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Rigidity of Bott-Samelson-Demazure-Hansen Variety for $PSO(2n+1,\mathbb{C})$

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Abstract. Let $G = PSO(2n+1,\mathbb{C})$ $(n \geq 3)$ and B be the Borel subgroup of G containing maximal torus T of G. Let w be an element of Weyl group W and $X(w)$ be the Schubert variety in the flag variety G/B corresponding to w. Let $Z(w,\underline{i})$ be the Bott-Samelson-Demazure-Hansen Variety (the desingularization of $X(w)$) corresponding to a reduced expression $\underline{i}$ of w.
In this article, we study the cohomology modules of the tangent bundle on $Z(w_0,\underline{i})$, where w_0 is the longest element of the Weyl group W. We describe all the reduced expressions of w_0 in terms of a Coxeter element such that all the higher cohomology modules of the tangent bundle on $Z(w_0,\underline{i})$ vanish.

Mathematics Subject Classification: 14M15

Key Words: Bott-Samelson-Demazure-Hansen variety, Coxeter element, tangent bundle.

1. Introduction

Let G be a simple algebraic group of adjoint type over the field $\mathbb{C}$ of complex numbers. We fix a maximal torus T of G and let $W = N_G(T)/T$ denote the Weyl group of G with respect to T. We denote by R the set of roots of G with respect to T and by R^+ a set of positive roots. Let B^+ be the Borel subgroup of G containing T with respect to R^+. Let w_0 denote the longest element of the Weyl group W. Let B be the Borel subgroup of G opposite to B^+ determined by T, i.e. $B = n_{w_0}B^+n_{w_0}^{-1}$, where n_{w_0} is a representative of w_0 in $N_G(T)$. Note that the set of roots associated to B is the set $R^- := -R^+$ of negative roots. We use the notation $\beta < 0$ for $\beta \in R^-$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of all simple roots in R^+, where n is the rank of G. The simple reflection in the Weyl group corresponding to a simple root α is denoted by s_α. For simplicity of notation, the simple reflection corresponding to a simple root α_i is denoted by s_i.

For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, \ldots, i_r)$, we denote by $Z(w,\underline{i})$ the desingularization of the Schubert variety $X(w)$, which is now known as Bott-Samelson-Demazure-Hansen variety. This was first introduced by Bott and Samelson in a differential geometric and topological

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context (see [2]). Demazure in [6] and Hansen in [8] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote any Bott-Samelson-Demazure-Hansen variety by a BSDH-variety.

The construction of the BSDH-variety $Z(w, \underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w. In [5], the automorphism groups of these varieties were studied. There, the following vanishing results of the tangent bundle $T_{Z(w,\underline{i})}$ on $Z(w, \underline{i})$ were proved (see [5, Section 3]):

(1) $H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 2$.

(2) If G is simply laced, then $H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 1$.

As a consequence, it follows that the BSDH-varieties are rigid for simply laced groups and their deformations are unobstructed in general (see [5, Section 3]). The above vanishing result is independent of the choice of the reduced expression $\underline{i}$ of w. While computing the first cohomology module $H^1(Z(w, \underline{i}), T_{Z(w,\underline{i})})$ for non simply laced group, we observed that this cohomology module very much depend on the choice of a reduced expression $\underline{i}$ of w.

It is a natural question to ask that for which reduced expressions $\underline{i}$ of w, the cohomology module $H^1(Z(w, \underline{i}), T_{Z(w,\underline{i})})$ does vanish. In [4], a partial answer is given to this question for $w = w_0$ when $G = PSp(2n, \mathbb{C})$. In this article, we give a partial answer to this question for $w = w_0$ when $G = PSO(2n + 1, \mathbb{C})$.

Recall that a Coxeter element is an element of the Weyl group having a reduced expression of the form $s_{i_1} s_{i_2} \cdots s_{i_n}$ such that $i_j \neq i_l$ whenever $j \neq l$ (see [11, p. 56, Section 4.4]). Note that for any Coxeter element c, there is a decreasing sequence of integers $n \geq a_1 > a_2 > \ldots > a_k = 1$ such that $c = \prod_{j=1}^{k} [a_j, a_{j-1} - 1]$, where $a_0 := n + 1$, $[i, j] := s_i s_{i+1} \cdots s_j$ for $i \leq j$.

