Journal of Lie Theory
Volume **29** (2019) 95–106

An Integral Transform Connecting Spherical Analysis on Harmonic NA Groups to that of Odd Dimensional Real Hyperbolic Spaces

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Communicated by A. Pasquale

In memory of Professeur Ahmed Intissar (1952-2017).

Abstract. The main aim of the present paper is to establish an integral transform connecting spherical analysis on harmonic NA groups to that of odd dimensional real hyperbolic spaces. Moreover, certain interesting integral identities for the Gauss hypergeometric functions have also been given.

Mathematics Subject Classification: 44A20, 33C05, 53C35.

Key Words and Phrases: Harmonic NA groups, hyperbolic spaces, Laplace-Beltrami operator, resolvent kernel, Gauss hypergeometric function, Jacobi equation, hyperbolic functions.

1. Introduction

Harmonic NA groups have been studied by several authors [3, 4, 5, 16, ...]. Recall that, as Riemannian manifolds, these solvable Lie groups include all symmetric spaces of noncompact type and rank one, namely the hyperbolic spaces $H^n_{\mathbb{F}}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) and $H^2_{\mathbb{O}}$, but that most of them are not symmetric, providing numerous counterexamples to the Lichnerowicz conjecture [13]. Despite the lack of symmetry, spherical analysis i.e. the analysis of radial functions on these spaces is quite similar to the hyperbolic space case. We shall emphasize that spherical analysis is again a particular case of the Jacobi function analysis [11].

These groups form a class of solvable Lie groups, equipped with a left-invariant metric. More precisely, given a group N of Heisenberg type, let $X = N \rtimes A$ be the semi-direct product of N obtained by letting $A = \mathbb{R}^*_+$ acts on N by the action $(V, Z) \in N \longmapsto (a^{\frac{1}{2}}V, aZ)$.

Introduced by Kaplan in [9], *H-type groups* (also called *Heisenberg type groups*) have attracted a considerable attention by various authors (see [3, 9, 10, 12, 17] ...). Recall briefly their structure. Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra, equipped with an inner product $\langle , \rangle$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ (so that $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$).

ISSN 0949–5932 / \$2.50

Let $J_Z \colon \mathfrak{v} \to \mathfrak{v}$ be the linear map defined by

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \quad (X, Y \in \mathfrak{v}; Z \in \mathfrak{z}).$$

Then $\mathfrak{n}$ is of *Heisenberg type* if the following identity is satisfied

$$J_Z^2 = -|Z|^2 I \quad \text{for every } Z \in \mathfrak{z}.$$

The corresponding (connected and) simply connected Lie groups N are called of Heisenberg type. We shall identify them with their Lie algebra $\mathfrak{n}$ via the exponential map. Thus multiplication in $N \equiv \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ reads

$$(X, Z)(X', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']),$$

and the Lebesgue measure $dXdZ$ is a Haar measure on N.
Let $m = \dim \mathfrak{v}$, $k = \dim \mathfrak{z}$ and $\sigma = \dim X = m + k + 1$. The number $Q = \frac{m}{2} + k$ is called the *homogeneous dimension* of N. The associated left-invariant (Riemann-Haar) measure on X is given by [3]

$$a^{-Q} dX dZ \frac{da}{a}.$$

In harmonic analysis, the role played by radial functions on X, in particular the spherical functions, is crucial. If L_X denote the Laplace-Beltrami operator of $X = NA$, then a spherical function φ on X is a radial eigenfunction of L_X such that $\varphi(e) = 1$.
The *radial part* of L_X in geodesic polar coordinates is given by

$$rad(L_X) = \frac{d^2}{dr^2} + \left\{\frac{m+k}{2}\coth(r/2) + \frac{k}{2}\tanh(r/2)\right\}\frac{d}{dr}, \tag{1}$$

where $r = d(x, y)$ is the geodesic distance between two points $x, y \in X = NA$.
It is well known that the *spherical resolvent kernel* $R_X(\lambda; x, y) := R_X(\lambda; r)$ of the Laplacian L_X on X can be described as the singular solution at $r = 0$ of the following equation of Jacobi type

$$\left(rad(L_X) + \tfrac{Q^2}{4} + \lambda^2\right) R_X(\lambda, r) = 0, \quad r > 0, \lambda \in \mathbb{C}, \tag{2}$$

