\sigma(x) := ux$  while a linear map  $r: V \rightarrow \mathfrak{g}$  is implemented by an element  $g_0 \in \mathfrak{g}$  via  $r(x) := g_0$ . Now, the axioms (G1), (G3) and (G4) which define the group  $\mathbb{G}_{\mathfrak{g}}^V(\leftarrow, \rightarrow, \theta, \{-, -\})$  from Theorem 3.2 are trivially fulfilled, while axiom (G2) comes down to the compatibility condition (28). Finally, the group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is metabelian due to its embedding in the metabelian group  $k^* \rtimes \mathfrak{g}$ . ■

**Example 4.2.** Let  $n \in \mathbb{N}^*$  be a positive integer and consider the extension of Lie algebras  $\mathfrak{h}^{2n+1} \subseteq \mathfrak{t}^{2n+2}$ , where  $\mathfrak{h}^{2n+1}$  is the  $(2n + 1)$ -dimensional Heisenberg Lie algebra from Example 3.5 and  $\mathfrak{t}^{2n+2}$  is the Lie algebra with basis  $\{x_1, \dots, x_n, y_1, \dots, y_n, w, u\}$  and bracket given for all  $i = 1, \dots, n$  by:  $[x_i, y_i] = w$ ,  $[u, x_i] = w + u$ ,  $[u, y_i] = w + u$ . Then there exists an isomorphism of groups:

$$\text{Gal}(\mathfrak{t}^{2n+2}/\mathfrak{h}^{2n+1}) \cong (k^*, \cdot).$$

First observe that the Lie algebra  $\mathfrak{t}^{2n+2}$  is isomorphic to  $\mathfrak{h}_{(\lambda, \Delta)}^{2n+1}$ , where the twisted derivation  $(\lambda, \Delta)$  of the Heisenberg Lie algebra  $\mathfrak{h}^{2n+1}$  is given by:  $\lambda(w) := 0$ ,  $\lambda(x_i) = \lambda(y_i) := 1$ ,  $\Delta(w) := 0$ ,  $\Delta(x_i) = \Delta(y_i) := w$ , for all  $i = 1, 2, \dots, n$ . Now a straightforward computation shows that  $\mathbb{G}_{\mathfrak{h}^{2n+1}}(\lambda, \Delta) = \{(\alpha, (\alpha - 1)w) \mid \alpha \in k^*\}$  and the map  $\varphi: (\mathbb{G}_{\mathfrak{h}^{2n+1}}(\lambda, \Delta), \cdot) \rightarrow k^*$  given by  $\varphi(\alpha, (\alpha - 1)w) = \alpha$  is a group isomorphism where  $\cdot$  is the multiplication defined by (3). The conclusion follows by Corollary 4.1. ■

We recall that an extension  $\mathfrak{g} \subseteq \mathfrak{h}$  of Lie algebras is called a flag extension [1, Definition 4.1] if there exists a finite chain of Lie subalgebras of  $\mathfrak{h}$

$$\mathfrak{g} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \dots \subset \mathfrak{h}_m = \mathfrak{h} \tag{30}$$

such that  $\mathfrak{h}_i$  has codimension 1 in  $\mathfrak{h}_{i+1}$ , for all  $i = 0, \dots, m - 1$ . Supersolvable Lie algebras provide examples of flag extensions. Based on this concept, we propose the following definition as the counterpart for Lie algebras of normal radical extensions of fields:

**Definition 4.3.** An extension  $\mathfrak{g} \subseteq \mathfrak{h}$  of Lie algebras is called a *radical extension* if there exists a chain of subalgebras as in equation (30) such that each  $\mathfrak{h}_{i-1}$  is invariant with respect to any element  $\tau \in \text{Gal}(\mathfrak{h}_i/\mathfrak{g})$ , i.e.  $\tau(\mathfrak{h}_{i-1}) \subseteq \mathfrak{h}_{i-1}$ , for all  $\tau \in \text{Gal}(\mathfrak{h}_i/\mathfrak{g})$  and  $i = 1, \dots, m$ .