In this paper we prove the following theorem.

Theorem 1.1. *Let $G = PSO(2n + 1, \mathbb{C})$ $(n \geq 3)$ and let $c \in W$ be a Coxeter element. Let $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ be a sequence corresponding to a reduced expression of w_0, where $\underline{i}^r$ $(1 \leq r \leq n)$ is a sequence of reduced expressions of c (see Lemma 2.8). Then, $H^j(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $j \geq 1$ if and only if $c = \prod_{j=1}^{k} [a_j, a_{j-1} - 1]$, where $a_0 := n + 1$ and $a_2 \neq n - 1$.*

By the above vanishing results, we conclude that if $G = PSO(2n + 1, \mathbb{C})$ $(n \geq 3)$ and $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ is a reduced expression of w_0 as above, then the BSDH-variety $Z(w_0, \underline{i})$ is rigid.

The main differences in the proof of the main theorem between the case of type C_n and the case of type B_n are as follows: When G is of type C_n, there is only one long simple root, namely α_n. Therefore, by [14, Corollary 5.6, p. 778], we have $H^1(w, \alpha_j) = 0$ for any $w \in W$ and for any $j \neq n$. But, if G is of type B_n, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are all long simple roots. So, we can not apply [14, Corollary 5.6, p. 778]. Hence, we need to study the cohomology modules $H^1(w, \alpha_j)$ $(w \in W$, $j \neq n - 1)$. While studying these modules, we prove that $H^1(w, \alpha_j) = 0$ for any $w \in W$ and for any $j \neq n-1$ (see Lemma 3.3). Also, in this article, we need to prove an additional statement, namely, $(s_1s_2\cdots s_n)^{r-1}s_1s_2\cdots s_{n-1}(\alpha_j)<0$ for $2\leq r\leq n$ and $n+1-r\leq j\leq n-1$ (see Lemma 5.9).

The organization of the paper is as follows: In Section 2, we recall some preliminaries on BSDH-varieties. We deal with the special case $G=PSO(2n+1,\mathbb{C})$ $(n\geq 3)$ in the later sections 3, 4, 5, 6 and 7. In Section 3, we prove $H^1(w,\alpha_j)=0$ for $j\neq n-1$ and $w\in W$. In Section 4 (respectively, Section 5) we compute the weight spaces of H^0 (respectively, H^1) of the relative tangent bundle of BSDH-varieties associated to some elements of the Weyl group. In Section 6, we prove some results on cohomology modules of the tangent bundle of BSDH varieties. In Section 7, we prove the main result using the results from the previous sections.

2. Preliminaries

In this section, we set up some notation and preliminaries. We refer to [3], [9], [10], [13] for preliminaries in algebraic groups and Lie algebras.

Let G be a simple algebraic group of adjoint type over $\mathbb{C}$ and T be a maximal torus of G. Let $W=N_G(T)/T$ denote the *Weyl group* of G with respect to T and we denote the set of roots of G with respect to T by R. Let B^+ be a *Borel subgroup* of G containing T. Let B be the Borel subgroup of G opposite to B^+ determined by T. That is, $B=n_0B^+n_0^{-1}$, where n_0 is a representative in $N_G(T)$ of the longest element w_0 of W. Let $R^+\subset R$ be the set of positive roots of G with respect to the Borel subgroup B^+. Note that the set of roots associated to B is equal to the set $R^-:=-R^+$ of negative roots.