where λ is a complex number such that $\Im m\lambda \geq 0$. In fact the resolvent kernel function of the shifted Laplacian $L_X + \frac{Q^2}{4}$ on X is well known to be given in terms of the Gauss Hypergeometric functions ${}_2F_1$ [3, 6]. More precisely, we have

$$\begin{aligned} R_X(\lambda, r) = {} & C_X(\lambda)\cosh^{-Q+i2\lambda}(r/2) \times \\ & \times {}_2F_1\Big(\frac{Q}{2} - i\lambda, \frac{Q}{2} - i\lambda - \frac{k-1}{2};\ 1 - i2\lambda;\ \cosh^{-2}(r/2)\Big) \end{aligned} \tag{3}$$

for $\Im m\lambda \geq 0$ and where $C_X(\lambda)$ is the constant given by

$$C_X(\lambda) = \pi^{-\sigma/2} 2^{-\sigma/2} \frac{\Gamma(\frac{Q}{2} - i\lambda)\Gamma(\frac{Q}{2} - i\lambda - \frac{k-1}{2})}{\Gamma(1 - i2\lambda)},$$

where the hypergeometric function ${}_2F_1(a,b;c;z)$ reads as

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{k!\Gamma(c+k)} z^k $$

and where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is Euler's Gamma-function.

Remark 1.1. For instance if Y is a real hyperbolic space of dimension N the resolvent kernel of the Laplacian operator L_Y is given as in (13) with $k=0$ and $m = N-1$, in particular where $\dim Y$ is odd, the resolvent kernel $R_Y(\lambda, r)$ can be given in terms of elementary function, see Section 2. ■

An integral transform relating the heat kernels on even dimensional hyperbolic spaces to the ones of odd dimension was obtained by Davies and Mandouvalos [7]. More precisely, in [7, p. 190], it has been established a recurrence relations relating the heat kernel, written in terms of the hyperbolic distance, on the real hyperbolic spaces H^{n+1} and H^{n+2} of dimension $n+1$ and $n+2$ respectively, given by the following integral transform

$$ K_{n+1}(t,\rho) = \frac{1}{\sqrt{2}} \int_\rho^\infty \frac{e^{(2n+1)t/4}}{(\cosh\mu - \cosh\rho)^{1/2}} K_{n+2}(t,\mu)\sinh(\mu) d\mu, \tag{4} $$

where $K_{n+1}(t,.)$ is the heat kernel associated to the hyperbolic Laplacian L_{n+1} for the $(n+1)$-dimensional hyperbolic space H^{n+1}.
The main aim of this paper is to show that all resolvent kernels of harmonic NA groups can be expressed as an integral transform of those odd dimension hyperbolic spaces. Namely the result to which is aimed this paper is to establish the following integral transform.

Theorem 1.2. *Let $X = NA$ be a harmonic group. Then the resolvent kernel $R_X(\lambda, r)$ of the Laplacian L_X as given in (3) can be expressed in terms of the resolvent kernel of odd dimensional hyperbolic spaces as*

$$ R_X(\lambda, r) = \int_r^\infty W_X(r,\rho) R_Y(\lambda,\rho)\sinh(\rho) d\rho\,, \tag{5} $$

where the kernel $W_X(r,\rho)$ is independent of $\lambda \in \mathbb{C}$, and is given by

$$ W_X(r,\rho) = \frac{2^{2-\sigma}\pi^{-\frac{m}{2}}}{\Gamma(k/2)} \cosh^{1-k}(r/2)\left(\cosh^2(\rho/2) - \cosh^2(r/2)\right)^{\frac{k}{2}-1}, \tag{6} $$

while $R_Y(\lambda,\rho)$ is the resolvent kernel of the real hyperbolic space Y of odd dimension $\dim Y = m + 2k + 1$.

The plan of the article is as follows. Inspired by the work of Davies-Mandouvalos [7], in Section 2, we have obtained a recurrence relation relating the real hyperbolic resolvent kernels for different dimensions. The Section 3 deals with some integral identities for the hypergeometric functions ${}_2F_1$. In the Section 4, we prove the main result (Theorem 1.2) of our paper. Finally, in Section 5, some concluding remarks are made.