If  $\mathfrak{g}$  has codimension 1 in  $\mathfrak{h}$ , then  $\mathfrak{h}/\mathfrak{g}$  is a radical extension. Based on Theorem 3.2 and Corollary 4.1, exactly as in the classical case of radical extensions of fields, we can prove the following:

**Theorem 4.4.** *Let  $\mathfrak{g} \subseteq \mathfrak{h}$  be a radical extension of finite dimensional Lie algebras. Then the Galois group  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is solvable.*

**Proof.** Consider a finite chain of subalgebras as in (30). We will proceed by induction on  $m$ . If  $m = 1$  the conclusion follows by Corollary 4.1. Now let  $m > 1$

and assume the statement to be true for  $m - 1$ , that is the group  $\text{Gal}(\mathfrak{h}_{m-1}/\mathfrak{g})$  is solvable. Then, the map:

$$\Gamma: \text{Gal}(\mathfrak{h}/\mathfrak{g}) \rightarrow \text{Gal}(\mathfrak{h}_{m-1}/\mathfrak{g}), \quad \Gamma(\tau) := \tau|_{\mathfrak{h}_{m-1}},$$

where  $\tau|_{\mathfrak{h}_{m-1}}$  is the restriction of  $\tau$  to  $\mathfrak{h}_{m-1}$ , is well defined since the extension is radical, the Lie algebras are finite dimensional and  $\Gamma$  is a morphism of groups. Now  $\text{Ker}(\Gamma) = \text{Gal}(\mathfrak{h}/\mathfrak{h}_{m-1})$  which is a metabelian (in particular solvable) group again by Corollary 4.1. Thus, we obtain an isomorphism of groups  $\text{Gal}(\mathfrak{h}/\mathfrak{g})/\text{Gal}(\mathfrak{h}/\mathfrak{h}_{m-1}) \cong \text{Im}(\Gamma)$ , and  $\text{Im}(\Gamma)$  is a solvable group as a subgroup in such a group. To conclude, we have obtained that  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is an extension of a solvable group  $\text{Im}(\Gamma)$  by the solvable group  $\text{Gal}(\mathfrak{h}/\mathfrak{h}_{m-1})$ , hence  $\text{Gal}(\mathfrak{h}/\mathfrak{g})$  is solvable too. ■

The compatibility condition (28) which describes the elements of the group  $\text{Gal}(\mathfrak{g}_{(\lambda, \Delta)}/\mathfrak{g})$  is crucial and deserves a thorough analysis. First, observe that  $(1, 0) \in \mathbb{G}_{\mathfrak{g}}(\lambda, \Delta)$ . On the other hand, if  $(u, g_0) \in \mathbb{G}_{\mathfrak{g}}(\lambda, \Delta)$ , for some  $u \neq 1$ , then (28) implies that  $\Delta$  is given by the formula  $\Delta(g) = (u-1)^{-1}(\lambda(g)g_0 - [g_0, g])$ , for all  $g \in \mathfrak{g}$ . A straightforward computation shows that the second compatibility condition of (25) is trivially fulfilled, being equivalent to the Jacobi identity. The center of a Lie algebra  $\mathfrak{g}$  will be denoted by  $Z(\mathfrak{g}) := \{g \in \mathfrak{g} \mid [g, -] = 0\}$ . Then  $Z(\mathfrak{g})$  is an abelian subgroup of  $(\mathfrak{g}, +)$  and it can be realized as a Galois group of the following type of Lie algebra extensions:

**Corollary 4.5.** *Let  $\mathfrak{g}$  be a Lie algebra and  $\Delta \in \text{Der}(\mathfrak{g})$  a derivation that is not inner. Then there exists an isomorphism of groups  $\text{Gal}(\mathfrak{g}_{(\Delta)}/\mathfrak{g}) \cong Z(\mathfrak{g})$ .*

**Proof.** Using (28) for  $\lambda := 0$ , we obtain that  $(u, g_0) \in \mathbb{G}_{\mathfrak{g}}(\Delta) := \mathbb{G}_{\mathfrak{g}}(0, \Delta)$  if and only if  $(u - 1)\Delta(g) = [g, g_0]$ , for all  $g \in \mathfrak{g}$ . Hence,  $(1, g_0) \in \mathbb{G}_{\mathfrak{g}}(\Delta)$  if and only if  $g_0 \in Z(\mathfrak{g})$ . On the other hand, since  $\Delta$  is not inner, it follows that  $\mathbb{G}_{\mathfrak{g}}(\Delta)$  does not contain elements of the form  $(u, g_0)$ , with  $u \neq 1$ . Now, we apply Corollary 4.1. ■