Let $S=\{\alpha_1,\ldots,\alpha_n\}$ denote the set of simple roots in R^+. For $\beta\in R^+$, we also use the notation $\beta>0$. The simple reflection in W corresponding to α_i is denoted by s_{α_i}. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h}\subset\mathfrak{g}$ be the Lie algebra of T and $\mathfrak{b}\subset\mathfrak{g}$ be the Lie algebra of B. Let $X(T)$ denote the group of all characters of T. We have $X(T)\otimes\mathbb{R}=\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}},\mathbb{R})$, the dual of the real form of $\mathfrak{h}$. The positive definite W-invariant form on $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}},\mathbb{R})$ induced by the Killing form of $\mathfrak{g}$ is denoted by $(\ ,\ )$. We use the notation $\langle\ ,\ \rangle$ to denote $\langle\mu,\alpha\rangle=2(\mu,\alpha)/(\alpha,\alpha)$, for every $\mu\in X(T)\otimes\mathbb{R}$ and $\alpha\in R$. We denote by $X(T)^+$ the set of dominant characters of T with respect to B^+. Let ρ denote the half sum of all positive roots of G with respect to T and B^+. For any simple root α, we denote the fundamental weight corresponding to α by ω_α. For $1\leq i\leq n$, let $h(\alpha_i)\in\mathfrak{h}$ be the fundamental coweight corresponding to α_i. That is ; $\alpha_i(h(\alpha_j))=\delta_{ij}$, where δ_{ij} is Kronecker delta.

For a simple root $\alpha\in S$, let $n_\alpha\in N_G(T)$ be a representative of s_α. We denote the *minimal parabolic subgroup* of G containing B and n_α by P_α. We recall that the BSDH-variety corresponds to a reduced expression $\underline{i}$ of $w=s_{i_1}s_{i_2}\cdots s_{i_r}$ defined by

$$Z(w,\underline{i})=\frac{P_{\alpha_1}\times P_{\alpha_2}\times\cdots\times P_{\alpha_{i_r}}}{B\times B\times\cdots\times B},$$

where the action of $B\times B\times\cdots\times B$ on $P_{\alpha_{i_1}}\times P_{\alpha_{i_2}}\times\cdots\times P_{\alpha_{i_r}}$ is given by

$$(p_1,p_2,\ldots,p_r)(b_1,b_2\ldots,b_r)=(p_1\cdot b_1,b_1^{-1}\cdot p_2\cdot b_2,\ldots,b_{r-1}^{-1}\cdot p_r\cdot b_r),$$

$p_j\in P_{\alpha_{i_j}}$, $b_j\in B$, and $\underline{i}=(i_1,i_2,\ldots,i_r)$ (see [6, Definition 1, p. 73], [3, Definition 2.2.1, p. 64]). We note that for each reduced expression $\underline{i}$ of w, $Z(w,\underline{i})$ is a smooth projective variety. We denote by ϕ_w the natural birational surjective morphism from $Z(w, \underline{i})$ to $X(w)$.

Let $f_r \colon Z(w, \underline{i}) \longrightarrow Z(ws_{i_r}, \underline{i'})$ denote the map induced by the projection

$$P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_{r-1}}},$$

where $\underline{i'} = (i_1, i_2, \ldots, i_{r-1})$. Then we observe that f_r is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$-fibration.

For a B-module V, let $\mathcal{L}(w, V)$ denote the restriction of the associated homogeneous vector bundle on G/B to $X(w)$. By abuse of notation, we denote the pull back of $\mathcal{L}(w, V)$ via ϕ_w to $Z(w, \underline{i})$ also by $\mathcal{L}(w, V)$, when there is no confusion. Since for any B-module V the vector bundle $\mathcal{L}(w, V)$ on $Z(w, \underline{i})$ is the pull back of the homogeneous vector bundle from $X(w)$, we conclude that the cohomology modules

$$H^j(Z(w, \underline{i}), \mathcal{L}(w, V)) \simeq H^j(X(w), \mathcal{L}(w, V))$$

for all $j \geq 0$ (see [3, Theorem 3.3.4(b)]) are independent of choice of reduced expression $\underline{i}$. Hence we denote $H^j(Z(w, \underline{i}), \mathcal{L}(w, V))$ by $H^j(w, V)$. In particular, if λ is character of B, then we denote the cohomology modules $H^j(Z(w, \underline{i}), \mathcal{L}_\lambda)$ by $H^j(w, \lambda)$.