2. Spherical analysis on the hyperbolic space

Spherical analysis on hyperbolic spaces was developed in [7, 8, 11, ...]. We recall briefly some of it in this section. The radial part (in geodesic polar coordinates) of the Laplace-Beltrami operator L_Y on a hyperbolic space Y of dimension n reads as

$$rad(L_Y) = \frac{d^2}{dr^2} + (n-1)\coth(r)\frac{d}{dr}, \qquad r > 0. \tag{7}$$

Let $\lambda \in \mathbb{C}$, the spherical functions $\varphi_Y(\lambda, .)$ on Y are normalized radial eigenfunctions of L_Y with eigenvalue $\nu = -(\sigma_Y^2 + \lambda^2)$ and $\varphi_Y(\lambda, 0) = 1$. We then have, that $\varphi_Y(\lambda, .)$ is the solution of the following differential equation

$$\left(\frac{d^2}{dr^2} + (n-1)\coth(r)\frac{d}{dr} + ((n-1)/2)^2 + \lambda^2\right)\varphi_Y(\lambda, r) = 0; \quad r > 0, \lambda \in \mathbb{C}, \tag{8}$$

continuous at $r = 0$, and since $\lim_{r\to 0^+} r\coth(r) = 1$, this equation has a regular singular point at $r = 0$. This is a Jacobi equation with parameters λ, $\alpha = \frac{n-2}{2}$ and $\beta = \frac{-1}{2}$ (for more group theoretic interpretations of Jacobi functions see Koornwinder's paper [11]). Therefore, the spherical functions $\varphi_Y(\lambda, r)$ are given by Jacobi functions in the following way:

$$\varphi_Y(\lambda, r) = \phi_\lambda^{(\frac{n-2}{2}, \frac{-1}{2})}(r). \tag{9}$$

Equivalently, in terms of the Gauss hypergeometric functions, we have

$$\varphi_Y(\lambda, r) = {}_2F_1\left(\frac{1}{2}\left((n-1)/2 - i\lambda\right), \frac{1}{2}\left((n-1)/2 + i\lambda\right); \frac{n}{2}; -\sinh^2(r)\right). \tag{10}$$

It can also be seen [11, p. 7] that for $\lambda \neq -i, -2i, ...$, a second solution of (8) on $(0, +\infty)$ is given by

$$\begin{aligned}\varphi_Y(\lambda, r) \;=\; &(\cosh r)^{i\lambda-(n-1)/2} \times \\ &{}_2F_1\left(\frac{1}{2}\left((n-1)/2 - i\lambda\right), \frac{1}{2}\Big((n-1)/2 - i\lambda\Big) + \frac{1}{2}; 1 - i\lambda; \cosh^{-2}(r)\right).\end{aligned} \tag{11}$$

It is known that the resolvent kernel $R_Y(\lambda, r)$ is a multiple of the fundamental solution at infinity (11) of (8) which reads as (see [2])

$$\begin{aligned}R_Y(\lambda, r) = &C_{n,\lambda}\,(\cosh r/2)^{2i\lambda-(n-1)} \times \\ &{}_2F_1\left((n-1)/2 - i\lambda, 1/2 - i\lambda; 1 - 2i\lambda; \cosh^{-2}(r/2)\right)\end{aligned} \tag{12}$$

where
$$C_{n,\lambda} = 2^{-(n-2i\lambda)}\pi^{-(n-1)/2}\Gamma((n-1)/2 - i\lambda)/\Gamma(1 - i\lambda). \tag{13}$$

In [7, p. 185], it has been established a recurrence relations relating the heat kernels $K(t, .)$ on the real hyperbolic spaces H^{n-1} and H^{n+1} of dimension $n-1$ and $n+1$ respectively, given by the following recurrence relation

$$K_{n+1}(t, \rho) = -\frac{e^{(1-n)t}}{2\pi\sinh\rho}\frac{\partial}{\partial\rho}K_{n-1}(t, \rho), \quad \rho > 0, \tag{14}$$

where $K_{n+1}(t,.)$ is the heat kernel associated to the hyperbolic Laplacian L_{n+1} for the $(n+1)$-dimensional hyperbolic space H^{n+1}.
In what follows, we give an analogous of this result relating the real hyperbolic resolvent kernels on the real hyperbolic spaces of dimension n and $n+2$ respectively. Then, we prove that for odd dimensional real hyperbolic spaces the resolvent kernel is given in terms of elementary functions. Namely, our main result of this section is the following:

