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The Left-Regular Representation of a Super Lie Group

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Abstract. With the usual definition of a super Hilbert space and a super unitary representation, it is easy to show that there are lots of super Lie groups for which the left-regular representation is not super unitary. I will show that weakening the definition of a super Hilbert space (by allowing the super scalar product to be non-homogeneous, not just even) will allow the left-regular representation of all (connected) super Lie groups to be super unitary (with an adapted definition). Along the way I will introduce a (super) metric on a supermanifold that will allow me to define super and non-super scalar products on function spaces.

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1. Introduction

The generally accepted definition of a super unitary representation of a super Lie group G is the one that can be found (among others) in [3, Def. 2, §2.3] and [1]. It starts with the interpretation of a super Lie group G as a super Harish-Chandra pair $(G_o, \mathfrak{g})$, in which $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a super Lie algebra (over $\mathbf{R}$) and G_o an ordinary Lie group acting on $\mathfrak{g}$ such that:

(A1) the Lie algebra of G_o is (isomorphic to) $\mathfrak{g}_0$;

(A2) the action of G_o preserves each $\mathfrak{g}_\alpha$ (the action is "even");

(A3) the restriction of the G_o action to $\mathfrak{g}_0$ is (isomorphic to) the adjoint action of G_o on it Lie algebra.

The next item to be defined is a super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$, which is usually taken to be a graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with scalar product $\langle\cdot,\cdot\rangle$ and super scalar product $\mathscr{S}$ (a graded symmetric non-degenerate sesquilinear form) satisfying the following conditions:

(B1) $\langle \mathcal{H}_0, \mathcal{H}_1 \rangle = 0$;

(B2) for all homogeneous $x, y \in \mathcal{H}$ we have $\mathscr{S}(x,y) = i^{|x|} \cdot \langle x, y\rangle$.

*This paper is, for reasons of length, an abridged version of my paper [17]; the sections on Fourier-Berezin decompositions in [17] have been omitted here.

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With these ingredients a super unitary representation of $(G_o, \mathfrak{g})$ on the super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ then is a couple (ρ_o, τ) in which ρ_o is an ordinary unitary representation of G_o on the Hilbert space $\mathcal{H}$ and $\tau\colon \mathfrak{g} \to \mathrm{End}\big(C^\infty(\rho_o)\big)$ an even super Lie algebra representation of $\mathfrak{g}$ on $C^\infty(\rho_o)$, the space of smooth vectors for ρ_o defined by

$$C^\infty(\rho_o) = \{\, \psi \in \mathcal{H} \mid g \mapsto \rho(g)\psi \text{ is a smooth map } G \to \mathcal{H} \,\}\,,$$

satisfying the conditions:

(C1) for each $g \in G_o$ the map $\rho_o(g)$ preserves each $\mathcal{H}_\alpha$ (the representation is "even");

(C2) for each $X \in \mathfrak{g}_0$ (the Lie algebra of G_o!) the map $\tau(X)$ is the restriction of the infinitesimal generator of $\rho_o\big(\exp(tX)\big)$ to $C^\infty(\rho_o)$;

(C3) for each $X \in \mathfrak{g}_\alpha$ the map $\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}$;

(C4) for all $g \in G_o$ and all $X \in \mathfrak{g}_1$ we have $\tau(g\cdot X) = \rho_o(g) \circ \tau(X) \circ \rho_o(g^{-1})$, where on the left we denote by $g \cdot X$ the action of G_o on $\mathfrak{g}$.

Actually, condition (B2) tells us that the super scalar product $\mathscr{S}$ can be deduced from the scalar product $\langle\cdot,\cdot\rangle$. Moreover, using (B2) one can show that condition (C3) is equivalent to the condition that a suitable multiple of $\tau(X)$ is essentially self-adjoint on $C^\infty(\rho_o)$. In other words, one can dispense completely with the notion of the super scalar product $\mathscr{S}$.

In the context of non-super Lie groups, any Lie group G admits a canonically defined unitary representation: the left-regular representation on $L^2(G)$, the space of square-integrable functions on G (with respect to an invariant (Haar) measure). Unfortunately, this no longer is true for super Lie groups when we adopt the above definition of a super unitary representation. One can easily show that there are lots of super Lie groups for which there is no way to turn the natural left-regular representation into a super unitary one according to the above definition. The easiest example is the additive (abelian) group of graded dimension $0|1$ usually denoted as $\mathbf{R}^{0|1}$.

Of course one could accept this fact as a particularity of super Lie groups, but I want to argue that we have another option: changing the definition of a super Hilbert space by allowing the super scalar product to be non-homogeneous (not only even). I will show that with this weaker definition (and an adapted version of a super unitary representation) one can turn the left-regular representation of any super Lie group into a super unitary one. More precisely, I propose to define a super scalar product to be any (not necessarily even or homogeneous) non-degenerate sesquilinear map $\mathscr{S}\colon \mathcal{H} \times \mathcal{H} \to \mathbf{C}$, and then to define a super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ by the condition

> the map $\mathscr{S}\colon \mathcal{H} \times \mathcal{H} \to \mathbf{C}$ is continuous with respect to the topology induced by the scalar product $\langle\cdot,\cdot\rangle$.

I also change the viewpoint of a super unitary representation by defining a super unitary representation of a super Lie group G (seen as a super manifold with a group structure such that multiplication and inverse are smooth operations, see also

Definition 2.1) on a super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ to be a couple $(\mathcal{D}, \rho)$ with the following properties. $\mathcal{D} \subset \mathcal{H}$ is a dense graded subspace and $\rho \colon G \to \mathrm{Aut}(\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}})$ (notation to be explained in the main text) a group homomorphism satisfying the following conditions.

(D1) For all $g \in G$ the map $\rho(g)$ preserves $\mathscr{S}$.

(D2) For all $\psi \in \mathcal{D}$ the map $\mathbf{F}_\psi \colon G \to \mathcal{D}$, $\mathbf{F}_\psi(g) = \rho(g)\psi$ is smooth.

(D3) For all $g \in \mathbf{B}G$ the map $\rho(g)$ preserves $\langle\cdot,\cdot\rangle$.

(D4) The couple $(\mathcal{D}, \rho)$ is maximal with respect to the previous conditions.

Before showing that this definition of a super unitary representation is equivalent to a lookalike of the standard definition given by the conditions (C1)–(C4) above, let me first make some comments. The space $\mathbf{B}G$ is the underlying non-super Lie group and condition (D3) says that it is unitary. It thus extends naturally to a unitary representation ρ_o of $\mathbf{B}G$ on $\mathcal{H}$. Looking at condition (D2) then says in particular that all elements $\psi \in \mathcal{D}$ belong to $C^\infty(\rho_o)$, i.e., $\mathcal{D} \subset C^\infty(\rho_o)$. Moreover, this condition says that $\mathcal{D}$ is a dense graded subspace of smooth vectors for the whole representation ρ. And then condition (D4) can be interpreted as saying that $\mathcal{D}$ is the whole space of smooth vectors for ρ. With these remarks in mind, the following result should not come as a total surprise.

Theorem 1.1 (a reformulation of Corollary 3.12)**.** *Giving a super unitary representation of a super Lie group G on a super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ in the form of a couple $(\mathcal{D}, \rho)$ satisfying the conditions* (D1)–(D4) *above is equivalent, in terms of the super Harish-Chandra pair $(G_o, \mathfrak{g}) \cong G$, to a triple $(\rho_o, \mathcal{D}, \tau)$ with the following properties. ρ_o is a unitary representation of $G_o \equiv \mathbf{B}G$ on $\mathcal{H}$, $\mathcal{D} \subset C^\infty(\rho_o)$ is a dense graded subspace of $\mathcal{H}$ and $\tau \colon \mathfrak{g} \to \mathrm{End}(\mathcal{D})$ an even graded Lie algebra morphism satisfying the conditions:*

(i) *for each $g \in G_o$ the map $\rho_o(g)$ preserves each $\mathcal{H}_\alpha$ (the representation is "even");*

(ii) *for each $X \in \mathfrak{g}_0$ (the Lie algebra of G_o!) the map $\tau(X)$ is the restriction of the infinitesimal generator of $\rho_o\big(\exp(tX)\big)$ to $\mathcal{D}$;*

(iii) *for each $X \in \mathfrak{g}_\alpha$ the map $\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}$;*

(iv) *for all $g \in G_o$ and all $X \in \mathfrak{g}_1$ we have $\tau(g \cdot X) = \rho_o(g) \circ \tau(X) \circ \rho_o(g^{-1})$;*

(v) *the couple $(\mathcal{D}, \tau)$ is maximal with respect to the above conditions.*

With these ingredients, the maps $\tau(X)$ with $X \in \mathfrak{g}_\alpha$ are the infinitesimal generators of the representation ρ.

Comparing this result with the conditions (C1)–(C4) shows that they really are a lookalike. The one exception is that τ acts on a dense subspace of $C^\infty(\rho_o)$ and not on the whole of it. But the change in the definition of a super Hilbert space has one far more important consequence: the maps $\tau(X)$ with $X \in \mathfrak{g}_1$ are no longer (essentially) self-adjoint (nor a suitable multiple)! Now this should not worry the reader, simply because we have no use for it: even if (a multiple) were (essentially) self-adjoint, the 1-parameter unitary group it generates plays no role, as the $\tau(X)$ with $X \in \mathfrak{g}_1$ are **odd** operators, and thus (as is obvious from the proof of the above equivalence) we are only interested in $\exp\big(\xi\tau(X)\big) \equiv \mathbf{1} + \xi\tau(X)$ with $\xi \in \mathcal{A}_1$ an **odd** parameter.

Once we have this new definition, we can state the main result of this paper: the left-regular representation of any connected super Lie group is super unitary in the new sense. But let us be a bit more precise. For any super Lie group G we define the natural left-regular representation ρ of G as being defined on the space $C^\infty(G)$ of smooth functions on G by the expression

$$(\rho_g \psi)(h) = \psi(g^{-1}h) .$$

The subspace $C_c^\infty(G)$ of compactly supported smooth functions is obviously an invariant subspace.

Theorem 1.2 (a reformulation of Theorem 7.13 and other results of Section 7)**.** *Let $\mathbf{g}$ be a (smooth) super metric on $\mathfrak{g}$, the super Lie algebra of G. Associated to $\mathbf{g}$ we have two structures.*

(i) *An ordinary scalar product/metric $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ on $C_c^\infty(G)$ (via a "canonical" Batchelor atlas of G) that is invariant under all ρ_g with $g \in \mathbf{B}G$; changing $\mathbf{g}$ does not change the topology induced by $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ on $C_c^\infty(G)$. We will denote the completion of $C_c^\infty(G)$ by $L^2(G)$.*

(ii) *A super scalar product $\mathscr{S}_{\mathbf{g}}$ on $C_c^\infty(G)$ (via a left-invariant super density on G) that is invariant under the action of all ρ_g and that is continuous with respect to the topology induced by $\langle \cdot, \cdot \rangle_{\mathbf{g}}$; changing $\mathbf{g}$ changes $\mathscr{S}_{\mathbf{g}}$ with a (positive real) multiplicative factor (Lemma* 7.6*). Moreover, the extension of $\mathscr{S}_{\mathbf{g}}$ to $L^2(G)$ remains non-degenerate (Corollary* 7.11*).*

When we define the subspace $\mathcal{D}_\rho \subset L^2(G)$ as

$$\mathcal{D}_\rho = \left\{\, \psi \in C^\infty(G; \mathcal{A}^{\mathbf{K}}) \mid \forall k \in \mathbf{N}\ \forall X_i \in \mathbf{B}\mathfrak{g} : X_1^R \cdots X_k^R \psi \in L^2(G) \,\right\} ,$$

where X_i^R denotes the right-invariant vector field on G associated to the Lie algebra element $X_i \in \mathfrak{g}$, then the couple $(\mathcal{D}_\rho, \rho)$ is a super unitary representation of G on the super Hilbert space $(L^2(G), \langle \cdot, \cdot \rangle_{\mathbf{g}}, \mathscr{S}_{\mathbf{g}})$; it is the only one with $C_c^\infty(G) \subset \mathcal{D}_\rho$. Its infinitesimal form τ is given by (minus) the action of right-invariant vector fields:

$$\tau(X) = -X^R \in \mathrm{End}(\mathcal{D}_\rho) .$$

Let me finish this introduction with an outline of this paper. I will be working with $\mathcal{A}$-manifold theory (my version of the geometric H^∞ version of DeWitt supermanifolds), which is equivalent to the theory of graded manifolds of Leites and Kostant (see [4], [8], [9], [13] [15], [18]). Any reader using a (slightly) different version of supermanifolds should be able to translate the results to her/his version of supermanifolds. Readers unfamiliar with my approach will find in the appendix of [16] an extremely succinct overview of $\mathcal{A}$-manifold theory. We start in Section 2 with the definition of super scalar products and super metrics. With these ingredients we are able to state, in Section 3, our weakened definition of a super Hilbert space and to prove our Equivalence Theorem 3.6.

In Section 4 we recall the definition of the Batchelor bundle associated to an $\mathcal{A}$-manifold M, we introduce the notion of a super metric on an $\mathcal{A}$-manifold, and we show how to deduce an ordinary metric on $C_c^\infty(M)$. This will allow us to obtain a Hilbert space as the completion of this space of compactly supported smooth functions; it will be the analogue of the space of square integrable functions on an ordinary non-super manifold with respect to a metric volume form. We also will show that a metric preserving diffeomorphism of M satisfying a "linearity" condition induces a unitary map of $C_c^\infty(M)$.

In Section 5 we will recall some basic facts about Berezin integration on $\mathcal{A}$-manifolds and we will show that any super metric on an $\mathcal{A}$-manifold M defines a natural super scalar product on the space $C_c^\infty(M)$ of compactly supported smooth functions on M (just as an ordinary metric does for an ordinary manifold).

In Section 6 we prove a "change of variables" formula for Berezin integration of functions depending upon odd parameters (more precisely, depending upon a point in an $\mathcal{A}$-manifold). This will allow us to justify that the natural super scalar product on $C_c^\infty(M)$ associated to an invariant super metric is invariant under left-translation by the group.

In Section 7 we apply our results to the left-regular representation of an $\mathcal{A}$-Lie group and prove our main result, Theorem 7.13, viz, that the natural (left-regular) representation of G on the space of compactly supported smooth functions (on G) admits a unique maximal extension to a super unitary representation $(\mathcal{D}_\rho, \rho)$. Moreover, the definition of $\mathcal{D}_\rho$ is a lookalike of the description of the space of smooth vectors of the left-regular representation of an ordinary Lie group, reinforcing our interpretation of the dense subspace in the definition of a super unitary representation as the set of smooth vectors of this representation. We finish with two interesting examples: the $a\xi + \beta$ group for which we explicitly show that $\mathcal{D}_\rho$ is smaller than $C^\infty(\rho_o)$, and the orthosymplectic group $\mathrm{OSp}(1, 2)$, for which the super scalar product is less "trivial" than in the previous example.

2. Super scalar products and related stuff

Conventions and Definitions 2.1. • The starting point of $\mathcal{A}$-manifold theory is a graded-commutative ring $\mathcal{A}$ which we will fix as being the exterior algebra of an infinite dimensional (real) vector space: $\mathcal{A} = \bigwedge \mathbf{R}^\infty$. We denote by $\mathbf{B}\colon \mathcal{A} \to \mathcal{A}$ (sic) the *canonical projection* onto $\mathbf{R} \equiv \bigwedge^0 V \subset \bigoplus_{k=0}^\infty \bigwedge^k V$, the direct summand; we will call $\mathbf{B}$ the *body map* and the image $\mathbf{B}(a) \in \mathbf{R}$ the *body of* $a \in \mathcal{A}$.

• In this paper we will deal with the two fields $\mathbf{R}$ and $\mathbf{C}$ and two associated graded rings: $\mathcal{A}$ (over $\mathbf{R}$) and its complexification $\mathcal{A}^{\mathbf{C}} \equiv \mathcal{A} \otimes_{\mathbf{R}} \mathbf{C}$ (over $\mathbf{C}$). Some results will be the same for all four objects, some only for two of them. In order not to have to repeat any statement twice (or sometimes even four times), we will introduce the symbol $\mathbf{K}$ for either $\mathbf{R}$ or $\mathbf{C}$, $\mathcal{A}^{\mathbf{K}}$ for either $\mathcal{A} \equiv \mathcal{A}^{\mathbf{R}}$ or $\mathcal{A}^{\mathbf{C}}$ and $\mathcal{K}$ for one of the four objects $\mathbf{R}$, $\mathbf{C}$, $\mathcal{A}$ or $\mathcal{A}^{\mathbf{C}}$.

• An $\mathcal{A}^{\mathbf{K}}$-vector space is a *graded bi-module* E over $\mathcal{A}^{\mathbf{K}}$ of the form $E = V \otimes_{\mathbf{K}} \mathcal{A}^{\mathbf{K}}$ with V a graded vector space over $\mathbf{K}$. The body map $\mathbf{B}\colon E \to V$ is defined by $\mathbf{B}(v \otimes \lambda) = v \cdot \mathbf{B}\lambda$. We will in particular identify V with $\mathbf{B}E$, thus writing $E = \mathbf{B}E \otimes_{\mathbf{K}} \mathcal{A}^{\mathbf{K}}$. In the same spirit we will identify $v \otimes \lambda$ with $v \cdot \lambda$, which is the same as saying that we identify $v \in \mathbf{B}E$ with $v \otimes 1 \in E$.

• If V is any graded vector space over $\mathbf{K}$ equipped with a norm, we define the *DeWitt topology* on the $\mathcal{A}^{\mathbf{K}}$-vector space $V \otimes \mathcal{A}^{\mathbf{K}}$ to be the coarsest topology for which the projection $\mathbf{B} \colon V \otimes \mathcal{A}^{\mathbf{K}} \to V$ is continuous. It should be noted that the DeWitt topology is not separated/Hausdorff and that for any open set $U \subset V \otimes \mathcal{A}^{\mathbf{K}}$ we have $\mathbf{B}U \subset U$ and $U = \mathbf{B}^{-1}(\mathbf{B}U)$.

• Now let E be a finite dimensional $\mathcal{A}$-vector space, let $e_1, \dots, e_p$, $f_1, \dots, f_q \in \mathbf{B}E$ be a homogeneous basis, i.e., $e_1, \dots, e_p$ is a basis of the vector space (over $\mathbf{R}$) $(\mathbf{B}E)_0 \subset E_0$ and $f_1, \dots, f_q$ a basis of $(\mathbf{B}E)_1 \subset E_1$, and let $U \subset E_0$ be open in E_0 (equipped with the relative topology). It follows that any element $e \in U$ is described uniquely by $p+q$ elements $x_1, \dots, x_p \in \mathcal{A}_0$, $\xi_1, \dots, \xi_q \in \mathcal{A}_1$ according to

$$e = \sum_{i=1}^{p} x_i e_i + \sum_{j=1}^{q} \xi_j f_j \,.$$

If $f \colon \mathbf{B}U \subset \mathbf{R}^p \to F$ is any smooth function (and F any normed vector space over $\mathbf{R}$), then we define the $\mathbf{G}$*-extension* $\mathbf{G}f \colon U \to F \otimes \mathcal{A}$ to be the function

$$(\mathbf{G}f)(x_1, \dots, x_p, \xi_1, \dots, \xi_q) = \sum_{k=0}^{\infty} \frac{1}{k!} \big((D^k f)(r_1, \dots, r_p)\big)\overbrace{(n, \dots, n)}^{k \text{ terms}} \,,$$

where $r_i = \mathbf{B}x_i \in \mathbf{R}$, $n_i = x_i - r_i \in \mathcal{A}_0$, $n = (n_1, \dots, n_p)$, $(D^k f)(r_1, \dots, r_p)$ the kth order derivative of f at $(r_1, \dots, r_p) \in \mathbf{B}U$ seen as k-linear symmetric map with values in F and $\big((D^k f)(r_1, \dots, r_p)\big)(n, \dots, n)$ the formal evaluation of $(D^k f)(r_1, \dots, r_p)$ in the even nilpotent coefficients n_i.

• In the context of $\mathcal{A}$-manifold theory we then have the theorem – but here we can take it as a definition – that any smooth function $f \colon U \to F \otimes \mathcal{A}$ is of the form

$$f(x_1, \dots, x_p, \xi_1, \dots, \xi_q) = \sum_{I \subset \{1, \dots, q\}} (\mathbf{G}f_I)(x_1, \dots, x_p) \cdot \xi^I \,,$$

for (unique) smooth functions $f_I \colon \mathbf{B}U \to F$, where ξ^I denotes the product

$$I = \{j_1, \dots, j_k\} \text{ with } 1 \le j_1 < \dots < j_k \le q \qquad \Longrightarrow \qquad \xi^I = \xi_{j_1} \cdots \xi_{j_k} \,,$$

with $\xi^{\emptyset} = 1$.

• Using this definition of smooth functions, it is fairly straightforward to develop $\mathcal{A}$-manifold theory in terms of coordinate charts and transition functions. In particular an $\mathcal{A}$-Lie group G will be an $\mathcal{A}$-manifold with the additional structure of a group in such a way that multiplication and inverse are smooth maps. Its graded Lie algebra $\mathfrak{g}$ (which is in particular an $\mathcal{A}$-vector space) then is isomorphic to the set of all left-invariant vector fields on G, a set that is isomorphic to the tangent space of G at the identity. Giving an $\mathcal{A}$-Lie group G then is equivalent to giving a super Harish-Chandra pair $(\mathbf{B}G, \mathbf{B}\mathfrak{g})$ by the following argument. Given G one restricts the adjoint action of G on $\mathfrak{g}$ to $\mathbf{B}G$, an action which preserves $\mathbf{B}\mathfrak{g}$, obtaining the super Harish-Chandra pair. Conversely, given a super Harish-Chandra pair $(G_o, \mathbf{B}\mathfrak{g})$, one first constructs the simply connected $\mathcal{A}$-Lie group G' corresponding to the graded

Lie algebra $\mathfrak{g}$ (see [15, §VI.9]) and one then uses the compatibility condition of the Harish-Chandra pair to prove that one can quotient by a central discrete subgroup to obtain an $\mathcal{A}$-Lie group G with $\mathbf{B}G = G_o$ whose associated Harish-Chandra pair is the given one.

• For a $\mathcal{K}$-vector space E we denote by $\mathrm{End}(E)$ the space of (right-) linear endomorphisms:

$$\mathrm{End}(E) = \{\, f : E \to E \mid f \text{ is right-linear over } \mathcal{K}\,\}$$

and by $\mathrm{Aut}(E)$ the space of automorphisms of E, i.e.the even invertible elements of $\mathrm{End}(E)$:

$$\mathrm{Aut}(E) = \{\, f \in \mathrm{End}(E) \mid f \text{ even and bijective}\,\}\,.$$

• In the non-super context there is a difference in terminology when dealing with real or complex vector spaces concerning maps that preserve a metric: in the real case one talks about orthogonal maps, and in the complex case of unitary ones. As most of our results apply to both cases, it would be rather cumbersome to distinguish these two cases, the more so when we use $\mathbf{K}$ to denote either $\mathbf{R}$ of $\mathbf{C}$. We thus will stick exclusively to the name "super unitary", even when we are considering graded vector spaces over $\mathcal{A}^{\mathbf{R}}$ (where one would expect the name super orthogonal). A similar remark applies to the terminology symmetric versus hermitian: the first is used in the real context, the second in the complex case. Being totally inconsistent, we stick to the name "symmetric" in both cases (we will mainly be concerned with graded skew-symmetric maps, which thus will include the ones that appear in the complex setting and should be called graded skew-hermitian).

• Let x be any graded object, which we can decompose as $x = x_0 + x_1$ with x_α homogeneous of degree α. We then define the *conjugation operator* $\mathfrak{C}$ by

$$\mathfrak{C}(x) \equiv \mathfrak{C}(x_0 + x_1) = x_0 - x_1\,.$$

For any $n \in \mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ we define (as usual) the operator $\mathfrak{C}^n$ to be either the identity (when n is even) or $\mathfrak{C}$ (when n is odd).

• For any graded homogeneous object x we will denote by $|x|$ its parity, i.e., $|x|$ is either 0 (when x is even) or 1 (when x is odd). However, we will use the same "operation" on finite sets I to denote their cardinal: $|\{a, b, c\}| = 3$. But we will use this nearly exclusively in combination with other parities, in which case we interpret $|I|$ as the cardinal of I modulo 2.

Now already in ordinary differential geometry we have two kind of metrics: normal, euclidean or Riemannian metrics on the one hand, and Lorentzian or pseudo-Riemannian metrics on the other, where the difference lies in the fact that the former is positive definite, while the latter is not. In this paper we will adopt the following naming convention. By a (hermitian) scalar product we will mean a non-degenerate symmetric sesquilinear form on an ordinary (ungraded) vector space over $\mathbf{K}$ (a Lorentzian metric), and by a metric we will mean an ordinary (euclidean) positive definite hermitian scalar product, again on an ungraded vector space over $\mathbf{K}$. When we add the qualification "super", we get a super scalar product and a super metric, both of which will be defined on graded vector spaces of any kind. The super scalar product has "only" the natural non-degeneracy condition, whereas a super metric has an additional condition reflecting the positive definiteness condition for euclidean metrics.

Classical Definitions 2.2. Let V be a vector space over $\mathbf{C}$.
• A *hermitian scalar product on* V is a non-degenerate symmetric sesquilinear form $\langle\ ,\ \rangle \colon V \times V \to \mathbf{C}$. More precisely, the map $\langle\ ,\ \rangle \colon V \times V \to \mathbf{C}$ should satisfy the three conditions.

(i) $\forall x, y, z \in V\ \ \forall r \in \mathbf{C}$: $\langle x, y + rz\rangle = \langle x, y\rangle + r\,\langle x, z\rangle$ (linearity in the second variable).

(ii) $\forall x, y, z \in V\ \ \forall r \in \mathbf{C}$: $\langle x + ry, z\rangle = \langle x, z\rangle + \overline{r}\ \langle y, z\rangle$ (anti-linearity in the first variable).

(iii) $\forall x, y \in V$: $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (symmetry).

(iv) $\forall x \in V : (\,\forall y \in V : \langle x, y\rangle = 0\,) \ \Rightarrow\ x = 0$ (non-degeneracy).

• A *metric on* V is a positive definite hermitian scalar product $\langle\ ,\ \rangle$ on V. More precisely, the hermitian scalar product should satisfy the additional condition

(v) $\forall x \in V : \langle x, x\rangle \geq 0$.

• If V is a vector space over $\mathbf{R}$, we can define a *scalar product* as a non-degenerate hermitian symmetric sesquilinear map $\langle\ ,\ \rangle \colon V \times V \to \mathbf{C}$ (i.e., satisfying exactly the conditions (i)-(iv) above, restricting the scalar r to $r \in \mathbf{R}$) satsifying the additional (reality) condition

(vi) $\forall x, y \in V$: $\langle x, y\rangle = \overline{\langle x, y\rangle}$.

This amounts to requiring that $\langle\ ,\ \rangle$ takes its values in $\mathbf{R}$. And as for a vector space over $\mathbf{C}$, a *metric* on a vector space over $\mathbf{R}$ is a positive definite scalar product.
• A *Hilbert space* is a vector space V (over $\mathbf{R}$ or $\mathbf{C}$) equipped with a metric $\langle\ ,\ \rangle$ such that V is complete with respect to the norm $\|x\| = \sqrt{\langle x, x\rangle}$.

Definition 2.3. Let V be a graded vector space over $\mathcal{K}$.
• In the case $\mathcal{K} = \mathbf{K}$, a *(graded, right-) sesquilinear form* on V is a map $\mathscr{S} \colon V \times V \to \mathbf{C}$ satisfying the conditions (i) and (ii) below; in the case $\mathcal{K} = \mathcal{A}^{\mathbf{K}}$, it is a map $\mathscr{S} \colon V \times V \to \mathcal{A}^{\mathbf{C}}$ satisfying these conditions.

(i) *Right-linearity* in the second variable: for all $v, w_1, w_2 \in V$ and all $\lambda \in \mathcal{K}$ we have
$$\mathscr{S}(v, w_1 + w_2\lambda) = \mathscr{S}(v, w_1) + \mathscr{S}(v, w_2) \cdot \lambda\,.$$

(ii) *Anti-right-linearity* in the first variable (see also Lemma 2.6 and Remark 2.7): for all $v_1, v_2, w \in V$ and all $\lambda \in \mathcal{K}$ we have
$$\mathscr{S}(v_1 + v_2\lambda, w) = \mathscr{S}(v_1, w) + \mathscr{S}(v_2, \overline{\lambda} w)\,.$$

• If $\mathscr{S}$ is a graded sesquilinear form on a graded vector space V over $\mathcal{A}^{\mathbf{K}}$, then the map $\mathbf{B}\mathscr{S} \colon \mathbf{B}V \times \mathbf{B}V \to \mathbf{C}$ defined by
$$(\mathbf{B}\mathscr{S})(v, w) = \mathbf{B}\big(\mathscr{S}(v, w)\big)$$
is a graded sesquilinear form on the graded vector space $\mathbf{B}V$ over $\mathbf{K}$.

• In the case $\mathcal{K} = \mathcal{A}^{\mathbf{K}}$, a graded sesquilinear form $\mathscr{S}$ on V is said to be *smooth* if it satisfies the condition

(iii) *Smoothness*: $\mathscr{S}(\mathbf{B}V, \mathbf{B}V) \subset \mathbf{C}$.

Another way to state this condition is that $\mathbf{B}\mathscr{S}$ equals the restriction of $\mathscr{S}$ to $\mathbf{B}V$.

• A graded sesquilinear form on V is said to be *graded symmetric* if it satisfies the condition

(iv) *Graded symmetry*: for all homogeneous $w, v \in V$: $\mathscr{S}(w,v) = (-1)^{|w||v|}\,\overline{\mathscr{S}(v,w)}$.

• A graded symmetric graded sesquilinear form on V is said to be *non-degenerate* if it satisfies the condition

(v) *Non-degeneracy*: $\forall v \in \mathbf{B}V:\ \ v \neq 0 \quad\Rightarrow\quad \exists w \in \mathbf{B}V : (\mathbf{B}\mathscr{S})(v,w) \neq 0$, where, in the case of $\mathcal{K} = \mathbf{K}$ the use of the body map should be ignored.

• A *super scalar product* on V is a graded symmetric non-degenerate graded sesquilinear form on V.

• A graded sesquilinear form $\mathscr{S}$ is said to be *homogeneous of degree* $\alpha \in \mathbf{Z}/2\mathbf{Z}$ if for all $\beta, \gamma \in \mathbf{Z}/2\mathbf{Z}$ we have the inclusion

$$\mathscr{S}(V_\beta, V_\gamma) \subset \mathcal{A}^{\mathbf{C}}_{\alpha+\beta+\gamma}\,,$$

where in the case $\mathcal{K} = \mathbf{K}$ we use the inclusion $\mathbf{C} \subset \mathcal{A}^{\mathbf{C}}$ to interpret $\mathscr{S}$ as taking values in $\mathcal{A}^{\mathbf{C}}$. Any hermitian form decomposes as the sum of two homogeneous hermitian forms $\mathscr{S} = \mathscr{S}_0 + \mathscr{S}_1$ when we define $\mathscr{S}_\alpha$ by

$$\mathscr{S}_\alpha(v,w) = \mathscr{S}(v_0,w_0)_\alpha + \mathscr{S}(v_0,w_1)_{\alpha+1} + \mathscr{S}(v_1,w_0)_{\alpha+1} + \mathscr{S}(v_1,w_1)_\alpha\,,$$

where $\mathscr{S}(x,y)_\beta$ denotes the homogeneous part of degree β of $\mathscr{S}(x,y) \in \mathcal{A}^{\mathbf{C}}$.

• A right-linear map $A\colon V \to V$ is said to be *graded skew-symmetric* with respect to a graded sesquilinear form $\mathscr{S}$ on V if we have, for any two $v, w \in V$:

$$\mathscr{S}(Av, w) + \mathscr{S}(v, A_0 w) + \mathscr{S}\big(\mathfrak{C}(v), A_1 w\big) = 0\,,$$

where $A = A_0 + A_1$ is the decomposition of A into its homogeneous parts.

Nota Bene 2.4. We explicitly do **not** require a super scalar product to be homogeneous!

Remark 2.5. • We have defined the notion of a hermitian form on a graded vector space over $\mathbf{R}$ or $\mathcal{A}$ as taking values in $\mathbf{C}$ respectively $\mathcal{A}^{\mathbf{C}}$. The reader might have expected that in those two cases it should take values in $\mathbf{R}$ respectively $\mathcal{A}$ only. Our definition does not exclude that possibility, but we will need later on the possibility (necessity!) that a super scalar product on a graded vector space over $\mathbf{R}$ or $\mathcal{A}$ is allowed to take values in these larger sets (see in particular Lemma 2.9 and Discussion 2.13).

• We have formulated the non-degeneracy condition in the case $\mathcal{K} = \mathcal{A}^{\mathbf{K}}$ in terms of $\mathbf{B}\mathscr{S}$. For smooth $\mathscr{S}$ one can show that this condition is equivalent to the same condition without the body maps. But we will need it this way because when we define a metric on an $\mathcal{A}$-manifold M, it will not be smooth in the sense given above on every tangent space.

• The name "smoothness" we used above for graded sesquilinear forms might look strange to some. However, when V is finite dimensional, it is the necessary and sufficient condition for such a map to be smooth in the category of $\mathcal{A}$-manifolds [15, III.1.27].

Lemma 2.6. *Let V be a graded vector space over $\mathcal{K}$ and let $\mathscr{S}$ be a graded sesquilinear form on V. Then we have the following "anti-linearity" property (for $v, w \in V$ and $\lambda \in \mathcal{K}$):*

$$\begin{aligned}\mathscr{S}(\lambda v, w) &= \overline{\lambda} \cdot \mathscr{S}_0(v,w) + \overline{\mathfrak{C}(\lambda)} \cdot \mathscr{S}_1(v,w) \\ &= \overline{\lambda_0} \cdot \mathscr{S}(v,w) + \overline{\lambda_1} \cdot \big(\mathscr{S}_0(v,w) - \mathscr{S}_1(v,w)\big)\,.\end{aligned}$$

Remark 2.7. Let $\mathscr{S}$ be a graded sesquilinear form on V and let $\mathscr{S}_\alpha$ be its homogeneous parts. It then follows from Lemma 2.6 that the (anti-) linearity conditions are such that $\mathscr{S}_\alpha$ acts as an object of parity α, meaning that interchanging a super object with $\mathscr{S}_\alpha$ gives an additional sign. As we write the symbol $\mathscr{S}$ to the left of the vectors on which we evaluate, it follows that the parity of a graded sesquilinear form is concentrated on the left. The way we formulated the condition of "anti-right-linearity" in Definition 2.3 avoids all mention of parity.

Lemma 2.8. *Let V be a graded vector space over $\mathcal{A}^{\mathbf{K}}$.*

(i) *If $\mathscr{S}$ is a graded sesquilinear form on V, then $\mathbf{B}\mathscr{S}$ is a graded sesquilinar form on $\mathbf{B}V$. And if $\mathscr{S}$ is homogeneous, graded symmetric or non-degenerate, so is $\mathbf{B}\mathscr{S}$.*

(ii) *Any graded sesquilinear form $\mathscr{S}^r$ on $\mathbf{B}V$ extends in a unique way to a smooth graded sesquilinear form $\mathscr{S}$ on V (i.e., $\mathbf{B}\mathscr{S} = \mathscr{S}^r$) by defining*

$$\begin{aligned}\mathscr{S}(v \otimes \lambda, w \otimes \mu) &= \mathscr{S}^r(v, w_0) \cdot \lambda_0 \cdot \mu + \mathscr{S}^r(v, w_1) \cdot \overline{\lambda}_0 \cdot \mu \\ &\qquad + \mathscr{S}^r(v, w_0) \cdot \lambda_1 \cdot \mu - \mathscr{S}^r(v, w_1) \cdot \overline{\lambda}_1 \cdot \mu \\ &\equiv \mathscr{S}^r(v, w_0) \cdot \lambda \cdot \mu + \mathscr{S}^r(v, w_1) \cdot \mathfrak{C}(\,\overline{\lambda}\,) \cdot \mu\,,\end{aligned}$$

for $v, w \in V$ and $\lambda, \mu \in \mathcal{K}$. If $\mathscr{S}^r$ is homogeneous, graded symmetric or non-degenerate on $\mathbf{B}V$, then so is its extension $\mathscr{S}$ to V.

Lemma 2.9. *Let $\mathscr{S}$ be an even super scalar product on a graded vector space V over $\mathcal{A}^{\mathbf{K}}$. Then:*

(i) $(\mathbf{B}\mathscr{S})(\mathbf{B}V_0, \mathbf{B}V_1) = 0$*;*

(ii) *the restriction of $\mathbf{B}\mathscr{S}$ to $\mathbf{B}V_0$ is a hermitian scalar product on the ungraded vector space $\mathbf{B}V_0$ over $\mathbf{K}$ in the sense of Definition 2.2;*

(iii) *the restriction of $-i \cdot \mathbf{B}\mathscr{S}$ to $\mathbf{B}V_1$ is a hermitian scalar product on the ungraded vector space $\mathbf{B}V_1$ over $\mathbf{K}$ in the sense of Definition 2.2.*

Definition 2.10. A super scalar product $\mathscr{S}$ on a graded vector space V over $\mathbf{C}$ or $\mathcal{A}^{\mathbf{C}}$ will be called a *super metric* if it is **even** and satisfies the two conditions

(i) $\forall v \in \mathbf{B}V_0 : (\mathbf{B}\mathscr{S})(v,v) \geq 0$ and

(ii) $\forall v \in \mathbf{B}V_1 : -i \cdot (\mathbf{B}\mathscr{S})(v,v) \geq 0$.

Remark 2.11. In the definition of a super metric we have made an arbitrary choice by requiring that $-i \cdot (\mathbf{B}\mathscr{S})(v,v) \geq 0$ for all $v \in \mathbf{B}V_1$. We could as well have required that $+i \cdot (\mathbf{B}\mathscr{S})(v,v) \geq 0$. However, there is another instance where we can make an arbitrary choice of a sign: in the definition of an equivalence of super Hilbert spaces, see Remark 3.4. If one wishes to have consistent results, these choices are not independent: fixing one choice imposes the other.

Corollary 2.12. *If $\mathscr{S}$ is a super metric on a graded vector space V over $\mathbf{C}$ or $\mathcal{A}^{\mathbf{C}}$, then $\mathbf{B}\mathscr{S}|_{\mathbf{B}V_0}$ and $-i \cdot \mathbf{B}\mathscr{S}|_{\mathbf{B}V_1}$ are metrics (in the sense of Definition 2.2) on the ungraded vector spaces (over $\mathbf{C}$) $\mathbf{B}V_0$ and $\mathbf{B}V_1$ respectively.*

Discussion 2.13. In Definition 2.10 we have defined the notion of a super metric for graded vector spaces over $\mathbf{C}$ or $\mathcal{A}^{\mathbf{C}}$. For graded vector spaces over $\mathbf{R}$ or $\mathcal{A}$ the definition of a super metric is similar, but with a slightly unexpected twist: they should still take values in $\mathbf{C}$ or $\mathcal{A}^{\mathbf{C}}$ respectively! There are several arguments in favor of this definition.

In the first place, defining a metric on a graded vector space over $\mathbf{R}$ as a non-degenerate graded symmetric bilinear form with values in $\mathbf{R}$ implies automatically that the odd dimension of this vector space should be even (as on the odd-odd part such a form defines a skew-symmetric form and thus non-degeneracy implies even dimension). In the second place, even when we restrict attention to graded vector spaces with an even dimension of the odd part, we will miss an important property, needed when one wants to apply partition of unity arguments: a linear combination with positive coefficients is not guaranteed to remain non-degenerate. And in the third place, we have seen that in the complex setting, the restriction of a super metric to the odd part of the graded vector space is i times an ordinary (hermitian!) metric.