We recall the following short exact sequence of B-modules from [5], we call it SES:

(1) $H^0(w, V) \simeq H^0(s_\gamma, H^0(s_\gamma w, V))$.

(2) $0 \to H^1(s_\gamma, H^0(s_\gamma w, V)) \to H^1(w, V) \to H^0(s_\gamma, H^1(s_\gamma w, V)) \to 0$.

Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let $\mathbb{C}_\lambda$ denote one dimensional B-module associated to λ. Here, we recall the following result due to Demazure [7, p. 271] on short exact sequence of B-modules:

Lemma 2.1. *Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let* $\mathrm{ev} : H^0(s_\alpha, \lambda) \longrightarrow \mathbb{C}_\lambda$ *be the evaluation map. Then we have*

(1) *If $\langle \lambda, \alpha \rangle = 0$, then $H^0(s_\alpha, \lambda) \simeq \mathbb{C}_\lambda$.*

(2) *If $\langle \lambda, \alpha \rangle \geq 1$, then $\mathbb{C}_{s_\alpha(\lambda)} \hookrightarrow H^0(s_\alpha, \lambda)$, and there is a short exact sequence of B-modules:*

$$0 \to H^0(s_\alpha, \lambda - \alpha) \longrightarrow H^0(s_\alpha, \lambda)/\mathbb{C}_{s_\alpha(\lambda)} \longrightarrow \mathbb{C}_\lambda \to 0.$$

Further more, $H^0(s_\alpha, \lambda - \alpha) = 0$ when $\langle \lambda, \alpha \rangle = 1$.

(3) *Let $n = \langle \lambda, \alpha \rangle$. As a B-module, $H^0(s_\alpha, \lambda)$ has a composition series*

$$0 \subseteq V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_0 = H^0(s_\alpha, \lambda)$$

such that $V_i/V_{i+1} \simeq \mathbb{C}_{\lambda - i\alpha}$ for $i = 0, 1, \ldots, n-1$ and $V_n = \mathbb{C}_{s_\alpha(\lambda)}$.

We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half sum of positive roots. As a consequence of exact sequences of Lemma 2.1, we can prove the following. Let $w \in W$, α be a simple root, and set $v = ws_\alpha$.

Lemma 2.2. *If $l(w) = l(v) + 1$, then we have*

(1) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(v, H^0(s_\alpha, \lambda))$ for all $j \geq 0$.*

(2) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*

(3) *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*

(4) *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.*

The following consequence of Lemma 2.2 will be used to compute the cohomology modules in this paper. Now onwards we will denote the Levi subgroup of $P_\alpha\,(\alpha \in S)$ containing T by L_α and the subgroup $L_\alpha \cap B$ by B_α.
Let $\pi\colon \tilde{G} \longrightarrow G$ be the universal cover. Let $\tilde{L}_\alpha$ (respectively, $\tilde{B_\alpha}$) be the inverse image of L_α (respectively, B_α).

Lemma 2.3. *Let V be an irreducible L_α-module. Let λ be a character of B_α. Then we have*

(1) *As L_α-modules, $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$.*

(2) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic as an L_α-module to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$. Further, we have for every $j \geq 1$: $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$.*

(3) *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$, and $H^1(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda})$.*

(4) *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 0$.*

Proof. (1) By [13, Proposition 4.8, p. 53, I] and [13, Proposition 5.12, p. 77, I], for all $j \geq 0$, we have the following isomorphism of L_α-modules:

$$H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda).$$

Proofs of (2), (3) and (4) follow from Lemma 2.2 by taking $w = s_\alpha$ and the fact that $L_\alpha/B_\alpha \simeq P_\alpha/B$. ∎

Recall the structure of indecomposable modules over B_α and $\widetilde{B}_\alpha$ (see [1, Corollary 9.1, p. 130]).

Lemma 2.4. (1) *Any finite dimensional indecomposable $\widetilde{B}_\alpha$-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.*

(2) *Any finite dimensional indecomposable B_α-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.*

Now onwards we will assume that $G = PSO(2n+1, \mathbb{C})$ $(n \geq 3)$. Note that longest element w_0 of the Weyl group W of G is equal to $-\mathrm{id}$. We recall the following Proposition from [15, Proposition 1.3, p. 858].