Proposition 2.1. *Let $R_n(\lambda, r)$ be the resolvent kernel for a real hyperbolic space H^n of dimension n. Then*

(i) *The resolvent kernel in* (13) *can be written as*

$$R_n(\lambda, r) = C_n(\lambda)\,(\cosh r)^{i\lambda-(n-1)/2} \times$$
$$\times\, {}_2F_1\left(\frac{1}{2}\left((n-1)/2 - i\lambda\right), \frac{1}{2}\left((n-1)/2 - i\lambda\right) + \frac{1}{2}; 1 - i\lambda; \cosh^{-2}(r)\right) \tag{15}$$

where

$$C_n(\lambda) = (4\pi^{n/2})^{-1}\,\frac{\Gamma\left(\frac{1}{2}((n-1)/2 - i\lambda)\right)\Gamma\left(\frac{1}{2}((n-1)/2 - i\lambda) + 1/2\right)}{\Gamma(1 - i\lambda)}. \tag{16}$$

(ii) *The following recurrence relation holds:*

$$\frac{-1}{2\pi \sinh r}\frac{\partial}{\partial r}\Big[R_n(\lambda, r)\Big] = R_{n+2}(\lambda, r). \tag{17}$$

(iii) *The resolvent kernel for odd dimensional real hyperbolic space of dimension $2m+1$ can be given in terms of elementary functions as*

$$R_{2m+1}(\lambda, r) = C_m(\lambda)\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \left(e^{ir\lambda}\right), \tag{18}$$

where $C_m(\lambda) = (-1)^{m+1}/2i\lambda(2\pi)^m$.

Proof. Making use of the identity [14, p. 50]

$${}_2F_1\left(2a, c - 1/2; 2c - 1; 2\sqrt{z}/(1+\sqrt{z})\right) = (1+\sqrt{z})^{2a}\,{}_2F_1(a, a+1/2; c; z)$$

and Legendre's duplication formula

$$\Gamma(2a) = (1/\sqrt{\pi})\Gamma(a)\Gamma(a+1/2)2^{2a-1}$$

for $z = \cosh^{-2}(r)$, $a = ((n-1)/2 - i\lambda)/2$ and $c = 1 - i\lambda$, the resolvent kernel in (13) becomes

$$R_n(\lambda, r) = C_n(\lambda)\cosh^{i\lambda-(n-1)/2}(r) \times$$
$$\times\, {}_2F_1\left(\frac{1}{2}\left((n-1)/2 - i\lambda\right), \frac{1}{2}\left((n-1)/2 - i\lambda\right) + \frac{1}{2}; 1 - i\lambda; \cosh^{-2}(r)\right)$$

where $C_n(\lambda) = (4\pi^{n/2})^{-1}\,\dfrac{\Gamma\left(\frac{1}{2}((n-1)/2 - i\lambda)\right)\Gamma\left(\frac{1}{2}((n-1)/2 - i\lambda) + 1/2\right)}{\Gamma(1 - i\lambda)}$.

This proves relation (i). Relation (ii) is obtained as follows. Recall that the real hyperbolic resolvent kernel of dimension n is given in terms of the hypergeometric function ${}_2F_1$ by (see [2])

$$R_n(\lambda, r) = C_{n,\lambda} (\cosh r/2)^{2i\lambda-(n-1)} \times \\ \times \, {}_2F_1\left((n-1)/2 - i\lambda, 1/2 - i\lambda; 1 - 2i\lambda; \cosh^{-2}(r/2)\right) \tag{19}$$

where $C_{n,\lambda} = 2^{-(n-2i\lambda)}\pi^{-(n-1)/2}\Gamma((n-1)/2 - i\lambda)/\Gamma(1-i\lambda)$.

By using the differential formula of the hypergeometric function ${}_2F_1$ [1, p. 557]

$$\frac{\partial^k}{\partial z^k}\left[z^{a+k-1}\, {}_2F_1(a,b;c;z)\right] = (a)_k\, z^{a-1}\, {}_2F_1(a+k,b;c;z),$$

where $(a)_k = a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol, for $k = 1$, we prove that the resolvent kernel $R_n(\lambda, r)$ in (19) verifies the recurrence relation

$$\frac{-1}{2\pi \sinh r}\frac{\partial}{\partial r}\left[R_n(\lambda, r)\right] = R_{n+2}(\lambda, r). \tag{20}$$

Finally, relation (iii) is obtained easily from (ii). In fact, thanks to the recurrence formula (20), we obtain the following expression for $R_n(\lambda, r)$ when n is odd:

$$R_{2m+1}(\lambda, r) = \left(\frac{-1}{2\pi \sinh r}\frac{\partial}{\partial r}\right)^m \left[R_1(\lambda, r)\right], \tag{21}$$

where $R_1(\lambda, r)$ is the the resolvent kernel on the real hyperbolic space of one dimensional H^1.