**Example 4.6.** Let  $n \in \mathbb{N}^*$  be a positive integer and consider  $\mathfrak{h}^{2n+1}$  to be the  $(2n + 1)$ -dimensional Heisenberg Lie algebra from Example 3.5. Then  $\Delta: \mathfrak{h}^{2n+1} \rightarrow \mathfrak{h}^{2n+1}$  given by  $\Delta(x_i) := y_i, \Delta(y_i) = \Delta(w) := 0$ , for all  $i = 1, 2, \dots, n$  is a derivation of  $\mathfrak{h}^{2n+1}$  that is not inner. Furthermore, we denote by  $\mathfrak{b}^{2n+2}$  the Lie algebra  $\mathfrak{h}_{(\Delta)}^{2n+1}$ : it has the  $k$ -basis  $\{x_1, \dots, x_n, y_1, \dots, y_n, w, z\}$  and bracket given for any  $i = 1, \dots, n$  by  $[x_i, y_i] = w, [z, x_i] = y_i$ . By applying Corollary 4.5 and taking into account that  $Z(\mathfrak{h}^{2n+1}) = kw \cong (k, +)$  we obtain that there exists an isomorphism of groups  $\text{Gal}(\mathfrak{b}^{2n+2}/\mathfrak{h}^{2n+1}) \cong (k, +)$ . ■

A Lie algebra  $\mathfrak{g}$  is called *sympathetic* if  $\mathfrak{g}$  is perfect, has trivial center and any derivation is inner. Semisimple Lie algebras over a field of characteristic zero are sympathetic and there is a sympathetic non-semisimple Lie algebra in dimension 25 (see [35]).

**Corollary 4.7.** *Let  $\mathfrak{g}$  be a symplectic Lie subalgebra of codimension 1 in a Lie algebra  $\mathfrak{h}$ . Then there exists an isomorphism of groups  $\text{Gal}(\mathfrak{h}/\mathfrak{g}) \cong k^*$ . In particular, if  $k$  is a field of characteristic zero then  $\text{Gal}(\mathfrak{gl}(m, k)/\mathfrak{sl}(m, k)) \cong k^*$ .*

**Proof.** Indeed, since  $\mathfrak{g}$  is perfect we obtain that  $\mathfrak{h} \cong \mathfrak{g}_{(\Delta)}$ , for some derivation  $\Delta$  of  $\mathfrak{g}$  [2, Proposition 2.1]. Let  $\delta \in \mathfrak{g}$  such that  $\Delta = [\delta, -]$ . By applying the compatibility condition (28) for  $\lambda := 0$  and  $\Delta := [\delta, -]$  we obtain that  $(u, g_0) \in \mathbb{G}_{\mathfrak{g}}(\Delta)$  if and only if  $[g, g_0 + (u - 1)\delta] = 0$ , for all  $g \in \mathfrak{g}$ . Since  $Z(\mathfrak{g}) = \{0\}$ , this is equivalent to the fact that  $g_0 = (1 - u)\delta$ . Hence  $\mathbb{G}_{\mathfrak{g}}(\Delta)$  consists of all elements of the form  $(u, (1 - u)\delta)$ , for any  $u \in k^*$  and there exists an isomorphism of groups  $\mathbb{G}_{\mathfrak{g}}(\Delta) \cong k^*$ . Now we apply Corollary 4.1. ■

All examples of Lie algebra extensions  $\mathfrak{h}/\mathfrak{g}$  presented so far have non-trivial Galois group. We end the paper with a Lie algebra extension whose Galois group is trivial:

**Example 4.8.** Let  $k$  be a field of characteristic  $\neq 2$  and consider  $\mathfrak{g}$  to be the perfect 5-dimensional Lie algebra with the basis  $\{e_1, e_2, e_3, e_4, e_5\}$  and bracket given by:

$$\begin{aligned} [e_1, e_2] &= e_3, & [e_1, e_3] &= -2e_1, & [e_1, e_5] &= [e_3, e_4] = e_4 \\ [e_2, e_3] &= 2e_2, & [e_2, e_4] &= e_5, & [e_3, e_5] &= -e_5 \end{aligned}$$

It was proven in [2, Example 3.7] that the derivation given in matrix form by  $\Delta := e_{11} - e_{41} - e_{22} + e_{53} - e_{44} - 2e_{55}$  is not inner, where  $e_{ij} \in \mathcal{M}_5(k)$  is the matrix having 1 in the  $(i, j)^{\text{th}}$  position and zeros elsewhere. On the other hand, a straightforward computation shows that  $Z(\mathfrak{g}) = \{0\}$ . By applying Corollary 4.5 it follows that the extension  $\mathfrak{g} \subseteq \mathfrak{g}_{(\Delta)}$  has trivial Galois group  $\{\text{Id}_{\mathfrak{g}_{(\Delta)}}\}$ . ■

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