Proposition 2.5. *Let $c \in W$ be a Coxeter element, let ω_i be the fundamental weight corresponding to the simple root α_i. Then there exists a least positive integer $h(i, c)$ such that $c^{h(i,c)}(\omega_i) = w_0(\omega_i)$.*

Lemma 2.6. *Let $c \in W$ be a Coxeter element. Then we have*

(1) $w_0 = c^n$.

(2) *For any sequence $\underline{i}^r$ $(1 \leq r \leq n)$ of reduced expressions of c; the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^r)$ is a reduced expression of w_0.*

Proof. Note that for $n \geq 3$ there is an isomorphism of Weyl group of B_n and Weyl group of C_n sending $s_i \mapsto s_i$ for $1 \leq i \leq n$. Proof of the lemma holds in the case of type C_n for $(n \geq 3)$ (see [4, Lemma 4.2, p. 441]). Therefore lemma holds for type B_n $(n \geq 3)$. ∎

Lemma 2.7. *Let $n \geq a_1 > a_2 > \cdots > a_{r-1} > a_r \geq 1$ be a decreasing sequence of integers. Then,*

$$w = \left(\prod_{j=a_1}^{n} s_j\right)\left(\prod_{j=a_2}^{n} s_j\right)\cdots\left(\prod_{j=a_{r-1}}^{n} s_j\right)\left(\prod_{j=a_r}^{n-1} s_j\right)$$

is a reduced expression of w.

Proof. Note that for $n \geq 3$ there is an isomorphism of Weyl group of B_n and Weyl group of C_n sending $s_i \mapsto s_i$ for $1 \leq i \leq n$. Proof of the lemma holds in the case of type C_n for $n \geq 3$ (see [4, Lemma 4.3,p. 441]). Therefore lemma holds for type B_n $(n \geq 3)$. ∎

Let c be a Coxeter element in W. We take a reduced expression $c = [a_1, n][a_2, a_1-1] \cdots [a_k, a_{k-1}-1]$, where $[i, j] = s_i s_{i+1} \cdots s_j$ for $i \leq j$ and $n \geq a_1 > a_2 > \cdots > a_k = 1$. Then we have following.

Lemma 2.8. (1) *For all $1 \leq i \leq k-1$,*

$$c^i = \left(\prod_{l_1=1}^{i}[a_{l_1}, n]\right)\left(\prod_{l_2=i+1}^{k}[a_{l_2}, a_{l_2-i}-1]\right)\left(\prod_{l_3=1}^{i-1}[a_{l_k}, a_{k-i+l_3}-1]\right).$$

(2) *For all $k \leq j \leq n$,*

$$c^j = \left(\prod_{l_1=1}^{k-1}[a_{l_1}, n]\right)\left([a_k, n]^{j+1-k}\right)\left(\prod_{l_2=1}^{k-1}[a_k, a_{l_2}-1]\right).$$

(3) *The expressions of c^i for $1 \leq i \leq n$ as in* (1) *and* (2) *are reduced.*

Proof. Note that for $n \geq 3$, there is an isomorphism of Weyl group of B_n and Weyl group of C_n sending $s_i \mapsto s_i$ for $1 \leq i \leq n$. Proof of the lemma holds in the case of type C_n for $n \geq 3$ (see [4, Lemma 4.4, p. 442]). Therefore lemma holds for type B_n $(n \geq 3)$. ∎

3. Cohomology modules $H^1(w, \alpha_j)$ where $j \neq n-1$ and $w \in W$

In this section, we prove that $H^1(w, \alpha_j) = 0$ for every $w \in W$ and $j \neq n-1$.