In addition, we observe that, for $n = 1$, the expression of the resolvent kernel $R_n(\lambda, r)$ given in (19) reduces to

$$R_1(\lambda, r) = C_{1,\lambda} (\cosh r/2)^{2i\lambda}\, {}_2F_1\left(-i\lambda, 1/2 - i\lambda; 1 - 2i\lambda; \cosh^{-2}(r/2)\right)$$

where $C_{1,\lambda} = 2^{2i\lambda-1}\Gamma(-i\lambda)/\Gamma(1-i\lambda)$.

Further, we use the well-known elementary expression of the hypergeometric function given by [14, p. 38]

$${}_2F_1\left(a, 1/2 + a; 2a + 1; z^2\right) = 2^{2a}\left(1 + \sqrt{1 - z^2}\right)^{-2a}$$

for $a = -i\lambda$ and $z = 1/\cosh(r/2)$, we arrive to the following identity

$$R_1(\lambda, r) = \frac{-1}{2i\lambda}e^{ir\lambda}.$$

Replacing $R_1(\lambda, r)$ by its expression in the recurrence formula (21), we obtain the announced relation (iii). This ends the proof. ■

3. Gauss hypergeometric functions ${}_2F_1$

The two expressions of the resolvent kernels in (3) and in (15) are given in terms of the hypergeometric function ${}_2F_1$. For this reason, to find a proof of Theorem 1.2, we should say something about hypergeometric functions.

Recall that the hypergeometric function ${}_2F_1$ is defined as

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{k!\Gamma(c+k)} z^k, \tag{22} $$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is Euler's Gamma-function. See [1], [14] or [18] for a general discussion of ${}_2F_1$'s and of more general series of this type.
In the following we prove some integral representation, connecting two hypergeometric functions. The corresponding kernel function can be seen as the kernel of the Fourier-Jacobi or Olevskii index transform studied, for instance, in the book of S. B. Yakubovich [19].
The most important properties of hypergeometric functions that we use all follow from the following lemma.

Lemma 3.1. *For $x > 1$, $\Re\mu > 0$ and $\Re b > 0$, we have*

$$ x^{-b}\, {}_2F_1(a,b;c;x^{-1}) = \frac{\Gamma(b+\mu)}{\Gamma(b)\Gamma(\mu)} \int_x^\infty y^{-b-\mu}(y-x)^{\mu-1}\, {}_2F_1(a,b+\mu;c;y^{-1})\, dy. \tag{23} $$

Proof. We use the well known series expansion for hypergeometric function:

$$ {}_2F_1(a,b+\mu;c;1/y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b+\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+\mu+k)}{k!\Gamma(c+k)} 1/y^k. \tag{24} $$

Note that this series converges absolutely because of the above assumptions. Inserting (24) in the right hand side of the formula (23) and integrating term by term making the change of variable $y = xt^{-1}$ and use the Euler's formula for the beta function

$$ \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, $$

we obtain at once the desired result. ■

Another integral connecting hypergeometric functions with different parameters are the following

Lemma 3.2. *For $x > 1$, $\Re\nu > 0$ and $\Re c > \Re\nu$, we have*

$$ x^{c-\nu}\, {}_2F_1(a,b;c;x^{-1}) = \frac{\Gamma(c)}{\Gamma(c-\nu)\Gamma(\nu)} \int_x^\infty y^{-c}(y-x)^{\nu-1}\, {}_2F_1(a,b;c-\nu;y^{-1})\, dy. $$

This can be proved in the same way that Lemma 3.1 was proved, i.e. expanding ${}_2F_1(a,b;c;y)$ in the series (22) and integrating term by term.
The other is a more complicated formula. In fact, in this lemma, we establish an integral identity between the Gauss hypergeometric functions ${}_2F_1(a,b;c;z)$ and ${}_2F_1(a+\mu,b+\nu;c;z)$, where μ and ν are given real numbers, given by the following lemma which will play a crucial role in the next section. Namely, we have