Definition 2.14. Let V be a graded vector space over $\mathbf{R}$ or $\mathcal{A}$. A *super metric on* V is a super metric $\mathscr{S}$ on V in the sense of Definition 2.10 with the additional *reality condition*:

$$\forall v, w \in \mathbf{B}V \text{ homogeneous: } (\mathbf{B}\mathscr{S})(v,w) = (-1)^{|v||w|}\, \overline{(\mathbf{B}\mathscr{S})(v,w)}\,.$$

This condition should be compared with condition (vi) in Definition 2.2 for metrics on ungraded vector spaces over $\mathbf{R}$ (see also [15, §IV.7]).

Lemma 2.15. *We collect here some interesting properties of super metrics on a graded vector space V over $\mathbf{R}$ or $\mathcal{A}$.*

(i) *Let V be a graded vector space over $\mathcal{A}$ and let $\mathscr{S}$ be a super metric on V, then $\mathbf{B}\mathscr{S}$ is a super metric on $\mathbf{B}V$.*

(ii) *Let V be a graded vector space over $\mathcal{A}$ and let $\mathscr{S}$ be a super metric on $\mathbf{B}V$. Then there exists a unique smooth super metric $\mathscr{S}^{\mathcal{A}}$ on V with $\mathbf{B}\mathscr{S}^{\mathcal{A}} = \mathscr{S}$.*

(iii) *If* V *is a graded vector space over* $\mathcal{A}$ *and* $\mathscr{S}$ *a super metric on* V*, then* $\mathbf{B}\mathscr{S}|_{\mathbf{B}V_0}$ *and* $-i\,\mathbf{B}\mathscr{S}|_{\mathbf{B}V_1}$ *are (ordinary) metrics in the sense of Definition* 2.2 *on the ungraded vector spaces (over* $\mathbf{R}$*)* $\mathbf{B}V_0$ *and* $\mathbf{B}V_1$ *respectively.*

(iv) *Let* $\mathscr{S}^a$*,* $a = 1, \dots, \ell$ *be super metrics on* V*, a graded vector space over* $\mathbf{R}$ *or* $\mathcal{A}$*, and let* $r_1, \dots, r_\ell$ *be elements in* $\mathbf{R}$ *respectively* $\mathcal{A}_0$ *such that* $\mathbf{B}r_a > 0$*. Then* $\sum_{a=1}^{\ell} r_a \cdot \mathscr{S}^a$ *is a super metric on* V*.*

3. Super Hilbert spaces and super unitary representations

Definition 3.1. As said in the introduction, I propose to weaken the definition of a super Hilbert space. Let us start with the definition of a *proto super Hilbert space*, by which I will mean a triple $(E, \langle\cdot,\cdot\rangle, \mathscr{S})$ in which E is a graded vector space over $\mathbf{K}$, $\langle\cdot,\cdot\rangle$ a metric on E seen as an ungraded vector space and $\mathscr{S}$ a super scalar product on E satisfying the following two conditions.

(SHS1) $\langle E_0, E_1\rangle = 0$.

(SHS2) $\mathscr{S}$ is continuous with respect to the topology induced by the metric $\langle\cdot,\cdot\rangle$.

And then a *super Hilbert space* will be a proto super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ such that $\mathcal{H}$ is complete with respect to the topology induced by the metric $\langle\cdot,\cdot\rangle$, i.e., the couple $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space in the usual sense. When $(E, \langle\cdot,\cdot\rangle, \mathscr{S})$ is a proto super Hilbert space, the couple $(E, \langle\cdot,\cdot\rangle)$ is a usual inner product space, one which we can complete to a Hilbert space $\mathcal{H}$. And as $\mathscr{S}$ is continuous on E, it extends to a continuous sesquilinear map on $\mathcal{H}$. However, non-degeneracy of the extension of $\mathscr{S}$ to $\mathcal{H}$ is not automatic and has to be checked separately before one can conclude that we have a super Hilbert space. The basic example of a super Hilbert space is $\mathbf{K}$ itself, with $\langle\cdot,\cdot\rangle$ and $\mathscr{S}$ given by

$$\langle x, y\rangle = \mathscr{S}(x, y) = \overline{x}\, y\,.$$

With this notion of a super Hilbert space, I now define a *proto super unitary representation* of an $\mathcal{A}$-Lie group G on a super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ as a couple $(\mathcal{D}, \rho)$ with the following properties. $\mathcal{D} \subset \mathcal{H}$ is a dense graded subspace and $\rho\colon G \to \mathrm{Aut}(\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}})$ a group homomorphism satisfying the following conditions.

(SUR1) for all $g \in G$, $\rho(g)$ preserves $\mathscr{S}$ (or more accurately: preserves the restriction to $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$ of the extension of $\mathscr{S}$ to $\mathcal{H} \otimes \mathcal{A}^{\mathbf{K}}$);

(SUR2) For all $\psi \in \mathcal{D}$ the map $\mathbf{F}_\psi\colon G \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$, $\mathbf{F}_\psi(g) = \rho(g)\psi$ is smooth.

(SUR3) For all $g \in \mathbf{B}G$ the map $\rho(g)$ preserves $\langle\cdot,\cdot\rangle$.

If $(\mathcal{D}, \rho)$ and $(\mathcal{D}', \rho')$ are two proto super unitary representations of G on the super Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$, then $(\mathcal{D}', \rho')$ *extends* $(\mathcal{D}, \rho)$ if $\mathcal{D} \subset \mathcal{D}'$ and for all $g \in G$ we have $\rho'(g)|_{\mathcal{D}} = \rho(g)$. We will say that a proto super unitary representation $(\mathcal{D}, \rho)$ is a *super unitary representation of* G *on* $\mathcal{H}$ if it does not admit a non-trivial extension.

Lemma 3.2. *If the couple $(\mathcal{D}, \rho)$ is a proto super unitary representation of G on $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$, then:*

(i) $\forall g \in \mathbf{B}G$ *the map* $\rho(g)$ *preserves* $\mathcal{D}$ *(thus justifying* (SUR3)*);*

(ii) *the group homomorphism* $\rho\colon \mathbf{B}G \to \mathrm{Aut}(\mathcal{D})$ *extends to a unitary representation* ρ_o *of* $\mathbf{B}G$ *on* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$*;*

(iii) *we have the inclusion* $\mathcal{D} \subset C^\infty(\rho_o)$*.*

Proof. If $(\mathcal{D}, \rho)$ satisfies (SUR2), the map $\mathbf{F}_\psi\colon G \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$ is smooth. But smoothness implies that when applied to an element of $\mathbf{B}G$ the result lies in $\mathbf{B}(\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}) = \mathcal{D}$ (see the appendix of [16] or [15, III.1.16(A2)]). It thus makes sense to require that $\rho(g)$ with $g \in \mathbf{B}G$ preserves $\langle\cdot,\cdot\rangle$ (and not its extension to $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$). Once we know that $\rho(g)$ with $g \in \mathbf{B}G$ preserves $\mathcal{D}$, condition (SUR3) says that it is unitary. It thus extends by continuity to a unitary representation ρ_o on the whole of $\mathcal{H}$. We then recall that the set $C^\infty(\rho_o)$ of smooth vectors for ρ_o is defined by

$$C^\infty(\rho_o) = \{\, \psi \in \mathcal{H} \mid \mathbf{F}_\psi|_{\mathbf{B}G} \text{ is smooth} \,\} .$$

Condition (SUR2), applied to the restriction of $\mathbf{F}_\psi$ to $\mathbf{B}G$, then tells us that we have the inclusion $\mathcal{D} \subset C^\infty(\rho_o)$. ■

In the non-super context, a unitary representation is a group homomorphism $\rho\colon G \to \mathrm{Aut}(\mathcal{H})$ such that each $\rho(g)$ preserves the metric on the Hilbert space $\mathcal{H}$. Associated we then have the space of smooth vectors $C^\infty(\rho) \subset \mathcal{H}$; they form a common dense domain of essential self-adjointness for the infinitesimal generators $\tau(X)$ defined as

$$\tau(X)\psi = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho\big(\exp(Xt)\big)\psi .$$

In the super context the interplay between a super unitary representation and its set of smooth vectors is a bit more complicated. For a super unitary representation $(\mathcal{D}, \rho)$ it is the set $\mathcal{D}$ that is/plays the role of the set of smooth vectors of this representation according to condition (SUR2) and the fact that there is no non-trivial extension. In Lemma 3.2 we have shown that we can extend the restriction of ρ to $\mathbf{B}G$ to a unitary representation ρ_o of $\mathbf{B}G$ on $\mathcal{H}$. Now there is a very simple reason why we cannot extend ρ to $\mathcal{H}$ on the whole of G: if $\mathcal{D}$ is the domain of definition of an odd generator $\tau(X)$, $X \in \mathbf{B}\mathfrak{g}_1$ (and nobody will be surprised that the domain of a generator is not the whole of $\mathcal{H}$), then its flow is given by $\exp\big(\xi\tau(X)\big)$ for an odd parameter ξ. But then the smoothness condition tells us that we have

$$\exp\big(\xi\tau(X)\big) = \mathbf{1} + \xi\tau(X)$$

and thus the domain of definition of this flow is the same as the domain of $\tau(X)$. In the conventional approach to a super unitary representation, the infinitesimal generators $\tau(X)$ for X odd are supposed to be defined on $C^\infty(\rho_o)$ and they (or a suitable multiple) are supposed to be essentially self-adjoint. However, the condition to be (essentially) self-adjoint is equivalent to the condition that it generates a 1-parameter unitary group of transformations, and for odd generators, this 1-parameter group has no rôle in the super representation, as we are interested only in $\exp\bigl(\xi\tau(X)\bigr)$ with ξ an odd parameter. Moreover, nobody will be surprised that the set of smooth vectors for a subgroup is larger than the set of smooth vectors for the whole group. And thus it should not come as a surprise that the alleged set of smooth vectors $\mathcal{D}$ for ρ might be smaller than the set of smooth vectors $C^\infty(\rho_o)$ for the subgroup $\mathbf{B}G \subset G$.

Definition 3.3. (Notions of equivalence) • Let $(E, \langle\cdot,\cdot\rangle_E, \mathscr{S}_E)$ and $(F,\ \langle\cdot,\cdot\rangle_F, \mathscr{S}_F)$ be two proto super Hilbert spaces. We will say that these two spaces are *equivalent* if there exists a linear bijection $A\colon E \to F$ satisfying the following conditions (where $A = A_0 + A_1$ is the decomposition of A into homogeneous parts).

(i) $\ker(A_0) + \ker(A_1) = E$ (and thus $E = \ker(A_0) \oplus \ker(A_1)$).

(ii) $\forall x, y \in E\colon\ \langle Ax, Ay\rangle_F = \langle x, y\rangle_E$.

(iii) $\mathscr{S}_F(Ax, Ay) = \begin{cases} \mathscr{S}_E(x,y) & x \in E\ ,\ y \in \ker(A_1) \\ \mathscr{S}_E(\mathfrak{C}x, y) & x \in \ker(A_1)\ ,\ y \in \ker(A_0) \\ i\,\mathscr{S}_E(\mathfrak{C}x, y) & x, y \in \ker(A_0)\,. \end{cases}$

Alternatively we will say that a map A satisfying these conditions is an *equivalence* between the two proto super Hilbert spaces.

• Let G be an $\mathcal{A}$-Lie group, let $(\mathcal{D}_E, \rho_E)$ be a proto super unitary representation of G on $(\mathcal{H}_E, \langle\cdot,\cdot\rangle_E, \mathscr{S}_E)$ and let $(\mathcal{D}_F, \rho_F)$ be a proto super unitary representation of (the same) G on $(\mathcal{H}_F, \langle\cdot,\cdot\rangle_F, \mathscr{S}_F)$. We will say that these two representations are *equivalent* is there exists an equivalence of proto super Hilbert spaces $A\colon \mathcal{D}_E \to \mathcal{D}_F$ intertwining the two representations:

$$\forall g \in G \quad : \quad \rho_F(g) \circ A = A \circ \rho_E(g)\,.$$

Remark 3.4. • One might have expected that an equivalence of two proto super Hilbert spaces should be an **even** linear bijection. However, the following argument should convince the reader that at least odd linear bijections are completely natural as equivalences for super unitary representations. Let E be a graded vector space over $\mathbf{R}$ of graded dimension $p|q$ and let $F = \prod E$ be the parity reversal of E of graded dimension $q|p$. It is immediate that the two groups $\mathrm{Aut}(E)$ and $\mathrm{Aut}(F)$ are isomorphic: it suffices to look at their matrix representation. It thus is quite natural to say that the two tautological representations on E and F are equivalent. But the equivalence is given by the parity reversal map, and thus is odd. And once we have accepted the idea of an odd equivalence, non-homogeneous ones appear naturally by taking direct sums. Condition (i) might be too restrictive, but it is compatible with the procedure of taking direct sums.

• The reader might have expected that our (non-homogeneous) linear bijection $A\colon E \to F$ should not only be compatible with the metrics as in condition (ii), but also with the super scalar products in the form of the condition

$$\forall x, y \in E \quad : \quad \mathscr{S}_F(Ax, Ay) = \mathscr{S}_E(x, y)\,.$$

However, just looking at an odd map A shows that such a condition is not compatible with the properties of a super scalar product. In particular graded symmetry (and reality in the case of $\mathbf{K} = \mathbf{R}$) are incompatible with this formula. On the other hand, when taking condition (i) into account, some elementary computations show that the given formula in condition (iii) is compatible with these properties (and that it nearly is imposed by them). Moreover, when we extend to $\mathcal{A}^{\mathbf{K}}$ vector spaces, it will be compatible with anti-right-linearity in the first variable. There is, though, one arbitrary choice I made: the use of the factor i in the case $x, y \in \ker(A_0)$. Choosing $-i$ instead will also produce a formula compatible with the properties of a super scalar product. However, the choice for $+i$ is imposed when we want that a super metric, Definition 2.10, is equivalent to a super metric (the opposite choice will turn the positive definite condition into negative definite, see also Remark 2.11).

• If $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ is a super Hilbert space and $\mathcal{D} \subset \mathcal{H}$ a dense graded subspace, then the triple $(\mathcal{D}, \langle\cdot,\cdot\rangle|_{\mathcal{D}}, \mathscr{S}|_{\mathcal{D}})$ is again a super Hilbert space and in particular the restriction of $\mathscr{S}$ to $\mathcal{D}$ remains non-degenerate (something that is not true for an arbitrary graded subspace!). This implies that the definition of equivalence of two proto super unitary representations makes sense.

Lemma 3.5. *Let $(\mathcal{D}_E, \rho_E)$ be a proto super unitary representation of G on the super Hilbert space $(\mathcal{H}_E, \langle\cdot,\cdot\rangle_E, \mathscr{S}_E)$ and let $(\mathcal{D}_F, \rho_F)$ be a proto super unitary representation of G on $(\mathcal{H}_F, \langle\cdot,\cdot\rangle_F, \mathscr{S}_F)$. Then the following properties are equivalent.*

(i) *There exists a homogeneous equivalence of proto super Hilbert spaces $A\colon \mathcal{D}_E \to \mathcal{D}_F$ intertwining ρ_E and ρ_F.*

(ii) *There exists a homogeneous equivalence of super Hilbert spaces $A\colon \mathcal{H}_E \to \mathcal{H}_F$ such that $A(\mathcal{D}_E) = \mathcal{D}_F$ and intertwining ρ_E and ρ_F.*

Theorem 3.6 (see also [5, Thm 4.18][1])**.** *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle, \mathscr{S})$ be a super Hilbert space, let G be a connected $\mathcal{A}$-Lie group and let $(\mathcal{D}, \rho)$ be a proto super unitary representation of G on $\mathcal{H}$. If we denote by $\mathfrak{g} \cong T_eG$ the graded Lie algebra of G, then the map $\tau\colon \mathbf{B}\mathfrak{g} \to \mathrm{End}(\mathcal{D})$ defined by* (*see* (SUR2))

$$\tau(X)\psi = X\mathbf{F}_\psi \tag{1}$$

is an even graded Lie algebra morphism with the following properties:

(i) *For each $X \in \mathbf{B}\mathfrak{g}_0$ the map $\tau(X)$ is the restriction of the infinitesimal generator of $\rho_o\big(\exp(tX)\big)$ to $\mathcal{D}$ (see Lemma 3.2).*

(ii) *For all $X \in \mathbf{B}\mathfrak{g}$ the map $\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}$.*

(iii) *For all $g \in \mathbf{B}G$ and all $X \in \mathbf{B}\mathfrak{g}_1$ we have*

$$\tau\big(\mathrm{Ad}(g)X\big) = \rho_o(g) \circ \tau(X) \circ \rho_o(g^{-1}) .$$

The triple $(\rho_o, \mathcal{D}, \tau)$ is called the infinitesimal form of $(\mathcal{D}, \rho)$.

[1] After the completion of this paper (which had a long gestation period), I became aware of the paper [5] and realized that I could and should have known of its existence. Their theorem 4.18 is, apart from inessential details, the same as my Theorem 3.6. Moreover, their proof is much tighter than mine. Credit for this result thus is theirs.

Conversely, if a triple $(\rho_o, \mathcal{D}, \tau)$ *satisfies the conditions* (i)–(iii) *with* ρ_o *an even unitary representation of* $\mathbf{B}G$ *on* $\mathcal{H}$ *(meaning that* $\rho_o(g)$ *preserves* $\mathcal{H}_\alpha$*),* $\mathcal{D} \subset C^\infty(\rho_o)$ *a dense graded subspace of* $\mathcal{H}$ *and invariant under the maps* $\rho_o(g)$*,* $g \in \mathbf{B}G$ *and* $\tau\colon \mathbf{B}\mathfrak{g} \to \mathrm{End}(\mathcal{D})$ *an even graded Lie algebra morphism, then there exists a unique group homomorphism* $\rho\colon G \to \mathrm{Aut}(\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}})$ *satisfying the conditions* (SUR1)–(SUR3) *such that the map* τ *is given by (1) and such that the restriction of* ρ *to* $\mathbf{B}G$ *is* ρ_o*.*

Proof of Theorem 3.6, direct part. Suppose that $(\mathcal{D}, \rho)$ is a proto super unitary representation of G. Our first task is to show that the map τ is well defined. According to (SUR2) the map $\mathbf{F}_\psi\colon G \to \mathcal{D}\otimes\mathcal{A}^{\mathbf{K}}$ defined as $\mathbf{F}_\psi(g) = \rho(g)\psi$ is smooth for all fixed $\psi \in \mathcal{D}$. As such we thus can apply the tangent vector $X \in \mathbf{B}\mathfrak{g} \cong \mathbf{B}T_eG$ to this function to obtain an element of $\mathcal{D}$, which shows that the element $\tau(X)\psi$ is well defined. Because $\rho(g)$ in linear, it is immediate that $\tau(X)\psi$ is right-linear in ψ for fixed $X \in \mathbf{B}\mathfrak{g}$, and thus we have indeed a map $\tau\colon \mathbf{B}\mathfrak{g} \to \mathrm{End}(\mathcal{D})$. Moreover, applying a tangent vector is a linear operation, from which it follows immediately that τ is left-linear in X. Now if X and ψ are homogeneous, $\mathbf{F}_\psi$ is homogeneous of the same parity as ψ and hence $\tau(X)\psi = X\mathbf{F}_\psi$ is homogeneous of parity $|X| + |\psi|$. This shows that τ is even.

Remains to prove that this τ is a graded Lie algebra morphism satisfying the properties (i)–(iii). To start with property (i) we take $X \in \mathbf{B}\mathfrak{g}_0$ and $\psi \in \mathcal{D}$ and we note that (more or less by definition of the action of a tangent vector on a (smooth) function) $\tau(X)\psi$ can be defined as the derivative at $t = 0$ of the smooth map $t \mapsto \rho\big(\exp(tX)\big)\psi$ from $\mathcal{A}_0$ to $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$. But this is a smooth map of a single **even** coordinate, so it must be the $\mathbf{G}$-extension of “the same” smooth map from $\mathbf{R} \to \mathcal{D}$ [15, III.2.15–17]. Now, as $\exp(tX) \in \mathbf{B}G$ for $t \in \mathbf{R}$ (and $X \in \mathbf{B}\mathfrak{g}_0$), we have (because ρ_o is the extension of $\rho|_{\mathbf{B}G}$ to $\mathcal{H}$, Lemma 3.2), the equality

$$\rho\big(\exp(tX)\big)\psi = \rho_o\big(\exp(tX)\big)\psi\,.$$

Property (i) follows immediately.

For property (ii) we take two homogeneous $\psi, \chi \in \mathcal{D}$ and we note that (by additivity) we only have to prove the graded skew-symmetry property for homogeneous X. We thus start with $X \in \mathbf{B}\mathfrak{g}_0$ and we consider the function $f\colon \mathbf{R} \to \mathbf{C}$ defined by

$$f(t) = \mathscr{S}\Big(\rho\big(\exp(tX)\big)\chi, \rho\big(\exp(tX)\big)\psi\Big)\,.$$

According to (SUR1) this is a constant function. But according to (SHS2) the map $\mathscr{S}$ is sesquilinear continuous, hence smooth (we are here working in the category of ordinary normed vector spaces). And thus, using (SUR2), the map f is smooth as composition of smooth maps. We thus can compute its derivative at $t = 0$, which should be zero, giving

$$\begin{aligned} 0 = f'(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathscr{S}\Big(\rho\big(\exp(tX)\big)\chi, \rho\big(\exp(tX)\big)\psi\Big) \\ &= \mathscr{S}(\tau(X)\chi, \psi) + \mathscr{S}(\chi, \tau(X)\psi)\,. \end{aligned}$$

This is graded skew-symmetry with respect to $\mathscr{S}$ for $X \in \mathbf{B}\mathfrak{g}_0$.

For $X \in \mathbf{B}\mathfrak{g}_1$ we start with the observation that for any smooth map $f\colon \mathcal{A}_1 \to E \otimes \mathcal{A}^{\mathbf{K}}$ for any graded vector space E (over $\mathbf{K}$), we necessarily have the equality

$$f(\xi) = f(0) + \xi \cdot f'(0)\,,$$

with $f(0)$, $f'(0) \in E$. When we apply this to the smooth (by (SUR2)) mapping $f(\xi) = \rho\big(\exp(\xi X)\big)\psi \equiv \mathbf{F}_\psi\big(\exp(\xi X)\big)$ for any $X \in \mathbf{B}\mathfrak{g}_1$ and $\psi \in \mathcal{D}$, we immediately find the equality

$$\rho\big(\exp(\xi X)\big)\psi = \psi + \xi \cdot \tau(X)\psi\,. \tag{2}$$

We then use (SUR1) and the properties of a super scalar product to compute:

$$\begin{aligned}
&\mathscr{S}\big(\chi, \psi\big) = \mathscr{S}\big(\rho\big(\exp(\xi X)\big)\chi, \rho\big(\exp(\xi X)\big)\psi\big) \\
&= \mathscr{S}\big(\chi + \xi \cdot \tau(X)\chi, \psi + \xi \cdot \tau(X)\psi\big) \\
&= \mathscr{S}\big(\chi, \psi\big) + \mathscr{S}\big(\xi \cdot \tau(X)\chi, \psi\big) + \mathscr{S}\big(\chi, \xi \cdot \tau(X)\psi\big) \\
&= \mathscr{S}\big(\chi, \psi\big) + \Big(\mathscr{S}\big(\tau(X)\chi, \psi\big) + (-1)^{|\chi|} \cdot \mathscr{S}\big(\chi, \tau(X)\psi\big)\Big) \cdot \xi \cdot (-1)^{1+|\chi|+|\psi|}\,.
\end{aligned}$$

As this is valid for all $\xi \in \mathcal{A}_1$, it follows (it is one of the essential features of our graded ring $\mathcal{A}$) that we have

$$\mathscr{S}\big(\tau(X)\chi, \psi\big) + (-1)^{|\chi|} \cdot \mathscr{S}\big(\chi, \xi \cdot \tau(X)\psi\big) = 0\,,$$

which is graded skew-symmetry with respect to $\mathscr{S}$ for $X \in \mathbf{B}\mathfrak{g}_1$, finishing the proof of property (ii).

For property (iii) we take $g \in \mathbf{B}G$, which guarantees that $\mathrm{Ad}(g)X \in \mathbf{B}\mathfrak{g}_1$ and $\rho(g^{-1})\psi \in \mathcal{D}$, not in $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$, so we can use (2) to make the computation

$$\begin{aligned}
&\psi + \xi \cdot \tau\big(\mathrm{Ad}(g)X\big)\psi = \rho\Big(\exp\big(\xi \mathrm{Ad}(g)X\big)\Big)\psi = \rho\big(g \cdot \exp(\xi X) \cdot g^{-1}\big)\psi \\
&= \rho(g)\Big(\rho\big(\exp(\xi X)\big)\big(\rho(g^{-1})\psi\big)\Big) = \rho(g)\Big(\rho(g^{-1})\psi + \xi \cdot \tau(X)\big(\rho(g^{-1})\psi\big)\Big) \\
&= \psi + \xi \cdot \big(\rho(g) \circ \tau(X) \circ \rho(g^{-1})\big)\psi\,.
\end{aligned}$$

Again because this is valid for all $\xi \in \mathcal{A}_1$, it follows that we have

$$\tau\big(\mathrm{Ad}(g)X\big)\psi = \big(\rho(g) \circ \tau(X) \circ \rho(g^{-1})\big)\psi\,.$$

As ρ_o is the extension of $\rho|_{\mathbf{B}G}$ to $\mathcal{H}$, it follows that for $g \in \mathbf{B}G$ and $\psi \in \mathcal{D}$ we have the equality $\rho(g)\psi = \rho_o(g)\psi$, proving property (iii).

And thus it only remains to show that τ is a graded Lie algebra morphism. In order to do so, we will need the following preliminary result. Let $A\colon \mathcal{D} \to \mathcal{D}$ be a homogeneous linear map, graded skew-symmetric with respect to $\mathscr{S}$ and let $X \in \mathbf{B}\mathfrak{g}_0$. Then we have the equality

$$\frac{\partial}{\partial x}\Big|_{x=0} A\Big(\rho\big(\exp(xX)\big)\psi\Big) = A\big(\tau(X)\psi\big)\,. \tag{3}$$

The reason that this is not as obvious as it looks, is because we do not assume that A is continuous. To circumvent that problem, we use the graded skew-symmetry of A and the continuity of $\mathscr{S}$ to compute, with an arbitrary $\chi \in \mathcal{D}$:

$$\begin{aligned}\mathscr{S}\Big(\frac{\partial}{\partial x}\Big|_{x=0} A\Big(\rho\big(\exp(xX)\big)\psi\Big), \chi\Big) &= \frac{\partial}{\partial x}\Big|_{x=0} \mathscr{S}\Big(A\Big(\rho\big(\exp(xX)\big)\psi\Big), \chi\Big) \\ &= -\frac{\partial}{\partial x}\Big|_{x=0} \mathscr{S}\Big(\rho\big(\exp(xX)\big)\mathfrak{C}^{|A|}(\psi), A\chi\Big) \\ &= -\mathscr{S}\Big(\tau(X)\big(\mathfrak{C}^{A}(\psi)\big), A\chi\Big) = \mathscr{S}\Big(A\big(\tau(X)\psi\big), \chi\Big)\,.\end{aligned}$$

The result then follows from the non-degeneracy property of $\mathscr{S}$.

Next we recall that the commutator of two vector fields can be realised as a second order derivative involving their flows: if the vector fields X and Y have flows ϕ_t^X and ϕ_s^Y, then we have

$$[X,Y]_m = \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} (\phi_{-t}^X \circ \phi_s^Y \circ \phi_t^X)(m) = \frac{\partial}{\partial t}\Big|_{t=0} (T\phi_{-t}^X)(Y|_{\phi_t^X(m)})\,.$$

This classical result remains valid for (smooth) homogeneous vector fields in the super context [15, V.5.15]. We thus can apply it to left-invariant vector fields associated to homogeneous $X, Y \in \mathbf{B}\mathfrak{g}$, whose flows are right-multiplication by $\exp(tX)$ respectively $\exp(sY)$ with t and s of the appropriate parity.

Having said this, we start with $\psi \in \mathcal{D}$, $X \in \mathbf{B}\mathfrak{g}_0$ and $Y \in \mathbf{B}\mathfrak{g}$ homogeneous and we compute:

$$\begin{aligned}\tau([X,Y])\psi &= \frac{\partial}{\partial x}\Big|_{x=0} \frac{\partial}{\partial y}\Big|_{y=0} \rho\big(\exp(xX)\,\exp(yY)\,\exp(-xX)\big)\psi \\ &= \frac{\partial}{\partial x}\Big|_{x=0} \frac{\partial}{\partial y}\Big|_{y=0} \rho\big(\exp(xX)\big)\rho\big(\exp(yY)\big)\rho\big(\exp(-xX)\big)\psi \\ {\scriptstyle x \in \mathbf{R} \text{ and } \rho_o = \rho|_{\mathbf{B}G}} \quad &= \frac{\partial}{\partial x}\Big|_{x=0} \frac{\partial}{\partial y}\Big|_{y=0} \rho_o\big(\exp(xX)\big)\rho\big(\exp(yY)\big)\rho\big(\exp(-xX)\big)\psi \\ {\scriptstyle \rho_o(g) \text{ is continuous}} \quad &= \frac{\partial}{\partial x}\Big|_{x=0} \rho_o\big(\exp(xX)\big)\tau(Y)\rho\big(\exp(-xX)\big)\psi \\ {\scriptstyle (3) \text{ with } A = \tau(Y)} \quad &= \tau(X)\big(\tau(Y)\psi\big) + \tau(Y)\big(\tau(-X)\psi\big) = \big[\tau(X), \tau(Y)\big]\psi\,.\end{aligned}$$

This proves the morphism property for $X \in \mathbf{B}\mathfrak{g}_0$ and $Y \in \mathbf{B}\mathfrak{g}$ homogeneous.

By bilinearity and graded skew-symmetry of the graded Lie bracket, it now suffices to show the morphism property for $X, Y \in \mathbf{B}\mathfrak{g}_1$, for which the computations start similarly, but now with odd coordinates:

$$
\begin{aligned}
\tau([X,Y])\psi &= \frac{\partial}{\partial\xi}\Big|_{\xi=0}\frac{\partial}{\partial\eta}\Big|_{\eta=0}\rho\big(\exp(\xi X)\big)\rho\big(\exp(\eta Y)\big)\rho\big(\exp(-\xi X)\big)\psi \\
&\overset{(2)}{=} \frac{\partial}{\partial\xi}\Big|_{\xi=0}\frac{\partial}{\partial\eta}\Big|_{\eta=0}\big(\mathbf{1}+\xi\cdot\tau(X)\big)\big(\mathbf{1}+\eta\cdot\tau(Y)\big)\big(\mathbf{1}+\xi\cdot\tau(-X)\big)\psi \\
&= \frac{\partial}{\partial\xi}\Big|_{\xi=0}\frac{\partial}{\partial\eta}\Big|_{\eta=0}\psi+\eta\cdot\tau(Y)\psi+\eta\xi\cdot\big(\tau(X)\tau(Y)+\tau(Y)\tau(X)\big)\psi \\
&= \big(\tau(X)\tau(Y)+\tau(Y)\tau(X)\big)\psi = \big[\tau(X),\tau(Y)\big]\psi\,,
\end{aligned}
$$

which shows the graded morphism property for $X,Y\in\mathbf{B}\mathfrak{g}_1$, finishing the proof that τ is a graded Lie algebra morphism. ■

In order to prove the "converse-part" of Theorem 3.6, we need a preliminary result, which in turn needs several preliminary definitions.

Definition 3.7. Let E be a finite dimensional graded vector space of dimension $d|n$ over $\mathcal{A}^{\mathbf{K}}$ with homogeneous basis $e_1,\dots,e_d,f_1,\dots,f_n$, i.e., $e_1,\dots,e_d$ is a basis of $\mathbf{B}E_0$ and $f_1,\dots,f_n$ is a basis of $\mathbf{B}E_1$. It is immediate that an element x of E_0 retains information about the odd basis vectors because x is determined by d elements of $\mathcal{A}^{\mathbf{K}}_0$ (its coordinates with respect to the even basis vectors e_i) and n elements of $\mathcal{A}^{\mathbf{K}}_1$ (its coordinates with respect to the odd basis vectors f_j). But sometimes we wish to retain only the even or only the odd basis vectors. We thus define the subspaces $E^{\text{odd dim.}}$ and $E^{\text{even dim.}}$ by

$$
E^{\text{even dim.}} = \Big\{\sum_{i=1}^d x_i\,e_i \;\Big|\; x_i\in\mathcal{A}^{\mathbf{K}}\Big\} \quad\text{and}\quad E^{\text{odd dim.}} = \Big\{\sum_{j=1}^n y_j\,f_j \;\Big|\; y_j\in\mathcal{A}^{\mathbf{K}}\Big\}\,.
$$

A more intrinsic definition would be

$$
\begin{aligned}
E^{\text{even dim.}} &= \mathbf{B}E_0\otimes\mathcal{A}^{\mathbf{K}} \equiv (\mathbf{B}E_0\oplus\{0\})\otimes\mathcal{A}^{\mathbf{K}} \\
E^{\text{odd dim.}} &= \mathbf{B}E_1\otimes\mathcal{A}^{\mathbf{K}} \equiv (\{0\}\oplus\mathbf{B}E_1)\otimes\mathcal{A}^{\mathbf{K}}\,,
\end{aligned}
$$

but the previous one is easier to understand. Both $E^{\text{even dim.}}$ and $E^{\text{odd dim.}}$ are graded subspaces of E and we have in particular the following equalities:

$$
E = E^{\text{even dim.}}\oplus E^{\text{odd dim.}} \qquad\text{and}\qquad E_0 = E_0^{\text{even dim.}}\oplus E_0^{\text{odd dim.}}\,.
$$

Construction 3.8. (Forgetting the odd coordinates) Let f be a (local) smooth function of d even coordinates and n odd ones. There thus exist ordinary smooth functions f_I, $I\subset\{1,\dots,n\}$, of d real coordinates such that (see Definition 2.1)

$$
f(x,\xi) = \sum_{I\subset\{1,\dots,n\}} \xi^I\,(\mathbf{G}f_I)(x)\,.
$$

The body map $\mathbf{B}$ is defined on smooth maps and then yields what remains when we "kill" all nilpotent elements. In particular we have $\mathbf{B}f = f_\emptyset$.

When we now apply $\mathbf{G}$, we get a function of d even coordinates, given by

$$(\mathbf{GB}f)(x) = (\mathbf{G}f_\emptyset)(x) = f(x,0)\,.$$

It follows that the combination of operations $\mathbf{B}$ and $\mathbf{G}$ first kills all nilpotent elements (and in particular the odd coordinates) and then reestablishes the nilpotent parts in the even coordinates. In other words, we get a smooth function of even coordinates only, which is the function f that has forgotten about the dependence on the odd coordinates. The important observation is that these two operations are "intrinsic" and thus preserve in particular equalities involving the composition of maps.

Now let M be any ordinary manifold of dimension d. We then can take an atlas with charts U_a, $a \in A$ and extend all coordinate changes φ^{ab} to functions $\mathbf{G}\varphi^{ab}$ between open sets of $\mathcal{A}_0^d$. This defines an $\mathcal{A}$-manifold $\mathbf{G}M$ of dimension $d|0$. Conversely we can start with an $\mathcal{A}$-manifold M of dimension $d|n$, take its body to obtain an ordinary manifold $\mathbf{B}M$ of dimension d on which we can apply the above construction to obtain an $\mathcal{A}$-manifold $\mathbf{GB}M$ of dimension $d|0$. Looking carefully at the construction of $\mathbf{GB}M$ will show that it boils down to start with an atlas with coordinate charts and then to replace the transition functions φ^{ab} by $\mathbf{GB}\varphi^{ab}$. Said differently, we take an atlas and then forget about the odd coordinates (setting them to zero) and we retain only the even coordinates. What should be obvious is that if the $\mathcal{A}$-manifold M is modelled on the graded vector space E, then $\mathbf{GB}M$ is modelled on $E^{\text{even dim.}}$. For that reason we also denote the $\mathcal{A}$-manifold $\mathbf{GB}M$ as $M^{\text{even dim.}}$.