Lemma 3.1. *Let $v \in W$ and $\alpha \in S$. Then $H^1(s_j, H^0(v, \alpha)) = 0$ for $j \neq n$.*

Proof. By [14, Corollary 5.6, p. 778] we have $H^1(w, \alpha_n) = 0$. Therefore, we may assume that α is a long simple root. If $H^1(s_j, H^0(v, \alpha))_\mu \neq 0$, then there exists an indecomposable $\tilde{L}_{\alpha_j}$-summand V of $H^0(v, \alpha)$ such that $H^1(s_j, V)_\mu \neq 0$. By Lemma 2.4, we have $V \simeq V' \otimes \mathbb{C}_\lambda$ for some character λ of $\tilde{B}_{\alpha_j}$ and for some irreducible $\tilde{L}_{\alpha_j}$-module V'. Since $H^1(s_j, V)_\mu \neq 0$ from Lemma 2.3(3) we have $\langle \lambda, \alpha_j \rangle \leq -2$. Since α is a long root, there exists $w \in W$ such that $w(\alpha) = \alpha_0$. Thus $H^0(v, \alpha) \subseteq H^0(vw, \alpha_0)$. Again, since α_0 is highest long root, $H^0(w_0, \alpha_0) = \mathfrak{g} \longrightarrow H^0(vw, \alpha_0)$ is surjective. Let μ' be the lowest weight of V. Then by the above argument μ' is a root. Therefore we have $\mu' = \mu_1 + \lambda$, where μ_1 is the lowest weight of V'. Hence, we have $\langle \mu', \alpha_j \rangle \leq -2$. Since α_j is a long root and μ' is a root, we have $\langle \mu', \alpha_j \rangle = -1, 0, 1$. This is a contradiction. Thus we have $H^1(s_j, H^0(v, \alpha))_\mu = 0$. ■

Lemma 3.2. *Let $v \in W$ and $\alpha_j \in S$ be such that $j \neq n-1$. Then we have $H^1(s_k, H^0(v, \alpha_j)) = 0$, for every $k = 1, 2, \ldots, n$.*

Proof. **Step 1:** $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$.

Case 1: Assume that $j = n$, choose an element $u \in W$ of minimal length such that $u^{-1}(\alpha_n) = \beta_0$, the highest short root. Then we have $H^0(v, \alpha_j) \subseteq H^0(vu, \beta_0)$. Since β_0 is dominant weight the natural restriction map

$$H^0(w_0, \beta_0) \longrightarrow H^0(vu, \beta_0)$$

is surjective. Hence $H^0(v, \alpha_j)_\mu \neq 0$ implies either $\mu = 0$ or μ is a short root. Therefore, we have $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$.

Case 2: Assume that $1 \leq j \leq n-2$. Note that if $H^0(v, \alpha_j)_\mu \neq 0$ then either $\mu = \alpha_j, 0$ or $\mu \leq -\alpha_j$ (see [4, Corollary 4.5, p. 678]). Hence $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$.

Step 2: If $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$ for some $1 \leq i \leq n-2$, then $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$.

If $v = \mathrm{id}$, we are done. So choose $1 \leq t \leq n$ such that $l(s_t v) = l(v) - 1$. Let $v' = s_t v$. Then $H^0(v, \alpha_j) = H^0(s_t, H^0(v', \alpha_j))$.

Case 1: Assume that $t = n$.

In this case $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = -2$. If $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$, then there is an indecomposable B_{α_n}-summand V of $H^0(v', \alpha_j)$ with highest weight $-\beta_i$. Since $\langle -\beta_i, \alpha_n \rangle = 2$, we have $H^0(s_t, V)_{-(\beta_i+\alpha_n)} \neq 0$. Therefore $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$.

Case 2: Assume that $t = n-1$.

In this case $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = 1$. If $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$, then there is an indecomposable B_{α_n}-summand V of $H^0(v', \alpha_j)$ with highest weight $-(\beta_i + 2\alpha_n)$. Thus by induction hypothesis we have $H^0(v', \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$. Since $\langle -(\beta_i + \alpha_n), \alpha_t \rangle = 0$, we have $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$.

Case 3: Assume that $1 \leq t \leq n-2$.