Key Lemma 3.3. *Let a, b, μ, ν be complex numbers such that $\Re a$, $\Re b$, $\Re\mu$, $\Re\nu > 0$. Then for every $x > 1$, the following identity holds*

$$x^{-b+\mu}\,{}_2F_1(a,b;c;x^{-1}) = \frac{\Gamma(a+\mu)\Gamma(b+\nu)}{\Gamma(a)\Gamma(b)\Gamma(\mu+\nu)} \times \\ \times \int_x^\infty W_{ab\mu\nu}(x,y) y^{-b-\nu}\,{}_2F_1(a+\mu, b+\nu; c; y^{-1})\,dy \tag{25}$$

where the kernel function $W_{ab\mu\nu}(x,y)$ is given by the following formula

$$W_{ab\mu\nu}(x,y) = (y-x)^{\mu+\nu-1}\,{}_2F_1(\mu, a-b+\mu; \mu+\nu; 1-y/x). \tag{26}$$

Proof. The proof will rely on Lemma 3.1. In fact, to prove our key integral formula (25) we will apply the above integral identity (23) and iterating it twice. That is, fixing μ and using the above lemma, we get

$$y^{-b}\,{}_2F_1(a+\mu, b; c; y^{-1}) = \frac{\Gamma(b+\nu)}{\Gamma(b)\Gamma(\nu)} \int_y^\infty z^{-b-\nu}(z-y)^{\nu-1}\,{}_2F_1(a+\mu, b+\nu; c; z^{-1})\,dz.$$

Therefore, multiplying both sides by $y^{b-a-\mu}(y-x)^{\mu-1}$ and integrating the both sides in y, we obtain

$$x^{-a}\,{}_2F_1(a,b;c;x^{-1}) = \frac{\Gamma(a+\mu)\Gamma(b+\nu)}{\Gamma(a)\Gamma(b)\Gamma(\mu)\Gamma(\nu)} \int_x^\infty y^{b-a-\mu}(y-x)^{\mu-1} \\ \times \int_y^\infty z^{-b-\nu}(z-y)^{\nu-1}\,{}_2F_1(a+\mu, b+\nu; c; z^{-1})\,dzdy.$$

Note that $z > y > x > 1$ and by Fubini's theorem the integral

$$\int_x^\infty y^{b-a-\mu}(y-x)^{\mu-1}\left(\int_y^\infty z^{-b-\nu}(z-y)^{\nu-1}\,{}_2F_1(a+\mu, b+\nu; c; z^{-1})\,dz\right)dy$$

can be transformed into the integral

$$\int_x^\infty z^{-b-\nu}\,{}_2F_1(a+\mu, b+\nu; c; z^{-1})\left(\int_x^z y^{b-a-\mu}(y-x)^{\mu-1}(z-y)^{\nu-1}dy\right)dz. \tag{27}$$

Setting $y = (1-t)x + tz$, $t \in [0,1]$, we obtain

$$\int_x^z y^{b-a-\mu}(y-x)^{\mu-1}(z-y)^{\nu-1}\,dy = x^{b-a-\mu}(z-x)^{\mu+\nu-1} \times \\ \times \int_0^1 t^{\mu-1}(1-t)^{\nu-1}\left(1-t(1-z/x)\right)^{-(a-b+\mu)}\,dt.$$

Using the integral representation of the hypergeometric function [1, p. 558]

$${}_2F_1(a', b'; c'; z') = \frac{\Gamma(c')}{\Gamma(b')\Gamma(c'-b')} \int_0^1 s^{b'-1}(1-s)^{c'-b'-1}\left(1-sz'\right)^{-a'}\,ds,$$

where $\Re(c') > \Re(b') > 0$, $a' = a-b+\mu$, $b' = \mu$, $c' = \mu+\nu$ and $z' = 1-z/x$, we get

$$\int_x^z y^{b-a-\mu}(y-x)^{\mu-1}(z-y)^{\nu-1}\,dy =$$
$$= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\, x^{b-a-\mu}(z-x)^{\mu+\nu-1} \times {}_2F_1(\mu, a-b+\mu; \mu+\nu; 1-z/x). \tag{28}$$