An important application of this notion is to an $\mathcal{A}$-Lie group G. An $\mathcal{A}$-Lie group G is modelled on its corresponding graded Lie algebra $\mathfrak{g} \cong T_eG$ over $\mathcal{A}$ and one can show that the map $\Phi\colon G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}} \to G$ defined by

$$\Phi(g,X) = g\cdot \exp(X) \tag{4}$$

is a diffeomorphism of $\mathcal{A}$-manifolds [15, VI.1.7]. This means that, as an $\mathcal{A}$-manifold, an $\mathcal{A}$-Lie group G is completely determined by the underlying ordinary Lie group $\mathbf{B}G$ and the odd part $\mathbf{B}\mathfrak{g}_1$ of its graded Lie algebra $\mathfrak{g}$, simply because we have the equalities

$$G^{\text{even dim.}} = \mathbf{GB}G \qquad \text{and} \qquad \mathfrak{g}^{\text{odd dim.}} = \mathbf{B}\mathfrak{g}_1 \otimes \mathcal{A}\,.$$

We will use the identification/diffeomorphism (4) quite often in the sequel. As such it will turn out to be important to know how the product of two group elements is given in this interpretation of G as a direct product. It is immediate that we have, for $g,h \in G^{\text{even dim.}}$ and $X,Y \in \mathfrak{g}_0^{\text{odd dim.}}$:

$$\Phi(g,x)\cdot\Phi(h,Y) = g\cdot\exp(X)\cdot h\cdot\exp(Y) = gh\cdot\exp\big(\mathrm{Ad}(h^{-1})X\big)\cdot\exp(Y)\,.$$

Now $\mathrm{Ad}(h^{-1})X$ and Y belong to $\mathfrak{g}_0^{\text{odd dim.}}$, but the product $\exp\big(\mathrm{Ad}(h^{-1})X\big)\cdot\exp(Y)$ does not necessarily lie in the image of $\mathfrak{g}_0^{\text{odd dim.}}$ under $\exp$. The purpose of Lemma 3.10 is to provide, for $X,Y \in \mathfrak{g}_0^{\text{odd dim.}}$, two elements $B^{(0)}(X,Y) \in \mathfrak{g}_0^{\text{even dim.}}$ and $B^{(1)}(X,Y) \in \mathfrak{g}_0^{\text{odd dim.}}$ such that we have

$$\exp(X)\cdot\exp(Y) = \exp\big(B^{(0)}(X,Y)\big)\cdot\exp\big(B^{(1)}(X,Y)\big)\,.$$

And then we can finish our computation of the group product in terms of the diffeomorphism Φ:

$$\Phi(g,X)\cdot\Phi(h,Y) = \Phi\Big(gh\,\exp\big(B^{(0)}(\mathrm{Ad}(h^{-1})X,Y)\big)\ ,\ B^{(1)}(\mathrm{Ad}(h^{-1})X,Y)\ \Big)$$

or equivalently, omitting the map Φ:

$$(g,X)\cdot(h,Y) = \Big(gh\,\exp\big(B^{(0)}(\mathrm{Ad}(h^{-1})X,Y)\big)\ ,\ B^{(1)}(\mathrm{Ad}(h^{-1})X,Y)\ \Big)\,.$$

Definition 3.9. Let $\mathfrak{g}$ be a Lie algebra (graded or not) and let $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ be a map. We will say that B is a *homogeneous algebraic expression in repeated commutators of degree* $k\geq 2$ if there exist constants $c_{i_1,\dots,i_k}\in\mathbf{R}$, $i_j=1,2$ such that for all $X_1,X_2\in\mathfrak{g}$ we have

$$\begin{aligned} B(X_1,X_2) &= \sum_{i_1,\dots,i_k=1}^{2} c_{i_1,\dots,i_k}\cdot\big(\mathrm{ad}(X_{i_k})\circ\cdots\circ\mathrm{ad}(X_{i_2})\big)(X_{i_1}) \\ &= \sum_{i_1,\dots,i_k=1}^{2} c_{i_1,\dots,i_k}\cdot\Big[X_{i_k},\Big[X_{i_{k-1}},\dots\big[X_{i_3},[X_{i_2},X_{i_1}]\big]\dots\Big]\Big]\,. \end{aligned} \tag{5}$$

We will say that B is an *algebraic expression in repeated commutators* (abbreviated as AERC) if there exist a sequence B_k, $k\geq 2$ of homogeneous algebraic expressions in repeated commutators B_k of degree k such that we have $B(X,Y)=\sum_{k\geq 2}B_k(X,Y)$. The typical example of an AERC is the Baker-Campbell-Hausdorff formula. This is the map $\mathcal{B}_{\mathrm{BCH}}\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that

$$\exp(X)\cdot\exp(Y) = \exp\big(X+Y+\mathcal{B}_{\mathrm{BCH}}(X,Y)\big)\,.$$

It starts as (see for instance [6, p. 31])

$$\begin{aligned} \mathcal{B}_{\mathrm{BCH}}(X,Y) = \tfrac{1}{2}\,[X,Y] + \tfrac{1}{12}\,\Big(\,\big[X,[X,Y]\big]+\big[Y,[Y,X]\big]\,\Big) \\ + \tfrac{1}{24}\,\Big[Y,\big[X,[Y,X]\big]\Big]+\cdots\,. \end{aligned}$$

Lemma 3.10. *Let $\mathfrak{g}$ be a graded Lie algebra of finite dimension $d|n$. Then there exist AERC's $B^{(0)}(X,Y)$ and $B^{(1)}(X,Y)$ with the following properties.*

(i) *Each term in $B^{(0)}$ contains only an even number of commutators (only even k in (5)) and each term in $B^{(1)}$ contains only an odd number of commutators (only odd k in (5)).*

(ii) *For $X,Y\in\mathfrak{g}_0^{\text{odd dim.}}$ we have the equality*

$$\exp(X)\cdot\exp(Y) = \exp\big(B^{(0)}(X,Y)\big)\cdot\exp\big(X+Y+B^{(1)}(X,Y)\big)\,.$$

Proof. The proof is a "simple" application of the Baker-Campbell-Hausdorff formula. We start by writing

$$B^{(0)}(X,Y)=\sum_{k=1}^{\infty}B_{2k}(X,Y)\quad\text{and}\quad B^{(1)}(X,Y)=\sum_{k=1}^{\infty}B_{2k+1}(X,Y)$$

and apply the Baker-Campbell-Hausdorff formula on both sides of the wanted equality

$$\exp(X) \cdot \exp(Y) = \exp\big(B^{(0)}(X,Y)\big) \cdot \exp\big(X + Y + B^{(1)}(X,Y)\big)\,.$$

This gives us the equality

$$\begin{aligned} \exp\big(X + Y + \mathcal{B}_{\mathrm{BCH}}(X,Y)\big) = \exp\Big(&B^{(0)}(X,Y) + X + Y + B^{(1)}(X,Y) \\ &+\mathcal{B}_{\mathrm{BCH}}\big(B^{(0)}(X,Y)\,,\, X + Y + B^{(1)}(X,Y)\big)\Big)\,. \end{aligned} \tag{6}$$

For an ordinary (non-super) Lie algebra we should worry about convergence and we will certainly not be allowed to deduce (without careful justification) the equality

$$\begin{aligned} \mathcal{B}_{\mathrm{BCH}}(X,Y) = &B^{(0)}(X,Y) + B^{(1)}(X,Y) \\ &+\mathcal{B}_{\mathrm{BCH}}\big(B^{(0)}(X,Y)\,,\, X + Y + B^{(1)}(X,Y)\big)\,. \end{aligned} \tag{7}$$

However, X and Y belong to $\mathfrak{g}_0^{\text{odd dim.}}$ and thus are of the form $\sum_i \xi_i X_i$ with $X_i \in \mathbf{B}\mathfrak{g}_1$ and $\xi_i \in \mathcal{A}_1$. In particular these coefficients are nilpotent. It follows immediately that any homogeneous AERC $f(X,Y)$ of degree k is identically zero for any $k > 2n$. The argument is that with respect to a basis $f_1, \dots, f_n$ of $\mathfrak{g}_1$ the two vectors X and Y are expressed with (at most) $2n$ different odd coefficients. And thus any term involving more than $2n$ terms must contain one of these odd coefficients twice (multiplicatively), proving that such a term is zero.
This has two consequences: first that all AERC's in X and Y only contain a finite number of homogeneous terms, so no convergence problems arise. And secondly that the arguments of the exponential function in (6) have zero body and thus lie in the neighbourhood where the exponential map is a diffeomorphism. We thus are allowed to deduce the equality (7), which we (re)write as

$$\begin{aligned} \mathcal{B}_{\mathrm{BCH}}(X,Y) = &\sum_{k\geq 2} B_k(X,Y) \\ &+ \mathcal{B}_{\mathrm{BCH}}\Big(\sum_{k\geq 1} B_{2k}(X,Y)\,,\, X + Y + \sum_{k\geq 1} B_{2k+1}(X,Y)\Big)\,. \end{aligned} \tag{8}$$

We then note that if B_k and B_ℓ are homogeneous AERC's of degree k and ℓ respectively, then

$$\big[B_k(X,Y), B_\ell(X,Y)\big]$$

is a homogeneous AERC of degree $k+\ell$. It follows that we can determine the homogeneous AERC's B_k iteratively from (8), starting with B_2. The reason this works is because if we equate the homogeneous AERC's of degree k in (8), we find on the left hand side the homogeneous AERC of degree k of $\mathcal{B}_{\mathrm{BCH}}(X,Y)$, which is known. On the right hand side we first find $B_k(X,Y)$, followed by all homogeneous AERC's of degree k coming from the $\mathcal{B}_{\mathrm{BCH}}$ term. But as each term of $\mathcal{B}_{\mathrm{BCH}}$ contains at least one commutator, it follows that the terms of degree k can only involve B_i's with $i < k$.
To see how this works, let us determine the first few terms. Looking at the term $\mathcal{B}_{\mathrm{BCH}}$ on the right hand side of (8), we note that its first argument starts at degree

2 and its second argument at degree 1 (the $X+Y$ term). As $\mathcal{B}_{\mathrm{BCH}}$ itself starts at degree 2, it follows that this term starts at degree 3. It follows that we can equate $B_2(X,Y)$ with the degree 2 term of $\mathcal{B}_{\mathrm{BCH}}(X,Y)$ on the left hand side of (8), i.e.,

$$B_2(X,Y) = \tfrac{1}{2}\,[X,Y]\,.$$

The only term of degree 3 appearing in the $\mathcal{B}_{\mathrm{BCH}}$ term on the right hand side of (8) comes from the degree 2 term of $\mathcal{B}_{\mathrm{BCH}}$ and is given by

$$\tfrac{1}{2}\left[B_2(X,Y), X+Y\right],$$

which gives us the equality

$$\tfrac{1}{12}\Big(\big[X,[X,Y]\big] + \big[Y,[Y,X]\big]\Big) = B_3(X,Y) + \tfrac{1}{2}\left[B_2(X,Y), X+Y\right].$$

As we already know $B_2(X,Y)$, this determines $B_3(X,Y)$ as

$$B_3(X,Y) = \tfrac{1}{3}\big[X,[X,Y]\big] - \tfrac{1}{6}\big[Y,[Y,X]\big]\,.$$

Careful analysis shows that the only term of degree 4 appearing in the $\mathcal{B}_{\mathrm{BCH}}$ term on the right hand side of (8) comes from the degree 3 term of $\mathcal{B}_{\mathrm{BCH}}$ applied to $B_2(X,Y)$ and $X+Y$; it is given by

$$\tfrac{1}{12}\big[X+Y,[X+Y,B_2(X,Y)]\big]\,.$$

This gives us the equality

$$\tfrac{1}{24}\Big[Y,\big[\,X,[Y,X]\,\big]\Big] = B_4(X,Y) + \tfrac{1}{12}\big[X+Y,[X+Y,B_2(X,Y)]\big]\,,$$

which tells us that $B_4(X,Y)$ is given by

$$B_4(X,Y) = \tfrac{1}{8}\Big[Y,\big[\,X,[Y,X]\,\big]\Big] - \tfrac{1}{24}\Big[X,\big[\,X,[X,Y]\,\big]\Big] + \tfrac{1}{24}\Big[Y,\big[\,Y,[Y,X]\,\big]\Big]\,.$$

As $B^{(0)}$ and $B^{(1)}$ satisfy by construction the properties (i) and (ii), this finishes the proof. ■

Proof of Theorem 3.6, converse part. Given a triple $(\rho_o, \mathcal{D}, \tau)$ satisfying conditions (i)–(iii), we have to define a representation ρ of G on $\mathcal{D}\otimes\mathcal{A}^{\mathbf{K}}$ satisfying the conditions (SUR1)–(SUR3) and to show that it is unique when τ is given by (1). The idea is to extend the representation ρ_o of $\mathbf{B}G$ in two stages to the sought-for representation ρ: first to $G^{\text{even dim.}}$ and then to G, and to prove at each stage that the extension preserves $\mathscr{S}$. In that way (SUR1) and (SUR3) are automatically satisfied. Proving (SUR2) then turns out not to be very hard, only the homomorphism property requires some work.

We start by proving that ρ_o preserves $\mathscr{S}$. For $\chi,\psi\in\mathcal{D}$ and $X\in\mathbf{B}\mathfrak{g}_0$ we consider the function $f\colon\mathbf{R}\to\mathbf{C}$ defined by

$$f(t) = \mathscr{S}\Big(\rho_o\big(\exp(tX)\big)\chi, \rho_o\big(\exp(tX)\big)\psi\Big)\,.$$

As $\mathscr{S}$ is sequilinear continuous, it is smooth (because, as in the proof of the direct part, we are here working in the category of ordinary normed vector spaces), so f is smooth by composition of smooth maps and its derivative is given by (using property (i) of τ)

$$f'(t) = \mathscr{S}\Big(\tau(X)\rho_o\big(\exp(tX)\big)\chi, \rho_o\big(\exp(tX)\big)\psi\Big) \\ + \mathscr{S}\Big(\rho_o\big(\exp(tX)\big)\chi, \tau(X)\rho_o\big(\exp(tX)\big)\psi\Big) .$$

By graded skew-symmetry of the $\tau(X)$ with respect to $\mathscr{S}$ (condition (ii) this is zero, and hence f is constant. But exp defines a diffeomorphism between a neighborhood of $0 \in \mathfrak{g}_0$ and $\mathbf{B}G$. And thus for all g in this neighborhood the map $\rho_o(g)$ preserves $\mathscr{S}$. By the representation property and connectedness of G it follows that it is true for all $g \in \mathbf{B}G$.

We next extend the representation ρ_o of $\mathbf{B}G$ to a map (also denoted by ρ_o) $\rho_o\colon G^{\text{even dim.}} \equiv \mathbf{GB}G \to \operatorname{End}\big(\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}\big)$ as follows. For each $\psi \in \mathcal{D}$ the map $\mathbf{F}_\psi\colon \mathbf{B}G \to \mathcal{D}$ is smooth by the assumption $\mathcal{D} \subset C^\infty(\rho_o)$. Hence its $\mathbf{G}$-extension (denoted by the same symbol) $\mathbf{F}_\psi\colon G^{\text{even dim.}} \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$ is a smooth map in the category of $\mathcal{A}$-manifolds [15, III.2.15&III.5.24]. It is elementary to show that for any $g \in G^{\text{even dim.}}$, the map $\psi \mapsto \mathbf{F}_\psi(g)$ is linear in ψ. For each $g \in G^{\text{even dim.}}$ we thus have a linear map (over $\mathbf{K}$) $\rho_o(g)\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$. Moreover, as $\rho_o(g)$ preserves the subspaces $\mathcal{H}_\alpha$, so does its $\mathbf{G}$-extension. Extending this $\rho_o(g)$ by $\mathcal{A}^{\mathbf{K}}$-linearity to $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$, we obtain an even right-linear map (over $\mathcal{A}^{\mathbf{K}}$)

$$\rho_o(g)\colon \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}} \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}} .$$

As we have the equality $\rho_o(gh)\psi = \rho_o(g)\big(\rho_o(h)\psi\big)$ for all $g, h \in \mathbf{B}G$, this property extends to the $\mathbf{G}$-extension, proving that we have a morphism of groups

$$\rho_o\colon G^{\text{even dim.}} \to \operatorname{Aut}\big(\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}\big) .$$

In other words, we have a representation of $G^{\text{even dim.}}$ on $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$.
In order to prove that this extension preserves $\mathscr{S}$ (or rather its extension to $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$), we argue as follows. For $\chi, \psi \in \mathcal{D}$ we have shown that the function $f\colon \mathbf{B}G \to \mathbf{C}$ defined as

$$f(g) = \mathscr{S}\big(\rho_o(g)\chi, \rho_o(g)\psi\big)$$

is constant $\mathscr{S}(\chi, \psi)$. This implies that in any local coordinate system $(r_1, \ldots, r_d)$ on $\mathbf{B}G$ all k-th order partial derivatives $\partial_{j_1} \ldots \partial_{j_k} f$ are zero. But we can also compute $\mathbf{G}f$, the $\mathbf{G}$-extension of f, directly from its definition in terms of the $\mathbf{G}$-extension of $\mathbf{F}_\chi$ and $\mathbf{F}_\psi$ and the extension of $\mathscr{S}$ to $\mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$. When we do this at a point with local coordinates $x_i \in \mathcal{A}_0$ with $x_i = r_i + n_i$, $r_i \in \mathbf{R}$ and n_i nilpotent (and even), we obtain:

$$
\begin{aligned}
(\mathbf{G}f)(x_1,\dots,x_d) &= \mathscr{S}\Big(\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i_1,\dots,i_k=1}^{d}\partial_{i_1}\dots\partial_{i_k}\mathbf{F}_\chi(r)\,n_{i_1}\cdots n_{i_k}\,,\\
&\qquad\qquad \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1,\dots,j_\ell=1}^{d}\partial_{j_1}\dots\partial_{j_\ell}\mathbf{F}_\psi(r)\,n_{j_1}\cdots n_{j_\ell}\Big)\\
&= \sum_{p=0}^{\infty}\frac{1}{p!}\,\partial_{i_1}\dots\partial_{i_p}\Big(\mathscr{S}\big(\mathbf{F}_\chi(r),\mathbf{F}_\psi(r)\big)\Big)\,n_{i_1}\cdots n_{i_p}\\
&\equiv \sum_{p=0}^{\infty}\frac{1}{p!}\,(\partial_{i_1}\dots\partial_{i_p}f)(r)\,n_{i_1}\cdots n_{i_p} = \mathscr{S}\big(\mathbf{F}_\chi(r),\mathbf{F}_\psi(r)\big) = \mathscr{S}\big(\chi,\psi\big)\,,
\end{aligned}
$$

which shows that indeed $\rho_o(g)$ preserves $\mathscr{S}$ for all $g \in G^{\text{even dim.}}$ as wanted.[2]

We now recall that the map $\Phi(g,X) = g\cdot\exp(X)$ (4) defines a diffeomorphism $G^{\text{even dim.}}\times\mathfrak{g}_0^{\text{odd dim.}}\to G$. We will use (mostly implicitly) this diffeomorphism to extend the representation ρ_o of $G^{\text{even dim.}}$ to a representation ρ of G on $\mathcal{D}\otimes\mathcal{A}^{\mathbf{K}}$ by defining ρ on such an element by

$$
\rho\big(g\cdot\exp(X)\big)\psi = \rho_o(g)\circ\exp\big(\tau(X)\big)\psi\,,
$$

where $\exp\big(\tau(X)\big)$ is defined by its power series:

$$
\exp\big(\tau(X)\big)\psi = \sum_{k=0}^{\infty}\frac{1}{k!}\,\tau(X)^k(\psi)\,.
$$

More precisely, any $X\in\mathfrak{g}_0^{\text{odd dim.}}$ is necessarily of the form

$$
X = \sum_{i=1}^{n} f_i\otimes\xi_i \equiv \sum_{i=1}^{n} f_i\cdot\xi_i
$$

with $f_1,\dots,f_n$ a basis of $\mathbf{B}\mathfrak{g}_1$ and $\xi_i\in\mathcal{A}_1$. The map $\tau(X)\in\mathrm{End}\big(\mathcal{D}\otimes\mathcal{A}^{\mathbf{K}}\big)$ then is defined by linearity as

$$
\tau(X) = \sum_{i=1}^{n}\tau(f_i)\cdot\xi_i \quad\text{or equivalently}\quad \tau(X)(\psi\otimes\lambda) = -\sum_{i=1}^{n}\xi_i\cdot\big(\tau(f_i)\psi\big)\otimes\lambda\,,
$$

where we used that τ is even and f_i odd, which explains the minus sign in the second formula. Because X contains n odd elements, it follows immediately that the power series for $\exp\big(\tau(X)\big)$ actually stops at $k=n$, any higher order containing at least the square of one of these odd elements, which yields zero. And finally we note that the map $\exp\big(\tau(X)\big)$ is even, simply because each map $\tau(f_i)$ is odd, as well as its coefficient ξ_i.

---

[2] The second equality is a combinatorial fact using that $\mathscr{S}$ is sesquilinear continuous (condition SHS**??**) and thus that we have, for any partial derivative, the equality $\partial_i\big(\mathscr{S}\big(f(x),g(x)\big)\big) = \mathscr{S}\big((\partial_i f)(x),g(t)\big)+\mathscr{S}\big(f(x),(\partial_i g)(x)\big)$. Writing out the details is a bit cumbersome, so I hope the reader will forgive me for omitting them.

In order to show that the maps $\mathbf{F}_\psi : G \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$, $g \mapsto \rho(g)\psi$ are smooth for $\psi \in \mathcal{D}$ we argue as follows. Using the diffeomorphism $G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}} \cong G$ we have to prove smoothness in the couple $(g, X) \in G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}} \cong G$. Now the expression $\exp\big(\tau(X)\big)\psi$ is given by

$$\exp\big(\tau(X)\big)\psi = \sum_{k=0}^{n} \frac{1}{k!}\,\tau(X)^k\psi = \sum_{I\subset\{1,\dots,n\}} \xi^I \psi_I \,,$$

with $\psi_I \in \mathcal{D}$ an element of the form (with $c_I \in \mathbf{R}$)

$$\psi_I = c_I\,\tau(f_{i_1}) \circ \cdots \circ \tau(f_{i_k})\psi \,.$$

It follows that $\mathbf{F}_\psi(g, X) = \rho(g, X)\psi$ (with $(g, X) \in G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}} \cong G$) is given by

$$\mathbf{F}_\psi(g, X) = \sum_{I\subset\{1,\dots,n\}} \xi^I \rho_o(g)\psi_I \,.$$

As $\rho_o(g)\psi$ is smooth in $g \in G^{\text{even dim.}}$, this shows that $\mathbf{F}_\psi$ is indeed a smooth function as required for (SUR2).

To show that this ρ is a homomorphism, we take the elements $g, h \in G^{\text{even dim.}}$ and $X, Y \in \mathfrak{g}_0^{\text{odd dim.}}$ and we compute (writing $Z = \mathrm{Ad}(h^{-1})X \in \mathfrak{g}_0^{\text{odd dim.}}$ in the second line):

$$\begin{aligned}
\rho\big(g\cdot\exp(X)\cdot h\cdot\exp(Y)\big) &= \rho\Big(gh\cdot\exp\big(\mathrm{Ad}(h^{-1})X\big)\cdot\exp(Y)\Big)\\
&\equiv \rho\big(gh\cdot\exp(Z)\cdot\exp(Y)\big)\\
&\overset{[3.10]}{=} \rho\Big(gh\cdot\exp\big(B^{(0)}(Z,Y)\big)\cdot\exp\big(Z+Y+B^{(1)}(Z,Y)\big)\Big)\\
&\overset{\text{def.}}{=} \rho_o\Big(gh\cdot\exp\big(B^{(0)}(Z,Y)\big)\Big)\circ\exp\Big(\tau\big(Z+Y+B^{(1)}(Z,Y)\big)\Big)\\
&= \rho_o(gh)\circ\rho_o\Big(\exp\big(B^{(0)}(Z,Y)\big)\Big)\circ\exp\Big(\tau\big(Z+Y+B^{(1)}(Z,Y)\big)\Big)\\
&= \rho_o(gh)\circ\exp\Big(\tau\big(B^{(0)}(Z,Y)\big)\Big)\circ\exp\Big(\tau\big(Z+Y+B^{(1)}(Z,Y)\big)\Big)\\
&= \rho_o(gh)\circ\exp\Big(B^{(0)}\big(\tau(Z),\tau(Y)\big)\Big)\\
&\qquad\qquad\circ\exp\Big(\tau(Z)+\tau(Y)+B^{(1)}\big(\tau(Z),\tau(Y)\big)\Big)\\
&\overset{[3.10]}{=} \rho_o(gh)\circ\exp\big(\tau(Z)\big)\circ\exp\big(\tau(Y)\big)\\
&= \rho_o(g)\circ\rho_o(h)\circ\exp\big(\tau(Z)\big)\circ\exp\big(\tau(Y)\big)\\
&= \rho_o(g)\circ\exp\big(\rho_o(h)\circ\tau(Z)\circ\rho_o(h)^{-1}\big)\circ\rho_o(h)\circ\exp\big(\tau(Y)\big)\\
&= \rho_o(g)\circ\exp\Big(\tau\big(\mathrm{Ad}(h)Z\big)\Big)\circ\rho_o(h)\circ\exp\big(\tau(Y)\big)\\
&= \rho_o(g)\circ\exp\big(\tau(X)\big)\circ\rho_o(h)\circ\exp\big(\tau(Y)\big)\\
&\overset{\text{def.}}{=} \rho\big(g\cdot\exp(X)\big)\circ\rho\big(h\cdot\exp(Y)\big)\,.
\end{aligned}$$

Let us explain this computation line by line.

line 1: We apply the definition of the Adjoint representation.

line 2: We abbreviate $\mathrm{Ad}(h^{-1})X$ to Z.

line 3: We apply Lemma 3.10.

line 4: We first recall that for any Lie superalgebra we have $[\mathbf{B}\mathfrak{g}_\alpha, \mathbf{B}\mathfrak{g}_1] \subset \mathbf{B}\mathfrak{g}_{1-\alpha}$. It follows directly that necessarily $B^{(0)}(Z,Y) \in \mathfrak{g}_0^{\text{even dim.}}$ and furthermore $B^{(1)}(Z,Y) \in \mathfrak{g}_0^{\text{odd dim.}}$ because $B^{(0)}$ only has repeated commutators of an even number of elements in $\mathfrak{g}_0^{\text{odd dim.}} \equiv (\mathbf{B}\mathfrak{g}_1 \otimes \mathcal{A})_0$ and $B^{(1)}$ only has repeated commutators of an odd number of such elements. In particular $\exp\bigl(B^{(0)}(Z,Y)\bigr)$ belongs to $G^{\text{even dim.}}$, so we can apply our definition of ρ.

line 5: We use that ρ_o is a homomorphism.

line 6: We use the fact that τ is the infinitesimal form of ρ_o on $\mathcal{D}$ by hypothesis (i) and that we have extended ρ_o to $G^{\text{even dim.}}$ by $\mathbf{G}$-extension.

line 7: We use the fact that τ is a graded Lie algebra homomorphism and that $B^{(0)}$ and $B^{(1)}$ are AERC's, so we can commute these two operations.

line 8: We again use Lemma 3.10, which is allowed because this result is independent of the particular $\mathcal{A}$-Lie group.

line 9: We use again that ρ_o is a homomorphism.

line 10: We use the definition of $\exp\bigl(\tau(Z)\bigr)$ as a power series (finite!).

line 11: This is our hypothesis (iii).

line 12: By definition of Z we have the equality $\mathrm{Ad}(h)Z = X$.

line 13: And finally we apply our definition of ρ backwards.

Now that we have the representation ρ, we have to prove that it preserves $\mathscr{S}$. As we already know this to be true for $g \in G^{\text{even dim.}}$, it suffices to prove that we have

$$\mathscr{S}\Bigl(\rho\bigl(\exp(X)\bigr)\chi, \rho\bigl(\exp(X)\bigr)\psi\Bigr) = \mathscr{S}(\chi,\psi)$$

for any $X \in \mathfrak{g}_0^{\text{odd dim.}}$. To do so, we start with the observation that the condition that $\tau(X)$ with $X \in \mathbf{B}\mathfrak{g}_1$ is graded skew-symmetric with respect to $\mathscr{S}$ (condition (ii)) implies that, for $\lambda \in \mathcal{A}_1$, the even map $\tau(\lambda X)$ is skew-symmetric with respect to $\mathscr{S}$ and thus for any $X \in \mathfrak{g}_0^{\text{odd dim.}}$ the (even) map $\tau(X)$ is skew-symmetric. We then compute:

$$\begin{aligned}
\mathscr{S}\bigl(\rho\bigl(\exp(X)\bigr)\chi, \rho\bigl(\exp(X)\bigr)\psi\bigr) &= \mathscr{S}\bigl(\exp\bigl(\tau(X)\bigr)\chi, \exp\bigl(\tau(X)\bigr)\psi\bigr) \\
&= \mathscr{S}\Bigl(\sum_{k=0}^\infty \frac{1}{k!}\,\tau(X)^k\chi, \sum_{\ell=0}^\infty \frac{1}{\ell!}\,\tau(X)^\ell\psi\Bigr) \\
&= \sum_{k=0}^\infty \sum_{\ell=0}^\infty \frac{1}{k!\,\ell!}\cdot \mathscr{S}(\tau(X)^k\chi, \tau(X)^\ell\psi) \\
\text{write } p=k+\ell \qquad &= \sum_{p=0}^\infty \sum_{k=0}^p \frac{1}{k!\,(p-k)!}\,(-1)^k\,\mathscr{S}(\chi, \tau(X)^p\psi) \\
&= \mathscr{S}(\chi,\psi) + \sum_{p=1}^\infty \frac{\mathscr{S}(\chi,\tau(X)^p\psi)}{p!}\cdot\Bigl(\sum_{k=0}^p \frac{p!}{k!\,(p-k)!}\,(-1)^k\cdot 1^{p-k}\Bigr) \\
&= \mathscr{S}(\chi,\psi)\,.
\end{aligned}$$

We now only have to prove uniqueness of ρ to finish the proof of Theorem 3.6. So let $\hat\rho\colon G \to \mathrm{Aut}(\mathcal{D}\otimes\mathcal{A}^{\mathbf{K}})$ be another map satisfying the conditions (SUR1)–(SUR3) with infinitesimal form τ and whose restriction to $\mathbf{B}G$ equals ρ_o. Hence $\hat\rho$ and ρ coincide on $\mathbf{B}G$. But then by (SUR2) the restriction of $\hat\rho$ to $G^{\text{even dim.}}$ must be the $\mathbf{G}$-extension of the restriction to $\mathbf{B}G$, hence $\hat\rho$ coincides with ρ on $G^{\text{even dim.}}$. And finally, for $X \in \mathbf{B}\mathfrak{g}_1$ and $\xi \in \mathcal{A}_1$ we must have by (SUR2) and the fact that τ is the infinitesimal form of $\hat\rho$:

$$\hat\rho\big(\exp(\xi X)\big)\psi = \psi + \xi\tau(X)\psi \overset{\text{def.}}{=} \rho\big(\exp(\xi X)\big)\psi\,.$$

As elements of the form $\exp(\xi X)$ with $X \in \mathbf{B}\mathfrak{g}_1$ and $\xi \in \mathcal{A}_1$ together with elements of $G^{\text{even dim.}}$ generate G, the homomorphism property shows that $\hat\rho$ and ρ coincide at all elements $(g,X) \in G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}} \cong G$. ■

Remark 3.11. The condition in Theorem 3.6 that G should be connected is needed only to prove, in the converse part, that ρ preserves $\mathscr{S}$ if τ satisfies condition (ii). If one knows by other means that the maps $\rho_o(g)$, $g \in \mathbf{B}G$ preserve $\mathscr{S}$, then one can drop the connectedness condition on G.

Corollary 3.12. *Providing the data $(\mathcal{D},\rho)$ of a proto super unitary representation of a (connected) $\mathcal{A}$-Lie group G on a super Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle,\mathscr{S})$ is equivalent to giving a triple $(\rho_o,\mathcal{D},\tau)$ in which ρ_o is an even unitary representation of $\mathbf{B}G$ on $(\mathcal{H},\langle\cdot,\cdot\rangle)$, $\mathcal{D}\subset C^\infty(\rho_o)$ a dense subspace of $\mathcal{H}$ invariant under all maps $\rho_o(g)$, $g\in\mathbf{B}G$ and $\tau\colon \mathbf{B}\mathfrak{g}\to\mathrm{End}(\mathcal{D})$ an even graded Lie algebra morphism satisfying the following properties.*

(i) *For each $X\in\mathbf{B}\mathfrak{g}_0$ the map $\tau(X)$ is the restriction of the infinitesimal generator of $\rho_o\big(\exp(tX)\big)$ to $\mathcal{D}$.*

(ii) *For all $X\in\mathbf{B}\mathfrak{g}$ the map $\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}$.*

(iii) *For all $g\in\mathbf{B}G$ and all $X\in\mathbf{B}\mathfrak{g}_1$ we have*

$$\tau\big(\mathrm{Ad}(g)X\big) = \rho_o(g)\circ\tau(X)\circ\rho_o(g^{-1})\,.$$

Moreover, the couple $(\mathcal{D},\rho)$ will be a super unitary representation if and only if the couple $(\mathcal{D},\tau)$ is maximal with respect to properties (i)–(iii)*, meaning that if $(\mathcal{D}',\tau')$ satisfies the same conditions with $\mathcal{D}\subset\mathcal{D}'$ and $\tau'(X)|_{\mathcal{D}}=\tau(X)$ for all $X\in\mathbf{B}\mathfrak{g}$, then $\mathcal{D}'=\mathcal{D}$ (and thus $\tau'=\tau$).*

4. The Batchelor bundle and metrics

Definition/Notation 4.1. Let $X_1,\dots,X_n$ be n objects that can be multiplied/ composed. Then for any $I\subset\{1,\dots,n\}$ we define

$$X^I = X_{i_1}\circ\cdots\circ X_{i_k}\,,$$

when $I=\{i_1,\dots,i_k\}$ with $i_1<i_2<\cdots<i_k$, with the convention that $X^\emptyset=1$ (or $X^\emptyset=\mathrm{id}$ in case of maps). When the set $\{1,\dots,n\}$ itself appears in an exponent or subscript, we will abbreviate it as $\underline{n}$, i.e., we define $\underline{n}$ to be shorthand for

$$\underline{n}=\{1,\dots,n\}\,.$$

4.1. We start with an $\mathcal{A}$-manifold

Let M be an $\mathcal{A}$-manifold modelled on the even part of an $\mathcal{A}$-vector space E of graded dimension $d|n$ and let $\mathcal{U} = \{ U_a \mid a \in A \}$ be an atlas of coordinate charts $\varphi_a \colon U_a \to O_a \subset E_0$. It is immediate that the maps $\mathbf{B}\varphi_a \colon \mathbf{B}U_a \to \mathbf{B}O_a \subset \mathbf{R}^p$ form an atlas for $\mathbf{B}M$. Denoting by (x^a, ξ^a) the even and odd coordinates on a chart U_a (i.e., the coordinates on $\varphi_a(U_a) = O_a \subset E_0$), the change of coordinates $\varphi^{ba} \equiv \varphi_b \circ \varphi_a^{-1}$ can be written as

$$(x^b, \xi^b) = (\varphi_b \circ \varphi_a^{-1})(x^a, \xi^a) \equiv \varphi^{ba}(x^a, \xi^a) \,.$$

In [2] (see also [15, §IV.8]) it is shown that for any $\mathcal{A}$-manifold there always exists an atlas such that the change of coordinate functions φ^{ba} have the following special form:

$$(x^b, \xi^b) = \varphi^{ba}(x^a, \xi^a) \quad \text{with} \quad x^b = \varphi_0^{ba}(x^a) \ \text{ and } \ \xi_i^b = A_{ij}^{ba}(x^a) \cdot \xi_j^a \,, \tag{9}$$

where φ_0^{ba} and A_{ij}^{ba} are smooth functions of the even coordinates x^a only. We will call such an atlas a *Batchelor atlas*.

Now let $\mathcal{U}$ be a Batchelor atlas (for the moment any atlas will do, but later on we will need a Batchelor one). Then the maps $\mathbf{B}\varphi_a \colon \mathbf{B}U_a \to \mathbf{B}O_a \subset \mathbf{R}^d$ form an atlas for the underlying ordinary manifold $\mathbf{B}M$. Moreover, we can interpret the functions $\mathbf{B}A_{ij}^{ba}$ as smooth maps $\mathbf{B}A^{ab} \colon \mathbf{B}U_a \cap \mathbf{B}U_b \to \mathrm{Gl}(n, \mathbf{R})$. As such they satisfy the cocycle condition

$$\sum_{j=1}^{n} (\mathbf{B}A_{ij}^{cb})(m)\, (\mathbf{B}A_{jk}^{ba})(m) = (\mathbf{B}A_{ik}^{ca})(m) \,,$$

simply because the φ^{ab} do. It follows that the functions $T^{ab} \colon \mathbf{B}U_a \cap \mathbf{B}U_b \to \mathrm{Gl}(n, \mathbf{R})$ defined as $T_{ij}^{ab}(m) = \big(\mathbf{B}A^{ab}(m)\big)_{ji}^{-1}$, i.e., T^{ba} is the inverse-transpose of $\mathbf{B}A^{ba}$, also satisfy the cocycle condition. We thus can use them to define a vector bundle $VM \to \mathbf{B}M$ with typical fiber $\mathbf{R}^n$. More precisely, $\pi \colon VM \to \mathbf{B}M$ is a vector bundle with local trivializing charts $\Psi_a \colon \pi^{-1}(\mathbf{B}U_a) \to \mathbf{B}U_a \times \mathbf{R}^n$ and transition functions

$$\Psi^{ba} = \Psi_b \circ \Psi_a^{-1} : (\mathbf{B}U_a \cap \mathbf{B}U_b) \times \mathbf{R}^n \to (\mathbf{B}U_a \cap \mathbf{B}U_b) \times \mathbf{R}^n$$

given by $\Psi^{ba}(m, v) = (m, w)$ with $w_i = \displaystyle\sum_{j=1}^{n} T_{ij}^{ba}(m)\, v_j \in \mathbf{R}$.

We will say that M *is oriented in the odd directions*, abbreviated as *o.o.d*, when the vector bundle $VM \to \mathbf{B}M$ is oriented. This is possible only if there exists an atlas for which all the matrices $\mathbf{B}A^{ab}$ have positive determinant, in which case there are two possible choices for such an orientation. When M is o.o.d, we will say that a Batchelor atlas is o.o.d when it satisfies the condition that all matrices $\mathbf{B}A^{ab}$ have positive determinant.

Now let F be any graded vector space over $\mathbf{K}$ equipped with a norm topology. We then can consider the space $C^\infty(M; F \otimes \mathcal{A}^{\mathbf{K}})$ of smooth functions on M with values in $F \otimes \mathcal{A}^{\mathbf{K}}$, which is a graded vector space over $\mathbf{K}$ (and when $F = \mathbf{K}$, it even is a graded $\mathbf{K}$-algebra). We will now argue that the choice of a Batchelor atlas allows us to create an isomorphism (of graded vector spaces or graded $\mathbf{K}$-algebras)

$$\sigma \colon C^\infty(M; F \otimes \mathcal{A}^{\mathbf{K}}) \to \Gamma^\infty\big(\bigwedge VM \otimes F \to \mathbf{B}M \big) , \tag{10}$$

i.e., between $C^\infty(M; F \otimes \mathcal{A}^{\mathbf{K}})$ and the space of smooth sections of the (non-super) exterior algebra bundle (tensored with F) $\bigwedge VM \otimes F \to \mathbf{B}M$. To do so, let $f \colon M \to F \otimes \mathcal{A}^{\mathbf{K}}$ be a smooth function. Then, for any coordinate chart U_a in the Batchelor atlas there exist smooth functions $f_I^a \colon \mathbf{B}O_a \to F$ such that we have

$$(f \circ \varphi_a^{-1})(x^a, \xi^a) = \sum_{I \subset \{1,\dots,n\}} (\xi^a)^I \, (\mathbf{G} f_I^a)(x^a) . \tag{11}$$

Using the functions f_I^a we then define the local section $\sigma(f) \colon \mathbf{B}U_a \to \bigwedge VM \otimes F$ by

$$(\Psi_a \circ \sigma(f) \circ \mathbf{B}\varphi_a^{-1})(x^a) = \Big(\, x^a \, , \sum_{I \subset \{1,\dots,n\}} e^I \otimes f_I^a(x^a) \, \Big) , \tag{12}$$

where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbf{R}^n$ and where e^I is defined as ξ^I by replacing the product in $\mathcal{A}$ by the wedge-product, see Definition 4.1. The definition of the transition functions for the bundle VM together with the fact that our atlas is a Batchelor atlas guarantee that these local sections glue together to form a global smooth section. It is then routine to show that σ is an isomorphism of graded vector spaces over $\mathbf{K}$ (or in case $F = \mathbf{K}$ of graded $\mathbf{K}$-algebras). Moreover, it follows directly from the definition of the DeWitt topology on M that σ preserves compact support, i.e., that σ also is an isomorphism between compactly supported smooth functions on M with values in $F \otimes \mathcal{A}^{\mathbf{K}}$ and compactly supported smooth sections of $\bigwedge VM \otimes F$.

Nota Bene 4.2. The Batchelor bundle (as well as the notion of "being oriented in the odd directions") can be defined without the use of a Batchelor atlas by noting that we have the equality

$$\mathbf{B}A_{ij}^{ba}(x^a) = \mathbf{B}\frac{\partial \xi_i^b}{\partial \xi_j^a}(x^a) .$$

However, the identification between $C^\infty(M; F \otimes \mathcal{A}^{\mathbf{K}})$ and $\Gamma^\infty\big(\bigwedge VM \otimes F \to \mathbf{B}M \big)$ depends upon the choice of a Batchelor atlas.

Nota Bene/Convention 4.3. From now on we will always assume, unless stated explicitly otherwise, that we have chosen a Batchelor atlas. If needed, we could extend it to a maximal atlas by including all coordinate charts that have the change of coordinate form (9) with all charts in the given Batchelor atlas.

An additional advantage of using a Batchelor atlas is that it allows us to extend a partition of unity for $\mathbf{B}M$ to a partition of unity of M. More precisely, let $\mathcal{U} = \{ U_a \mid a \in A \}$ be a Batchelor atlas for M and let the family $\rho_a \colon \mathbf{B}U_a \to [0,1]$ be a (smooth) partition of unity subordinated to the cover $\{ \mathbf{B}U_a \mid a \in A \}$ of $\mathbf{B}M$. Then the family $\mathbf{G}\rho_a \colon U_a \to \mathcal{A}_0$ is a partition of unity subordinated to the cover $\mathcal{U}$. This works because when expressing these functions in another chart, we never get an additional dependence on odd coordinates via a change of coordinates in the Batchelor atlas.

4.2. An intermezzo on matrices

Let E be a finite dimensional graded vector space over $\mathcal{K}$ of graded dimension $d|n$, let $e_1, \dots, e_d, e_{d+1}, \dots, e_{d+n}$ be a homogeneous basis of E (recall Definition 3.7 that this means that $e_1, \dots, e_d$ is a basis of $\mathbf{B}E_0$ (these are thus even vectors) and $e_{d+1}, \dots, e_{d+n}$ a basis of $\mathbf{B}E_1$ (and thus odd vectors)). If $f\colon E \to E$ is a right-linear map, we can associate to f its matrix $f_{ij} \in \mathrm{Gl}(d|n, \mathcal{K})$ (the set of square matrices of size $d+n$ with values in $\mathcal{K}$) defined by

$$f(e_i) = \sum_{j=1}^{d+n} e_j\, f_{ji}\,.$$

As we know that the first p basis elements are even, we can decompose this matrix into four by writing

$$\begin{pmatrix} f_{11} & \dots & f_{1,p+q} \\ \vdots & & \vdots \\ f_{p+q,1} & \dots & f_{p+q,p+q} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{13}$$

with A a square matrix of size $p\times p$ and the other three of appropriate corresponding sizes. If f is even, A and D have even entries, whereas B and C have odd ones.

