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Discrete Subgroups of a Locally Compact Group with Jointly Discrete Chabauty Neighborhoods

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Abstract. Let G be a locally compact group. We denote by $\mathcal{SUB}(G)$ the space of closed subgroups of G equipped with the *Chabauty topology*. A discrete subgroup Γ of G is said to admit a *jointly discrete Chabauty neighborhood* if there exists an identity neighborhood U in G and an open neighborhood Ω of Γ in $\mathcal{SUB}(G)$ such that every closed subgroup $L \in \Omega$ satisfies $L \cap U = \{e\}$. Recently, T. Gelander and A. Levit proved that every lattice in a semi-simple analytic group admits a jointly discrete Chabauty neighborhood. In this paper, we prove that G is a Lie group if and only if the trivial subgroup $\{e\}$ admits a jointly discrete Chabauty neighborhood, if and only if every discrete subgroup of G admits a jointly discrete Chabauty neighborhood.

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1. Introduction and main result

Let G be a locally compact group with identity element e. We denote by $\mathcal{SUB}(G)$ the space of closed subgroups of G equipped with the *Chabauty topology*; this is a compact space. In this space, each closed subgroup H of G has a neighborhood base consisting of sets

$$\mathcal{U}_G(H; K, W) \stackrel{\text{def}}{=} \{L \in \mathcal{SUB}(G) \mid L \cap K \subseteq WH \text{ and } H \cap K \subseteq WL\}, \tag{1}$$

where K ranges through the set $\mathcal{K}(G)$ of all compact subsets of G and W through the filter $\mathcal{U}(e)$ of all neighborhoods of the identity (The statement "U is a neighborhood of a point x" is used in the Bourbaki sense throughout this paper; i.e., U is any subset which contains an open subset containing x). In particular, the trivial subgroup $H = \{e\}$ has a neighborhood base consisting of sets

$$\mathcal{U}_G(H; K, W) = \{L \in \mathcal{SUB}(G) \mid L \cap K \subseteq W\}, \tag{2}$$

where $K \in \mathcal{K}(G)$ and $W \in \mathcal{U}(e)$.

The Chabauty topology is named after Claude Chabauty, who introduced it in [7] to generalize Mahler's compactness criterion to lattices in locally compact groups.

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In [3] it is introduced in a measure theoretic manner. The space $\mathcal{SUB}(G)$ has been used in several contexts to obtain approximation properties of the collection of closed subgroups of a locally compact group, and is an essential ingredient in the emerging theory of invariant random subgroups ([1]).

Following [8, Definition 7.1], we say that a discrete subgroup Γ of a locally compact group G admits a *jointly discrete Chabauty neighborhood* if there exists an identity neighborhood U in G and an open neighborhood Ω of Γ in $\mathcal{SUB}(G)$ such that every closed subgroup $L \in \Omega$ satisfies $L \cap U = \{e\}$. Recently, T. Gelander and A. Levit proved the following theorem ([8, Theorem 7.2]).

Theorem 1.1. *Every lattice Γ in a semi-simple analytic group G admits a jointly discrete Chabauty neighborhood.*

The main result of the present paper is a generalization of Theorem 1.1.

Theorem 1.2. *Let G be a locally compact group. The following conditions are equivalent:*

(1) *Every discrete subgroup of G admits a jointly discrete Chabauty neighborhood.*

(2) *The trivial subgroup $\{e\}$ admits a jointly discrete Chabauty neighborhood.*

(3) *G is a Lie group.*

A family $\{H_\lambda \mid \lambda \in I\}$ of subgroups of topological group G is called *uniformly discrete* if $H_\lambda \cap V = \{e\}$ for some neighborhood V of the identity e and all λ ([16, Definition 1.1], see also [9, page 48]). As a consequence of Theorem 1.2 we obtain the following result of S. P. Wang ([16, Lemma 1.3]).

Corollary 1.3. *Let (H_n) be a sequence of closed subgroups of a Lie group G converging to a discrete subgroup H in $\mathcal{SUB}(G)$; then $\{H_n \mid n \geq n_0\}$, n_0 a certain integer, is uniformly discrete.*

For further applications see [9] and [10, pages 25–27, Theorem 5.5].

2. Proof of Theorem 1.2

Lemma 2.1. *A locally compact group G is a Lie group if and only if it has an open subgroup which is a Lie group.*

Proof. Let H be an open subgroup of G which is a Lie group. We have

$$G_0 = H_0, \tag{3}$$

where G_0 (resp. H_0) denotes the identity component of G (resp. of H). Since H_0 is open in H, then G_0 is open in G and so the factor group G/G_0 is discrete. On the other hand, from (3) we deduce that G_0 is a Lie group. Consequently, by Theorem 7 of [13], G is a Lie group. The converse is trivial, since any Lie group is an open subgroup of itself. ∎

Proposition 2.2. *Let G be a locally compact group and let $\mathcal{F}$ be a filter basis of closed subgroups of G. If $\mathcal{F}$ converges to e then the net $(N)_{N\in\mathcal{F}}$ converges to $\{e\}$ in $\mathcal{SUB}(G)$.*