Substituting (28) into the integral in (27), we obtain

$$x^{-b+\mu}\,{}_2F_1(a,b;c;x^{-1}) =$$
$$= \frac{\Gamma(a+\mu)\Gamma(b+\nu)}{\Gamma(a)\Gamma(b)\Gamma(\mu+\nu)} \int_x^\infty W_{ab\mu\nu}(x,y)y^{-b-\nu}\,{}_2F_1(a+\mu, b+\nu; c; y^{-1})\,dy$$

where $W_{ab\mu\nu}(x,y)$ is given by

$$W_{ab\mu\nu}(x,y) = (y-x)^{\mu+\nu-1}\,{}_2F_1(\mu, a-b+\mu; \mu+\nu; 1-y/x).$$

Hence the result of the key integral identity holds. ■

Remark 3.4. We shall notice that interchanging a and b together with μ and ν we get similarly the following integral identity:

$$x^{-a+\nu}\,{}_2F_1(a,b;c;x^{-1}) =$$
$$= \frac{\Gamma(a+\mu)\Gamma(b+\nu)}{\Gamma(a)\Gamma(b)\Gamma(\mu+\nu)} \int_x^\infty \widetilde{W}_{ab\mu\nu}(x,y)y^{-a-\mu}\,{}_2F_1(a+\mu, b+\nu; c; y^{-1})\,dy$$

where the kernel function $\widetilde{W}_{ab\mu\nu}(x,y)$ is given by the following formula

$$\widetilde{W}_{ab\mu\nu}(x,y) = (y-x)^{\mu+\nu-1}\,{}_2F_1(\nu, a-b+\nu; \mu+\nu; 1-y/x) = \left(\frac{y}{x}\right)^{a+\mu-(b+\nu)} W_{ab\mu\nu}(x,y).$$

4. Proof of Theorem 1.2

Let $R_X(\lambda, r)$ be the the resolvent kernel function given in (3) by

$$R_X(\lambda, r) = C_X(\lambda)\cosh^{-Q+i2\lambda}(r/2) \times$$
$$\times\ {}_2F_1\Big(\frac{Q}{2} - i\lambda,\ \frac{Q}{2} - i\lambda - \frac{k-1}{2};\ 1-i2\lambda;\ \cosh^{-2}(r/2)\Big) \tag{29}$$

for $\Im m\lambda \geq 0$ and where $C_X(\lambda)$ is the constant given by

$$C_X(\lambda) = \pi^{-\sigma/2}2^{-\sigma/2}\frac{\Gamma(\frac{Q}{2} - i\lambda)\Gamma(\frac{Q}{2} - i\lambda - \frac{k-1}{2})}{\Gamma(1-i2\lambda)},$$

where the hypergeometric function ${}_2F_1(a,b;c;z)$ reads as

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{k!\Gamma(c+k)}\, z^k.$$

To establish this integral representation, we begin by representing the resolvent kernel $R_X(\lambda, r)$ after substitution $x = \cosh^2(r/2)$ as follows

$$R_X(\lambda,r) =: G_X(\lambda,x) = C_X(\lambda)\, x^{-\frac{Q-i2\lambda}{2}} \times \times\, {}_2F_1\left(\frac{Q}{2}-i\lambda,\, \frac{Q}{2}-i\lambda-\frac{k-1}{2};\, 1-i2\lambda;\, x^{-1}\right) \tag{30}$$

and appealing to the key integral formula (25) with $a=\frac{Q}{2}-i\lambda, b=\frac{Q}{2}-i\lambda-\frac{k-1}{2}$ and $c=1-i2\lambda$, so that μ and ν are given respectively by $\mu=\frac{1}{2}$ and $\nu=\frac{k-1}{2}$, we have the following integral representation

$$G_X(\lambda,x) = C(\sigma,Q,\lambda)\, x^{-\frac{k}{2}} \times \times \int_x^\infty K_k(x,y)\, y^{-\frac{Q}{2}+i\lambda}\, {}_2F_1\left(\frac{Q}{2}-i\lambda, \frac{Q}{2}-i\lambda+\frac{1}{2}; 1-i2\lambda; y^{-1}\right) dy \tag{31}$$

where we denote by $K_k(x,y)$ the function of type

$$K_k(x,y) = (y-x)^{\frac{k}{2}-1}\, {}_2F_1\left(\frac{1}{2}, \frac{k}{2}; \frac{k}{2}; 1-y/x\right)$$

and where
$$C(\sigma,Q,\lambda) = \pi^{\frac{1-\sigma}{2}}\, 2^{1-\sigma-Q+2i\lambda}\, \frac{\Gamma(Q-2i\lambda)}{\Gamma(k/2)}.$$