In this case $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = -1, 0$ or 1.
Assume that $i = t$. Then we have $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = -1$. If $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$, then there is an indecomposable B_{α_t}-summand V of $H^0(v', \alpha_j)$ with highest weight $-(\beta_{i+1} + 2\alpha_n)$. Therefore we have $H^0(v', \alpha_j)_{-(\beta_{i+1}+2\alpha_n)} \neq 0$. It is clear from Step 1 that $t + 1 \leq n - 2$. Therefore by induction $H^0(v', \alpha_j)_{-(\beta_{t+1}+\alpha_n)} \neq 0$. Since $\langle -(\beta_{t+1} + \alpha_n), \alpha_t \rangle = 1$, we have $H^0(v, \alpha_j)_{-(\beta_t+\alpha_n)} \neq 0$.
Assume that $1 \leq t \leq i - 2$ or $i + 1 \leq t \leq n - 2$. Then we get directly $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = 0$. Thus $H^0(v, \alpha_j)_{-(\beta_t+2\alpha_n)} = H^0(v', \alpha_j)_{-(\beta_t+2\alpha_n)} \neq 0$. Therefore by induction $H^0(v, \alpha_j)_{-(\beta_t+\alpha_n)} \neq 0$. Since $\langle -(\beta_i + \alpha_n), \alpha_t \rangle = 0$, we have $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$.
Assume that $i = t + 1$. Since $\langle -(\beta_i + \alpha_n), \alpha_t \rangle = 1$, then there is an indecomposable B_{α_t}-summand V of $H^0(v', \alpha_j)$ such that: $V = \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_{i-1}+2\alpha_n)}$ or $V = \mathbb{C}_{-(\beta_i+2\alpha_n)}$. Then we have $H^0(v', \alpha_j)_{-(\beta_t+2\alpha_n)} \neq 0$. Therefore by induction $H^0(v', \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$. Since $\langle -(\beta_t + \alpha_n), \alpha_t \rangle = 1$, we have $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$. Hence the proof of Step 2.

Proof of the Lemma:

Case 1: Assume that $k \neq n$. Then by Lemma 3.1 we have $H^1(s_k, H^0(v, \alpha_j)) = 0$.

Case 2: Assume that $k = n$. By Step 1 we see that $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$. Note that if β is a root such that $H^0(v, \alpha_j)_\beta \neq 0$ and $\langle \beta, \alpha_n \rangle = -2$, then we have $\beta = -(\beta_i + 2\alpha_n)$ for some $1 \leq i \leq n - 2$.
By Step 2 if $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$ for some $1 \leq i \leq n-2$, then $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$. Therefore $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ is an indecomposable B_{α_n}-summand of $H^0(v, \alpha_j)$. By Lemma 2.4, $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ is isomorphic to $V \otimes \mathbb{C}_{-\omega_n}$, where V is an irreducible $\tilde{L}_{\alpha_n}$-module. Therefore we have $H^1(s_k, \mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}) = 0$ by Lemma 2.3(4). Thus our result follows. ■

Lemma 3.3. *Let w be an element of W and α_j be an element of S such that $j \neq n - 1$. Then $H^1(w, \alpha_j) = 0$.*

Proof. We will prove by induction on length of w. If length of w is 0, then $w = \mathrm{id}$. Thus it follows trivially. Now suppose $w \in W$ such that $l(w) \geq 1$. Then there exists a simple root $\alpha \in S$ such that $l(s_\alpha w) = l(w) - 1$. Then using SES:

$$0 \longrightarrow H^1(s_\alpha, H^0(s_\alpha w, \alpha_j)) \longrightarrow H^1(w, \alpha_j) \longrightarrow H^0(s_\alpha, H^1(s_\alpha w, \alpha_j)) \longrightarrow 0.$$

From the above SES using the induction hypothesis and Lemma 3.2, we finally get $H^1(w, \alpha_j) = 0$ for $j \neq n - 1$. ■

4. Cohomology module H^0 of the relative tangent bundle

In this section we describe the weights of H^0 of the relative tangent bundle.