For a left-linear map $f\colon E \to E$ we can also define its matrix, but now its definition is given by

$$\iota(e_i)f = \sum_{i=1}^{p+q} f_{ij}\, e_j\,.$$

But here again we can decompose this matrix in four submatrices as in (13), and again if f is even, A and D will have even entries, whereas B and C will have odd entries.

Yet another way to obtain a matrix is with a sesquilinear map $S\colon E \times E \to \mathcal{A}^{\mathbf{C}}$. In that case we define the matrix $S_{ij} \in \mathrm{Gl}(d|n, \mathcal{A}^{\mathbf{C}})$ by

$$S_{ij} = S(e_i, e_j)\,. \tag{14}$$

And again we can decompose this matrix in four submatrices as in (13), and once again: if S is even, A and D will have even entries, whereas B and C will have odd entries.

This decomposition of a matrix into four submatrices becomes quite natural in the setting of tangent maps. Consider a smooth map $\Phi\colon E_0 \to E_0$ (or only defined on an open subset of E_0) and denote by (x, ξ) the even and odd coordinates on E according to $v = \sum_{i=1}^d x_i\, e_i + \sum_{j=1}^n \xi_j\, e_{d+j}$. We then can write

$$\Phi(x, \xi) = \big(F(x, \xi), \phi(x, \xi)\big)\,,$$

with F an even function (with d components) and ϕ an odd function (with n components). With these notations, the matrix of its tangent map (the Jacobian matrix) at a point $v \in E_0$ is given by

$$\operatorname{matrix}(T_v\Phi) = \begin{pmatrix} \dfrac{\partial F}{\partial x}(v) & \dfrac{\partial \phi}{\partial x}(v) \\ \dfrac{\partial F}{\partial \xi}(v) & \dfrac{\partial \phi}{\partial \xi}(v) \end{pmatrix} \tag{15}$$

or explicitly in terms of a (tangent) vector:

$$\begin{aligned} \iota\Big(\sum_{i=1}^{d} t_i\, e_i + \sum_{j=1}^{n} \tau_j\, e_{d+j}\Big) T_v\Phi &= \begin{pmatrix} t & \tau \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial F}{\partial x}(v) & \dfrac{\partial \phi}{\partial x}(v) \\ \dfrac{\partial F}{\partial \xi}(v) & \dfrac{\partial \phi}{\partial \xi}(v) \end{pmatrix} \cdot \begin{pmatrix} e_{1,\dots,d} \\ e_{d+1,\dots,d+n} \end{pmatrix} \\ &= \sum_{p=1}^{d}\Big(\sum_{i=1}^{d} t_i \cdot \frac{\partial F_p}{\partial x_i}(v) + \sum_{j=1}^{n} \tau_j \cdot \frac{\partial F_p}{\partial \xi_j}(v)\Big) \cdot e_p \\ &\qquad + \sum_{q=1}^{n}\Big(\sum_{i=1}^{d} t_i \cdot \frac{\partial \phi_q}{\partial x_i}(v) + \sum_{j=1}^{n} \tau_j \cdot \frac{\partial \phi_q}{\partial \xi_j}(v)\Big) \cdot e_{d+q}\,. \end{aligned} \tag{16}$$

When Φ represents the change of (local) coordinates on an $\mathcal{A}$-manifold M, it is customary to denote the coordinates on the source by (x, ξ) and to change the name at the target, say to (y, η). In particular one writes $\Phi(x, \xi) = (y, \eta) = \big(y(x, \xi), \eta(x, \xi)\big)$. In that case the homogeneous basis is given by the tangent vectors $(\partial_{x_i}, \partial_{\xi_j})$ or $(\partial_{y_i}, \partial_{\eta_j})$. And then (16) takes the form of the associated change of basis in the tangent space T_mM:

$$\begin{pmatrix} \dfrac{\partial}{\partial x}\Big|_m \\ \dfrac{\partial}{\partial \xi}\Big|_m \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y}{\partial x}(x,\xi) & \dfrac{\partial \eta}{\partial x}(x,\xi) \\ \dfrac{\partial y}{\partial \xi}(x,\xi) & \dfrac{\partial \eta}{\partial \xi}(x,\xi) \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial}{\partial y}\Big|_m \\ \dfrac{\partial}{\partial \eta}\Big|_m \end{pmatrix}\,. \tag{17}$$

When both local coordinate systems belong to a Batchelor atlas, we have in particular $\partial_\xi y = 0$.

4.3. We add a metric to obtain a Hilbert space

We define a *super metric* $\mathbf{g}$[3] on the $\mathcal{A}$-manifold M to be a smooth assignment of a super metric $\mathbf{g}(m)$ in the sense of Definition 2.14 at T_mM for all $m \in M$ (see also [15, §IV.7]).[4] Here the smoothness condition means that in terms of local coordinates $(x_1, \dots, x_{d+n})$ (even and odd together) the matrix elements

$$\mathbf{g}_{ij}(m) = \mathbf{g}(m)\big(\,\partial_{x_i}|_m\,,\, \partial_{x_j}|_m\,\big)\;, \quad i,j = 1,\dots,n+d \tag{18}$$

[3]As is usual, I use the letter g to denote a metric on a manifold. But as a generic element of a group G is also denoted by the letter g, this might be confusing when both appear in the same formula (even when most of the time it will be obvious who is what). I thus write a metric in bold face, providing a visual distinction for these two objects.

[4]Note that this definition of a super metric differs significantly from the more standard definition of a super metric as given for instance in [7] or [10]

are smooth functions of m. In any local chart with coordinates (x,ξ) we then can define the four matrices $\mathbf{g}^{\alpha\beta}(x,\xi)$, $\alpha,\beta\in\mathbf{Z}/2\mathbf{Z}$ (see the previous subsection) by

$$\mathbf{g}^{00}_{pq}(x,\xi)=\mathbf{g}\Big(\frac{\partial}{\partial x_p}\Big|_{(x,\xi)},\frac{\partial}{\partial x_q}\Big|_{(x,\xi)}\Big),\quad \mathbf{g}^{01}_{pq}(x,\xi)=\mathbf{g}\Big(\frac{\partial}{\partial x_p}\Big|_{(x,\xi)},\frac{\partial}{\partial \xi_q}\Big|_{(x,\xi)}\Big)$$
$$\mathbf{g}^{10}_{pq}(x,\xi)=\mathbf{g}\Big(\frac{\partial}{\partial \xi_p}\Big|_{(x,\xi)},\frac{\partial}{\partial x_q}\Big|_{(x,\xi)}\Big),\quad \mathbf{g}^{11}_{pq}(x,\xi)=\mathbf{g}\Big(\frac{\partial}{\partial \xi_p}\Big|_{(x,\xi)},\frac{\partial}{\partial \xi_q}\Big|_{(x,\xi)}\Big)\,.$$

The matrices $\mathbf{g}^{00}$ and $\mathbf{g}^{11}$ are even of size $d\times d$ and $n\times n$ respectively, whereas the matrices $\mathbf{g}^{10}$ and $\mathbf{g}^{01}$ are odd and of size $n\times d$ and $d\times n$ respectively. Now if (x^a,ξ^a) and (x^b,ξ^b) are two coordinate systems, we get two sets of matrices $\mathbf{g}^{a,\alpha\beta}$ and $\mathbf{g}^{b,\alpha\beta}$. Using (17) it is not hard to show that these two sets are related by

$$\begin{pmatrix}\mathbf{g}^{a,00} & \mathbf{g}^{a,01}\\ \mathbf{g}^{a,10} & \mathbf{g}^{a,11}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial x^b}{\partial x^a} & \dfrac{\partial \xi^b}{\partial x^a}\\[2ex] \dfrac{\partial x^b}{\partial \xi^a} & \dfrac{\partial \xi^b}{\partial \xi^a}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{g}^{b,00} & \mathbf{g}^{b,01}\\ \mathbf{g}^{b,10} & \mathbf{g}^{b,11}\end{pmatrix}\cdot\begin{pmatrix}\Big(\dfrac{\partial x^b}{\partial x^a}\Big)^t & \Big(\dfrac{\partial x^b}{\partial \xi^a}\Big)^t\\[2ex] -\Big(\dfrac{\partial \xi^b}{\partial x^a}\Big)^t & \Big(\dfrac{\partial \xi^b}{\partial \xi^a}\Big)^t\end{pmatrix}. \tag{19}$$

Now if we realise that $\partial_{x^a}\xi^b$ and $\partial_{\xi^a}x^b$ are odd matrices, taking the body map of the above equality gives us the two equalities

$$\mathbf{B}\mathbf{g}^{a,00}=\mathbf{B}\frac{\partial x^b}{\partial x^a}\cdot\mathbf{B}\mathbf{g}^{b,00}\cdot\mathbf{B}\Big(\frac{\partial x^b}{\partial x^a}\Big)^t\ \text{ and }\ \mathbf{B}\mathbf{g}^{a,11}=\mathbf{B}\frac{\partial \xi^b}{\partial \xi^a}\cdot\mathbf{B}\mathbf{g}^{b,11}\cdot\mathbf{B}\Big(\frac{\partial \xi^b}{\partial \xi^a}\Big)^t. \tag{20}$$

This shows in the first place that $\mathbf{B}\mathbf{g}^{a,00}$ defines a global metric $\mathbf{g}_{\mathbf{B}M}$ on $\mathbf{B}M$. We then recall that the matrices $\mathbf{B}A^{ba}=\mathbf{B}\partial_{\xi^a}\xi^b$ are the inverse transpose of the transition functions of the Batchelor vector bundle $VM\to\mathbf{B}M$. Taking the definition of a super metric into account, we conclude that the matrices $(-i\,\mathbf{B}\mathbf{g}^{a,11})^{-1}$ define an ordinary metric $\mathbf{g}_{VM}$ on (the fibers of) VM. More precisely, in the trivializing chart U_a the metric $\mathbf{g}_{VM}(m)$ is defined by the equation

$$\sum_{k=1}^n \mathbf{g}_{VM}(m)(e_j,e_k)\cdot\mathbf{B}\big(-i\,\mathbf{g}^{a,11}_{k\ell}(m)\big)=\delta_{j\ell}\,, \tag{21}$$

where, as before, the e_i denote the canonical basis of the typical fiber $\mathbf{R}^n$.

Following [19, p. 79] we extend this scalar product to the exterior algebra bundle $\bigwedge VM$ by setting the scalar product of homogeneous elements of different degrees to zero, and by setting, in the trivializing chart U_a,

$$\mathbf{g}_{VM}(m)(v_1\wedge\cdots\wedge v_k,w_1\wedge\cdots\wedge w_k)=\det\big(\mathbf{g}_{VM}(m)(v_p,w_q)\big)\,.$$

It then follows that if $v_1,\dots,v_n$ is an orthonormal basis of the fiber $\pi^{-1}(m)\cong\mathbf{R}^n$ above $m\in\mathbf{B}M$ of the bundle VM, then the monomials $v^I\equiv v_{i_1}\wedge\cdots\wedge v_{i_k}$, $I\subset\{1,\dots,n\}$ (Definition 4.1) form an orthonormal basis of $\bigwedge\pi^{-1}(m)\subset\bigwedge VM$.

Now let F be a graded vector space over $\mathbf{K}$ and $\langle\cdot,\cdot\rangle_F$ an ordinary metric on F (seen without grading), thus providing a norm topology on F. We then can extend the metric $\mathbf{g}_{VM}$ on the fibers of $\bigwedge VM$ to an ordinary metric $\mathbf{g}^F_{VM}$ on the fibers of $\bigwedge VM\otimes F$ by defining

$$\mathbf{g}^F_{VM}(\omega\otimes e,\omega'\otimes e')=\mathbf{g}_{VM}(\omega,\omega')\cdot\langle e,e'\rangle_F\,.$$

Using this metric $\mathbf{g}^F_{VM}$ on the fibers of $\bigwedge VM \otimes F$ and the metric $\mathbf{g}_{\mathbf{B}M}$ on $\mathbf{B}M$, we can define in a natural way an (ordinary) metric $\langle \cdot, \cdot \rangle$ on the space $\Gamma^\infty_c\big(\bigwedge VM \otimes F \to M\big)$ of compactly supported smooth sections of $\bigwedge VM \otimes F$ as follows. Let us denote by $\mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}$ the metric volume density associated to the metric $\mathbf{g}_{\mathbf{B}M}$ given in a local coordinate system $x^a_1, \dots, x^a_d$ by

$$\mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(x^a) = \sqrt{\big|\det\big(\mathbf{B}\mathbf{g}^{a,00}(x^a)\big)\big|}\ \mathrm{d}\lambda^{(d)}(x^a)\,.$$

For any two sections $\chi, \psi \in \Gamma^\infty_c\big(\bigwedge VM \otimes F \to M\big)$ we then define $\langle \chi, \psi \rangle_{\mathbf{g}}$ by

$$\langle \chi, \psi \rangle_{\mathbf{g}} = \int_{\mathbf{B}M} \mathbf{g}^F_{VM}(m)\big(\chi(m), \psi(m)\big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(m)\,. \tag{22}$$

It is then routine to show that this is a metric in the usual sense of Definition 2.2.

Remark 4.4. Just as for the Batchelor bundle we do not need a Batchelor atlas to define the metrics $\mathbf{g}_{\mathbf{B}M}$ and $\mathbf{g}_{VM}$.

Definition 4.5. With these preparations we now use the isomorphism σ to transport the (ordinary) metric $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ on $\Gamma^\infty_c\big(\bigwedge VM \otimes F \to M\big)$ to $C^\infty_c(M; F \otimes \mathcal{A}^{\mathbf{K}})$, the space of compactly supported smooth functions on M (with values in $F \otimes \mathcal{A}^{\mathbf{K}}$). By abuse of notation we will denote this metric on $C^\infty_c(M; F \otimes \mathcal{A}^{\mathbf{K}})$ also by $\langle \cdot, \cdot \rangle_{\mathbf{g}}$. This would result in the confusing definition for $f, g \in C^\infty_c(M; F \otimes \mathcal{A}^{\mathbf{K}})$:

$$\langle f, g \rangle_{\mathbf{g}} = \langle \sigma(f), \sigma(g) \rangle_{\mathbf{g}}\,,$$

where on the right hand side $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ denotes the metric defined in (22), and on the left hand side the induced metric on $C^\infty_c(M; F \otimes \mathcal{A}^{\mathbf{K}})$.

We then define the (ordinary) graded Hilbert space $L^2(M; F; \mathbf{g})$ as the completion of the metric space $\big(C^\infty_c(M; F \otimes \mathcal{A}^{\mathbf{K}}), \langle \cdot, \cdot \rangle_{\mathbf{g}}\big)$. The extended metric (scalar product) on $L^2(M; F; \mathbf{g})$ will still be denoted by $\langle \cdot, \cdot \rangle_{\mathbf{g}}$, while the metric on F is understood implicitly.

Lemma 4.6. *If the homogeneous parts of F are orthogonal with respect to $\langle \cdot, \cdot \rangle_F$, i.e., $\langle F_0, F_1 \rangle_F = 0$, then the homogeneous parts of $C^\infty_c(M; F \otimes \mathcal{A}^{\mathbf{K}})$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbf{g}}$.*

Proof. Let $f, g \in C^\infty_c(M; F \otimes \mathcal{A}^{\mathbf{K}})$ be such that f is even and g odd. In any local coordinate chart (in the Batchelor atlas) we thus have

$$f(x, \xi) = \sum_{I \subset \{1, \dots, n\}} \xi^I\, f_I(x) \qquad \text{and} \qquad g(x, \xi) = \sum_{J \subset \{1, \dots, n\}} \xi^J\, g_J(x)\,.$$

The corresponding local expressions for $\chi = \sigma(f)$ and $\psi = \sigma(g)$ are given by (12):

$$\chi(x) = \Big(x, \sum_{I \subset \{1, \dots, n\}} e^I \otimes f_I(x)\Big) \qquad \text{and} \qquad \psi(x) = \Big(x, \sum_{J \subset \{1, \dots, n\}} e^J \otimes g_J(x)\Big)\,.$$

The fact that f is even implies that the parity of $f_I(x)$ equals $|I|$, the parity of I; similarly for g odd the parity of $|g_J(x)|$ equals $|J|+1$. Now we have

$$\mathbf{g}^F_{VM}(\chi(x),\psi(x)) = \sum_{I,J\subset\{1,\dots,n\}} \mathbf{g}_{VM}(e^I,e^J)\cdot\langle f_I(x),g_J(x)\rangle_F\,.$$

But for $|I|\neq|J|$ we have $\mathbf{g}_{VM}(e^I,e^J)=0$ (homogeneous elements of different degree are orthogonal), and for $|I|=|J|$ we have $\langle f_I(x),g_J(x)\rangle_F=0$ because then $f_I(x)$ and $g_J(x)$ are of opposite parity in F. The conclusion is that for even f and odd g we have $\mathbf{g}^F_{VM}(\chi(x),\psi(x))=0$, proving that $\langle f,g\rangle_{\mathbf{g}}=0$ as wanted. ■

Lemma 4.7. *Let* $\mathbf{g}$ *be a super metric on* M *and suppose we are given an atlas. Then there exists a cover* $\mathcal{U}=\{U_b\,|\,b\in A\}$ *by local coordinate charts* U_b *in (or better: compatible with) the given atlas on which the matrices* $-i\,\mathbf{B}\mathbf{g}^{b,11}$ *are constant the identity matrix. We will say that the resulting atlas* $\mathcal{U}$ *is adapted to* $\mathbf{g}$*. If the initial atlas is a Batchelor atlas and/or o.o.d, then so will be the resulting atlas.*

Proof. Let U_a be any local chart in the given atlas with local coordinates (x^a,ξ^a) and let $H^a(x)$ be the matrix defined by

$$H^a_{pq}(x^a) = -i\,\mathbf{B}\mathbf{g}^{a,11}\left(\frac{\partial}{\partial\xi^a_p}\Big|_{(x^a,\xi^a)}\,,\ \frac{\partial}{\partial\xi^a_q}\Big|_{(x^a,\xi^a)}\right).$$

As $\mathbf{g}$ is a super metric, it follows that this is a positive definite symmetric bilinear form. By a simple Gram-Schmid orthogonalisation process we can find a smooth matrix valued function $S\colon\mathbf{B}U_a\to\mathrm{Gl}(n,\mathbf{R})$ with positive determinant (it will actually be a triangular matrix) such that

$$\sum_{j,k=1}^n S_{jp}(x)\cdot H^a_{jk}(x^a)\cdot S_{kq}(x)=\delta_{pq}\,.$$

This suggests we introduce the chart U_b with the change of coordinates φ^{ba} defined as $U_b=U_a$ and $(x^b,\xi^b)=\varphi^{ba}(x^a,\xi^a)$ with $x^b=x^a$ and

$$\xi^a_p=\sum_q(\mathbf{G}S)(x^a)_{pq}\,\xi^b_q\equiv\sum_q(\mathbf{G}S)(x^b)_{pq}\,\xi^b_q\,.$$

As we have $\mathbf{B}\partial_{\xi^b}\xi^a=S(x^a)$, it follows from (20) that we have $H^b_{pq}(x^b)=\delta_{pq}$, i.e., the matrix valued function H^b is constant the identity matrix on U_b. As the change of coordinates φ^{ba} is linear in the odd coordinates, it is immediate that if the initial atlas is a Batchelor atlas, then so is the new one. And because the determinant of S is positive, it follows immediately that if the initial atlas is o.o.d, then so is the new one. ■

Lemma 4.8. *Let* $\mathcal{U}=\{U_a\mid a\in A\}$ *be a Batchelor atlas adapted to the super metric* $\mathbf{g}$ *on* M*, see Lemma 4.7, and let* $\chi,\psi\in C^\infty_c(M;F\otimes\mathcal{A}^{\mathbf{K}})$ *be smooth functions. Let furthermore* $\rho_a\colon U_a\to[0,1]$ *be a smooth partition of unity subordinated to* $\mathbf{B}\mathcal{U}$*. Then the metric* $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ *on* $C^\infty_c(M;F\otimes\mathcal{A}^{\mathbf{K}})$ *is given by*

$$\langle\chi,\psi\rangle_{\mathbf{g}}=\sum_{a\in A}\int_{\mathbf{B}U_a}\mathrm{d}\lambda^{(d)}(x^a)\;\rho_a(x^a)\cdot\sqrt{\left|\det\!\big(\mathbf{B}\mathbf{g}^{a,00}(x^a)\big)\right|}\cdot\sum_{I\subset\{1,\dots,n\}}\big\langle\chi_{a,I}(x^a),\psi_{a,I}(x^a)\big\rangle_F\,,$$

where (x^a, ξ^a) *denotes a local system of coordinates on* U_a *and where the local expression of* χ *(respectively* ψ*) is given by*

$$\chi(x^a, \xi^a) = \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I\, \chi_{a,I}(x^a)$$

for smooth functions $\chi_{a,I} : \mathbf{B}U_a \to F$ *.*

Proof. As the atlas is adapted to the super metric, the matrices $-i\,\mathbf{B}\mathbf{g}^{a,11}$ are constant the identity. This implies that the canonical basis $e_1, \dots, e_n$ of the typical fiber of VM is an orthonormal basis. We thus have

$$\begin{aligned}
\langle \chi, \psi\rangle_{\mathbf{g}} &= \int_{\mathbf{B}M} \mathbf{g}^F_{VM}(m)\big(\chi(m), \psi(m)\big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(m) \\
&= \sum_{a\in A} \int_{\mathbf{B}U_a} \rho_a(m)\cdot \mathbf{g}^F_{VM}(m)\big(\chi(m), \psi(m)\big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(m) \\
&= \sum_{a\in A} \int_{\mathbf{B}U_a} \mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot \sqrt{\big|\det\big(\mathbf{B}\mathbf{g}^{a,00}(x^a)\big)\big|} \\
&\qquad\qquad \cdot \sum_{I,J\subset\{1,\dots,n\}} \mathbf{g}_{VM}(e^I, e^J)\cdot \langle \chi_{a,I}(x^a), \psi_{a,J}(x^a)\rangle_F \\
&= \sum_{a\in A} \int_{\mathbf{B}U_a} \mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot \sqrt{\big|\det\big(\mathbf{B}\mathbf{g}^{a,00}(x^a)\big)\big|} \\
&\qquad\qquad \cdot \sum_{I\subset\{1,\dots,n\}} \langle \chi_{a,I}(x^a), \psi_{a,I}(x^a)\rangle_F\,.
\end{aligned}$$

■

4.4. And then we add a diffeomorphism

Definition 4.9. Let $\Phi : M \to M$ be a diffeomorphism. We will say that Φ *is linear in the odd coordinates (with respect to a Batchelor atlas)*, if for any $m \in M$, any coordinate system (x, ξ) around m and any coordinate system (y, η) around $\Phi(m)$ (coordinate systems in the fixed Batchelor atlas) this diffeomorphism is of the form

$$\Phi(x,\xi) = (y,\eta) \qquad \text{with} \qquad y = \Phi_0(x) \quad , \quad \eta_i = \sum_{j=1}^n \Phi_{1,ij}(x)\,\xi_j \tag{23}$$

for (local) smooth functions Φ_0 and $\Phi_{1,ij}$ of the even coordinates x only (the condition thus is slightly stronger than the name suggests, as the even coordinates y do not depend upon the odd coordinates ξ). It is immediate that $\Phi_0 : \mathbf{B}M \to \mathbf{B}M$ is a diffeomorphism of the underlying ordinary (i.e., non-super) manifold $\mathbf{B}M$ and that $\Phi_1(x)$ is an invertible $n\times n$ matrix depending smoothly on x.
We will say that Φ *preserves the super metric* $\mathbf{g}$ if for any $m \in M$ and any pair of tangent vectors $v, w \in T_mM$ we have the equality

$$\mathbf{g}_{\Phi(m)}\big(T_m\Phi(v), T_m\Phi(w)\big) = \mathbf{g}_m(v,w)\,. \tag{24}$$

Construction 4.10. If a diffeomorphism $\Phi : M \to M$ is linear in the odd coordinates, it defines a bundle isomorphism $\widehat{\Phi} : VM \to VM$ as follows. Let (x, ξ)

and (y,η) be as in (23) and let (x,v) with $v \in \mathbf{R}^n$ be a point in VM above $\mathbf{B}x \cong \mathbf{B}m \in \mathbf{B}M$ in the local trivialization associated to the coordinate chart (x,ξ). Then we define $\widehat{\Phi}(x,v)$ by

$$\widehat{\Phi}(x,v) = (y,w) \qquad \text{with} \qquad y = \Phi_0(x)\,, \quad w_i = \sum_{j=1}^{n} \big(\Phi_1(x)^{-1}\big)_{ji}\, v_j\,.$$

Note that we use the inverse-transpose of the matrix $\Phi_1(x)$ (just as we did in the definition of VM), which makes this definition a valid definition on VM.

Associated to this bundle isomorphism $\widehat{\Phi}\colon VM \to VM$ we have the induced bundle isomorphism (denoted by the same symbol, taking the trivial action on F) $\widehat{\Phi}\colon \bigwedge VM \otimes F \to \bigwedge VM \otimes F$. And this bundle isomorphism induces an isomorphism $\widehat{\Phi}^*$ of the space of smooth sections $\Gamma\big(\bigwedge VM \otimes F \to \mathbf{B}M\big)$ as follows. For any smooth section $s\colon \mathbf{B}M \to \bigwedge VM \otimes F$ we define

$$(\widehat{\Phi}^* s)(m) = \widehat{\Phi}^{-1}\Big(s\big(\Phi_0(m)\big)\Big)\,.$$

It is then routine to show that the identification σ (10) between super smooth functions on M with values in $F \otimes \mathcal{A}^{\mathbf{K}}$ and smooth sections of $\bigwedge VM \otimes F \to \mathbf{B}M$ intertwines the usual pull-back Φ^* of super smooth functions with $\widehat{\Phi}^*$: for any $f \in C^\infty(M; F \otimes \mathcal{A}^{\mathbf{K}})$ we have the equality

$$\sigma(\Phi^* f) = \widehat{\Phi}^*\big(\sigma(f)\big)\,.$$

Proposition 4.11. *Let $\Phi\colon M \to M$ be a diffeomorphism that preserves the super metric $\mathbf{g}$ and is linear in the odd coordinates (with respect to a given Batchelor atlas). Then we have the following properties.*

(i) $\Phi\colon \mathbf{B}M \to \mathbf{B}M$ *preserves the ordinary metric* $\mathbf{g}_{\mathbf{B}M}$.

(ii) *The induced vector bundles isomorphism $\widehat{\Phi}\colon VM \to VM$ preserves the ordinary metric $\mathbf{g}_{VM}$.*

(iii) *The pull-back operation $\Phi^*\colon C_c^\infty(M; F \otimes \mathcal{A}^{\mathbf{K}})$ is an even map preserving the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$. It thus induces an even unitary map on $L^2(M;F;\mathbf{g})$.*

Proof. Using coordinate systems (x,ξ) and (y,η) as in (23) and using that Φ is linear in the odd coordinates, we have (see (16))

$$\iota(\begin{pmatrix} \dfrac{\partial}{\partial x} \\[2mm] \dfrac{\partial}{\partial \xi} \end{pmatrix}) T_m\Phi = \begin{pmatrix} \dfrac{\partial \Phi_0}{\partial x}(x) & \dfrac{\partial \Phi_1}{\partial x}(x)\cdot \xi \\[2mm] \mathbf{0} & \Phi_1(x) \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial}{\partial y} \\[2mm] \dfrac{\partial}{\partial \eta} \end{pmatrix}\,.$$

Substituting this in the invariance condition of $\mathbf{g}$ and then taking the body map gives us (just as (20)) the equalities

$$\mathbf{B}\mathbf{g}^{x,00}(x) = \mathbf{B}\frac{\partial \Phi_0}{\partial x} \cdot \mathbf{B}\mathbf{g}^{y,00}\big(\Phi_0(x)\big) \cdot \big(\mathbf{B}\frac{\partial \Phi_0}{\partial x}\big)^t$$

$$\mathbf{B}\mathbf{g}^{x,11}(x) = \mathbf{B}\Phi_1(x) \cdot \mathbf{B}\mathbf{g}^{y,11}\big(\Phi_0(x)\big) \cdot \mathbf{B}\big(\Phi_1(x)\big)^t\,.$$

This means, taking the definition of the bundle isomorphism $\widehat{\Phi}$ into account, that the diffeomorphism $\Phi_0 \colon \mathbf{B}M \to \mathbf{B}M$ preserves the metric $\mathbf{g}_{\mathbf{B}M}$ on $\mathbf{B}M$ and that the bundle isomorphism $\widehat{\Phi}$ preserves the metric $\mathbf{g}_{VM}$ on the fibers of VM. It is immediate that the induced/extended action on $\bigwedge VM \otimes F$ (with trivial action on F) preserves the metric $\mathbf{g}^F_{VM}$. The following routine computation then shows that $\widehat{\Phi}^*$ preserves the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ on $\Gamma^\infty_c\big(\bigwedge VM \otimes F \to M\big)$.

$$
\begin{aligned}
\big\langle \widehat{\Phi}^*\chi, \widehat{\Phi}^*\psi\big\rangle_{\mathbf{g}} &= \int_{\mathbf{B}M} \mathbf{g}^F_{VM}(m)\big((\widehat{\Phi}^*\chi)(m), (\widehat{\Phi}^*\psi)(m)\big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(m) \\
&= \int_{\mathbf{B}M} \mathbf{g}^F_{VM}(m)\Big(\widehat{\Phi}^{-1}\Big(\chi\big(\Phi_0(m)\big)\Big), \widehat{\Phi}^{-1}\Big(\psi\big(\Phi_0(m)\big)\Big)\Big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(m) \\
\scriptstyle\text{inv. of } \mathbf{g}^F_{VM} \quad &= \int_{\mathbf{B}M} \mathbf{g}^F_{VM}\big(\Phi_0(m)\big)\Big(\chi\big(\Phi_0(m)\big), \psi\big(\Phi_0(m)\big)\Big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(m) \\
\scriptstyle\text{inv. of } \mathbf{g}_{\mathbf{B}M} \quad &= \int_{\mathbf{B}M} \mathbf{g}^F_{VM}\big(\Phi_0(m)\big)\Big(\chi\big(\Phi_0(m)\big), \psi\big(\Phi_0(m)\big)\Big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}\big(\Phi_0(m)\big) \\
&= \int_{\mathbf{B}M} \mathbf{g}^F_{VM}(m)\big(\chi(m), \psi(m)\big)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}M}}(m) = \langle \chi, \psi\rangle_{\mathbf{g}}\,.
\end{aligned}
$$

As σ intertwines $\widehat{\Phi}^*$ with Φ^* and as Φ^* preserves the parity of functions, it is immediate that Φ^* preserves parity and the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$. ■

5. Densities, integration on $\boldsymbol{\mathcal{A}}$-manifolds and super scalar products

Definition 5.1. Let E be a finite dimensional graded vector space over $\mathcal{K}$ and let $f\colon E \to E$ be an automorphism, i.e., an even bijective (right- and left-) linear map. In order to define $\mathrm{Ber}(f)$, the *Berezinian of* f, we decompose its matrix with respect to a homogeneous basis of E into four submatrices as in (13):

$$
\mathrm{matrix}(f) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
$$

And then $\mathrm{Ber}(f)$ is defined by the formula

$$
\mathrm{Ber}(f) = \mathrm{Det}(A - BD^{-1}C)\cdot \mathrm{Det}(D)^{-1}\,,
$$

which is independent of the choice of the chosen homogeneous basis. We thus obtain a map $\mathrm{Ber}\colon \mathrm{Aut}(E) \to \mathcal{K}_0^*$ defined on the group of all automorphisms of E and taking values in $\mathcal{K}_0^*$ (the invertible elements in the even part of $\mathcal{K}$). It is a group homomorphism.

We also introduce $\mathrm{Ber}_\pi(f)$, a variant of the Berezinian defined as

$$
\mathrm{Ber}_\pi(f) = |\mathrm{Det}(A - BD^{-1}C)|\cdot \mathrm{Det}(D)^{-1}\,.
$$

It is, as Ber, a homomorphism $\mathrm{Aut}(E) \to \mathcal{K}_0^*$. Note that in case $\mathcal{K} = \mathcal{A}^{\mathbf{K}}$, the absolute value is only defined for elements $x \in \mathcal{K}_0^*$, i.e., for elements with $\mathbf{B}x \neq 0$, in which case its definition reads

$$
|x| = |\mathbf{B}x| \cdot \frac{x}{\mathbf{B}x}\,.
$$

This is the standard $\mathbf{G}$-extension of the smooth function "absolute value" defined on $\mathbf{K}^* \equiv \mathbf{K}\setminus\{0\}$.

Definition 5.2. (Berezin integration) Let E be a finite dimensional graded vector space over $\mathcal{A}$, let $U \subset E_0$ be an open set and let $f\colon U \to \mathcal{A}^{\mathbf{K}}$ be a smooth map. This implies that there exist smooth functions $f_I\colon U \to \mathcal{A}^{\mathbf{K}}$, $I \subset \{1,\dots,n\}$ such that we have

$$f(x,\xi) = \sum_{I\subset\{1,\dots,n\}} \xi^I\,(\mathbf{G}f_I)(x)\,,$$

where (x,ξ) are the even and odd coordinates on E_0. We then define the *Berezin integral of f over U* as the element in $\mathbf{K}$ defined by

$$\int_U \mathrm{d}(x,\xi)\ f(x,\xi) = \int_{\mathbf{B}U} f_{\underline{n}}(x)\ \mathrm{d}\lambda^{(d)}(x)\,,$$

provided of course that the function $f_{\underline{n}}\colon \mathbf{B}U \to \mathbf{K}$ is Lebesgue integrable over $\mathbf{B}U$ (with $\underline{n} = \{1,\dots,n\}$ and $\lambda^{(d)}$ denoting the Lebesgue measure on $\mathbf{R}^d$).

Proposition 5.3 ([9], see also [14] and [18, Thm. 4.6.1])**.** *Let E be a graded vector space of graded dimension $d|n$, let $\varphi\colon U \to V$ be a (super) diffeomorphism between two open subsets of E_0, and let $(x,\xi) = (x_1,\dots,x_d,\xi_1,\dots,\xi_n)$ denote the even and odd coordinates on E_0. Then for any function $f\colon V \to \mathcal{A}^{\mathbf{K}}$ with compact support we have the equality*

$$\int_U \mathrm{d}(x,\xi)\ \mathrm{Ber}_\pi\big(T_{(x,\xi)}\varphi\big)\cdot f\big(\varphi(x,\xi)\big) = \int_V \mathrm{d}(y,\eta)\ f(y,\eta)\,. \tag{25}$$

Remark 5.4. The use of Ber_π instead of the ordinary Berezinian Ber in (25) allows us to ignore whether φ is orientation preserving (in the even coordinates) or not.

Definition 5.5. Let M be an $\mathcal{A}$-manifold of graded dimension $d|n$ modelled on the graded vector space E (over $\mathcal{A}$).

- The *frame bundle* $\mathcal{F}M \to M$ is the bundle whose fibers $\mathcal{F}_m$ consist of all homogeneous bases of the tangent space T_mM (see Definition 3.7). It is a principal $\mathrm{Gl}(d|n,\mathcal{A})$ bundle over M. The right-action of $\mathrm{Gl}(d|n,\mathcal{A})$ on $\mathcal{F}_mM$ is defined as follows. For $(v_1,\dots,v_{d+n}) \in \mathcal{F}_mM$, i.e., a basis of T_mM, and $(A_{ij}) \in \mathrm{Gl}(d|n,\mathcal{A})$ we have

$$(v_i)\cdot(A_{jk}) = (w_\ell) \qquad \text{with} \qquad w_i = \sum_{j=1}^{d+n} v_j\,A_{ji}\,.$$

- A *density* on M is a map $\nu\colon \mathcal{F}M \to \mathcal{A}^{\mathbf{K}}$ satisfying the condition that for all $v = (v_i) \in \mathcal{F}_mM$ and for all $A = (A_{jk}) \in \mathrm{Gl}(d|n,\mathcal{A})$ we have the equality

$$\nu\big(v\cdot A\big) = \mathrm{Ber}_\pi(A)\cdot\nu(v)\,.$$

A density can be seen as a section of the *density "line" bundle* $\mathscr{D}(M)$ associated to the principal $\mathrm{Gl}(d|n,\mathcal{A})$ bundle $\mathcal{F}M$ by the representation $\mathrm{Gl}(d|n,\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}^{\mathbf{K}})$, $A \mapsto \mathrm{Ber}_\pi(A)^{-1}$ on $\mathcal{A}^{\mathbf{K}}$. It thus has $\mathcal{A}^{\mathbf{K}}$ as typical fiber.

Construction 5.6. Let ν be a smooth density with compact support on an $\mathcal{A}$-manifold M of graded dimension $d|n$. We then can define the integral of ν over M, denoted as $\int_M \nu$ as follows. We choose an open cover $\mathcal{U} = \{ U_a \mid a \in A \}$ of coordinate charts with (local, even and odd) coordinates x^a, ξ^a and a (smooth) partition of unity ρ_a associated to this cover. We then define $\int_M \nu$ by

$$\int_M \nu = \sum_{a \in A} \int_{U_a} \mathrm{d}(x^a, \xi^a)\ \rho_a(x^a, \xi^a) \cdot \nu\Big(\frac{\partial}{\partial x^a}\Big|_{(x^a,\xi^a)}, \frac{\partial}{\partial \xi^a}\Big|_{(x^a,\xi^a)} \Big) .$$

Remark 5.7. When we use a Batchelor atlas, we can "simplify" the partition of unity used in the definition of $\int_M \nu$. It then suffices to choose first a partition of unity $\hat\rho_a$ associated to the cover $\{ \mathbf{B}U_a \mid a \in A \}$ of $\mathbf{B}M$, and then to extend these maps by $\mathbf{G}$-extension to a partition of unity ρ_a defined as

$$\rho_a(x^a, \xi^a) = (\mathbf{G}\hat\rho)(x^a)$$

independent of the odd coordinates.

Proposition 5.8. *The value of $\int_M \nu$ is independent of the choices of a coordinate cover and the partition of unity.*

Lemma 5.9. *The map $\nu \mapsto \int_M \nu$ defined on $\Gamma_c^\infty\big(\mathscr{D}(M)\big)$, the set of smooth densities with compact support on M (smooth sections with compact support of the density bundle $\mathscr{D}(M)$) is a homogeneous (right-) linear map of parity n (when $d|n$ is the dimension of M).*

Lemma 5.10. *Let $\varphi \colon M \to M$ be a diffeomorphism, let ν be a density with compact support and let $\varphi^*\nu$ be the density defined by*

$$(\varphi^*\nu)(v_m) = \nu\big(\varphi_*(v_m)\big) \equiv \nu\big((T_m\varphi)(v_m)\big) ,$$

where v_m denotes any frame at $m \in M$ (and thus $\varphi_* v_m$ a frame at $\varphi(m)$). Then $\int_M \nu = \int_M \varphi^*\nu$.

Proposition 5.11. *Let M be o.o.d and $\mathbf{g}$ a metric on M.*

(i) *The density $\nu_{\mathbf{g}}$ defined on any coordinate chart U_a with local coordinates (x^a, ξ^a) by*

$$\nu_{\mathbf{g}}\Big(\frac{\partial}{\partial x^a}\Big|_m, \frac{\partial}{\partial \xi^a}\Big|_m \Big) = \sqrt{\big|\mathrm{Ber}\big(\mathbf{g}^a_{ij}(m)\big)\big|}$$

is a globally defined density which is everywhere invertible, i.e., trivializing.