Using the identity [14, p. 38] $(1+z)^a = {}_2F_1(-a,b;b;-z)$ for $a=-\frac{1}{2}$, $b=\frac{k}{2}$ and $z=-(1-y/x)$, we get

$$K_k(x,y) = (y-x)^{\frac{k}{2}-1}\, (x/y)^{1/2}.$$

Then equation (31) becomes

$$G_X(\lambda,x) = C(\sigma,Q,\lambda)\, x^{\frac{1-k}{2}} \times \times \int_x^\infty (y-x)^{\frac{k}{2}-1}\, y^{-\frac{Q+1}{2}+i\lambda}\, {}_2F_1\left(\frac{Q}{2}-i\lambda, \frac{Q}{2}-i\lambda+\frac{1}{2}; 1-i2\lambda; y^{-1}\right) dy. \tag{32}$$

Considering the integral (4) by replacement of variable $x=\cosh^2(r/2)$, we arrive at the following identity

$$R_X(\lambda,r) = C(\sigma,Q,\lambda)\, \cosh^{1-k}(r/2) \int_r^\infty \left(\cosh^2(\rho/2)-\cosh^2(r/2)\right)^{\frac{k}{2}-1} \sinh(\rho/2) \times \times (\cosh\rho/2)^{-Q+i2\lambda}\, {}_2F_1\left(\frac{Q}{2}-i\lambda, \frac{Q}{2}-i\lambda+\frac{1}{2}; 1-i2\lambda; \cosh^{-2}(\rho/2)\right) d\rho.$$

Consequently one has the integral transform

$$R_X(\lambda,r) = \int_r^\infty W_X(r,\rho)\, R_Y(\lambda,\rho)\, \sinh(\rho/2)\, d\rho$$

where we denote by $W_X(r,\rho)$ the kernel (independent of $\lambda\in\mathbb{C}$) given by

$$W_X(r,\rho) = \frac{2^{2-\sigma}\pi^{-\frac{m}{2}}}{\Gamma(k/2)} \cosh^{1-k}(r/2) \left(\cosh^2(\rho/2)-\cosh^2(r/2)\right)^{\frac{k}{2}-1}$$

and where $R_Y(\lambda,\rho)$ is the resolvent kernel of the real hyperbolic space Y of odd dimension $\dim Y = m+2k+1$. This completes the proof. ∎

5. Concluding remarks

To finish this paper we have to mention that for complex and quaternionic hyperbolic spaces $S = G/K = NA$ (which are only a very small subclass of harmonic NA groups) there are some homogenous vector bundles V_τ over them and that the resolvent kernels of the Laplacian $L_{S,\tau}$ on $G/K = NA$ acting on sections of such vector bundles are given by:

$$R_{S,\tau}(\lambda, r) = C_S(\lambda)\,(\cosh r)^{-l+i\lambda}\ {}_2F_1\left(\frac{l-i\lambda}{2}, \frac{l-i\lambda}{2} - \frac{\tau}{2}; 1-i\lambda; \cosh^{-2}(r)\right) \quad (33)$$

for $\Im m\lambda \geq 0$, where the constant $C_S(\lambda)$ is given explicitly by

$$C_S(\lambda) = \pi^{-(dimN+1)/2}\frac{\Gamma((l-i\lambda)/2)\Gamma((l-i\lambda)/2)}{4\Gamma(1-i\lambda)},$$

where $l = (\dim N + \dim Z)/2$ and Z denotes the center of N.

Then there is also a general integral transform with kernel $W_{S,\tau}(r,\rho)$ that connects $R_{S,\tau}(\lambda, r)$ to the resolvent $R_Y(\lambda,\rho)$ as in Theorem 1.2. The detail is left for a forthcoming paper in relation with this subject.

We hope that we can use this explicit integral transform to solve other problems in the spherical analysis on the NA harmonic groups.

Acknowledgements. The authors would like to thank Professor A. Boussejra for useful discussions. They also would like to thank the referee for careful reading and useful suggestions.

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Received November 12, 2017
and in final form October 11, 2018