Proof. Let $\mathcal{U}_G(\{e\};K,U)$ be a neighborhood of $\{e\}$ in $\mathcal{SUB}(G)$, where $K\in\mathcal{K}(G)$ and $U\in\mathcal{U}(e)$. As the filter basis $\mathcal{F}$ converges to e, there exists $N\in\mathcal{F}$ such that $N\subseteq U$. It is clear that for any $M\subseteq N$ we have $M\in\mathcal{U}_G(\{e\};K,U)$. Then $\lim_{N\in\mathcal{F}} N=\{e\}$. ■

Definition 2.3 (Pro-Lie group)**.** A topological group G is *pro-Lie* if every neighborhood of the identity in G contains a compact normal subgroup N such that G/N is a Lie group.

Note that pro-Lie groups are locally compact. For a topological group G, let $\mathcal{N}(G)$ be the set of compact normal subgroups N of G such that G/N is a Lie group. As $\mathcal{N}(G)$ is stable under finite intersections, it is a filter basis of G.

Remark 2.4 (Remark 3.3 of [12])**.** For a locally compact group G, the following statements are equivalent:

(1) G is a pro-Lie group.

(2) The filter basis $\mathcal{N}(G)$ converges to e.

We denote by $\mathcal{SUB}_{\mathrm{d}}(G)$ the subspace of all discrete subgroups of a locally compact group G.

Proposition 2.5. *For every locally compact group G, the following statements are equivalent:*

(1) *G is a Lie group.*

(2) *$\mathcal{SUB}_{\mathrm{d}}(G)$ is open in $\mathcal{SUB}(G)$.*

(3) *The trivial subgroup $\{e\}$ is an interior point of $\mathcal{SUB}_{\mathrm{d}}(G)$.*

Proof. $(1)\Longrightarrow(2)$: See Proposition E.1.5 of [2] (see also [3], [6, Theorem 1.3.1.4] or [5, Proposition 3.4 (iii)]).

The implication $(2)\Longrightarrow(3)$ is trivial.

$(3)\Longrightarrow(1)$: Let Ω be an open set in $\mathcal{SUB}(G)$ such that $\{e\}\in\Omega\subseteq\mathcal{SUB}_{\mathrm{d}}(G)$. Let A be an open almost connected subgroup of G (that is, the quotient group A/A_0, modulo the connected component A_0 of the identity, is compact; see Lemma 6 of [4]). By Proposition 1 of [15], the identity mapping $\phi\colon \mathcal{SUB}(A)\to\mathcal{SUB}(G)$ is continuous and then $\phi^{-1}(\Omega)$ is an open subset of $\mathcal{SUB}(A)$. As a consequence of $\phi^{-1}(\mathcal{SUB}_{\mathrm{d}}(G))=\mathcal{SUB}_{\mathrm{d}}(A)$ we have $\{e\}\in\phi^{-1}(\Omega)\subseteq\mathcal{SUB}_{\mathrm{d}}(A)$. As A is a pro-Lie group ([14, page 175, Theorem 4.6]), then by Remark 2.4 and Proposition 2.2, the net $(N)_{N\in\mathcal{N}(A)}$ converges to $\{e\}$ in $\mathcal{SUB}(A)$ and therefore there exists $N\in\mathcal{N}(A)$ such that $N\in\phi^{-1}(\Omega)$. Then N is discrete and so, by Theorem 7 of [13], A is a Lie group. As A is open in G, G is also a Lie group (Lemma 2.1). ■

Proof of Theorem 1.2. The implication $(1) \Longrightarrow (2)$ is trivial.

$(2) \Longrightarrow (3)$: This follows from Proposition 2.5.

$(3) \Longrightarrow (1)$: Let Γ be a discrete subgroup of G. Let U be a compact symmetric neighborhood of e which does not contain any non-trivial subgroup (see Proposition 2.17 of [11]) and such that

$$U^3 \cap \Gamma = \{e\}. \tag{4}$$

Let $\Delta = \mathcal{U}_G(\Gamma; U^2, U)$ be a neighborhood of Γ in $\mathcal{SUB}(G)$ and let $H \in \Delta$. As

$$\Delta \overset{\text{def}}{=} \left\{ L \in \mathcal{SUB}(G) \mid L \cap U^2 \subseteq U\Gamma,\ \Gamma \cap U^2 \subseteq UL \right\}$$

then $(H \cap U)^2 \subseteq H \cap U^2 \subseteq U\Gamma \cap U^2$. Using (4) we deduce that $U^2 \cap U\Gamma = U$ and so

$$(H \cap U)^2 = H \cap U. \tag{5}$$

On the other hand, since U is symmetric then

$$(H \cap U)^{-1} = H \cap U. \tag{6}$$

Consequently, by (5) and (6) we deduce that $H \cap U$ is a subgroup of G contained in U and then $H \cap U = \{e\}$. Thus, there exists an open neighborhood $\Omega \subseteq \Delta$ of Γ such that every closed subgroup $L \in \Omega$ satisfies $L \cap U = \{e\}$. ∎

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