(ii) *Given a Batchelor atlas, the density $\nu_{B,\mathbf{g}}$ defined on any coordinate chart U_a in the Batchelor atlas with local coordinates (x^a, ξ^a) by*

$$\nu_{B,\mathbf{g}}\Big(\frac{\partial}{\partial x^a}\Big|_m, \frac{\partial}{\partial \xi^a}\Big|_m \Big) = \sqrt{\big|\mathrm{Ber}\big(\mathbf{GBg}^a_{ij}(m)\big)\big|}$$

is a globally defined trivializing density (*see also Remark* 3.8).

Proof. (i): If U_a and U_b are two local coordinates charts, then we have seen in (17) and (19) that the bases $\partial_{x^a}, \partial_{\xi^a}$ and $\partial_{x^b}, \partial_{\xi^b}$ are related by

$$\begin{pmatrix} \frac{\partial}{\partial x^a}\Big|_m \\ \frac{\partial}{\partial \xi^a}\Big|_m \end{pmatrix} = \begin{pmatrix} \frac{\partial x^b}{\partial x^a} & \frac{\partial \xi^b}{\partial x^a} \\ \frac{\partial x^b}{\partial \xi^a} & \frac{\partial \xi^b}{\partial \xi^a} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x^b}\Big|_m \\ \frac{\partial}{\partial \xi^b}\Big|_m \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \frac{\partial x^b}{\partial x^a} & \frac{\partial \xi^b}{\partial x^a} \\ \frac{\partial x^b}{\partial \xi^a} & \frac{\partial \xi^b}{\partial \xi^a} \end{pmatrix} = \mathrm{matrix}(T_m\varphi^{ba})$$

and that the matrices $\mathbf{g}^a_{ij}$ and $\mathbf{g}^b_{ij}$ are related by

$$\begin{pmatrix} \mathbf{g}^{a,00} & \mathbf{g}^{a,01} \\ \mathbf{g}^{a,10} & \mathbf{g}^{a,11} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^b}{\partial x^a} & \frac{\partial \xi^b}{\partial x^a} \\ \frac{\partial x^b}{\partial \xi^a} & \frac{\partial \xi^b}{\partial \xi^a} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{g}^{b,00} & \mathbf{g}^{b,01} \\ \mathbf{g}^{b,10} & \mathbf{g}^{b,11} \end{pmatrix} \cdot \begin{pmatrix} \left(\frac{\partial x^b}{\partial x^a}\right)^t & \left(\frac{\partial x^b}{\partial \xi^a}\right)^t \\ -\left(\frac{\partial \xi^b}{\partial x^a}\right)^t & \left(\frac{\partial \xi^b}{\partial \xi^a}\right)^t \end{pmatrix} .$$

In terms of the action of $\mathrm{Gl}(d|n,\mathcal{A})$ on the frame bundle $\mathcal{F}M$ (the action is on the right rather than on the left) the bases $\partial_{x^a}, \partial_{\xi^a}$ and $\partial_{x^b}, \partial_{\xi^b}$ are related by the matrix

$$\begin{pmatrix} \left(\frac{\partial x^b}{\partial x^a}\right)^t & \left(\frac{\partial x^b}{\partial \xi^a}\right)^t \\ -\left(\frac{\partial \xi^b}{\partial x^a}\right)^t & \left(\frac{\partial \xi^b}{\partial \xi^a}\right)^t \end{pmatrix} .$$

As we have (for even matrices) the equality

$$\begin{aligned} \mathrm{Ber}\left(\begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}\right) &= \det(A^t + C^t\, D^{t-1}\, B^t) \cdot \det(D^t)^{-1} \\ &= \det(A - B\, D^{-1}\, C) \cdot \det(D)^{-1} = \mathrm{Ber}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right), \end{aligned}$$

it follows that we have the equality

$$\mathrm{Ber}\big(\mathbf{g}^a_{ij}(m)\big) = \mathrm{Ber}(T_m\varphi^{ba})^2 \cdot \mathrm{Ber}\big(\mathbf{g}^b_{ij}(m)\big) . \tag{26}$$

As we assume that M and the chosen Batchelor atlas are o.o.d , we have in particular that $\mathrm{Ber}(T_m\varphi^{ba})^2 = \mathrm{Ber}_\pi(T_m\varphi^{ba})^2$ and $\mathbf{B}\mathrm{Ber}_\pi(T_m\varphi^{ba}) > 0$. It follows that we have

$$\begin{aligned} \nu_{\mathbf{g}}\Big(\frac{\partial}{\partial x^a}\Big|_m, \frac{\partial}{\partial \xi^a}\Big|_m\Big) &= \sqrt{\big|\mathrm{Ber}\big(\mathbf{g}^a_{ij}(m)\big)\big|} = \mathrm{Ber}_\pi(T_m\varphi^{ba}) \cdot \sqrt{\big|\mathrm{Ber}\big(\mathbf{g}^b_{ij}(m)\big)\big|} \\ &= \mathrm{Ber}_\pi(T_m\varphi^{ba}) \cdot \nu_{\mathbf{g}}\Big(\frac{\partial}{\partial x^b}\Big|_m, \frac{\partial}{\partial \xi^b}\Big|_m\Big) . \end{aligned}$$

As this is the right behaviour, $\nu_{\mathbf{g}}$ is indeed a well defined global density on M. As we obviously have $\mathbf{B}\nu_{\mathbf{g}} \neq 0$, it is a trivializing section.

(ii): The case of the density $\nu_{B,\mathbf{g}}$ is very similar; we only have to change (26) to

$$\begin{aligned} \mathrm{Ber}\big(\mathbf{GB}\mathbf{g}^a_{ij}(m)\big) &= \mathbf{GB}\big(\mathrm{Ber}(T_m\varphi^{ba})^2 \cdot \mathrm{Ber}\big(\mathbf{GB}\mathbf{g}^b_{ij}(m)\big) \\ &= \mathrm{Ber}(T_m\varphi^{ba})^2 \cdot \mathrm{Ber}\big(\mathbf{GB}\mathbf{g}^b(m)\big) , \end{aligned}$$

where the last equality follows from the fact that for two charts in a Batchelor atlas we have $\mathrm{Ber}(T_m\varphi^{ba}) = \mathbf{GB}\mathrm{Ber}(T_m\varphi^{ba})$ (as $T_m\varphi^{ba}$ does not depend upon the odd coordinates). ■

Notation/Definition 5.12. For any subset $I \subset \{1,\dots,n\}$ we define the set $I^c \subset \{1,\dots,n\}$ as $I^c = \{1,\dots,n\} \setminus I$. Now let $I, J \subset \{1,\dots,n\}$ be any two disjoint subsets for which we write $I = \{i_1,\dots,i_k\}$ and $J = \{j_1,\dots,j_\ell\}$ with $i_1 < i_2 < \cdots < i_k$ and $j_1 < \cdots < j_\ell$. We then define the integer $\varepsilon(I,J)$ as the number of transpositions needed to put the sequence

$$i_1, i_2, \dots, i_k, j_1, \dots, j_\ell$$

in increasing order.

Proposition 5.13. *Let $(E, \langle\cdot,\cdot\rangle_E, \mathscr{S}_E)$ be a proto super Hilbert over $\mathbf{K}$ and let M be an $\mathcal{A}$-manifold of graded dimension $d|n$ equipped with a trivializing density ν. Then the map $\mathscr{S}_\nu \colon C_c^\infty(M, E \otimes \mathcal{A}^{\mathbf{K}}) \times C_c^\infty(M, E \otimes \mathcal{A}^{\mathbf{K}}) \to \mathbf{C}$ given by*

$$\mathscr{S}_\nu(\chi, \psi) = \int_M \nu(m) \cdot \mathscr{S}_E\big(\chi(m), \psi(m)\big)$$

defines a super scalar product $\mathscr{S}_\nu$ on $C_c^\infty(M, E \otimes \mathcal{A}^{\mathbf{K}})$, the space of compactly supported smooth functions on M with values in $E \otimes \mathcal{A}^{\mathbf{K}}$. If $\mathscr{S}_E$ is homogeneous of parity α, $\mathscr{S}_\nu$ is homogeneous of parity $\alpha + n$.

Proof. That it defines a graded symmetric sesquilinear form is immediate. That it indeed takes values in $\mathbf{C}$ is a direct consequence of the fact that Berezin integration of a $\mathcal{A}^{\mathbf{K}}$-valued density on M yields an element of $\mathbf{K}$ and that (the extension of) $\mathscr{S}_E$ takes values in $\mathcal{A}^{\mathbf{C}}$. As ν is even, the conclusion concerning the homogeneity/parity of $\mathscr{S}_\nu$ follows immediately from the fact that the parity of $\int_M$ is $n \bmod 2$, see Lemma 5.9. Remains non-degeneracy.

Let $\chi \colon M \to E \otimes \mathcal{A}^{\mathbf{K}}$ be a smooth function with compact support. If χ is not (identically) zero, there exists $m \in M$ such that $\chi(m) \neq 0$. Now choose a local coordinate chart $m \in U \subset M$ with local coordinates x_i, ξ_j. On U there exist smooth functions $\chi_I \colon \mathbf{B}U \to E$ such that χ is given by

$$\chi(x, \xi) = \sum_{I \subset \{1,\dots,n\}} \xi^I \cdot \mathbf{G}\chi_I(x)\,.$$

As there exists $m \in U$ with $\chi(m) \neq 0$, not all functions χ_I can be identically zero, say χ_I for some $I \subset \{1,\dots,n\}$ which we now consider fixed. It follows that there exists $m_o \in \mathbf{B}U$ with $0 \neq \chi_I(m_o) \in E$. Because $\mathscr{S}_E$ is non-degenerate, there exists $v \in E$ such that $\mathscr{S}_E\big(\chi_I(m_o), v\big) \neq 0$. Taking a suitable multiple of v, we may assume $\big|\mathscr{S}_E\big(\chi_I(m_o), v\big)\big| = 1$.[5]

Now $\mathscr{S}_E$ is continuous, so the map $m \mapsto \mathscr{S}_E\big(\chi_I(m), v\big)$ (on $\mathbf{B}U$) is smooth and non-zero at $m_o \in \mathbf{B}U$. We thus can choose a smooth function $\rho \colon \mathbf{B}U \to [0,1] \subset \mathbf{R}$ with compact support such that $\rho(m_o) = 1$ and such that $\rho(m) \neq 0$ implies $\big|\mathscr{S}_E\big(\chi_I(m), v\big)\big| > \frac{1}{2}$. We then define the (smooth) function $\psi \colon M \to E \otimes \mathcal{A}^{\mathbf{K}}$ by

[5] We need the absolute value here because $\mathscr{S}_E$ takes values in $\mathbf{C}$ and E might be a vector space over $\mathbf{R}$.

$$\psi(m) = \begin{cases} \dfrac{(\mathbf{G}\rho)(x)}{\big(\mathbf{G}\mathscr{S}_E(\chi_I, v)\big)(x)\cdot \nu(\partial_x|_m, \partial_\xi|_m)} \cdot \mathfrak{C}^{|I|}(v)\cdot \xi^{I^c} & \text{for } (x,\xi)\cong m \in U \\ 0 & \text{otherwise.} \end{cases}$$

It follows immediately that we have (no longer writing the $\mathbf{G}$-extension)

$$\begin{aligned} \mathscr{S}_\nu(\chi,\psi) &= \int_M \nu(m)\cdot \mathscr{S}_E\big(\chi(m),\psi(m)\big) \\ &= \int_U \mathrm{d}(x,\xi)\ \nu(\partial_x|_m,\partial_\xi|_m)\cdot \frac{\mathscr{S}_E\bigg(\displaystyle\sum_{J\subset\{1,\dots,n\}} \chi_J(x)\,\xi^J, \mathfrak{C}^{|I|}(v)\,\xi^{I^c}\bigg)\cdot\rho(x)}{\mathscr{S}_E\big(\chi_I(x),v\big)\cdot \nu(\partial_x|_m,\partial_\xi|_m)} \\ &= \int_U \mathrm{d}(x,\xi)\ \rho(x)\cdot\xi^I\cdot\xi^{I^c} = (-1)^{\varepsilon(I,I^c)}\cdot \int_{\mathbf{B}U} \mathrm{d}\lambda^{(p)}(x)\ \rho(x) \neq 0\,, \end{aligned}$$

where for the third equality we used that $J=I$ is the only term contributing to the integral. This proves non-degeneracy. ■

Definition 5.14. Let M be o.o.d and $\mathbf{g}$ a metric on M. Then we define the super scalar products $\mathscr{S}_{\mathbf{g}}$ and $\mathscr{S}_{B,\mathbf{g}}$ on $C_c^\infty(M, E\otimes\mathcal{A}^{\mathbf{K}})$ as those associated to the trivializing densities $\nu_{\mathbf{g}}$ and $\nu_{B,\mathbf{g}}$ (defined in Proposition 5.11) according to Proposition 5.13.

Lemma 5.15. *Let $\mathcal{U} = \{U_a \mid a\in A\}$ be an o.o.d Batchelor atlas adapted to the super metric $\mathbf{g}$ on M, Lemma 4.7, and let $\chi,\psi\in C_c^\infty(M;E\otimes\mathcal{A}^{\mathbf{K}})$ be smooth functions. Let furthermore $\rho_a\colon \mathbf{B}U_a\to[0,1]$ be a smooth partition of unity subordinated to $\mathbf{B}\mathcal{U}$. Then the super scalar product $\mathscr{S}_{B,\mathbf{g}}$ on $C_c^\infty(M;E\otimes\mathcal{A}^{\mathbf{K}})$ is given by*

$$\begin{aligned} \mathscr{S}_{B,\mathbf{g}}(\chi,\psi) = \sum_{a\in A}\int_{\mathbf{B}U_a} \mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot \sqrt{\big|\det\big(\mathbf{B}\mathbf{g}^{a,00}_{ij}(x^a)\big)\big|} \\ \cdot \sum_{I\subset\{1,\dots,n\}} (-1)^{\varepsilon(I,I^c)}\,\mathscr{S}_E\Big(\mathfrak{C}^{|I|}\big(\chi_{a,I}(x^a)\big), \mathfrak{C}^n\big(\psi_{a,I^c}(x^a)\big)\Big)\,, \end{aligned}$$

where (x^a,ξ^a) denotes a local system of coordinates on U_a and where the local expression of χ (respectively ψ) is given by

$$\chi(x^a,\xi^a) = \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I\,\mathbf{G}\chi_{a,I}(x^a)$$

for smooth functions $\chi_{a,I}\colon \mathbf{B}U_a\to E$.

Proof. As the atlas is adapted to the super metric, the matrices $-i\,\mathbf{B}\mathbf{g}^{a,11}$ are constant the identity. We thus have

$$
\begin{aligned}
\mathscr{S}_{B,\mathbf{g}}(\chi,\psi) &= \int_M \nu_{B,\mathbf{g}}(m)\cdot \mathscr{S}_E\big(\chi(m),\psi(m)\big)\\
&\overset{[5.6]}{=} \sum_{a\in A}\int_{U_a} \mathrm{d}(x^a,\xi^a)\ (\mathbf{G}\rho_a)(x^a)\cdot \nu_{B,\mathbf{g}}\Big(\frac{\partial}{\partial x^a}\Big|_{(x^a,\xi^a)}, \frac{\partial}{\partial \xi^a}\Big|_{(x^a,\xi^a)}\Big)\\
&\qquad\qquad \cdot \sum_{I,J\subset\{1,\dots,n\}} \mathscr{S}_E\big((\xi^a)^I\,\chi_{a,I}(x^a), (\xi^A)^J\,\psi_{a,J}(x^a)\big)\\
&\overset{[5.11]}{=} \sum_{a\in A}\int_{U_a} \mathrm{d}(x^a,\xi^a)\ (\mathbf{G}\rho_a)(x^a)\cdot \sqrt{\big|\mathrm{Ber}\big(\mathbf{G}\mathbf{B}\mathbf{g}^a_{ij}(x^a)\big)\big|}\\
&\qquad\qquad \cdot \sum_{I,J\subset\{1,\dots,n\}} (\xi^a)^I\,(\xi^a)^J\,\mathscr{S}_E\big(\mathfrak{C}^{|I|}\,\chi_{a,I}(x^a), \mathfrak{C}^{I+J}\,\psi_{a,J}(x^a)\big)\\
&= \sum_{a\in A}\int_{\mathbf{B}U_a} \mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot \sqrt{\big|\det\big(\mathbf{B}\mathbf{g}^{a,00}_{ij}(x^a)\big)\big|}\\
&\qquad\qquad \cdot \sum_{I\subset\{1,\dots,n\}} (-1)^{\varepsilon(I,I^c)}\mathscr{S}_E\big(\mathfrak{C}^{|I|}\,\chi_{a,I}(x^a), \mathfrak{C}^{n}\,\psi_{a,I^c}(x^a)\big)\ ,
\end{aligned}
$$

where for the third equality we first used the extension of $\mathscr{S}_E$ to $E\otimes\mathcal{A}^{\mathbf{K}}$ by right-linearity, see Lemma 2.8(ii), to extract the powers of ξ^a, giving us the conjugations on $\chi_{a,I}$ and $\psi_{a,J}$, and then that $\mathscr{S}_E$ takes values in $\mathbf{C}$, which is even, allowing to put the powers of ξ^a to the left. For the fourth equality we used $-i\,\mathbf{B}\mathbf{g}^{a,11}=\mathbf{1}$ to simplify the Berezinian and we applied the definition of Berezin integration, Definition 5.2. ■

Proposition 5.16. *The super scalar product $\mathscr{S}_{B,\mathbf{g}}$, see Proposition* 5.14 *and Definition* 5.13*, is continuous with respect to the (ordinary) metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$, see* (22) *and Lemma* 4.8*.*

Proof. We use the descriptions in terms of an adapted Batchelor atlas given in Lemma 4.8 and Lemma 5.15. We first note that according to Lemma 4.8 the norm-squared of an element $\chi\in C^\infty_c(M;E\otimes\mathcal{A}^{\mathbf{K}})$ is given by

$$
\|\chi\|^2_{\mathbf{g}} = \sum_{a\in A}\int_{\mathbf{B}U_a} \mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot\sqrt{\big|\det\big(\mathbf{B}\mathbf{g}^{a,00}(x^a)\big)\big|}\cdot \sum_{I\subset\{1,\dots,n\}} \|\chi_{a,I}(x^a)\|^2_E\ ,
$$

As E is a proto super Hilbert space, we have $\langle E_0,E_1\rangle_E=0$, which implies in particular that we have (for $\chi,\psi\in E$) the equality

$$
\langle\mathfrak{C}\chi,\mathfrak{C}\psi\rangle_E \equiv \langle\chi_0-\chi_1,\psi_0-\psi_1\rangle_E = \langle\chi_0,\psi_0\rangle_E+\langle\chi_1,\psi_1\rangle_E = \langle\chi,\psi\rangle_E\ ,
$$

and thus in particular $\|\mathfrak{C}\chi\|_E=\|\chi\|_E$. Moreover, $\mathscr{S}_E$ is continuous with respect to the topology induced by the norm, which means that there exists a constant $C\geq 0$ such that

$$
|\mathscr{S}_E(\chi,\psi)| \leq C\cdot\|\chi\|_E\cdot\|\psi\|_E\ .
$$

With these preparations we now turn our attention to $|\mathscr{S}_{B,\mathbf{g}}(\chi,\psi)|$ (with $\chi,\psi\in C^\infty_c(M;E\otimes\mathcal{A}^{\mathbf{K}})$), for which we obtain, using Lemma 5.15, the following estimates:

$$
\begin{aligned}
|\mathscr{S}_{B,\mathbf{g}}(\chi,\psi)| \leq \sum_{a\in A}\int_{\mathbf{B}U_a} &\mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot\sqrt{\left|\det\bigl(\mathbf{B}\mathbf{g}^{a,00}_{ij}(x^a)\bigr)\right|}\\
&\cdot\sum_{I\subset\{1,\dots,n\}}\left|\mathscr{S}_E\bigl(\mathfrak{C}^I\,\chi_{a,I}(x^a),\mathfrak{C}^n\,\psi_{a,I^c}(x^a)\bigr)\right|\\
\leq \sum_{a\in A}\int_{\mathbf{B}U_a} &\mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot\sqrt{\left|\det\bigl(\mathbf{B}\mathbf{g}^{a,00}_{ij}(x^a)\bigr)\right|}\\
&\cdot\sum_{I\subset\{1,\dots,n\}} C\cdot\|\mathfrak{C}^I\,\chi_{a,I}(x^a)\|_E\cdot\|\mathfrak{C}^n\,\psi_{a,I^c}(x^a)\|_E\\
= \sum_{a\in A}\int_{\mathbf{B}U_a} &\mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot\sqrt{\left|\det\bigl(\mathbf{B}\mathbf{g}^{a,00}_{ij}(x^a)\bigr)\right|}\\
&\cdot\sum_{I\subset\{1,\dots,n\}} C\cdot\|\chi_{a,I}(x^a)\|_E\cdot\|\psi_{a,I^c}(x^a)\|_E\,.
\end{aligned}
$$

We now use the standard argument for the Cauchy-Schwarz inequality, i.e., the positivity of the quadratic function $g(t)$ defined as

$$
\begin{aligned}
g(t) = \sum_{a\in A}\int_{\mathbf{B}U_a} &\mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot\sqrt{\left|\det\bigl(\mathbf{B}\mathbf{g}^{a,00}_{ij}(x^a)\bigr)\right|}\\
&\cdot\sum_{I\subset\{1,\dots,n\}}\bigl(\,\|\chi_{a,I}(x^a)\|_E + t\cdot\|\psi_{a,I^c}(x^a)\|_E\bigr)^2\\
= \|\chi\|^2_{\mathbf{g}} + t^2\cdot\|\psi\|^2_{\mathbf{g}} + 2t\sum_{a\in A}\int_{\mathbf{B}U_a} &\mathrm{d}\lambda^{(d)}(x^a)\ \rho_a(x^a)\cdot\sqrt{\left|\det\bigl(\mathbf{B}\mathbf{g}^{a,00}_{ij}(x^a)\bigr)\right|}\\
&\cdot\sum_{I\subset\{1,\dots,n\}}\|\chi_{a,I}(x^a)\|_E\cdot\|\psi_{a,I^c}(x^a)\|_E\,,
\end{aligned}
$$

to finally obtain the estimate

$$
|\mathscr{S}_{B,\mathbf{g}}(\chi,\psi)| \leq C\cdot\|\chi\|_{\mathbf{g}}\cdot\|\psi\|_{\mathbf{g}}\,,
$$

which shows that $\mathscr{S}_{B,\mathbf{g}}$ is continuous with respect to the topology on the space $C^\infty_c(M;E\otimes\mathcal{A}^{\mathbf{K}})$ induced by the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$. ■

Corollary 5.17. *If $(E,\langle\cdot,\cdot\rangle_E,\mathscr{S}_E)$ is a proto super Hilbert space (over $\mathbf{K}$), then $C^\infty_c(M;E\otimes\mathcal{A}^{\mathbf{K}})$ equipped with the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ and the super scalar product $\mathscr{S}_{B,\mathbf{g}}$ also is a proto super Hilbert space.*

6. Integrating along fibers

It frequently happens that we have a function f on a product space $E\times F$ and that we want to integrate only over the F-part in such a way that the result is a function on E. In the ordinary context one usually speaks about this as integrals depending upon parameters and several results as to when the resulting function on E is continuous or smooth are available. In the super context we have to be a bit careful because Berezin integration breaks down a smooth function of even and odd variables into smooth functions of real variables only (that then are integrated). Without care, we thus would not get a (smooth) function of the remaining even and odd variables on E. The purpose of this section is to provide a rigorous framework for such integrals depending upon even and odd parameters and to show that we have a change-of-variables formula for such integrals. We then use it to show that if a super metric $\mathbf{g}$ on an $\mathcal{A}$-manifold is invariant under the action of an $\mathcal{A}$-Lie group, then the super scalar product $\mathscr{S}_{\mathbf{g}}$ is invariant under the induced group action on the space of functions.

Discussion 6.1. Let E be a finite dimensional graded vector space of dimension $p|q$, let F be a finite dimensional graded vector space of dimension $d|n$ and let $U \subset E_0 \times F_0$ be an open set. When we use (even and odd) coordinates (s, σ) on E_0 and (x, ξ) on F_0, any smooth function $f\colon U \to \mathcal{A}^{\mathbf{K}}$ is determined by $2^q \times 2^n = 2^{q+n}$ ordinary smooth functions $f_{I,J}\colon \mathbf{B}U \to \mathbf{K}$ according to (note the order of the products of coordinates ξ and σ)

$$f(s, \sigma, x, \xi) = \sum_{I \subset \{1,\dots,n\}} \sum_{J \subset \{1,\dots,q\}} \xi^I \, \sigma^J \, \big(\mathbf{G}(f_{I,J})\big)(s, x) \,.$$

Associated to U we define the subset $U_{(s,\sigma)} \subset E_0$ by

$$U_{(s,\sigma)} = \{\, (x, \xi) \in E_0 \mid (s, \sigma, x, \xi) \in U \,\} \,.$$

We now assume that the function f is such that its restriction to $U_{(s,\sigma)}$ has compact support for all (s, σ). With these ingredients we then define the function $\mathcal{I}_f\colon \pi_E(U) \to \mathcal{A}^{\mathbf{K}}$ (where $\pi_E\colon E_0 \times F_0 \to E_0$ denotes the canonical projection) as integration of f over the fibers F_0 of U by

$$\begin{aligned}
\mathcal{I}_f(s, \sigma) &= \int_{U_{(s,\sigma)}} \mathrm{d}(x, \xi) \; f(s, \sigma, x, \xi) \\
&\stackrel{\text{def.}}{=} \sum_{J \subset \{1,\dots,q\}} \sigma^J \int_{\mathbf{B}U_{(s,\sigma)}} \mathrm{d}\lambda^{(d)}(x) \; f_{\underline{n},J}(s, x) \,.
\end{aligned}$$

A slightly better but less clear definition would be the following (which clarifies some obscure points in the above formula). For $s \in \mathbf{B}E_0 \cong \mathbf{R}^p$ we define the set $\mathbf{B}U_s = \{\, x \in \mathbf{B}F_0 \cong \mathbf{R}^d \mid (s, x) \in \mathbf{B}U \,\}$ (which is the same as $\mathbf{B}U_{(s,\sigma)}$, as the map $\mathbf{B}$ discards any dependence on nilpotent and thus odd coordinates). We then define the (ordinary!) smooth functions $\mathcal{I}_{f,J}\colon \mathbf{B}\pi_E(U) \to \mathbf{K}$ by

$$\mathcal{I}_{f,J}(s) = \int_{\mathbf{B}U_s} f_{\underline{n},J}(s, x) \; \mathrm{d}\lambda^{(d)}(x) \,.$$

That these functions are smooth is because of our assumption that the restriction of f to all $U_{(s,\sigma)}$ has compact support (and of course because the functions $f_{I,J}$ are smooth). With these functions we then define the (super) smooth function $\mathcal{I}_f$ by

$$\mathcal{I}_f(s, \sigma) = \sum_{J \subset \{1,\dots,q\}} \sigma^J \, \big(\mathbf{G}(\mathcal{I}_{f,J})\big)(s) \,.$$

We now want to show that this partial integration (not to be confused with integration by parts) or fiberwise integration has a nice transformation property under a special kind of (local) diffeomorphisms of $E_0 \times F_0$: those that preserve the "fibers."

Definition 6.2. Let $U, V \subset E_0 \times F_0$ be two open sets and let $\Phi\colon U \to V$ be a (local) diffeomorphism. Using the product structure $E_0 \times F_0$ we thus can "decompose" Φ into two components $\Phi_E\colon U \to E_0$ and $\Phi_F\colon U \to F_0$ according to

$$\Phi(u,v) = \big(\Phi_E(u,v), \Phi_F(u,v)\big)\,.$$

We will say that Φ is a *fiber-diffeomorphism* if Φ_E is independent of v. A more sophisticated way would be to say that Φ is a fiber-diffeomorphism if there exists a diffeomorphism $\Phi_E\colon \pi_E(U) \to \pi_E(V)$ such that the following diagram is commutative:

$$\begin{tikzcd} U \arrow[r, "\Phi"] \arrow[d, "\pi_E"'] & V \arrow[d, "\pi_E"] \\ \pi_E(U) \arrow[r, "\Phi_E"'] & \pi_E(V)\,. \end{tikzcd} \tag{27}$$

When we use coordinates (even and odd) on $E_0 \times F_0$ and when we denote those on U by $\big((s,\sigma),(x,\xi)\big) \cong (s,\sigma,x,\xi)$ ((s,σ) being coordinates on E_0 and (x,ξ) on F_0) and those on V by (t,τ,y,η) (they are the same coordinates, but we use different symbols to differentiate between source and target space), then a fiber-diffeomorphism has the form

$$\Phi(s,\sigma,x,\xi) = (t,\tau,y,\eta)\,,$$

with

$$t = t(s,\sigma)\,, \quad \tau = \tau(s,\sigma)\,, \quad y = y(s,\sigma,x,\xi)\,, \quad \eta = \eta(s,\sigma,x,\xi)\,.$$

If $\Phi\colon U \to V$ is a fiber-diffeomorphism, the restriction $\Phi_{(s,\sigma)}$ of Φ to a slice $U_{(s,\sigma)}$ is a "diffeomorphism" $\Phi_{(s,\sigma)}\colon U_{(s,\sigma)} \to V_{(t,\tau)}$, where $(t,\tau) \in E_0$ is determined by $(t,\tau) = \Phi_E(s,\sigma)$. Of course it is not a real diffeomorphism, it only is a homeomorphism with inverse $(\Phi^{-1})_{(t,\tau)}$, the restriction of Φ^{-1} to the slice $V_{(t,\tau)}$.
However, in many ways it behaves/can behave like a diffeomorphism. In particular we can attribute a tangent map to it (this will be a particular case of a generalized tangent map as described in [15, V.3.19]). If $\Phi\colon U \to V$ is a fiber-diffeomorphism, then its tangent map is given by (see (15) and (17))

$$T\Phi(\begin{pmatrix} \dfrac{\partial}{\partial g} \\ \dfrac{\partial}{\partial \chi} \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial \xi} \end{pmatrix}) = \begin{pmatrix} \dfrac{\partial t}{\partial g} & \dfrac{\partial \tau}{\partial g} & \dfrac{\partial y}{\partial g} & \dfrac{\partial \eta}{\partial g} \\ \dfrac{\partial t}{\partial \chi} & \dfrac{\partial \tau}{\partial \chi} & \dfrac{\partial y}{\partial \chi} & \dfrac{\partial \eta}{\partial \chi} \\ 0 & 0 & \dfrac{\partial y}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ 0 & 0 & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial \eta}{\partial \xi} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial}{\partial t} \\ \dfrac{\partial}{\partial \tau} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial \eta} \end{pmatrix}\,.$$

Obviously the matrix elements are smooth functions on U. According to [15, V.3.19] the tangent map of $\Phi_{(s,\sigma)}$ is described by the matrix

$$T\Phi_{(s,\sigma)}(\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial \xi} \end{pmatrix}) = \begin{pmatrix} \dfrac{\partial y}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial \eta}{\partial \xi} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial \eta} \end{pmatrix} .$$

For fixed (s,σ) the matrix elements of $\Phi_{(s,\sigma)}$ are not (necessarily) smooth functions, but they are smooth in the complete set of variables (s,σ,x,ξ).

Lemma 6.3. *Let $U,V,W \subset E_0 \times F_0$ be three open sets and $\Phi\colon U \to V$ and $\Psi\colon V \to W$ two fiber-diffeomorphisms. Then $\Psi \circ \Phi\colon U \to W$ is a fiber diffeomorphism and we have the following properties concerning the generalized tangent maps:*

$$T(\Psi \circ \Phi)_{(s,\sigma)} = T\Psi_{\Phi_E(s,\sigma)} \circ T\Phi_{(s,\sigma)}$$

and

$$\mathrm{Ber}_{(\pi)}\big(T\Psi \circ \Phi)_{(s,\sigma)}\big) = \mathrm{Ber}_{(\pi)}\big(T\Psi_{\Phi_E(s,\sigma)}\big) \cdot \mathrm{Ber}_{(\pi)}\big(T\Phi_{(s,\sigma)}\big) .$$

Proof. It is immediate that the composition of two fiber-diffeomorphisms again is a fiber-diffeomorphism. When we denote the coordinates on W by (g,χ,z,ζ) (still the same, just changing the names again), we have for the full tangent maps

$$T(\Psi \circ \Phi) = (T\Psi) \circ (T\Phi) ,$$

which in terms of the matrix representation gives

$$T(\Psi \circ \Phi)(\begin{pmatrix} \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial \sigma} \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial \xi} \end{pmatrix}) = \begin{pmatrix} \dfrac{\partial t}{\partial s} & \dfrac{\partial \tau}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial \eta}{\partial s} \\ \dfrac{\partial t}{\partial \sigma} & \dfrac{\partial \tau}{\partial \sigma} & \dfrac{\partial y}{\partial \sigma} & \dfrac{\partial \eta}{\partial \sigma} \\ 0 & 0 & \dfrac{\partial y}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ 0 & 0 & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial \eta}{\partial \xi} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial g}{\partial t} & \dfrac{\partial \chi}{\partial t} & \dfrac{\partial z}{\partial t} & \dfrac{\partial \zeta}{\partial t} \\ \dfrac{\partial g}{\partial \tau} & \dfrac{\partial \chi}{\partial \tau} & \dfrac{\partial z}{\partial \tau} & \dfrac{\partial \zeta}{\partial \tau} \\ 0 & 0 & \dfrac{\partial z}{\partial y} & \dfrac{\partial \zeta}{\partial y} \\ 0 & 0 & \dfrac{\partial z}{\partial \eta} & \dfrac{\partial \zeta}{\partial \eta} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial}{\partial g} \\ \dfrac{\partial}{\partial \chi} \\ \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial \zeta} \end{pmatrix} .$$

It follows immediately that we have

$$\begin{pmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial \zeta}{\partial x} \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial \zeta}{\partial \xi} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial \eta}{\partial \xi} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial z}{\partial y} & \dfrac{\partial \zeta}{\partial y} \\ \dfrac{\partial z}{\partial \eta} & \dfrac{\partial \zeta}{\partial \eta} \end{pmatrix} .$$

This, together with the multiplicative property of the Berezinian $\mathrm{Ber}_{(\pi)}$, proves the second part of the lemma. ■

Proposition 6.4. *Let $\Phi\colon U \to V$ be a fiber-diffeomorphism with*

$$\Phi(s,\sigma,x,\xi) = \big(\Phi_E(s,\sigma), \Phi_F(s,\sigma,x,\xi)\big) \equiv (t,\tau,y,\eta)$$

and let $f\colon V \to \mathcal{A}^{\mathbf{K}}$ be a smooth function with compact support on all slices $V_{(t,\tau)}$. Then we have the equality

$$\begin{aligned}\int_{V_{(t,\tau)}} \mathrm{d}(y,\eta)\ f(t,\tau,y,\eta) &\equiv \int_{V_{(t,\tau)}} \mathrm{d}(y,\eta)\ f\big(\Phi_E(s,\sigma),y,\eta\big) \\ &= \int_{U_{(s,\sigma)}} \mathrm{d}(x,\xi)\ f\big(\Phi(s,\sigma,x,\xi)\big)\cdot \mathrm{Ber}_\pi(T_{(x,\xi)}\Phi_{(s,\sigma)})\,. \end{aligned} \tag{28}$$

Proof. The proof of this "change of variables formula with parameters" is a straightforward generalization of the proof of the analogous statement for the standard "change of variables formula" (without parameters) as given in [18, Thm 4.6.1] (this "exercice" has been done in detail in [17]). ■

Once we have this result, we are in business. We can define "partial" densities on locally trivial fiber bundles, integration of this kind of density along the fibers and their behavior under fiber-diffeomorphisms. We will do so only for direct products, but the generalization to locally trivial fiber bundles should be obvious.

Definition 6.5. Let G and M be two $\mathcal{A}$-manifolds of graded dimensions $p|q$ and $d|n$ respectively.

- An *M-frame at* $(g,m) \in G\times M$ is an ordered homogeneous basis of the subspace $\{0\}\times T_mM = \ker(T_{(g,m)}\pi_G) \subset T_{(g,m)}(G\times M)$, where $\pi_G\colon G\times M \to G$ denotes the canonical projection. The *M-frame bundle* $\mathcal{F}^M(G\times M) \to G\times M$ is the bundle whose fibers $\mathcal{F}^M_{(g,m)}(G\times M)$ consist of all M-frames at (g,m). Said differently, we have an "isomorphism"

$$\mathcal{F}^M_{(g,m)} \cong \mathcal{F}_m M\,.$$

It is a principal $\mathrm{Gl}(d|n,\mathcal{A})$ bundle over $G\times M$. The right-action of $\mathrm{Gl}(d|n,\mathcal{A})$ on $\mathcal{F}^M(G\times M)$ is defined as follows. For $(v_1,\dots,v_{d+n}) \in \mathcal{F}^M_{(g,m)}(G\times M)$, i.e., a basis of T_mM, and $(A_{ij}) \in \mathrm{Gl}(d|n,\mathcal{A})$ we have

$$(v_i)\cdot(A_{jk}) = (w_\ell) \qquad \text{with} \qquad w_i = \sum_{j=1}^{d+n} v_j\,A_{ji}\,.$$

- An *M-fiber-density* on $G\times M$ is a map $\nu\colon \mathcal{F}^M(G\times M) \to \mathcal{A}^{\mathbf{K}}$ satisfying the condition that for all $v=(v_i) \in \mathcal{F}^M_{(g,m)}(G\times M)$ and for all $A=(A_{jk}) \in \mathrm{Gl}(d|n,\mathcal{A})$ we have the equality

$$\nu\big(v\cdot A\big) = \mathrm{Ber}_\pi(A)\cdot \nu(v)\,.$$

An M-fiber-density can be seen as a section of the *M-fiber-density "line" bundle* $\mathscr{D}^M(G\times M)$ associated to the principal $\mathrm{Gl}(d|n,\mathcal{A})$ bundle $\mathcal{F}^M(G\times M)$ by the representation $\mathrm{Gl}(d|n,\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}^{\mathbf{K}})$, $A \mapsto \mathrm{Ber}_\pi(A)^{-1}$ on $\mathcal{A}^{\mathbf{K}}$. It thus has $\mathcal{A}^{\mathbf{K}}$ as typical fiber.

Construction 6.6. Let ν be a smooth M-fiber-density on $G \times M$ that has compact support on each slice $\{g\} \times M$, i.e., for all $g \in G$ the set

$$\{ m \in M \mid \nu(g,m) \neq 0 \} \subset M$$

has compact closure. Associated to such an M-fiber-density we associate a function on G with values in $\mathcal{A}^{\mathbf{K}}$ which we will denote by $\int_M \nu$. This function is defined by the following procedure. Let $\mathcal{U} = \{ U_a \mid a \in A \}$ be a cover of M by coordinate charts with (local, even and odd) coordinates (x^a, ξ^a), let ρ_a be a partition of unity associated to this cover, and let $\mathcal{V} = \{V_b \mid b \in B\}$ be a cover of G by coordinate charts with coordinates (s, σ). For a point $g \in V_b$ with coordinates $g \cong (s^b, \sigma^b)$ we define the value $\big(\int_M \nu\big)(g) \equiv \big(\int_M \nu\big)(s,\sigma) \in \mathcal{A}^{\mathbf{K}}$ by

$$\Big(\int_M \nu\Big)(s^b,\sigma^b) = \sum_{a\in A} \int_{\{g\}\times U_a} \mathrm{d}(x^a,\xi^a)\ \rho_a(x^a,\xi^a) \\ \cdot \nu\Big(\frac{\partial}{\partial x^a}\Big|_{(s^b,\sigma^b,x^a,\xi^a)}, \frac{\partial}{\partial \xi^a}\Big|_{(s^b,\sigma^b,x^a,\xi^a)}\Big) .$$

Note that the (s^b,σ^b)-dependence enters via ν when we evaluate it on tangent vectors at (s^b,σ^b,x^a,ξ^a). The following two results then are a direct application of Proposition 6.4 (together with the definition).

Lemma 6.7. *The smooth function $\int_M \nu \colon G \to \mathcal{A}^{\mathbf{K}}$ is well defined, independent of the choices of the local coordinate charts on G and M and independent of the chosen partition of unity.*

Lemma 6.8. *Let G and H be $\mathcal{A}$-manifolds of dimension $p|q$, let M and N be $\mathcal{A}$-manifolds of dimension $d|n$, let ν be a (smooth) fiber-density on $H \times N$, and let $\Phi \colon G\times M \to H\times N$ be a fiber-diffeomorphism,i.e., a diffeomorphism such that there exists a diffeomorphism $\Phi_1 \colon G \to H$ making the following diagram commutative (see (27))*

$$\begin{array}{ccc} G\times M & \xrightarrow{\ \Phi\ } & H\times N \\ {\scriptstyle \pi_G}\big\downarrow & & \big\downarrow{\scriptstyle \pi_H} \\ G & \xrightarrow[\ \Phi_1\ ]{} & H\,. \end{array}$$

(i) *If $v = (v_i)_{i=1}^{d+n}$ is an M-frame at (g,m), then $T_{(g,m)}\Phi(v)$ is an N-frame at $\Phi(g,m) \in H\times N$.*

(ii) *The map $\Phi^*\nu \colon \mathcal{F}^M(G\times M) \to \mathcal{A}^{\mathbf{K}}$ defined by*

$$v \in \mathcal{F}^M_{(g,m)} \quad \Rightarrow \quad (\Phi^*\nu)(v) = \nu\big(T_{(g,m)}\Phi(v)\big)$$

is a fiber-density on $G\times M$.

(iii) *If the restriction of ν to any slice $\{h\}\times N$ has compact support, then the restriction of $\Phi^*\nu$ to any slice $\{g\}\times M$ has compact support and we have the equality (of smooth functions on G)*

$$\int_M \Phi^*\nu = \Big(\int_N \nu\Big)\circ\Phi_1 \quad \textit{or equivalently} \quad \Big(\int_M \Phi^*\nu\Big)(g) = \Big(\int_N \nu\Big)\big(\Phi_1(g)\big) .$$

With the above preparations, we can now attack the question of invariant super metrics and the invariance of the associated super scalar products on $C_c^\infty(M;\mathcal{A}^{\mathbf{K}})$. Let M be an $\mathcal{A}$-manifold of dimension $d|n$, G an $\mathcal{A}$-Lie group and $\Phi\colon G\times M\to M$ a smooth left-action of G on M. Let furthermore $\mathbf{g}$ be a super metric on M and let ν be a smooth density on M. There are several (more or less equivalent) ways to define the notion of invariance of ν and/or $\mathbf{g}$ under the G action. The most direct way is to use the generalized tangent map [15, V.3.19]. For a fixed $g\in G$, the map $\Phi_g\colon M\to M$, $m\mapsto\Phi(g,m)$ is a bijection, but for $g\notin\mathbf{B}G$ it will (in general) not be smooth. Since we still want to define a tangent map $T\Phi_g$, we circumvent this problem by defining the (even linear) map $T_m\Phi_g\colon T_mM\to T_{\Phi(g,m)}M$ as follows: we make the identification $T_gG\times T_mM\cong T_{(g,m)}(G\times M)$ and then we define

$$v\in T_mM \quad\Rightarrow\quad T_m\Phi_g(v)=T_{(g,m)}\Phi(0,v)\,.$$

With this definition of a generalized tangent map, it becomes easy to define when $\mathbf{g}$ or ν is invariant.

We will say that $\mathbf{g}$ *is invariant under the G-action* if for all possible choices we have (compare with (24))

$$\mathbf{g}_{\Phi(g,m)}\big(T_m\Phi_g(v),T_m\Phi_g(w)\big)=\mathbf{g}_m(v,w)\,, \tag{29}$$

an equality that we could abbreviate to $(T_m\Phi_g)^*\mathbf{g}_{\Phi(g,m)}=\mathbf{g}_m$ or even shorter as $\Phi_g^*\mathbf{g}=\mathbf{g}$. In the same way we will say that ν *is invariant under the G-action* if for all possible choices we have

$$(v_i)_{i=1}^{d+n}\text{ a basis of }T_mM \quad\Rightarrow\quad \nu_{\Phi(g,m)}\Big(\big(T_m\Phi_g(v_i)\big)_{i=1}^{d+n}\Big)=\nu_m\big((v_i)_{i=1}^{d+n}\big)\,,$$

an equality that we could abbreviate as $(T_m\Phi_g)^*\nu_{\Phi(g,m)}=\nu_m$ or even shorter as $\Phi_g^*\nu=\nu$ (see also Lemma 5.10).

For readers that feel uncomfortable with the use of a generalized tangent map we can cast the definition of invariance of ν in terms of fiber-diffeomorphisms as follows (a similar definition is possible for the invariance of $\mathbf{g}$). One starts by extending the density ν on M in the obvious way to an M-fiber-density $\hat\nu$ on $G\times M$ (independent of $g\in G$) and then to require that we have the equality

$$\Psi^*\hat\nu=\hat\nu\,,$$

where $\Psi\colon G\times M\to G\times M$ is the fiber-diffeomorphism defined as

$$\Psi(g,m)=\big(g,\Phi(g,m)\big)\equiv\big(g,\Phi_g(m)\big)\,.$$

When one looks carefully at this definition, one will see that it boils down to exactly the same formula as the one given by the generalized tangent map.

Proposition 6.9. *Let M be an o.o.d $\mathcal{A}$-manifold, let $\mathbf{g}$ be a super metric on M and let $\Phi\colon G\times M\to M$ be a smooth left-action of a connected $\mathcal{A}$-Lie group G on M. If this G-action preserves $\mathbf{g}$, then the density $\nu_{\mathbf{g}}$ of Proposition* 5.11(i) *is invariant under the G-action.*

Proof. Let (x^a, ξ^a) be an o.o.d coordinate system around $m \in M$ and let (x^b, ξ^b) be a coordinate system around $\Phi_g(m)$. We thus can define the matrix of the even linear map $T_m\Phi_g$ in terms of the bases $(\partial_{x^a}, \partial_{\xi^a})$ of T_mM and $(\partial_{x^b}, \partial_{\xi^b})$ of $T_{\Phi(g,m)}M$ (a matrix that can be seen as part of the matrix of $T_{(g,m)}\Phi$), which we can write more or less symbolically as (see also (15), (16) and (17))

$$(T_m\Phi_g)(\partial_{x^a}, \partial_{\xi^a}) = \mathrm{matrix}(T_m\Phi_g) \cdot (\partial_{x^b}, \partial_{\xi^b}) \,. \tag{30}$$

We can also define the matrix of $\mathbf{g}$ in these bases as in (18), giving matrices $\mathbf{g}^a_{ij}(m)$ and $\mathbf{g}^b_{ij}\big(\Phi_g(m)\big)$. Using the invariance property (29) of $\mathbf{g}$ and applying the Berezinian, we then obtain the equality

$$\mathrm{Ber}\big(\mathrm{matrix}(T_m\Phi_g)\big)^2 \cdot \mathrm{Ber}\big(\mathbf{g}^b_{ij}(\Phi(g,m))\,\big) = \mathrm{Ber}\big(\mathbf{g}^a_{ij}(m)\big) \,. \tag{31}$$

Next we note that, by a continuity argument, connectedness of G and the fact that M is o.o.d , we have

$$\forall g \in G \quad : \quad \mathbf{B}\mathrm{Ber}_\pi\big(\mathrm{matrix}(T_m\Phi_g)\big) > 0 \,. \tag{32}$$

And finally we recall that the definition of $\nu_{\mathbf{g}}$ is given by

$$\nu_{\mathbf{g}}\Big(\frac{\partial}{\partial x^a}\Big|_m, \frac{\partial}{\partial \xi^a}\Big|_m\Big) = \sqrt{\big|\mathrm{Ber}\big(\mathbf{g}^a_{ij}(m)\big)\big|} \,.$$

We thus can compute:

$$\begin{aligned}
\nu_{\mathbf{g}}|_m\Big(\frac{\partial}{\partial x^a}\Big|_m, \frac{\partial}{\partial \xi^a}\Big|_m\Big) \quad &= \quad \sqrt{\big|\mathrm{Ber}\big(\mathbf{g}^a_{ij}(m)\big)\big|} \\
&\overset{(31)}{=} \quad \big|\,\mathrm{Ber}\big(\mathrm{matrix}(T_m\Phi_g)\big)\,\big| \cdot \sqrt{\big|\mathrm{Ber}\big(\mathbf{g}^b_{ij}(\Phi(g,m))\big)\big|} \\
&\overset{(32)}{=} \quad \mathrm{Ber}_\pi\big(\mathrm{matrix}(T_m\Phi_g)\big) \cdot \sqrt{\big|\mathrm{Ber}\big(\mathbf{g}^b_{ij}(\Phi(g,m))\big)\big|} \\
&= \quad \mathrm{Ber}_\pi\big(\mathrm{matrix}(T_m\Phi_g)\big) \cdot \nu_{\mathbf{g}}|_m\Big(\frac{\partial}{\partial x^b}\Big|_{\Phi(g,m)}, \frac{\partial}{\partial \xi^b}\Big|_{\Phi(g,m)}\Big) \\
&\overset{(30)}{=} \quad \nu_{\mathbf{g}}|_{\Phi(g,m)}\Big(T_m\Phi_g\frac{\partial}{\partial x^a}\Big|_m, T_m\Phi_g\frac{\partial}{\partial \xi^a}\Big|_m\Big) \,,
\end{aligned}$$

which shows that $\nu_{\mathbf{g}}$ is invariant under Φ_g as claimed. ∎

Lemma 6.10. *Let G and M be two $\mathcal{A}$-manifolds and $F\colon G \times M \to \mathcal{A}^{\mathbf{K}}$ a smooth function. Then for any fixed $g \in G$ the map $F_g\colon M \to \mathcal{A}^{\mathbf{K}}$, $F_g(m) = F(g,m)$ belongs to $C^\infty(M; \mathcal{A}^{\mathbf{K}}) \otimes \mathcal{A}^{\mathbf{K}}$.*

Proof. Let $p|q$ be the dimension of G and $d|n$ the dimension of M. Let $g \in G$ be fixed, choose a coordinate chart V around g with local coordinates (s, σ) and choose a cover $\mathcal{U} = \{\, U_a \mid a \in A \,\}$ of M by coordinate charts with local coordinates (x^a, ξ^a) on U_a. Then, by definition of a smooth function, there exist ordinary smooth functions $F^a_{I,J}\colon \mathbf{B}V \times \mathbf{B}U_a \to \mathbf{K}$ such that we have

$$F(s, \sigma, x^a, \xi^a) = \sum_{I \subset \{1,\dots,n\}} \sum_{J \subset \{1,\dots,q\}} \sigma^J \cdot (\xi^a)^I \cdot (\mathbf{G}F^a_{I,J})(s, x^a) \,.$$

We now assume that $p = d = 1$ in order to get less elaborate formulæ (the general case is similar but simply far more elaborate to write).
Writing $s = t + \alpha$ with $t = \mathbf{B}s \in \mathbf{R}$ and $\alpha \in \mathcal{A}_0$ even and nilpotent, and similarly $x^a = y^a + \beta^a$ with $y^a = \mathbf{B}x^a \in \mathbf{R}$ and β^a even and nilpotent, we have (by definition of the $\mathbf{G}$-extension):

$$\begin{aligned} F(s,\sigma,x^a,\xi^a) &= \sum_{I\subset\{1,\dots,n\}} \sum_{J\subset\{1,\dots,q\}} \sigma^J \cdot (\xi^a)^I \cdot \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\partial^{i+j} F^a_{I,J}}{(\partial s)^j\,(\partial x^a)^i}(t,y^a) \cdot \frac{\alpha^j \cdot \beta^i}{j!\,i!} \\ &= \sum_{J\subset\{1,\dots,q\}} \sigma^J \cdot \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} \cdot \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I \cdot \sum_{i=0}^{\infty} \frac{\partial^i}{(\partial x^a)^i} \frac{\partial^j F^a_{I,J}}{(\partial s)^j}(t,y^a) \cdot \frac{\beta^i}{i!}\,. \end{aligned} \tag{33}$$

Looking at this formula suggests to introduce the ordinary smooth functions $f^a_{I,J,t,j}$ defined as

$$f^a_{I,J,t,j}(y^a) = \frac{\partial^j F^a_{I,J}}{(\partial s)^j}(t,y^a)$$

and the (super) smooth functions $f^a_{J,t,j} : U_a \to \mathcal{A}^{\mathbf{K}}$ as

$$f^a_{J,t,j}(x^a,\xi^a) = \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I \cdot \sum_{i=0}^{\infty} (\mathbf{G}f^a_{I,J,t,j})(x^a)\,,$$

with which we can write

$$\begin{aligned} F(s,\sigma,x^a,\xi^a) &= \sum_{J\subset\{1,\dots,q\}} \sigma^J \cdot \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} \cdot \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I \cdot \sum_{i=0}^{\infty} \frac{\partial^i f^a_{I,J,t,j}}{(\partial x^a)^i}(y^a) \cdot \frac{\beta^i}{i!} \\ &= \sum_{J\subset\{1,\dots,q\}} \sigma^J \cdot \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} \cdot \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I \cdot (\mathbf{G}f^a_{I,J,t,j})(x^a) \\ &= \sum_{J\subset\{1,\dots,q\}} \sum_{j=0}^{\infty} \frac{\sigma^J \cdot \alpha^j}{j!} \cdot f^a_{J,t,j}(x^a,\xi^a)\,, \end{aligned}$$

which belongs, for fixed g with coordinates (s,σ), indeed to $C^\infty(U_a,\mathcal{A}^{\mathbf{K}}) \otimes \mathcal{A}^{\mathbf{K}}$ (remember, the infinite sum over j is actually finite because α is nilpotent).
In order to show that the functions $f^a_{J,t,j}(x^a,\xi^a)$ glue together to global smooth functions on M, we note that there is an intrinsic way to define these (local) functions. Looking at (33) it is immediate that when we derive with respect to the σ-coordinates and then evaluate at $\sigma = 0$, we have

$$\big((\partial_\sigma)^J F\big)(s,\sigma=0,x^a,\xi^a) = \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I \cdot \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\partial^{i+j} F^a_{I,J}}{(\partial s)^j\,(\partial x^a)^i}(t,y^a) \cdot \frac{\alpha^j \cdot \beta^i}{j!\,i!}\,.$$

Similarly, deriving with respect to s and evaluating at $s=t$, i.e., at $\alpha=0$, we find

$$\begin{aligned}
\big((\partial_s)^k(\partial_\sigma)^J F\big)&(s=t,\sigma=0,x^a,\xi^a)\\
&= \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I\cdot\sum_{i=0}^\infty\sum_{j=0}^\infty \frac{\partial^{i+j+k}F^a_{I,J}}{(\partial s)^{j+k}\,(\partial x^a)^i}(t,y^a)\cdot\frac{(\alpha=0)^j\cdot\beta^i}{j!\,i!}\\
&= \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I\cdot\sum_{i=0}^\infty \frac{\partial^{i+k}F^a_{I,J}}{(\partial s)^{k}\,(\partial x^a)^i}(t,y^a)\cdot\frac{\beta^i}{i!}\\
&= \sum_{I\subset\{1,\dots,n\}} (\xi^a)^I\cdot\sum_{i=0}^\infty \frac{\partial^{i}}{(\partial x^a)^i}\,\frac{\partial^k F^a_{I,J}}{(\partial s)^{k}}(t,y^a)\cdot\frac{\beta^i}{i!} = f^a_{J,t,k}(x^a,\xi^a)\,.
\end{aligned}$$

Now ∂_s and ∂_σ are well defined vector fields on $V\times M\subset G\times M$. It follows that the functions $(\partial_s)^k(\partial_\sigma)^JF$ are well defined smooth functions on $V\times M$. We then invoke the property of super smooth functions that says that restriction of some of the variables to real values leaves a super smooth functions of the remaining variables [15, III.1.23-g]. Hence the functions $f_{J,t,k}\colon M\to\mathcal{A}^{\mathbf{K}}$ defined by

$$f_{J,t,k}(m)=\big((\partial_s)^k(\partial_\sigma)^JF\big)(s=t,\sigma=0,m)$$

are well defined smooth functions on M. It thus follows that for fixed $g\in G$ with coordinates (s,σ) we have

$$F_g(m)=F(s,\sigma,m)=\sum_{J\subset\{1,\dots,q\}}\sum_{j=0}^\infty\frac{\sigma^J\cdot\alpha^j}{j!}\cdot f_{J,t,j}(m)\,,$$

proving that the function F_g belongs to $C^\infty(M;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}$. ■

Corollary 6.11. *Let G be an $\mathcal{A}$-Lie group acting on an $\mathcal{A}$-manifold M.*

(i) *For each fixed $g\in G$ the map $\rho(g)\colon C^\infty_{(c)}(M;\mathcal{A}^{\mathbf{K}})\to C^\infty_{(c)}(M;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}$ defined by $\big(\rho(g)\psi\big)(m)=\psi(g^{-1}m)$ extends to an even linear map*

$$\rho(g)\colon C^\infty_{(c)}(M;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}\to C^\infty_{(c)}(M;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}.$$

(ii) *The map $\rho\colon G\to\mathrm{Aut}\big(C^\infty_{(c)}(M;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}\big)$, $g\mapsto\rho(g)$ is a group homomorphism.*

Corollary 6.12. *Let M be an o.o.d $\mathcal{A}$-manifold, let $\mathbf{g}$ be a super metric on M and let $\Phi\colon G\times M\to M$ be the smooth action of a connected $\mathcal{A}$-Lie group G on M. If $\mathbf{g}$ is invariant under the G-action, then the super scalar product $\mathscr{S}_{\mathbf{g}}$ on $C^\infty_c(M;\mathcal{A}^{\mathbf{K}})$ and extended to $C^\infty_c(M;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}$, see Lemma 2.8, is invariant under the representation ρ of G on $C^\infty_c(M;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}$ defined by $\big(\rho(g)\psi\big)(m)=\psi\big(\Phi(g^{-1},m)\big)$. More precisely, for all $g\in G$ we have*

$$\mathscr{S}_{\mathbf{g}}\big(\rho(g)\chi,\rho(g)\psi\big)=\mathscr{S}_{\mathbf{g}}(\chi,\psi)\,.$$

Proof. Instead of proving this property directly, let us work backwards. We first note that $\rho(g)\psi$ is the pull-back of ψ by the bijection $\Phi_{g^{-1}}$. According to the definition we thus have:

$$\mathscr{S}_{\mathbf{g}}(\rho(g)\chi, \rho(g)\psi) = \int_M \overline{\Phi^*_{g^{-1}}\chi} \cdot \Phi^*_{g^{-1}}\psi \ \nu_{\mathbf{g}}\,.$$

But $\nu_{\mathbf{g}}$ is invariant under the G-action Proposition 6.9, so we have $\nu_{\mathbf{g}} = \Phi^*_{g^{-1}}\nu_{\mathbf{g}}$. We "thus" can apply Lemma 5.10 to conclude that we have

$$\begin{aligned}\mathscr{S}_{\mathbf{g}}(\rho(g)\chi, \rho(g)\psi) &= \int_M \overline{\Phi^*_{g^{-1}}\chi} \cdot \Phi^*_{g^{-1}}\psi \cdot \nu_{\mathbf{g}} \\ &= \int_M \Phi^*_{g^{-1}}\big(\,\overline{\chi} \cdot \psi \cdot \nu\,\big) \stackrel{5.10}{=} \int_M \overline{\chi} \cdot \psi \cdot \nu = \mathscr{S}_{\mathbf{g}}(\chi, \psi)\,.\end{aligned}$$

However, Lemma 5.10 does not apply, as for generic g the map $\Phi_{g^{-1}}$ is not a diffeomorphism, only for $g \in \mathbf{B}G$ we are sure it will be a diffeomorphism [15, III.1.23-g]. The way out (as most of the time) is to consider all g at the same time. We extend in the obvious way the density $\nu_{\mathbf{g}}$ on M to an M-fiber-density $\hat\nu_{\mathbf{g}}$ on $G \times M$ and we extend the functions χ and ψ in the obvious way to functions on $G \times M$. We furthermore define the fiber-diffeomorphism $\Psi\colon G \times M \to G \times M$ by $\Psi(g, m) = \big(g, \Phi(g^{-1}, m)\big)$. It follows immediately that we have

$$(\Psi^*\chi)(g, m) = (\Phi^*_{g^{-1}}\chi)(m)$$

and similarly for ψ. Moreover, the invariance of $\nu_{\mathbf{g}}$ can be stated as the property $\Psi^*\hat\nu_{\mathbf{g}} = \hat\nu_{\mathbf{g}}$. With these preparations, the sought for equality is a direct consequence of Lemma 6.8(iii). We start with the observation that the smooth function on G defined by

$$\int_M \overline{\chi} \cdot \psi \cdot \hat\nu_{\mathbf{g}}\,,$$

(i.e., by integrating the M-fiber-density $\overline{\chi} \cdot \psi \cdot \hat\nu_{\mathbf{g}}$, which is independent of $g \in G$) is constant equal to $\mathscr{S}_{\mathbf{g}}(\chi, \psi)$. On the other hand, as a function on G, the values $\mathscr{S}_{\mathbf{g}}\big(\rho(g)\chi, \rho(g)\psi\big)$ can be obtained by integrating the M-fiber-density $\overline{\Psi^*\chi} \cdot \Psi^*\psi \cdot \hat\nu_{\mathbf{g}}$. And then:

$$\begin{aligned}\mathscr{S}_{\mathbf{g}}\big(\rho(g)\chi, \rho(g)\psi\big) &= \Big(\int_M \overline{\Psi^*\chi} \cdot \Psi^*\psi \cdot \hat\nu_{\mathbf{g}}\Big)(g) \stackrel{\text{inv.}}{=} \Big(\int_M \Psi^*\big(\overline{\chi} \cdot \psi \cdot \hat\nu_{\mathbf{g}}\big)\Big)(g) \\ &\stackrel{6.8(\text{iii})}{=} \Big(\int_M \overline{\chi} \cdot \psi \cdot \hat\nu_{\mathbf{g}}\Big)(g) = \mathscr{S}_{\mathbf{g}}(\chi, \psi)\,,\end{aligned}$$

which now is an equality between smooth functions on G. ■

7. Left-regular representations

For an ordinary (non super) Lie group G, the left-regular representation can be defined as follows. One starts with a metric at $T_eG \cong \mathfrak{g}$, extends it to a left-invariant metric $\mathbf{g}$ on G and determines the associated left-invariant (Haar) measure ν_o on G. This determines the pre-Hilbert space $C^\infty_c(G; \mathbf{K})$ of compactly supported smooth functions on G equipped with the scalar product

$$\langle\!\langle \chi, \psi \rangle\!\rangle = \int_G \overline{\chi} \cdot \psi \ \nu_o\,. \tag{34}$$

The associated Hilbert space is its completion and gives $\mathcal{H} = L^2(G;\mathbf{K};\nu_o)$. If no confusion is possible, we will abbreviate this as $\mathcal{H} = L^2(G)$. The representation ρ on $\mathcal{H}$ is defined as usual by the formula

$$\big(\rho(g)\psi\big)(h) = \psi(g^{-1}h)\,,$$

and the fact that $\mathbf{g}$ and especially ν_o are left-invariant then guarantees that this representation is unitary.

For an $\mathcal{A}$-Lie group G of dimension $d|n$ we will follow the same scheme. We start with a super metric on $T_eG \cong \mathfrak{g}$, we extend it by left-translation to a left-invariant super metric $\mathbf{g}$ on G and we determine the associated trivializing density $\nu_{\mathbf{g}}$ on G, as well as the ordinary metrics $\mathbf{g}_{\mathbf{B}G}$ on $\mathbf{B}G$ and $\mathbf{g}_{VG}$ on the fibers of the vector bundle $VG \to \mathbf{B}G$ (see Section 4). The first allows us to define the super scalar product $\mathscr{S}_{\mathbf{g}}$ on $C^\infty_c(G;\mathcal{A}^{\mathbf{K}})$ and the last two allow us to define the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ on $C^\infty_c(G;\mathcal{A}^{\mathbf{K}})$, turning the latter into a pre-Hilbert space. We also have the representation ρ of G on $C^\infty_{(c)}(G;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}$ defined by (see Corollary 6.11)

$$\big(\rho(g)\psi\big)(h) = \psi(g^{-1}h)$$

and the super scalar product $\mathscr{S}_{\mathbf{g}}$ (extended to $C^\infty_c(G;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}$) is invariant under this action according to Corollary 6.12.

In order to prove that this can be turned into a super unitary representation of G, we need some more information on the structure of $C^\infty_{(c)}(G;\mathcal{A}^{\mathbf{K}})$. For that we will rely heavily upon the identification/diffeomorphism $\Phi\colon G^{\text{even dim.}}\times\mathfrak{g}_0^{\text{odd dim.}} \to G$, $\Phi(g,X) = g\cdot\exp(X)$ given in (4). And in order to "simplify" notation, we will most of the time not write the symbol Φ. This identification provides us with a canonical choice of a Batchelor atlas: choose any atlas for $G^{\text{even dim.}}$ (in which only even coordinates appear) and take the direct product of any chart in this atlas with $\mathfrak{g}_0^{\text{odd dim.}}$ to obtain a Batchelor atlas of $G^{\text{even dim.}}\times\mathfrak{g}_0^{\text{odd dim.}}$. The isomorphism Φ then gives us the Batchelor atlas for G. As any other choice of an isomorphism between $G^{\text{even dim.}}\times\mathfrak{g}_0^{\text{odd dim.}}$ and G is either less natural (for instance writing $\mathfrak{g}^{\text{odd dim.}}$ as a direct product of two subspaces $\mathfrak{g}^{\text{odd dim.}} = \mathfrak{h}\times\mathfrak{h}'$ and then using the isomorphism $\Phi'\colon G^{\text{even dim.}}\times\mathfrak{g}_0^{\text{odd dim.}}\to G$ defined by $\Phi'(g,X) = g\cdot\exp(X_{\mathfrak{h}})\cdot\exp(X_{\mathfrak{h}'})$ where $X = (X_{\mathfrak{h}},X_{\mathfrak{h}'})\in\mathfrak{h}_0\times\mathfrak{h}'_0$ is the decomposition of $X\in\mathfrak{g}_0^{\text{odd dim.}}$), or simply weird, we will stick to this one and its associated Batchelor atlas.

The first consequence is that G is o.o.d and thus that we indeed are allowed to apply Proposition 5.11 to obtain $\nu_{\mathbf{g}}$. We now fix (once and for all) a basis $e_1,\dots,e_d$ of $\mathbf{B}\mathfrak{g}_0$ and a basis $f_1,\dots,f_n$ of $\mathbf{B}\mathfrak{g}_1$. This will give us global odd coordinates $\xi_1,\dots,\xi_n$ on $\mathfrak{g}_0^{\text{odd dim.}}$ via $X = \sum_{i=1}^n \xi_i\,f_i$. As we have a direct product, it follows immediately that any smooth function $f\colon G\cong G^{\text{even dim.}}\times\mathfrak{g}_0^{\text{odd dim.}}\to\mathcal{A}^{\mathbf{C}}$ determines 2^n unique smooth functions $f_I\colon\mathbf{B}G\to\mathbf{C}$, $I\subset\{1,\dots,n\}$ such that we have

$$f(g,\xi) = \sum_{I\subset\{1,\dots,n\}} \xi^I\cdot(\mathbf{G}f_I)(g)\,.$$

We thus have an identification

$$C^\infty_{(c)}(G;\mathcal{A}^{\mathbf{C}}) \cong \big(\,C^\infty_{(c)}(\mathbf{B}G)\,\big)^{2^n}, \quad f\cong(f_I)_{I\subset\{1,\dots,n\}}\,. \tag{35}$$

In order to determine the (ordinary) scalar product associated to a left-invariant metric $\mathbf{g}$ on G, we need a more precise control over the left and right-invariant vector fields on G in terms of our product structure $G \cong G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}}$. For that we will need a refinement of Lemma 3.10 in the special case when Y contains a single odd parameter.

Definition 7.1. We define the functions $f, h, b, b_-, b_+ : \mathbf{R} \to \mathbf{R}$ by

$$f(t) = \frac{\mathrm{e}^t - 1}{t}, \quad h(t) = \frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1}, \quad b(t) = \frac{t(\mathrm{e}^t + 1)}{2(\mathrm{e}^t - 1)},$$
$$b_-(t) = b(t) - \tfrac{1}{2}\, t\, h(t) = \frac{t}{\sinh(t)}, \quad b_+(t) = b(t) + \tfrac{1}{2}\, t\, h(t) = \frac{t\, \cosh(t)}{\sinh(t)}.$$

All five have convergent power series in a neighbourhood of $0 \in \mathbf{R}$. Moreover h is an odd function, whereas b, $b_\pm$ are even functions. Their leading terms are given as

$$h(t) = \tfrac{1}{2}\, t - \tfrac{1}{24}\, t^3 + \cdots, \quad b(t) = 1 + \tfrac{1}{12}\, t^2 - \tfrac{1}{720}\, t^4 + \cdots,$$
$$b_-(t) = 1 - \tfrac{1}{6}\, t^2 + \tfrac{7}{360}\, t^4 + \cdots, \quad b_+(t) = 1 + \tfrac{1}{3}\, t^2 - \tfrac{1}{45}\, t^4 + \cdots.$$

For future use we note that we have the following relations among these functions:

$$b(t) = \frac{1}{f(t)} + \tfrac{1}{2}t, \quad b(t) \cdot h(t) = \tfrac{1}{2}\, t, \quad b_+(t) = b(2t). \tag{36}$$

Lemma 7.2. *Let $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_1$ and $\tau \in \mathcal{A}_1$. Then we have the following equalities:*

$$\begin{aligned} \exp(X + \tau Y) &= \exp(X) \cdot \exp\Big(\tau\, f\big(-\mathrm{ad}(X)\big) Y\Big) \\ &= \exp\Big(\tau\, f\big(\mathrm{ad}(X)\big) Y\Big) \cdot \exp(X), \end{aligned}$$

$$\begin{aligned} \exp(X) \cdot \exp(\tau Y) &= \exp\Big(X + \tfrac{1}{2}\, \tau\, [X, Y] + \tau\, b\big(\mathrm{ad}(X)\big) Y\Big) \\ &= \exp\Big(\tau\, h\big(\mathrm{ad}(X)\big) Y\Big) \cdot \exp\Big(X + \tau\, b_+\big(\mathrm{ad}(X)\big) Y\Big) \end{aligned}$$

$$\begin{aligned} \exp(\tau Y) \cdot \exp(X) &= \exp\Big(X - \tfrac{1}{2}\, \tau\, [X, Y] + \tau\, b\big(\mathrm{ad}(X)\big) Y\Big) \\ &= \exp\Big(-\tau\, h\big(\mathrm{ad}(X)\big) Y\Big) \cdot \exp\Big(X + \tau\, b_-\big(\mathrm{ad}(X)\big) Y\Big). \end{aligned}$$

Proof. These results can be proven by using the Baker-Campbell-Hausdorff formula, but it is faster to use the derivative of the exponential map (see [15, VI.3.15] or [6, p.24] for proof in the non-super setting), simply because any Taylor expansion with an odd parameter stops at order two ($\tau^2 = 0$). We thus can compute:

$$\begin{aligned} \exp(X + \tau Z) &= \exp(X) + (T_X \exp)(\tau Z) \\ &= \exp(X) + (T_e L_{\exp(X)}) \Big(\frac{1 - \mathrm{e}^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\Big)(\tau Z) \end{aligned}$$

$$
\begin{aligned}
&= (L_{\exp(X)} \circ \exp)(0) + (T_e L_{\exp(X)} \circ T_0 \exp)\Big(\big(f(-\mathrm{ad}(X))(\tau Z)\Big) \\
&= (L_{\exp(X)} \circ \exp)\Big(0 + \tau\big(f(-\mathrm{ad}(X)\big) Z\Big) \\
&= \exp(X) \cdot \exp\Big(\tau f\big(-\mathrm{ad}(X)\big) Z\Big) \,,
\end{aligned}
$$

which is the first equality to be proven. Taking inverses on both sides (and replacing X and Z by their opposites) one finds the second equality.

We now recall that if A is an endomorphism of a finite dimensional vector space and if f is a polynomial or a converging power series in a single variable, i.e., $f(t) = \sum_k c_k\, t^k$, then we can define the endomorphism $f(A)$ simply by $f(A)v = \sum_k c_k\, A^k(v)$. Moreover, if f and g are two such functions, we have the equality

$$
f(A) \circ g(A) = (f \cdot g)(A) \,.
$$

We will use this property repeatedly in the upcoming computations.

Writing $f\big(-\mathrm{ad}(X)\big)Z = Y$ we have (using (36))

$$
Z = \Big(\frac{1}{f\big(-\mathrm{ad}(X)\big)}\Big) Y = \Big(\tfrac{1}{2}\,\mathrm{ad}(X) + b\big(\mathrm{ad}(X)\big)\Big) Y
$$

and thus

$$
\begin{aligned}
\exp(X) \cdot \exp(\tau Y) &= \exp\Big(X + \tfrac{1}{2}\,\tau\,\mathrm{ad}(X)(Y) + \tau\, b\big(\mathrm{ad}(X)\big)(Y)\Big) \\
&= \exp\Big(X + \tfrac{1}{2}\,\tau\,[X,Y] + \tau\, b\big(\mathrm{ad}(X)\big)(Y)\Big) \,, \qquad (37)
\end{aligned}
$$

which is the third equality to be proven. Taking inverses and using that b is an even function we also find

$$
\exp(\tau Y) \cdot \exp(X) = \exp\Big(X - \tfrac{1}{2}\,\tau\,\mathrm{ad}(X)(Y) + \tau\, b\big(\mathrm{ad}(X)\big)(Y)\Big) \,, \qquad (38)
$$

which is the fifth equality. Writing, with $\varepsilon = \pm 1$,

$$
\begin{aligned}
X_\varepsilon &= X + \tau\Big(b\big(\mathrm{ad}(X)\big)(Y) - \tfrac{1}{2}\,\varepsilon\, h\big(\mathrm{ad}(X)\big)(\,[X,Y]\,)\Big) \\
Y_\varepsilon &= \varepsilon\, h\big(\mathrm{ad}(X)\big)(Y)
\end{aligned}
$$

we have, according to (37), $\exp(X_\varepsilon) \cdot \exp(\tau Y_\varepsilon) = \exp(Z_\varepsilon)$ with

$$
\begin{aligned}
Z_\varepsilon &= X_\varepsilon + \tfrac{1}{2}\,\tau\,\mathrm{ad}(X_\varepsilon) Y_\varepsilon + \tau\, b(\mathrm{ad}(X_\varepsilon)) Y_\varepsilon \\
&\overset{(\tau^2=0)}{=} X_\varepsilon + \tfrac{1}{2}\,\tau\,\mathrm{ad}(X) Y_\varepsilon + \tau\, b(\mathrm{ad}(X)) Y_\varepsilon \\
&= \Big(X + \tau\, b\big(\mathrm{ad}(X)\big) Y - \tfrac{1}{2}\,\varepsilon\,\tau\, h\big(\mathrm{ad}(X)\big)[X,Y]\Big) \\
&\qquad + \tfrac{1}{2}\,\varepsilon\,\tau\,\mathrm{ad}(X) h\big(\mathrm{ad}(X)\big) Y + \varepsilon\,\tau\, b\big(\mathrm{ad}(X)\big) h\big(\mathrm{ad}(X)\big) Y \\
&= X + \tau\Big(b\big(\mathrm{ad}(X)\big) + \varepsilon\, b\big(\mathrm{ad}(X)\big) \circ h\big(\mathrm{ad}(X)\big)\Big) Y \\
&\overset{(36)}{=} X + \tau\Big(b\big(\mathrm{ad}(X)\big) Y + \tfrac{1}{2}\,\varepsilon\,\mathrm{ad}(X) Y\Big) \,.
\end{aligned}
$$

Comparing this with (37) and (38) we thus find:

$$\begin{aligned}\exp(X_{+1})\cdot\exp(\tau Y_{+1}) &= \exp(X)\cdot\exp(\tau Y)\\ \exp(X_{-1})\cdot\exp(\tau Y_{-1}) &= \exp(\tau Y)\cdot\exp(X)\,.\end{aligned}$$

The second of these equalities is the sixth equality to be proven, whereas taking inverses in the first gives us the fourth equality to be proven. ■

Lemma 7.3. *In the decomposition $G = G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}}$ given by the map $(g,v) \mapsto g\cdot\exp(v)$, we have the natural identification of the tangent space as*

$$T_{(g,v)}\big(G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}}\big) \cong T_gG^{\text{even dim.}} \times \mathfrak{g}^{\text{odd dim.}}\,.$$

In this identification the left- and right-invariant vector fields take the following form.

(i) *For $X \in \mathbf{B}T_eG^{\text{even dim.}} \equiv \mathbf{B}\mathfrak{g}_0$ (and thus even) we have*

$$\begin{aligned}TL_{(g,v)}X|_{(e,0)} &= \big(TL_gX|_e\,,\ [v,X]|_v\big) \cong TL_gX|_e + [v,X]|_v\\ TR_{(g,v)}X|_{(e,0)} &= \big(TR_gX|_e\,,\ 0|_v\big) \cong TR_gX|_e + 0|_v\,.\end{aligned}$$

(ii) *For $Y \in \mathbf{B}\mathfrak{g}^{\text{odd dim.}} \equiv \mathbf{B}\mathfrak{g}_1$ (and thus odd) we have*

$$\begin{aligned}TL_{(g,v)}Y|_{(e,0)} &= \Big(\ TL_g\big(h\big(\mathrm{ad}(v)\big)Y|_e\big)\ ,\ b_+\big(\mathrm{ad}(v)\big)Y\ \Big)\\ &\cong TL_g\Big(h\big(\mathrm{ad}(v)\big)Y|_e\Big) + \Big(b_+\big(\mathrm{ad}(v)\big)Y\Big)\Big|_v\\ TR_{(g,v)}Y|_{(e,0)} &= \Big(\ -TR_g\big(h\big(\mathrm{ad}(\mathrm{Ad}(g)v)\big)Y|_e\big)\ ,\ \big(b_-\big(\mathrm{ad}(v)\big)\mathrm{Ad}(g^{-1})Y\,\big)\ \Big)\\ &\cong -TR_g\Big(h\big(\mathrm{ad}(\mathrm{Ad}(g)v)\big)Y|_e\Big) + \Big(b_-\big(\mathrm{ad}(v)\big)\mathrm{Ad}(g^{-1})Y\Big)\Big|_v\,.\end{aligned}$$

Note that these formulæ make sense: for $X \in \mathbf{B}\mathfrak{g}_0$ and $v \in \mathfrak{g}_0^{\text{odd dim.}}$ we have $[v,X] \in \mathfrak{g}_0^{\text{odd dim.}}$. Moreover, as h is odd and $b_\pm$ even, we have indeed, for any $Y \in \mathbf{B}\mathfrak{g}_1 \subset \mathfrak{g}^{\text{odd dim.}}$ that $h\big(\mathrm{ad}(v)\big)Y$ belongs to $\mathfrak{g}^{\text{even dim.}}$ and $b_\pm\big(\mathrm{ad}(v)\big)Y$ to $\mathfrak{g}^{\text{odd dim.}}$.

Proof. For (i) we compute:

$$\begin{aligned}TL_{(g,v)}X|_{(e,0)} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big((g,v)\cdot\big(\exp(tX),0\big)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(g\cdot\exp(v)\cdot\exp(tX)\Big)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(g\cdot\exp(tX)\cdot\exp\big(\mathrm{Ad}\big(\exp(-tX)\big)v\big)\Big)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(g\cdot\exp(tX)\ ,\ \exp\big(-\mathrm{ad}(tX)\big)v\Big)\\ &= TL_gX|_e - \mathrm{ad}(X)v|_v = TL_gX|_e + [v,X]|_v\,,\ \text{and}\end{aligned}$$

$$\begin{aligned}TR_{(g,v)}X|_{(e,0)} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(\big(\exp(tX),0\big)\cdot(g,v)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big(\big(\exp(tX)\cdot g,v\big)\Big)\\ &= \big(TR_gX|_e,0\big) \cong TR_gX|_e + 0|_v\end{aligned}$$

For (ii) we compute:

$$\begin{aligned}
TL_{(g,v)}Y|_{(e,0)} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\big((g,v)\cdot(e,\tau Y)\big) = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(g\cdot\exp(v)\cdot\exp(\tau Y)\big) \\
&\overset{(7.2)}{=} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(g\cdot\exp\big(\tau h\big(\mathrm{ad}(v)\big)Y\big)\cdot\exp\big(v+\tau\, b_+\big(\mathrm{ad}(v)\big)Y\big)\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(g\cdot\exp(\tau h(\mathrm{ad}(v))Y)\ ,\ v+\tau\, b_+\big(\mathrm{ad}(v)\big)Y\Big) \\
&= TL_g\Big(h\big(\mathrm{ad}(v)\big)Y\Big)\Big|_e + \Big(b_+\big(\mathrm{ad}(v)\big)Y\Big)\Big|_v\ ,\ \text{and} \\
TR_{(g,v)}Y|_{(e,0)} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\big((e,\tau Y)\cdot(g,v)\big) = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\exp(\tau Y)\cdot g\cdot\exp(v)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\big(g\cdot\exp(\tau\mathrm{Ad}(g^{-1})Y)\cdot\exp(v)\big) \\
&\overset{(7.2)}{=} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\exp\big(-\tau h\big(\mathrm{ad}(\mathrm{Ad}(g)v)\big)Y\big)\cdot g\cdot\exp\big(v+\tau\, b_-\big(\mathrm{ad}(v)\big)\mathrm{Ad}(g^{-1})Y\big)\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\exp(-\tau h(\mathrm{ad}(\mathrm{Ad}(g)v))Y)\cdot g\ ,\ v+\tau\, b_-\big(\mathrm{ad}(v)\big)\mathrm{Ad}(g^{-1})Y\Big) \\
&= -TR_g\Big(h\big(\mathrm{ad}(\mathrm{Ad}(g)v)\big)Y\Big)\Big|_e + \Big(b_-\big(\mathrm{ad}(v)\big)\mathrm{Ad}(g^{-1})Y\Big)\Big|_v\ . \qquad \blacksquare
\end{aligned}$$

Notation 7.4. In general the Lie algebra of a Lie group G "is" the space of left-invariant vector fields on G, which can be identified with the tangent space at the identity. However, in the sequel we will want to distinguish this dual nature of an element of the Lie algebra: a (left-invariant) vector field on G or an (abstract) element of an (abstract) vector space $\mathfrak{g} \cong T_eG$. In order to distinguish these two aspects, we will denote by $\vec{X}$ the left-invariant vector field on G whose value at the identity $e \in G$ is X: $\vec{X}|_e = X$. And we will denote by X^R the *right-invariant* vector field on G whose value at e is X.

Now let $x_1, \dots, x_d$ be local even coordinates on $G^{\text{even dim.}}$, then, together with our global odd coordinates $\xi_1, \dots, \xi_n$ on $\mathfrak{g}_0^{\text{odd dim.}}$, i.e., $(\xi_1, \dots, \xi_n)$ corresponds to $v = \sum_{i=1}^n \xi_i f_i \in \mathfrak{g}_0^{\text{odd dim.}}$, the couple (x, ξ) becomes a local system of coordinates on $G \cong G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}}$. In terms of these local coordinates we now can express the left-invariant vector fields as follows. We start with the observation that there exists an invertible matrix $C(x)$ on the local chart for $G^{\text{even dim.}}$ such that for the basis e_i of $\mathfrak{g}_0$ we have (with x being the local coordinates of g)

$$\vec{e}_i|_g \equiv TL_g(\vec{e}_i|_e) = \sum_j \partial_{x_j}\, C_{ji}(x)\,.$$

There also exist matrices $A(\xi)$, $B(\xi)$ and $H(\xi)$ (only B being necessarily square) depending only upon the odd coordinates such that

$$[v, e_i] = \sum_j f_j\, A_{ji}(\xi)\,,\quad b_+\big(\mathrm{ad}(v)\big)f_j = \sum_k f_k\, B_{kj}(\xi)\,,\quad h\big(\mathrm{ad}(v)\big)f_j = \sum_i e_i\, H_{ij}(\xi).$$

We now recall that in the identification

$$T_{(g,v)}\big(G^{\text{even dim.}}\times\mathfrak{g}_0^{\text{odd dim.}}\big)\cong T_gG^{\text{even dim.}}\times\mathfrak{g}^{\text{odd dim.}}$$

the basis vector f_i gets identified with the tangent vector $\partial_{\xi_i}\big|_{(g,v)}$. Applying Lemma 7.3 then immediately gives

$$\begin{aligned}TL_{(g,v)}\vec{e}_i|_{(e,0)}&=\sum_j\partial_{x_j}\,C_{ji}(x)+\sum_k\partial_{\xi_k}\,A_{ki}(\xi)\\TL_{(g,v)}\vec{f}_j|_{(e,0)}&=\sum_j\partial_{x_k}\,C_{ki}(x)\,H_{ij}(\xi)+\sum_k\partial_{\xi_k}\,B_{kj}(\xi)\,.\end{aligned}$$

We can write this in matrix form as

$$\begin{aligned}TL_{(g,v)}\big(\vec{e}_\bullet|_{(e,0)}\quad\partial_\xi|_{(e,0)}\big)&=\big(\partial_x\big|_{(g,v)}\quad\partial_\xi\big|_{(g,v)}\big)\cdot\begin{pmatrix}C(x)&C(x)\cdot H(\xi)\\A(\xi)&B(\xi)\end{pmatrix}\\&=\big(\partial_x\big|_{(g,v)}\quad\partial_\xi\big|_{(g,v)}\big)\cdot\begin{pmatrix}C(x)&\mathbf{0}\\\mathbf{0}&\mathbf{1}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{1}&H(\xi)\\A(\xi)&B(\xi)\end{pmatrix}\\&=\big(\vec{e}_\bullet|_g\quad\partial_\xi\big|_{(g,v)}\big)\cdot\begin{pmatrix}\mathbf{1}&H(\xi)\\A(\xi)&B(\xi)\end{pmatrix}\,.\end{aligned}\tag{39}$$

We now use the "standard" super metric on $\mathfrak{g}\cong T_{(e,0)}G$, i.e., $\mathbf{g}_{(e,0)}(e_j,e_k)=\delta_{jk}$ and $\mathbf{g}_{(e,0)}(f_j,f_k)=i\delta_{jk}$, which we extend by left-translation to a global left-invariant super metric $\mathbf{g}$ on G. It then follows from the definition of the matrix $\mathbf{g}_{ij}|_{(x,\xi)}=\mathbf{g}(\partial_i|_{(x,\xi)},\partial_j|_{(x,\xi)})$ (where ∂_i denotes "all" partial derivatives/basis tangent vectors at $(x,\xi)\in G$) and (39) that this matrix satisfies the equation (compare with (19), but beware: the definition of the matrix defining the change of basis is not the same, there is a change from left to right; see also [15, II.4.10])

$$\begin{aligned}&\begin{pmatrix}\mathbf{1}&\mathbf{0}\\\mathbf{0}&i\,\mathbf{1}\end{pmatrix}=\\&\begin{pmatrix}{}^tC(x)&-\,{}^tA(\xi)\\{}^tH(\xi)\cdot{}^tC(x)&{}^tB(\xi)\end{pmatrix}\cdot\begin{pmatrix}\mathbf{g}^{00}_{(x,\xi)}&\mathbf{g}^{01}_{(x,\xi)}\\\mathbf{g}^{10}_{(x,\xi)}&\mathbf{g}^{11}_{(x,\xi)}\end{pmatrix}\cdot\begin{pmatrix}C(x)&C(x)\cdot H(\xi)\\A(\xi)&B(\xi)\end{pmatrix}\,.\end{aligned}\tag{40}$$

As we have $\mathbf{B}B(\xi)=\mathbf{1}$, it follows easily that the metric $\mathbf{B}\mathbf{g}^{00}$ on $\mathbf{B}G$ and the metric $-i\,\mathbf{B}\mathbf{g}^{11}$ on the Batchelor bundle $B=\mathbf{B}G\times\mathbf{B}\mathfrak{g}_1\to\mathbf{B}G$ (see (20)) are as follows: $\mathbf{B}\mathbf{g}^{00}$ is a left-invariant metric on $\mathbf{B}G$ that is the standard metric on $\mathfrak{g}_0$ in terms of the basis $e_\bullet$ and $-i\,\mathbf{B}\mathbf{g}^{11}$ is the standard metric on $\mathfrak{g}_1$ in terms of the basis $f_\bullet$.

Corollary 7.5. *Let $\nu_{\mathbf{g}}$ and $\nu_{B,\mathbf{g}}$ be the two densities on G associated to the left-invariant metric $\mathbf{g}$ as defined in Proposition 5.11. Then in the local coordinate system x,ξ we have*

$$\begin{aligned}\nu_{\mathbf{g}}\Big(\frac{\partial}{\partial x}\Big|_{(x,\xi)},\frac{\partial}{\partial\xi}\Big|_{(x,\xi)}\Big)&=\big|\mathrm{Det}\big(C(x)\big)\big|^{-1}\cdot\frac{\mathrm{Det}\big(B(\xi)\big)}{\mathrm{Det}\big(\mathbf{1}-H(\xi)\cdot B(\xi)^{-1}\cdot A(\xi)\big)}\\\nu_{B,\mathbf{g}}\Big(\frac{\partial}{\partial x}\Big|_{(x,\xi)},\frac{\partial}{\partial\xi}\Big|_{(x,\xi)}\Big)&=\big|\mathrm{Det}\big(C(x)\big)\big|^{-1}\,.\end{aligned}$$

Moreover, the density ν_o *on* $\mathbf{B}G$ *defined in the local coordinates* x *(now with* $x \in \mathbf{R}^d$*) by*

$$\nu_o\Big(\frac{\partial}{\partial x}\Big|_{(x)}\Big) = \big|\mathrm{Det}\big(C(x)\big)\big|^{-1}$$

is the left-invariant density (volume form or Haar Measure) on $\mathbf{B}G$ *associated to the left-invariant metric* $\mathbf{Bg}^{00}$ *on* $\mathbf{B}G$*.*

Proof. According to Proposition 5.11 these densities are determined by the Berezinian of the matrix associated to the super metric $\mathbf{g}$. But according to (40) we have the relation

$$\mathrm{Ber}(\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & i\,\mathbf{1} \end{pmatrix}) = \mathrm{Ber}(\begin{pmatrix} \mathbf{g}^{00}_{(x,\xi)} & \mathbf{g}^{01}_{(x,\xi)} \\ \mathbf{g}^{10}_{(x,\xi)} & \mathbf{g}^{11}_{(x,\xi)} \end{pmatrix}) \cdot \Big(\mathrm{Ber}(\begin{pmatrix} C(x) & C(x)\cdot H(\xi) \\ A(\xi) & B(\xi) \end{pmatrix})\Big)^2 .$$

It follows immediately that we have

$$\begin{aligned} \mathrm{Ber}(\begin{pmatrix} \mathbf{g}^{00}_{(x,\xi)} & \mathbf{g}^{01}_{(x,\xi)} \\ \mathbf{g}^{10}_{(x,\xi)} & \mathbf{g}^{11}_{(x,\xi)} \end{pmatrix}) &= \left(\frac{\det\big(B(\xi)\big)}{\det\big(C(x) - C(x)\cdot H(\xi)\cdot B(\xi)^{-1}\cdot A(\xi)\big)}\right)^2 \\ &= \left(\frac{\det\big(B(\xi)\big)}{\det\big(C(x)\big)\cdot\det\big(\mathbf{1} - H(\xi)\cdot B(\xi)^{-1}\cdot A(\xi)\big)}\right)^2 . \end{aligned}$$

Applying Proposition 5.11 immediately gives the desired results (using that $B(\xi)$ starts with the identity, and thus we don't need absolute values). ■

Lemma 7.6. *Changing the initial "standard" super metric on* $T_eG \cong \mathfrak{g}$ *in* (40) *changes the densities* $\nu_{\mathbf{g}}$*,* $\nu_{B,\mathbf{g}}$ *and* ν_o *in Corollary* 7.5 *with a constant real multiple. These densities are thus unique up to multiplication by real numbers.*

Definition 7.7. For any $\mathcal{A}$-Lie group G equipped with a super metric $\mathbf{g}$, we define $L^2(\mathbf{B}G, \mathbf{K}, \nu_o)$ as the completion of $C^\infty_c(\mathbf{B}G; \mathbf{K})$ with respect to the metric $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ induced by the invariant volume form $\nu_o \equiv \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}G}}$ determined by the metric $\mathbf{g}_{\mathbf{B}G}$ (22), Corollary 7.5 (a volume form which is an invariant Haar measure), i.e., we have for $\chi, \psi \in L^2(\mathbf{B}G, \mathbf{K}, \nu_o)$ (see also (34)):

$$\langle\!\langle \chi, \psi \rangle\!\rangle = \int_{\mathbf{B}G} \overline{\chi(g)} \cdot \psi(g)\ \mathrm{Vol}_{\mathbf{g}_{\mathbf{B}G}} .$$

As the metric $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ is unique up to a real multiple, and as the field $\mathbf{K}$ is most of the time understood implicitly, we will shorten the name of this Hilbert space $L^2(\mathbf{B}G, \mathbf{K}, \nu_o)$ to $L^2(\mathbf{B}G)$. We finally denote by $\rho_{\mathbf{B}G}$ the left-regular representation of $\mathbf{B}G$ on $L^2(\mathbf{B}G)$ defined (as usual) by $\big(\rho_{\mathbf{B}G}(g)\psi\big)(h) = \psi(g^{-1}h)$.

Corollary 7.8. *The metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ and the super scalar product $\mathscr{S}_{B,\mathbf{g}}$ on the space $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$ induced by the left-invariant super metric $\mathbf{g}$ on G are given in the identification $C_c^\infty(G;\mathcal{A}^{\mathbf{K}}) \cong \big(C_c^\infty(\mathbf{B}G;\mathbf{K})\big)^{2^n}$ (35) by*

$$\langle\chi,\psi\rangle_{\mathbf{g}} = \sum_{I\subset\{1,\ldots,n\}} \int_{\mathbf{B}G} \overline{\chi_I(m)}\cdot\psi_I(m)\ \nu_o(m) \equiv \sum_{I\subset\{1,\ldots,n\}} \langle\!\langle \chi_I,\psi_I\rangle\!\rangle$$

and

$$\mathscr{S}_{B,\mathbf{g}}(\chi,\psi) \sum_{I\subset\{1,\ldots,n\}} (-1)^{\varepsilon(I,I^c)}\,\langle\!\langle \chi_I,\psi_{I^c}\rangle\!\rangle\,. \tag{41}$$

The completion $\mathcal{H}$ of $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$ thus is (isomorphic to)

$$\mathcal{H} \cong \big(L^2(\mathbf{B}G,\mathbf{K},\nu_o)\big)^{2^n} \equiv \big(L^2(\mathbf{B}G)\big)^{2^n}\,.$$

Proof. The expression for $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ is a direct consequence of Lemma 4.8. For $\mathscr{S}_{B,\mathbf{g}}$ we invoke Lemma 5.15, which gives us

$$\begin{aligned}
\mathscr{S}_{B,\mathbf{g}}(\chi,\psi) &= \int_M \nu_{B,\mathbf{g}}(m)\cdot\overline{\chi(m)}\cdot\psi(m)\\
&= \sum_{a\in A}\int_{U_a} \mathrm{d}\lambda^{(d)}(x^a)\,\mathrm{d}\xi^{(n)}\ \rho_a(x^a)\cdot\big|\mathrm{Det}\big(C(x)\big)\big|^{-1}\cdot\overline{\chi_a(x,\xi)}\cdot\psi_a(x,\xi)\\
&= \sum_{I,J\subset\{1,\ldots,n\}}\sum_{a\in A}\int_{\mathbf{B}U_a} \mathrm{d}\lambda^{(d)}(x^a)\,\mathrm{d}\xi^{(n)}\ \big|\mathrm{Det}\big(C(x)\big)\big|^{-1}\cdot\rho_a(x^a)\\
&\qquad\qquad\cdot\overline{\chi_{a,I}(x^a)}\cdot\psi_{a,J}(x^a)\cdot\xi^I\cdot\xi^J\\
&= \sum_{I\subset\{1,\ldots,n\}} (-1)^{\varepsilon(I,I^c)}\cdot\int_{\mathbf{B}G}\overline{\chi_I(g)}\cdot\psi_{I^c}(g)\ \nu_o(g)\ \equiv \sum_{I\subset\{1,\ldots,n\}} (-1)^{\varepsilon(I,I^c)}\,\langle\!\langle\chi_I,\psi_{I^c}\rangle\!\rangle. \qquad \blacksquare
\end{aligned}$$

Nota Bene 7.9. Changing the standard metric on $\mathfrak{g}$ to an arbitrary one means changing the left hand side of (40) to the matrix

$$\begin{pmatrix} \mathbf{h}_0 & \mathbf{0}\\ \mathbf{0} & i\,\mathbf{h}_1\end{pmatrix},$$

where $\mathbf{h}_0$ is an ordinary metric on $\mathfrak{g}_0$ and $\mathbf{h}_1$ an ordinary metric on $\mathfrak{g}_1$. We already know that this will change ν_o, and thus $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, with a (positive real) multiple (more precisely, the factor $\big|\sqrt{\det(\mathbf{h}_0)}\ \big|$), but we do not yet know how such a change affects the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$. When we change the basis $f_1,\ldots,f_n$ of $\mathfrak{g}_1$ to an orthonormal one with respect to $\mathbf{h}_1$, the matrix representation of $\mathbf{h}_1$ becomes as in (40). But such a basis change corresponds to a global linear change of the odd coordinates $\xi_1,\ldots,\xi_n$. In the identification $C_c^\infty(G;\mathcal{A}^{\mathbf{K}}) \cong \big(C_c^\infty(\mathbf{B}G;\mathbf{K})\big)^{2^n}$ (35) this corresponds to an isomorphism $A\colon \big(C_c^\infty(\mathbf{B}G)\big)^{2^n} \to \big(C_c^\infty(\mathbf{B}G)\big)^{2^n}$ of the form

$$A\big((f_I)_{I\subset\{0,\ldots,n\}}\big) = A\big((\hat f_I)_{I\subset\{0,\ldots,n\}}\big) \qquad\text{with}\qquad \hat f_I = \sum_{I\subset\{0,\ldots,n\}} A_{IJ} f_J\,,$$

for constants $A_{IJ} \in \mathbf{R}$. Essentially this is the isomorphism of $\bigwedge \mathfrak{g}_1$ associated to the isomorphism/change of basis of $\mathfrak{g}_1$.[6] As the metric $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ has he form given in Corollary 7.8 when the basis of $\mathfrak{g}_1$ is orthonormal, it takes the form

$$\langle \chi, \psi \rangle_{\mathbf{g}} = \sum_{I,J \subset \{1,\dots,n\}} B_{IJ} \,\langle\!\langle \chi_I, \psi_J \rangle\!\rangle$$

for some positive definite symmetric matrix B_{IJ} after this global linear change of the odd coordinates. Another way to arrive at this conclusion is to say that we start with a positive definite symmetric matrix $\mathbf{h}_1$ representing a metric on $\mathbf{B}\mathfrak{g}_1 \cong \mathbf{R}^n$, which we then extend to a metric B on $\bigwedge \mathbf{B}\mathfrak{g}_1 \cong \bigwedge \mathbf{R}^n \cong \mathbf{R}^{2^n}$ (see also Section 4.3). Now any two metrics on the finite dimensional vector space $\mathbf{R}^{2^n}$ are equivalent, so the induced topology is the same. It follows immediately that changing the original metric on $\mathfrak{g}$ to another one changes the metric $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ to an equivalent metric on $C_c^\infty(G)$, and thus the topology on this space, as well as its completion, are independent of the choice of the initial metric on $\mathfrak{g}$.

Definition 7.10. We will denote by $L^2(G; \mathcal{A}^{\mathbf{K}}; \mathbf{g})$ the completion of the pre-Hilbert space $C_c^\infty(G, \mathcal{A}^{\mathbf{K}})$ with respect to the metric $\langle \cdot, \cdot \rangle_{\mathbf{g}}$. And as for the non-super case (and with the same arguments), we will shorten this to $L^2(G)$ if no confusion is possible.

The explicit formula for $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ in the identification of the space $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$ with $\big(C_c^\infty(\mathbf{B}G; \mathbf{K})\big)^{2^n}$ gave us a natural identification $L^2(G) \cong \big(L^2(\mathbf{B}G)\big)^{2^n}$. Looking at (41) then confirms that $\mathscr{S}_{B,\mathbf{g}}$ is continuous with respect to the metric on $C_c^\infty(G; \mathcal{A}^{\mathbf{K}}) \subset L^2(G)$, Proposition 5.16, and thus extends to $L^2(G)$. Moreover, it is also immediate from (41) that the extension to $\mathcal{H}$ is (remains) non-degenerate, so it defines a super scalar product on $\mathcal{H}$.

In order to get a good understanding of the super scalar product $\mathscr{S}_{\mathbf{g}}$, see Definition 5.14, we define the function $\Delta \colon \mathfrak{g}_0^{\text{odd dim.}} \to \mathcal{A}_0$ by

$$\Delta(\xi) = \frac{\mathrm{Det}\big(B(\xi)\big)}{\mathrm{Det}\big(\mathbf{1} - H(\xi) \cdot B(\xi)^{-1} \cdot A(\xi)\big)} .$$

As it depends only upon the odd coordinates ξ, there exist real constants $\Delta_I \in \mathbf{R}$, $I \subset \{1, \dots, n\}$ such that

$$\Delta(\xi) = \sum_{I \subset \{1,\dots,n\}} \xi^I \, \Delta_I .$$

Moreover, because Δ takes values in $\mathcal{A}_0$, $\Delta_I = 0$ whenever I contains an odd number of elements. The definition of $\mathscr{S}_{\mathbf{g}}$ then tells us that we have

[6] Or its inverse; I am too lazy to figure out which one, as it is of no importance for the conclusion I want to obtain.

$$
\begin{aligned}
\mathscr{S}_{\mathbf{g}}(\chi,\psi) &= \int_M \nu_{\mathbf{g}}(m)\cdot\overline{\chi(m)}\cdot\psi(m)\\
&= \sum_{a\in A}\int_{U_a} \mathrm{d}\lambda^{(d)}(x^a)\,\mathrm{d}\xi^{(n)}\ \rho_a(x^a)\cdot\frac{\big|\mathrm{Det}\big(C(x)\big)\big|^{-1}\cdot\mathrm{Det}\big(B(\xi)\big)}{\mathrm{Det}\big(\mathbf{1}-H(\xi)\cdot B(\xi)^{-1}\cdot A(\xi)\big)}\\
&\qquad\qquad\cdot\overline{\chi_a(x,\xi)}\cdot\psi_a(x,\xi)\\
&= \sum_{I,J,K\subset\{1,\dots,n\}}\ \sum_{a\in A}\int_{U_a}\mathrm{d}\lambda^{(d)}(x^a)\,\mathrm{d}\xi^{(n)}\ \big|\mathrm{Det}\big(C(x)\big)\big|^{-1}\cdot\rho_a(x^a)\\
&\qquad\qquad\cdot\overline{\chi_{a,I}(x^a)}\cdot\psi_{a,J}(x^a)\cdot\Delta_K\cdot\xi^I\cdot\xi^J\cdot\xi^K\\
&= \sum_{I,J,K\subset\{1,\dots,n\}}\Bigg(\sum_{a\in A}\int_{\mathbf{B}U_a}\mathrm{d}\lambda^{(d)}(x^a)\ \big|\mathrm{Det}\big(C(x)\big)\big|^{-1}\cdot\rho_a(x^a)\\
&\qquad\qquad\cdot\overline{\chi_{a,I}(x^a)}\cdot\psi_{a,J}(x^a)\Bigg)\cdot\int_{\mathcal{A}_1^n}\mathrm{d}\xi^{(n)}\ \Delta_K\cdot\xi^I\cdot\xi^J\cdot\xi^K\,.
\end{aligned}
$$

This suggests that we define the numbers $\mathbf{S}_{IJ}\in\mathbf{R}$ by

$$\mathbf{S}_{IJ} = \int \mathrm{d}\xi^{(n)}\ \Delta_{(I\cup J)^c}\cdot\xi^I\cdot\xi^J\cdot\xi^{(I\cup J)^c}\,,$$

which allows us to obtain finally

$$\mathscr{S}_{\mathbf{g}}(\chi,\psi) = \sum_{I,J\subset\{1,\dots,n\}}\mathbf{S}_{IJ}\cdot\langle\!\langle\chi_I,\psi_J\rangle\!\rangle\,. \tag{42}$$

Corollary 7.11. *The matrix $(\mathbf{S}_{IJ})_{I,J\subset\{1,\dots,n\}}$ (of size $2^n\times 2^n$) is invertible and we have the implication*

$$\big[\ I\cap J\neq\emptyset \text{ or } |I|+|J| = n+1 \pmod 2\ \big] \qquad\Longrightarrow\qquad \mathbf{S}_{IJ}=0\,.$$

Moreover,the super scalar product $\mathscr{S}_{\mathbf{g}}$ is continuous with respect to the topology induced by $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ on $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})\subset L^2(G)$ and extends to a continuous super scalar product of parity n on $L^2(G)$.

Proof. Looking at the definition of $\mathbf{S}_{IJ}$ shows that it is necessarily zero whenever $I\cap J\neq\emptyset$ or whenever $\Delta_{(I\cup J)^c}=0$. But Δ is an even function, so $\Delta_K=0$ whenever $|K|=1$. And thus $\mathbf{S}_{IJ}=0$ when $I\cap J\neq\emptyset$ and when $I\cap J=\emptyset$ it will also be zero when $|(I\cup J)^c|=1$, i.e., whenever $|I|+|J|=n+1$ as claimed.

To prove invertibility, we suppose that we have a vector of numbers $c_I\in\mathbf{R}$, not all zero, such that $\sum_{J\subset\{1,\dots,n\}}\mathbf{S}_{IJ}\,c_J=0$ for all I. We then choose $I\subset\{1,\dots,n\}$ such that $c_I\neq 0$ and $c_J=0$ for all $J\subset\{1,\dots,n\}$ with a smaller cardinal than I (such I need not be unique). By hypothesis we have in particular

$$0=\sum_{J\subset\{1,\dots,n\}}\mathbf{S}_{I^cJ}\,c_J=\sum_{J\subset I}\mathbf{S}_{I^cJ}\,c_J\,,$$

where the second equality is a consequence of the fact that $\mathbf{S}_{I^cJ}=0$ whenever $I^c\cap J\neq\emptyset$. But by assumption $c_J=0$ when the cardinal of J is smaller than the cardinal of I. It thus follows that we must have $\mathbf{S}_{I^cI}\, c_I = 0$, i.e., $\mathbf{S}_{I^cI} = 0$. But $\Delta(0) = \Delta_\emptyset = 1$ and thus $\mathbf{S}_{I^cI} = (-1)^{\varepsilon(I^c,I)} \neq 0$. This contradiction shows that the kernel of the matrix $\mathbf{S}$ is zero and thus this matrix is invertible.

The only part concerning the super scalar product $\mathscr{S}_{\mathbf{g}}$ that is not directly obvious is non-degeneracy of the extension of $\mathscr{S}$ to $L^2(G)$. To do so, we take $0 \neq \chi \in L^2(G)$ and we define ψ by

$$\psi_I = \sum_{J \subset \{1,\dots,n\}} (\mathbf{S}^{-1})_{IJ}\chi_J\,.$$

We then have

$$\begin{aligned}\mathscr{S}_{\mathbf{g}}(\chi,\psi) &= \sum_{I,K\subset\{1,\dots,n\}} \mathbf{S}_{IK}\,\langle\!\langle \chi_I, \psi_K\rangle\!\rangle = \sum_{I,J,K\subset\{1,\dots,n\}} \mathbf{S}_{IK}\,(\mathbf{S}^{-1})_{KJ}\,\langle\!\langle \chi_I, \chi_J\rangle\!\rangle \\ &= \sum_{I\subset\{1,\dots,n\}} \langle\!\langle \chi_I, \chi_I\rangle\!\rangle = \langle \chi, \chi\rangle_{\mathbf{g}} \neq 0\,,\end{aligned}$$

proving non-degeneracy of $\mathscr{S}_{\mathbf{g}}$ on the whole of $L^2(G)$. ∎

Corollary 7.12. *The two triples* $\big(L^2(G), \langle\cdot,\cdot\rangle_{\mathbf{g}}, \mathscr{S}_{B,\mathbf{g}}\big)$ *and* $\big(L^2(G), \langle\cdot,\cdot\rangle_{\mathbf{g}}, \mathscr{S}_{\mathbf{g}}\big)$ *are super Hilbert spaces.*

We now turn our attention to the representation ρ of G on $C^\infty_{(c)}(G;\mathcal{A}^{\mathbf{K}})\otimes\mathcal{A}^{\mathbf{K}}$ defined by (see Corollary 6.11)

$$\big(\rho(g)\psi\big)(h) = \psi(g^{-1}h)\,.$$

In terms of the identification $G \cong G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}}$ the left-translation by an element $g \in G^{\text{even dim.}}$ is given by $L_g(h,X) = (gh,X)$.

It follows immediately that the maps L_g with $g \in \mathbf{B}G \subset G^{\text{even dim.}}$ are diffeomorphisms that are linear in the odd coordinates (in the Batchelor atlas defined by the above identification). We thus can apply Proposition 4.11 (also because $\mathbf{g}$ is left-invariant) and conclude that the maps $\rho(g)$ with $g \in \mathbf{B}G$ determine unitary maps on $C^\infty_c(G;\mathcal{A}^{\mathbf{K}}) \otimes \mathcal{A}^{\mathbf{K}}$. Extending them to the completion $L^2(G)$, we thus obtain a unitary representation ρ_o of $\mathbf{B}G$ on $L^2(G)$, see Lemma 3.2. In terms of the identification $\psi \in L^2(G) \cong (\psi_I)_{I\subset\{1,\dots,n\}} \in \big(L^2(\mathbf{B}G)\big)^{2^n}$ the representation ρ_o is given by

$$\big(\rho_o(g)\psi\big)_I(h) = \psi_I(g^{-1}h)\,,$$

i.e., ρ_o consists of 2^n copies of the left-regular representation $\rho_{\mathbf{B}G}$ of $\mathbf{B}G$.

Before turning our attention to the (full) representation ρ on G, we first recall the usual definition of a fundamental vector field associated to a smooth left-action $\Phi\colon G\times M \to M$ of a Lie group G on a manifold M (a definition that is also valid in the context of $\mathcal{A}$-manifolds [15, §VI.5]). For any $X \in \mathfrak{g}$, the associated *fundamental vector field* X^M *on* M is defined by

$$X^M|_m = \frac{d}{dt}\Big|_{t=0} \Phi(\mathrm{e}^{-tX}, m)\,.$$

Because of the minus sign, this map is a Lie algebra morphism: for any $X, Y \in \mathfrak{g}$ we have $[X,Y]^M = \big[X^M, Y^M\big]$. When we apply this to the multiplication map on G:

$m: G \times G \to G$, the associated fundamental vector field X^G is the right-invariant vector field on G whose value at e is $-X$ (which confirms that we need the minus sign to get a Lie algebra morphism). When we apply it to the map/left-action $\Phi: G \times G \to G$ defined by $\Phi(g,h) = h \cdot g^{-1}$ (which is essentially the right-action of G on itself), the associated fundamental vector field X^G is exactly the left-invariant vector field $\vec{X}$.

Looking at the (full) representation ρ of G on $C^\infty(G;\mathcal{A}^{\mathbf{K}}) \otimes \mathcal{A}^{\mathbf{K}}$, we know that its infinitesimal form is given by the fundamental vector fields of the left-action of G on itself. In particular for $X \in \mathbf{B}\mathfrak{g}$ homogeneous we have (with $|t| = |X|$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(\rho\big(\exp(Xt)\big)\psi\big)(h) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi\big(\exp(-tX)h\big) = -(X^R\psi)(h)\,.$$

We thus define the map $\tau: \mathbf{B}\mathfrak{g} \to \mathrm{End}\big(C^\infty(G;\mathcal{A}^{\mathbf{K}})\big)$ by

$$\tau(X)\psi = -X^R\psi\,,$$

which is an even graded Lie algebra morphism preserving the subspace $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$. We now recall a result by Poulsen [12, p114] that says that $C^\infty(\rho_{\mathbf{B}G})$ is given by [7]

$$C^\infty(\rho_{\mathbf{B}G}) = \{\, \psi \in C^\infty(\mathbf{B}G;\mathbf{K}) \mid \forall k \in \mathbf{N}\ \forall X_i \in \mathbf{B}\mathfrak{g}_0 : X_1^R \cdots X_k^R\psi \in L^2(\mathbf{B}G)\,\}\,.$$

Inspired by this result, we define the space $\mathcal{D}_\rho$ by

$$\mathcal{D}_\rho = \big\{\, \psi \in C^\infty(G;\mathcal{A}^{\mathbf{K}}) \mid \forall k \in \mathbf{N}\ \forall X_i \in \mathbf{B}\mathfrak{g} : X_1^R \cdots X_k^R\psi \in L^2(G)\,\big\}\,.$$

Using the identifications

$$C^\infty_{(c)}(G;\mathcal{A}^{\mathbf{K}}) \cong \big(C^\infty_{(c)}(\mathbf{B}G;\mathbf{K})\big)^{2^n} \qquad \text{and} \qquad L^2(G) \cong \big(L^2(\mathbf{B}G)\big)^{2^n}\,,$$

and knowing that ρ_o is 2^n copies of the left-regular representation $\rho_{\mathbf{B}G}$, it follows easily that we have

$$C_c^\infty(G;\mathcal{A}^{\mathbf{K}}) \subset C^\infty(\rho_o) \cong \big(C^\infty(\rho_{\mathbf{B}G})\big)^{2^n} \subset L^2(G)\,.$$

It then follows easily that we have the alternative description

$$\mathcal{D}_\rho = \{\, \psi \in C^\infty(\rho_o) \mid \forall k \in \mathbf{N}\ \forall X_i \in \mathbf{B}\mathfrak{g}_1 : X_1^R \cdots X_k^R\psi \in L^2(G)\,\}\,, \tag{43}$$

and that we have the inclusions $C_c^\infty(G;\mathcal{A}^{\mathbf{K}}) \subset \mathcal{D}_\rho \subset C^\infty(\rho_o)$, proving that $\mathcal{D}_\rho$ is dense in $\mathcal{H}$. Before stating our main result, we note that obviously $\mathcal{D}_\rho$ is the maximal domain that is invariant under all maps $\tau(X)$, $X \in \mathbf{B}\mathfrak{g}$.

Theorem 7.13. *The couple $\big(C_c^\infty(G;\mathcal{A}^{\mathbf{K}}),\rho\big)$ is a proto super unitary representation of G on the super Hilbert space $\big(L^2(G), \langle\cdot,\cdot\rangle_{\mathbf{g}}, \mathscr{S}_{\mathbf{g}}\big)$ whose infinitesimal form is the triple $\big(\rho_o, C_c^\infty(G;\mathcal{A}^{\mathbf{K}}),\tau\big)$. Moreover, the couple $(\mathcal{D}_\rho,\rho)$ is the unique super unitary representation of G on $L^2(G)$ that extends $\big(C_c^\infty(G;\mathcal{A}^{\mathbf{K}}),\rho\big)$; its infinitesimal form is the triple $(\rho_o,\mathcal{D}_\rho,\tau)$.*

[7] There is a slight abuse of notation in identifying $\mathbf{B}\mathfrak{g}_0$ with the Lie algebra of $\mathbf{B}G$ and by identifying the right-invariant vector fields X^R, $X \in \mathbf{B}\mathfrak{g}_0$ with the right-invariant vector fields on $\mathbf{B}G$.

The proof of Theroem 7.13 will be broken down into a series of smaller results in such a way that at the end it suffices to put these smaller results together to obtain a full proof.

Lemma 7.14. *The triple* $\big(\rho_o, C_c^\infty(G;\mathcal{A}^{\mathbf{K}}),\tau\big)$ *is a proto super unitary representation in its infinitesimal form of* G *on the super Hilbert space* $\big(L^2(G),\langle\cdot,\cdot\rangle_{\mathbf{g}},\mathscr{S}_{\mathbf{g}}\big)$.

Proof. Looking at the "definition" of a proto super unitary representation in infinitesimal form, Corollary 3.12, it is immediate that ρ_o and the maps $\tau(X)$ preserve the space $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$. We thus have to prove the properties (i)–(iii).
For the first property we recall that ρ_o consists of 2^n copies of $\rho_{\mathbf{B}G}$ and that the right-invariant vector fields X^R (for $X\in\mathbf{B}\mathfrak{g}_0$) "are" 2^n copies of the corresponding right-invariant vector fields on $\mathbf{B}G$, see Lemma 7.3(i). As these right-invariant vector fields on $\mathbf{B}G$ are the generators of the left-regular representation $\rho_{\mathbf{B}G}$, it follows immediately that the maps $\tau(X)$, $X\in\mathbf{B}\mathfrak{g}_0$ are the infinitesimal generators of ρ_o, proving (i).
By the same argument it follows that the maps $\tau(X)$ with $X\in\mathbf{B}\mathfrak{g}_0$ are skew-symmetric with respect to $\langle\cdot,\cdot\rangle_{\mathbf{g}}$. It then suffices to look at the form (42) of $\mathscr{S}_{\mathbf{g}}$ (and to use the fact that X^R consists of 2^n copies of the corresponding right-invariant vector field on $\mathbf{B}G$) to see that $\tau(X)$ is (graded) skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$. On the other hand, for $X\in\mathbf{B}\mathfrak{g}_1$ we can take $\xi\in\mathcal{A}_1$ and consider the map $F\colon\mathcal{A}_1\times G\to\mathcal{A}^{\mathbf{K}}$ defined by

$$F(\xi,g)=\psi\big(\exp(-\xi X)\,g\big)$$

for any $\psi\in C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$. As this is a smooth map and as $\xi^2=0$, we have

$$F(\xi,g)=F(0,g)+\xi\cdot\frac{\partial F}{\partial\xi}(0,g)\,.$$

But by definition $(\partial_\xi F)(0,g)$ is given by $(X^R\psi)(g)$, which belongs to $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$. We thus have, for any $\psi\in C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$ the equality

$$\rho\big(\exp(\xi X)\big)\psi=\psi+\xi\,\tau(X)\psi\,.$$

Using that ρ preserves $\mathscr{S}_{\mathbf{g}}$ on $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$, see Corollary 6.12, we deduce from this, just as in the proof of (the direct part of) Theorem 3.6 (where we used (SUR2) to obtain the analogous equality (2)) that $\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$. And thus all maps $\tau(X)$, $X\in\mathbf{B}\mathfrak{g}$ are graded skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$, proving (ii).
As an immediate consequence of the properties of a fundamental vector field associated to a smooth action (here the left-action of G on itself) we have

$$\forall g\in\mathbf{B}g\quad:\quad\tau\big(\mathrm{Ad}(g)X\big)=\rho(g)\circ\tau(X)\circ\rho(g^{-1})\,,$$

which is condition (iii) of Corollary 3.12. ■

Lemma 7.15. *If a proto super unitary representation* $(\rho_o,\mathcal{D},\tau)$ *is an extension of the (infinitesimal) proto super unitary representation* $\big(\rho_o, C_c^\infty(G;\mathcal{A}^{\mathbf{K}}),\tau\big)$ *then its integrated form is* $(\mathcal{D},\rho)$.

Proof. If $(\rho_o, \mathcal{D}, \tau)$ is an infinitesimal proto super unitary representation, then according to Corollary 3.12 there exists a unique proto super unitary representation $(\mathcal{D}, \hat\rho)$ whose infinitesimal form is $(\rho_o, \mathcal{D}, \tau)$. We thus have to show that we necessarily have $\hat\rho(g) = \rho(g)|_{\mathcal{D}}$. In order to do so, we follow more or less the proof of (the converse part of) Theorem 3.6 in which the representation $\hat\rho$ is constructed out of τ. By definition of $\hat\rho$ we have, for $g \in \mathbf{B}G$, $\hat\rho(g) = \rho_o(g) = \rho(g)$ and thus $\hat\rho = \rho$ when restricted to $\mathbf{B}G$. The next step in the construction of $\hat\rho$ is to extend ρ_o by $\mathbf{G}$-extension to $G^{\text{even dim.}}$. Here we have to be careful, because for fixed $\psi \in \mathcal{D}$, the $\mathbf{G}$-extension is done with respect to the smooth map $\mathbf{F}_\psi\colon \mathbf{B}G \to L^2(G)$, $g \mapsto \rho_o(g)\psi$, not with respect to the smooth map $(g,h) \mapsto \big(\rho_o(g)\psi\big)(h) = \psi(g^{-1}h)$, which takes values in $\mathcal{A}^{\mathbf{K}}$. We thus first recall the above cited result of Poulsen that says that $C^\infty(\rho_{\mathbf{B}G})$ is given by

$$C^\infty(\rho_{\mathbf{B}G}) = \{\, \psi \in C^\infty(\mathbf{B}G; \mathbf{K}) \mid \forall k \in \mathbf{N}\ \forall X_i \in \mathbf{B}\mathfrak{g}_0 : X_1^R \cdots X_k^R \psi \in L^2(\mathbf{B}G) \,\}\,,$$

and in particular that the elements of $C^\infty(\rho_{\mathbf{B}G})$ are smooth functions. It then is easy to deduce that (still in the context of ordinary non-super Lie groups and their left-regular representation) we have, for $X \in \mathbf{B}\mathfrak{g}_0$ and $\psi \in C^\infty(\rho_{\mathbf{B}G})$:

$$\vec{X}\mathbf{F}_\psi = \mathbf{F}_{\tau(X)\psi}\,, \tag{44}$$

simply because $\tau(X) = -X^R$ is the fundamental vector field associated to the left-action of G on itself. We next recall that ρ_o, the representation of $\mathbf{B}G$ on $L^2(G) \cong \big(L^2(\mathbf{B}G)\big)^{2^n}$ is just 2^n copies of $\rho_{\mathbf{B}G}$ and that the right-invariant vector field X^R boils down to 2^n copies of the restriction of X^R to $L^2(\mathbf{B}G)$ (see Lemma 7.3(i)). It follows that (44) remains valid for $\psi \in C^\infty(\rho_o)$, i.e., we have for all $\psi \in C^\infty(\rho_o)$ and all $g, h \in G$ the equality

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \hat\rho\big(g\,\exp(tX)\big)\psi\Big)(h) = \Big(\hat\rho(g)\big(\tau(X)\psi\big)\Big)(h)\,.$$

Now there is a general statement concerning the flow of a smooth vector field X on a manifold M that says that if ϕ_t denotes the flow of X and if, for a smooth function $f\colon M \to V$ (with V some normed vector space) we define the function $F\colon \mathbf{R} \to V$ by $F(t) = f\big(\phi_t(m)\big)$ for some $m \in M$, then we have for the k-th order derivative:

$$F^{(k)}(t) = (X^k f)\big(\phi_t(m)\big)\,.$$

Moreover, this statement remains valid for $\mathcal{A}$-manifolds and smooth even homogeneous vector fields (by definition of a flow, see [15, Def. V.4.3/9]).

In order to apply this in our situation, we consider a left-invariant vector field $\vec{X}$ on $G^{\text{even dim.}}$ with $X \in \mathbf{B}\mathfrak{g}_0$, whose flow is given by $g \mapsto g\,\exp(tX)$. We also consider the smooth function $\mathbf{F}_\psi\colon G^{\text{even dim.}} \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$ obtained by $\mathbf{G}$-extension from the smooth function with the same name $\mathbf{F}_\psi\colon \mathbf{B}G \to \mathcal{D}$. We then define, for any $g \in G^{\text{even dim.}}$, the function $F\colon \mathbf{R} \to \mathcal{D} \otimes \mathcal{A}^{\mathbf{K}}$ by $F(t) = (\mathbf{F}_\psi)\big(g\,\exp(tX)\big)$, and we find

$$F^{(k)}(t) = \big((\vec{X})^k \mathbf{F}_\psi\big)\big(g\,\exp(tX)\big) = \mathbf{F}_{\tau(X)^k\psi}\big(g\,\exp(tX)\big)\,.$$

By the Taylor expansion and the definition/construction of $\hat\rho$ we thus can make the computation for even nilpotent $n\in\mathcal{A}_0$ (which yields a finite Taylor series):

$$\begin{aligned}\hat\rho\big(g\,\exp(nX)\big)\psi &= \mathbf{F}_\psi\big(g\,\exp(nX)\big) = F(n) = \sum_{k=0}^\infty \frac{n^k}{k!}\,F^{(k)}(0)\\ &= \sum_{k=0}^\infty \frac{n^k}{k!}\,\mathbf{F}_{\tau(X)^k\psi}(g) = \sum_{k=0}^\infty \frac{n^k}{k!}\,\hat\rho(g)\big(\tau(X)^k\psi\big)\,.\end{aligned}$$

We now suppose that we know that, for our fixed $g\in G^{\text{even dim.}}$, we have the equality $\hat\rho(g)=\rho(g)$. Then we may conclude that we have, for all $h\in G$:

$$\begin{aligned}\Big(\hat\rho\big(g\,\exp(nX)\big)\psi\Big)(h) &= \sum_{k=0}^\infty \frac{n^k}{k!}\,\Big(\hat\rho(g)\big(\tau(X)^k\psi\big)\Big)(h) = \sum_{k=0}^\infty \frac{n^k}{k!}\,\big(\tau(X)^k\psi\big)(g^{-1}h)\\ &= \psi\big(\exp(-nX)\,g^{-1}h\big) = \Big(\rho\big(g\,\exp(nX)\big)\psi\Big)(h)\,,\end{aligned}$$

simply because the flow of the vector field $\tau(X)=-X^R$ on G is given by the map $h\mapsto\exp(-tX)h$. The conclusion is that, if we know $\hat\rho(g)=\rho(g)$, then we also know $\hat\rho\big(g\,\exp(nX)\big)=\rho\big(g\,\exp(nX)\big)$. Starting with $g\in\mathbf{B}G\subset G^{\text{even dim.}}$ for which we know it is true, we conclude (by taking successive basis elements $X=e_i$ for $\mathbf{B}\mathfrak{g}_0$) that it is true for all $g\in G^{\text{even dim.}}$.

To finish, we take $X\in\mathbf{B}\mathfrak{g}_1$ and $\xi\in\mathcal{A}_1$. Then by definition of $\hat\rho$ we have

$$\hat\rho\big(\exp(\xi X)\big)\psi = \psi+\xi\tau(X)\psi$$

and thus for any $g\in G$:

$$\begin{aligned}\Big(\hat\rho\big(\exp(\xi X)\big)\psi\Big)(g) &= \big(\psi+\xi\tau(X)\psi\big)(g) = \psi(g)-\xi(X^R\psi)(g)\\ &= \psi\big(\exp(-\xi X)\,g\big) = \Big(\rho\big(\exp(\xi X)\big)\psi\Big)(g)\,,\end{aligned}$$

where the third equality is a consequence of the property of the flow of the even vector field ξX^R whose flow is given by $g\mapsto\exp(t\xi X)\,g$. We thus may conclude that for any $X\in\mathbf{B}\mathfrak{g}_1$ and any $\xi\in\mathcal{A}_1$ we have $\hat\rho\big(\exp(\xi X)\big)=\rho\big(\exp(\xi X)\big)$. But $G^{\text{even dim.}}$ and elements of the form $\exp(\xi X)$ generate the whole group G and thus $\hat\rho=\rho$ as claimed. ∎

Lemma 7.16. *Let $\psi\in L^2(G)\cap C^\infty(G;\mathcal{A}^{\mathbf{K}})$.*

(i) *If $\chi\in L^2(G)$ is such that $\operatorname{supp}(\chi)\cap\operatorname{supp}(\psi)=0$, then $\mathscr{S}_{\mathbf{g}}(\chi,\psi)=\langle\chi,\psi\rangle_{\mathbf{g}}=0$.*

(ii) *If for some open subset $U\subset G$ we have the property*

$$\forall\chi\in C_c^\infty(G;\mathcal{A}^{\mathbf{K}})\quad:\quad \operatorname{supp}(\chi)\subset U\quad\Rightarrow\quad \mathscr{S}_{\mathbf{g}}(\chi,\psi)=0\,,$$

then $\psi|_U$, the restriction of ψ to U, is zero.

Proof. If we have $\operatorname{supp}(\chi) \cap \operatorname{supp}(\psi) = \emptyset$, then in particular for any $I, J \subset \{1, \ldots, n\}$ we also have $\operatorname{supp}(\chi_I) \cap \operatorname{supp}(\psi_J) = \emptyset$. It follows directly that we have

$$\langle\!\langle \chi_I, \psi_J \rangle\!\rangle = \int_{\mathbf{B}G} \overline{\chi_I(g)} \cdot \psi_J(g) \operatorname{Vol}_{\mathbf{g}_{\mathbf{B}G}} = 0 .$$

Applying the explicit expressions for $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ Corollary 7.8 and $\mathscr{S}_{\mathbf{g}}$ (42) then show the first point.

For the second point we fix $I_o \subset \{1, \ldots, n\}$, take an arbitrary $\chi_o \in C_c^\infty(\mathbf{B}G; \mathbf{K})$ with $\operatorname{supp}(\chi_o) \subset U$ and define $\chi \in C_c^\infty(G; \mathcal{A}^{\mathbf{K}}) \cong \big(C_c^\infty(\mathbf{B}G; \mathbf{K})\big)^{2^n}$ by $\chi_{I_o} = \chi_o$ and $\chi_I = 0$ for $I \neq I_o$. We then have $\operatorname{supp}(\chi) \subset U$ and we have

$$0 = \mathscr{S}_{\mathbf{g}}(\chi, \psi) = \sum_{I, J \subset \{1, \ldots, n\}} \mathbf{S}_{IJ} \cdot \langle\!\langle \chi_I, \psi_J \rangle\!\rangle = \sum_{J \subset \{1, \ldots, n\}} \mathbf{S}_{I_o J} \cdot \langle\!\langle \chi_o, \psi_J \rangle\!\rangle .$$

As this is true for an arbitrary I_o and as the matrix $\mathbf{S}_{IJ}$ is invertible, see Corollary 7.11, we may conclude that we have

$$\forall I_o, J \subset \{1, \ldots, n\} \quad : \quad 0 = \langle\!\langle \chi_o, \psi_J \rangle\!\rangle = \int_{\mathbf{B}G} \overline{\chi_o(g)} \cdot \psi_J(g) \operatorname{Vol}_{\mathbf{g}_{\mathbf{B}G}} .$$

The results then follows from standard measure theory and the fact that the functions are smooth. ■

Lemma 7.17. *Let $(\mathcal{D}, \hat\tau)$ be such that $\mathcal{D} \subset L^2(G)$ is a graded subspace satisfying $C_c^\infty(G; \mathcal{A}^{\mathbf{K}}) \subset \mathcal{D} \subset C^\infty(G; \mathcal{A}^{\mathbf{K}})$ and $\hat\tau \colon \mathbf{B}\mathfrak{g} \to \operatorname{End}(\mathcal{D})$ an even graded Lie algebra morphism satisfying the conditions*

(i) *for all $X \in \mathbf{B}\mathfrak{g}$ the restriction of $\hat\tau(X)$ to $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$ equals $\tau(X)$ and*

(ii) *for all $X \in \mathbf{B}\mathfrak{g}$ the map $\hat\tau(X)$ is graded skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$.*

Then $\hat\tau = \tau$ and in particular $\mathcal{D}$ is invariant under all maps $\tau(X)$, $X \in \mathbf{B}\mathfrak{g}$.

Proof. Let us start with the remark that by Lemma 7.14 the maps $\tau(X)$ are graded skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$ on $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$. It thus makes sense to require that the restriction of $\hat\tau$ to $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$ equals τ.

We now fix $X \in \mathbf{B}\mathfrak{g}$ homogeneous, $\psi \in \mathcal{D}$ and $g_o \in G$ and we choose $\phi \in C_c^\infty(G; \mathcal{A})$ and an open neighborhood $U \subset G$ of g_o such that $\phi(g) = 1$ for all $g \in U$. Such ϕ (with compact support!) and U exist because G is locally compact. As ϕ has compact support and ψ is smooth, the product $\phi \cdot \psi$ belongs to $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$. Because $\hat\tau$ and τ are graded skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$ on $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$, we have for any $\chi \in C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$ with $\operatorname{supp}(\chi) \subset U$ the equality

$$\mathscr{S}_{\mathbf{g}}\big(\chi, \tau(X)(\phi \cdot \psi) - \hat\tau(X)\psi\big) = -\mathscr{S}_{\mathbf{g}}\big(\tau(X)\mathfrak{C}^{|X|}(\chi), \phi \cdot \psi - \psi\big) .$$

But for all $h \in \operatorname{supp}(\chi)$ we have $\phi(h) = 1$ and thus, because $\tau(X) = -X^R$ is a differential operator, we have

$$\operatorname{supp}\big(\tau(X)\mathfrak{C}^{|X|}(\chi)\big) \cap \operatorname{supp}(\phi \cdot \psi - \psi) \subset \operatorname{supp}(\chi) \cap \operatorname{supp}(\phi \cdot \psi - \psi) = \emptyset .$$

It then follows from Lemma 7.16 that we have

$$\mathscr{S}_{\mathbf{g}}\big(\tau(X)\mathfrak{C}^{|X|}(\chi), \phi \cdot \psi - \psi\big) = 0\,.$$

We thus have shown that

$$\forall \chi \in C_c^\infty(G; \mathcal{A}^{\mathbf{K}}) \quad : \quad \mathrm{supp}(\chi) \subset U \;\Rightarrow\; \mathscr{S}_{\mathbf{g}}\big(\chi, \tau(X)(\phi \cdot \psi) - \hat\tau(X)\psi\big) = 0\,.$$

Invoking Lemma 7.16 again we obtain that $\tau(X)(\phi \cdot \psi) - \hat\tau(X)\psi$ is identically zero on U. For all $g \in U$ we thus have

$$\begin{aligned}\big(\hat\tau(X)\psi\big)(g) &= \big(\tau(X)(\phi\cdot\psi)\big)(g) = -\big(X^R(\phi\cdot\psi)\big)(g) \\ &= -(X^R\phi)(g)\cdot\psi(g) + \phi(g)\cdot\big(\tau(X)\psi\big)(g)\,.\end{aligned}$$

But ϕ is identically 1 on U so $X^R\phi$ is identically 0 on U. And thus in particular $\big(\hat\tau(X)\psi\big)(g_o) = \big(\tau(X)\psi\big)(g_o)$. As g_o was arbitrary, we have shown $\hat\tau(X)\psi = \tau(X)\psi$. And as $\psi \in \mathcal{D}$ was arbitrary, we have shown $\hat\tau(X) = \tau(X)$ for all homogeneous $X \in \mathbf{B}\mathfrak{g}$. ∎

Lemma 7.18. *The maps $\tau(X)$, $X \in \mathbf{B}\mathfrak{g}$ are graded skew-symmetric on $\mathcal{D}_\rho$.*

Proof. It is obvious from the definition of $\mathcal{D}_\rho$ that this graded subspace is invariant under all maps $\tau(X)$, so it makes sense to investigate whether they are graded skew-symmetric. By linearity it suffices to prove this only for our fixed basis $e_1, \ldots, e_d$, $f_1, \ldots, f_n$ of $\mathbf{B}\mathfrak{g}$. As we already have seen, the maps $\tau(e_i)$ are the generators of ρ_o and are thus (graded) skew symmetric on $C^\infty(\rho_o) \supset \mathcal{D}_\rho$ (see the proof of Lemma 7.14).

In order to prove that a $\tau(f_j)$ is graded skew-symmetric, we use a change of coordinates that will allow us to "rectify" the right-invariant vector field f_j^R. Besides the identification $\Phi\colon G^{\text{even dim.}} \times \mathfrak{g}_0^{\text{odd dim.}} \to G$ introduced in Section 3, the map $\widehat\Phi_j\colon \mathcal{A}_1^n \times G^{\text{even dim.}} \to G$ defined by

$$\begin{aligned}&\widehat\Phi_j\big((\xi_1, \ldots, \xi_n), g\big) = \\ &\qquad \exp(\xi_j\, f_j)\cdot\exp(\xi_1\, f_1)\cdots\exp(\xi_{j-1}\, f_{j-1})\cdot\exp(\xi_{j+1}\, f_{j+1})\cdot\exp(\xi_n\, f_n)\cdot g\end{aligned}$$

is also a diffeomorphism. And in these coordinates (the odd ξ_i and even coordinates on $G^{\text{even dim.}}$) the right-invariant vector field f_j^R is given by $f_j^R = \partial_{\xi_j}$. We then choose a sequence of smooth maps with compact support $\zeta_n\colon \mathbf{B}G \to [0,1]$ such that for all $g \in \mathbf{B}G$ we have $\lim_{n\to\infty}\zeta_n(g) = 1$. Such a sequence exists because any ordinary Lie group is σ-compact. Denoting by the same symbol ζ_n the extension of these maps first to $G^{\text{even dim.}}$, then to $\mathcal{A}_1^n \times G^{\text{even dim.}}$ and finally to G via the map $\widehat\Phi_j$, we obtain the property,

$$\lim_{n\to\infty}\zeta_n(g)\cdot\psi(g) = \psi(g)\,,$$

where $\psi\colon G \to \mathcal{A}^{\mathbf{K}}$ is any function. The important points for us are the following:

(i) if ψ is smooth, then $\zeta_n \cdot \psi$ belongs to $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$,

(ii) if ψ belongs to $L^2(G)$, then $\zeta_n \cdot \psi$ converges to ψ in $L^2(G)$,

(iii) if ψ is smooth, then $f_j^R(\zeta_n \cdot \psi) = \zeta_n \cdot (f_j^R \psi)$.

We then choose $\chi, \psi \in \mathcal{D}_\rho \subset L^2(G) \cap C^\infty(G; \mathcal{A}^{\mathbf{K}})$ and we compute:

$$
\begin{aligned}
\mathscr{S}_{\mathbf{g}}(f_j^R \chi, \psi) &= \mathscr{S}_{\mathbf{g}}(\lim_{m\to\infty} \zeta_m \, f_j^R \chi, \lim_{n\to\infty} \zeta_n \, \psi) = \lim_{m\to\infty} \lim_{n\to\infty} \mathscr{S}_{\mathbf{g}}(\zeta_m \, f_j^R \chi, \zeta_n \, \psi) \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \mathscr{S}_{\mathbf{g}}(f_j^R(\zeta_m \, \chi), \zeta_n \, \psi) = \lim_{m\to\infty} \lim_{n\to\infty} -\mathscr{S}_{\mathbf{g}}(\zeta_m \, \mathfrak{C}\chi, f_j^R(\zeta_n \, \psi)) \\
&= \lim_{m\to\infty} \lim_{n\to\infty} -\mathscr{S}_{\mathbf{g}}(\zeta_m \, \mathfrak{C}\chi, \zeta_n \, f_j^R \psi)) \\
&= -\mathscr{S}_{\mathbf{g}}(\lim_{m\to\infty} \zeta_m \, \mathfrak{C}\chi, \lim_{n\to\infty} \zeta_n \, f_j^R \psi)) = -\mathscr{S}_{\mathbf{g}}(\mathfrak{C}\chi, f_j^R \psi)) \,,
\end{aligned}
$$

where for the first and last equality we used (ii) above, for the second and sixth we used the continuity of $\mathscr{S}_{\mathbf{g}}$, for the third and fifth we used (iii) above, and for the fourth equality we used (i) above and the fact that f_j^R is graded skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$ on $C_c^\infty(G; \mathcal{A}^{\mathbf{K}})$, see Lemma 7.14, and that f_j^R is odd. The final result then says that $f_j^R \equiv -\tau(f_j)$ is graded skew-symmetric with respect to $\mathscr{S}_{\mathbf{g}}$ on $\mathcal{D}_\rho$ as claimed. ■

Proof of Theorem 7.13. By Lemma 7.14 we know that $\big(\rho_o, C_c^\infty(G; \mathcal{A}^{\mathbf{K}}), \tau\big)$ is a proto super unitary representation in infinitesimal form and then by Lemma 7.15 we know that its integrated form is $\big(C_c^\infty(G; \mathcal{A}^{\mathbf{K}}), \rho\big)$, proving the first part.

For the second part, we start by showing that $\big(\rho_o, \mathcal{D}_\rho, \tau\big)$ is a proto super unitary representation in infinitesimal form. For that we note that we already know that $\tau \colon \mathbf{B}\mathfrak{g} \to \mathrm{End}(\mathcal{D}_\rho)$ is an even graded Lie algebra morphism that preserves $\mathscr{S}_{\mathbf{g}}$. And as the $\tau(X)$ are fundamental vector fields, we also have condition (iii) of Corollary 3.12. As $\mathcal{D}_\rho \subset C^\infty(\rho_o)$, it also follows immediately that condition (i) of Corollary 3.12 is satisfied. It thus remains to prove that $\mathcal{D}_\rho$ is invariant under ρ_o. For that we recall the description (43) of $\mathcal{D}_\rho$, the fact that $C^\infty(\rho_o)$ is invariant under ρ_o and property (iii) of Corollary 3.12 to conclude that indeed $\mathcal{D}_\rho$ is invariant under ρ_o.

Once we know that $\big(\rho_o, \mathcal{D}_\rho, \tau\big)$ is a proto super unitary representation in infinitesimal form, we apply Lemma 7.15 again to conclude that $(\mathcal{D}_\rho, \rho)$ is a proto super unitary representation. Now suppose $(\rho_o, \mathcal{D}, \hat{\tau})$ is any maximal extension of $\big(\rho_o, C_c^\infty(G; \mathcal{A}^{\mathbf{K}}), \tau\big)$. Then by Lemma 7.17 we must have $\hat{\tau} = \tau$ and $\mathcal{D}$ is invariant under all maps $\tau(X)$, $X \in \mathbf{B}\mathfrak{g}$. But $\mathcal{D}_\rho$ is the maximal graded subspace invariant under all these maps, so we must have $\mathcal{D} \subset \mathcal{D}_\rho$. By maximality of $(\mathcal{D}, \hat{\tau})$ we conclude that we have $\mathcal{D} = \mathcal{D}_\rho$, proving that $\big(\rho_o, \mathcal{D}_\rho, \tau\big)$ is the unique maximal extension of $\big(\rho_o, C_c^\infty(G; \mathcal{A}^{\mathbf{K}}), \tau\big)$. And thus $(\mathcal{D}_\rho, \rho)$ is the unique super unitary representation extending $\big(C_c^\infty(G; \mathcal{A}^{\mathbf{K}}), \rho\big)$. ■

8. Two examples

Our first example is the group G of affine transformations of the odd line $\mathcal{A}_1$ consisting of the transformations $\zeta \mapsto a\zeta + \beta$ with $(a, \beta) \in \mathcal{A}_0 \times \mathcal{A}_1$ with $\mathbf{B}a > 0$. For computational reasons we change coordinates and we realize this group as $G = E_0$ with E a graded vector space of dimension $1|1$ with group law

$$(x, \xi) \cdot (y, \eta) = (x + y, \eta + \mathrm{e}^{-y} \, \xi) \,.$$

This corresponds to the matrix representation (acting on a vector $(\zeta, 1)$)

$$\begin{pmatrix} e^x & e^x\xi \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^y & e^y\eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{x+y} & e^{x+y}\,(\eta + e^{-y}\,\xi) \\ 0 & 1 \end{pmatrix} .$$

In order to establish the left-regular representation in detail, we start with a basis of the left-invariant vector fields $e, f \in \mathbf{B}\mathfrak{g}$ given by

$$\vec{e}\,|_{(x,\xi)} = \partial_x\big|_{(x,\xi)} - \xi\,\partial_\xi\big|_{(x,\xi)} \qquad \text{and} \qquad \vec{f}\,|_{(x,\xi)} = \partial_\xi\big|_{(x,\xi)} .$$

The corresponding right-invariant vector fields are given by

$$e^R\big|_{(x,\xi)} = \partial_x\big|_{(x,\xi)} \qquad \text{and} \qquad f^R\big|_{(x,\xi)} = e^{-x}\,\partial_\xi\big|_{(x,\xi)} .$$

A direct computation shows that, in the coordinates (x, ξ), the matrix of the invariant metric $\mathbf{g}$ is given by

$$\mathrm{matrix}(\mathbf{g}_{(x,\xi)}) = \begin{pmatrix} 1 & i\xi \\ -i\xi & i \end{pmatrix} .$$

It follows immediately that the invariant measure on $\mathbf{B}G = \mathbf{R}$ is the Lebesgue measure and that the we have $\Delta(\xi) \equiv 1$. We thus have $L^2(G) \cong L^2(\mathbf{R})^2$ with identification (for smooth elements)

$$\psi \in L^2(G) \quad \leftrightarrow \quad (\psi_0, \psi_1) \in L^2(\mathbf{R})^2\,, \quad \psi(x,\xi) = \psi_0(x) + \xi\,\psi_1(x) .$$

In terms of this identification the metric $\langle\cdot,\cdot\rangle_{\mathbf{g}}$ and the super scalar product $\mathscr{S}_{\mathbf{g}}$ are given by

$$\langle\chi,\psi\rangle_{\mathbf{g}} = \int_{\mathbf{R}} \overline{\chi_0(x)}\,\psi_0(x) + \overline{\chi_1(x)}\,\psi_1(x)\ \mathrm{d}\lambda^{(1)}(x) \equiv \langle\!\langle \chi_0, \psi_0\rangle\!\rangle + \langle\!\langle \chi_1, \psi_1\rangle\!\rangle$$

and

$$\mathscr{S}_{\mathbf{g}}(\chi,\psi) = \langle\!\langle \chi_0, \psi_1\rangle\!\rangle + \langle\!\langle \chi_1, \psi_0\rangle\!\rangle .$$

The left-regular representation is given by

$$\big(\rho(y,\eta)\psi\big)(x,\xi) = \psi\big((y,\eta)^{-1}\cdot(x,\xi)\big) = \psi(x-y, \xi - e^{y-x}\,\eta) .$$

Infinitesimally we get (taking derivatives with respect to y and η at $(y,\eta) = (0,0)$):

$$\begin{aligned} \big(\tau(e)\psi\big)(x,\xi) &= -(\partial_x\psi)(x,\xi) = (-e^R\psi)(x,\xi) \\ \big(\tau(f)\psi\big)(x,\xi) &= -e^{-x}\,(\partial_\xi\psi)(x,\xi) = (-f^R\psi)(x,\xi) . \end{aligned}$$

Now the standard left-regular representation of $\mathbf{R}$ has as set of smooth vectors all those smooth functions on $\mathbf{R}$ for which all order derivatives belong to $L^2(\mathbf{R})$:

$$C^\infty(\rho_{\mathbf{B}G}) \equiv C^\infty(\rho_{\mathbf{R}}) = \{\, f \in C^\infty(\mathbf{R}) \mid \forall k \in \mathbf{N} : f^{(k)} \in L^2(\mathbf{R})\,\} .$$

This set is not invariant under multiplication by the function $\phi(x) = e^{-x}$, which leads us to the definition of the set $C^\infty_\phi(\rho_{\mathbf{R}})$ as

$$C^\infty_\phi(\rho_{\mathbf{R}}) = \{\, f \in C^\infty(\rho_{\mathbf{R}}) \mid \phi\cdot f \in C^\infty(\rho_{\mathbf{R}})\,\} .$$

It then is not hard to show that the maximal domain $\mathcal{D}_\rho$ for the super unitary (left-regular) representation is given by

$$\mathcal{D}_\rho = C^\infty(\rho_{\mathbf{R}}) \times C^\infty_\phi(\rho_{\mathbf{R}}) \subset L^2(\mathbf{R}) \times L^2(\mathbf{R}) \cong L^2(G)\,.$$

Our second example is the group $\mathrm{OSp}(1,2)$, the (connected) subgroup of the group $\mathrm{Gl}(1,2) = \mathrm{Aut}(V)$ (with V a graded vector space over $\mathbf{R}$ of dimension $\dim(V) = 1|2$) given by

$$\mathrm{OSp}(1,2) = \left\{ \begin{pmatrix} 1+\alpha\beta & \alpha & \beta \\ a\beta - b\alpha & a & b \\ c\beta - d\alpha & c & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathcal{A}_0,\ \alpha,\beta \in \mathcal{A}_1\,, \\ ad - bc = 1 - \alpha\beta \end{matrix} \right\}.$$

Its dimension thus is $3|2$ with $\mathbf{B}\mathrm{OSp}(1,2) = \mathrm{Sl}(2,\mathbf{R})$. Its interest lies in the fact that the function $\Delta(\xi)$ is not constant, implying that the two super scalar products $\mathscr{S}_{\mathbf{g}}$ and $\mathscr{S}_{B,\mathbf{g}}$ are not the same. A basis e_1, e_2, e_3, f_1, f_2 for its graded Lie algebra $\mathfrak{g}$ is given by the matrices (see [11])

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$f_1 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

According to the general theory given in Section 7, we have to compute $\mathrm{ad}(v)$ for $v = \xi f_1 + \eta f_2 \in \mathfrak{g}_0^{\text{odd dim.}}$ and $\mathrm{Ad}(g)v$ for $g \in \mathbf{B}G \equiv \mathrm{Sl}(2,\mathbf{R})$, as well as the series $h\big(\mathrm{ad}(v)\big)$ and $b_\pm\big(\mathrm{ad}(v)\big)$. Using the explicit matrices for this basis, one readily finds that the matrix of $\mathrm{ad}(\xi f_1 + \eta f_2)$ with respect to this basis is given by

$$\mathrm{matrix}\big(\mathrm{ad}(\xi f_1 + \eta f_2)\big) = \begin{pmatrix} 0 & 0 & 0 & -\eta & -\xi \\ 0 & 0 & 0 & -2\xi & 0 \\ 0 & 0 & 0 & 0 & 2\eta \\ \xi & -\eta & 0 & 0 & 0 \\ -\eta & 0 & -\xi & 0 & 0 \end{pmatrix}.$$

Its square is a diagonal matrix given by

$$\mathrm{matrix}\big(\mathrm{ad}(\xi f_1 + \eta f_2)^2\big) = \mathrm{diag}(2\xi\eta, 2\xi\eta, 2\xi\eta, -3\xi\eta, -3\xi\eta)$$

and all higher order powers are zero, simplifying the computations of the series $h\big(\mathrm{ad}(v)\big)$ and $b_\pm\big(\mathrm{ad}(v)\big)$. We now recall that the matrices A, B and H are defined as follows: A is the lower-left block of $\mathrm{ad}(\xi f_1 + \eta f_2)$, B is the lower right block of $b_+\big(\mathrm{ad}(v)\big) = \mathbf{1} + \frac{1}{3}\mathrm{ad}(\xi f_1 + \eta f_2)^2$ and H is the upper-right block of the equation $h\big(\mathrm{ad}(v)\big) = \frac{1}{2}\mathrm{ad}(\xi f_1 + \eta f_2)$. We thus get

$$\begin{pmatrix} \mathbf{1} & H \\ A & B \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2}\eta & -\frac{1}{2}\xi \\ 0 & 1 & 0 & -\xi & 0 \\ 0 & 0 & 1 & 0 & \eta \\ \xi & -\eta & 0 & 1-\xi\eta & 0 \\ -\eta & 0 & -\xi & 0 & 1-\xi\eta \end{pmatrix}.$$

This means that the function $\Delta(\xi)$ is given as

$$\Delta(\xi,\eta) = \frac{\mathrm{Det}\big(B(\xi,\eta)\big)}{\mathrm{Det}\big(\mathbf{1} - H(\xi,\eta)\cdot B(\xi)^{-1}\cdot A(\xi,\eta)\big)} = 1 + \xi\eta\,.$$

In the identification

$$\mathcal{H} \cong \big(L^2(\mathrm{Sl}(2,\mathbf{R})\big)^4\,, \quad \psi(x,\xi,\eta) = \sum_{I\subset\{1,2\}} \xi^I\,\psi_I(x) \cong (\psi_\emptyset,\psi_\xi,\psi_\eta,\psi_{\xi\eta})$$

we thus find (see (42))

$$\mathscr{S}_{\mathbf{g}}(\chi,\psi) = \langle\!\langle \chi_\emptyset,\psi_\emptyset\rangle\!\rangle + \langle\!\langle \chi_\emptyset,\psi_{\xi\eta}\rangle\!\rangle + \langle\!\langle \chi_\xi,\psi_\eta\rangle\!\rangle - \langle\!\langle \chi_\eta,\psi_\xi\rangle\!\rangle + \langle\!\langle \chi_{\xi\eta},\psi_\emptyset\rangle\!\rangle\,,$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denotes the usual metric on $L^2\big(\mathrm{Sl}(2,\mathbf{R})\big)$ given by an invariant Haar measure.

Our next task is to establish the right-invariant vector fields in terms of the decomposition $\mathrm{OSp}(1,2) \cong \mathbf{G}\,\mathrm{Sl}(2,\mathbf{R}) \otimes \mathcal{A}_1^2$ as given by Lemma 7.3:

$$TR_{(g,v)}Y\big|_{(e,0)} = -TR_g\Big(h\big(\mathrm{ad}(\mathrm{Ad}(g)v)\big)Y\big|_e\Big) + \Big(b_-\big(\mathrm{ad}(v)\big)\mathrm{Ad}(g^{-1})Y\Big)\Big|_v$$

For $g \in \mathrm{OSp}(1,2)^{\text{even dim.}} \cong \mathbf{G}\,\mathrm{Sl}(2,\mathbf{R})$ with

$$g = \begin{pmatrix} 1 & 0 & 0\\ 0 & a & b\\ 0 & c & d\end{pmatrix}$$

one easily computes

$$\mathrm{Ad}(g) = \begin{pmatrix} ad+bc & -ac & bd & 0 & 0\\ -2ab & a^2 & -b^2 & 0 & 0\\ 2cd & -c^2 & d^2 & 0 & 0\\ 0 & 0 & 0 & a & -b\\ 0 & 0 & 0 & -c & d\end{pmatrix}.$$

With $Y = y_1f_1 + y_2f_2 \cong \big(\begin{smallmatrix} y_1\\ y_2\end{smallmatrix}\big)$, $y_i \in \mathbf{R}$ and $b_-\big(\mathrm{ad}(v)\big) = \mathbf{1} - \frac{1}{6}\mathrm{ad}(v)^2$ we thus find (in terms of the basis $e_\bullet$, $f_\bullet$)

$$\begin{aligned} b_-\big(\mathrm{ad}(v)\big)\mathrm{Ad}(g^{-1})Y &= \begin{pmatrix} 1+\frac12\xi\eta & 0\\ 0 & 1+\frac12\xi\eta\end{pmatrix}\cdot\begin{pmatrix} d & b\\ c & a\end{pmatrix}\cdot\begin{pmatrix} y_1\\ y_2\end{pmatrix}\\ &= (1+\tfrac12\xi\eta)\cdot\begin{pmatrix} dy_1+by_2\\ cy_1+ay_2\end{pmatrix}\end{aligned}$$

and (only noting the components with respect to the basis vectors $e_\bullet$ because the components with respect to $f_\bullet$ are necessarily zero)

$$h\big(\mathrm{ad}(\mathrm{Ad}(g)v)\big)Y = \begin{pmatrix} -\frac12(d\eta-c\xi) & -\frac12(a\xi-b\eta)\\ -(a\xi-b\eta) & 0\\ 0 & d\eta-c\xi\end{pmatrix}\cdot\begin{pmatrix} y_1\\ y_2\end{pmatrix}\,,$$

which means that we have

$$-h\bigl(\mathrm{ad}(\mathrm{Ad}(g)v)\bigr)Y = \tfrac{1}{2}\bigl((d\eta - c\xi)y_1 + (a\xi - b\eta)y_2\bigr)e_1 \\ + (a\xi - b\eta)y_1 e_2 - (d\eta - c\xi)y_2 e_3 \,.$$

Combining the results we find for the right-invariant vector fields associated to the basis f_1, f_2 of $\mathbf{B}\mathfrak{g}_1$ at the point g with coordinates $\bigl((a,b,c,d),(\xi,\eta)\bigr) \in \mathbf{G}\mathrm{Sl}(2,\mathbf{R}) \times \mathbf{B}\mathfrak{g}_1$:

$$f_1^R|_g = \tfrac{1}{2}(d\eta - c\xi)\, e_1^R|_g + (a\xi - b\eta)\, e_2^R|_g + (1 + \tfrac{1}{2}\xi\eta)d\,\partial_\xi + (1 + \tfrac{1}{2}\xi\eta)c\,\partial_\eta$$

$$f_2^R|_g = \tfrac{1}{2}(a\xi - b\eta)\, e_1^R|_g + (c\xi - d\eta)\, e_3^R|_g + (1 + \tfrac{1}{2}\xi\eta)b\,\partial_\xi + (1 + \tfrac{1}{2}\xi\eta)a\,\partial_\eta \,.$$

A direct computation (using that the operators/right-invariant vector fields e_i^R are skew-symmetric on $L^2\bigl(\mathrm{Sl}(2,\mathbf{R})\bigr)$) shows that the operators f_i^R indeed preserve $\mathscr{S}_{\mathbf{g}}$ (on the space of compactly supported functions $C_c^\infty(G;\mathcal{A}^{\mathbf{K}})$) and that the term $\langle\!\langle \chi_\emptyset, \psi_\emptyset \rangle\!\rangle$ (due to the function $\Delta(\xi)$) is essential for this result.

Note added in proof

My definition of a super Hilbert space is less restrictive than the one proposed in [5, Def. 3.2]. However, although my description of the super Hilbert space associated to the left-regular representation of any super Lie group does not fit their definition, a simple modification (needed for $\mathrm{OSp}(1,2)$ but not for the group of affine transformations of $\mathcal{A}_1$) of the metric $\langle\cdot,\cdot\rangle$ will make it fit. And after such a modification even the representation of $\mathbf{B}G$ will remain unitary with respect to the modified metric. On the other hand, as we have seen, we can not maintain the idea that the full representation is defined on the set of smooth vectors of ρ_o as required in [5]. Other examples investigated since the completion of this paper suggest that my definition of a super Hilbert space leaves probably too much freedom in the choice of the metric $\langle\cdot,\cdot\rangle$, while on the other hand my definition of a super unitary representation is still too restrictive. I am now inclined to look for a metric $\langle\cdot,\cdot\rangle$ directly linked to the super scalar product $\mathscr{S}$ along the lines suggested in [5] but without the restriction that $\mathscr{S}$ should be homogeneous. And then to drop, in the definition of a super unitary representation, the condition (SUR3) that the restriction of ρ to $\mathbf{B}G$ preserves $\langle\cdot,\cdot\rangle$.

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