Notation: Let c be a Coxeter element of W. We consider a reduced expression $c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j] = s_i s_{i+1} \cdots s_j$ for $i \leq j$ and $n \geq a_1 > a_2 > \cdots > a_k = 1$. Let $\beta_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1}$ for all $1 \leq i \leq n - 1$. For $1 \leq r \leq k$, let $n \geq a_1 > a_2 > a_3 > \cdots > a_r \geq 1$ be a decreasing sequence of integers.

Let $$w_r = \left(\prod_{j=a_1}^{n} s_j\right)\left(\prod_{j=a_2}^{n} s_j\right)\left(\prod_{j=a_3}^{n} s_j\right)\cdots\left(\prod_{j=a_{r-1}}^{n} s_j\right)\left(\prod_{j=a_r}^{n-1} s_j\right)$$

and let $$\tau_r = \left(\prod_{j=a_1}^{n} s_j\right)\left(\prod_{j=a_2}^{n} s_j\right)\left(\prod_{j=a_3}^{n} s_j\right)\cdots\left(\prod_{j=a_{r-1}}^{n} s_j\right)\left(\prod_{j=a_r}^{n-2} s_j\right).$$

Note that $l(w_r) = l(\tau_r) + 1$.

Lemma 4.1. *Assume that $r \geq 3$.*

(1) *Let $v = s_{a_{r-1}}s_{a_{r-1}+1}\cdots s_n s_{a_r}s_{a_r+1}\cdots s_{n-1}$. Then we have*

$$H^0(v, \alpha_{n-1}) =$$
$$= \bigoplus_{i=a_r}^{a_{r-1}-1}\Big(\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\Big)$$
$$\oplus \bigoplus_{i=a_{r-1}}^{n-2}\big(\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

(2) *Let $v' = s_1\cdots s_n s_1\cdots s_{n-1}$. Then we have*

$$H^0(v', \alpha_{n-1}) =$$
$$= \bigoplus_{i=1}^{n-2}\big(\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Proof. Proof of (1): Let $u = s_{a_r}s_{a_r+1}\cdots s_{n-1}$. By using SES we have

$$H^0(u, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \Big(\bigoplus_{j=a_r}^{n-1}\mathbb{C}_{-\beta_j}\Big).$$

Since $\langle -\beta_j, \alpha_n\rangle = 2$ for all $a_r \leq j \leq n-1$, by using SES we have

$$H^0(s_n u, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \Big(\bigoplus_{j=a_r}^{n-1}(\mathbb{C}_{-\beta_j} \oplus \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)})\Big). \tag{1}$$

Since $\mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}}$ is an indecomposable $B_{\alpha_{n-1}}$-module (see [5, p. 11] and [14, p. 8]), by Lemma 2.4, we have $\mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}} = V \otimes \mathbb{C}_{-\omega_{n-1}}$, where V is the two dimensional irreducible representation of $\tilde{L}_{\alpha_{n-1}}$. Therefore by Lemma 2.3(4) we have $H^0(\tilde{L}_{\alpha_{n-1}}/\tilde{B}_{\alpha_{n-1}}, \mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}}) = 0$, and, for all $a_r \leq j \leq n-2$,

$$\langle -(\beta_{n-1}+\alpha_n), \beta_{n-1}\rangle = -1, \qquad \langle -(\beta_{n-1}+2\alpha_n), \beta_{n-1}\rangle = 0,$$
$$\langle -\beta_j, \beta_{n-1}\rangle = -1, \quad \langle -(\beta_j+\alpha_n), \beta_{n-1}\rangle = 0 \quad \langle -(\beta_j+2\alpha_n), \beta_{n-1}\rangle = 1 \tag{2}$$

By (2), we have $$H^0(s_{n-1}s_n u, \alpha_{n-1}) =$$
$$= \Big(\bigoplus_{j=a_r}^{n-2}\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\Big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \tag{3}$$

Claim: For $a_{r-1} \leq k \leq n-2$,

$$H^0(s_k s_{k+1} \cdots s_n u, \alpha_{n-1}) =$$
$$= \bigoplus_{j=a_r}^{k-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_k)}\big) \oplus$$
$$\oplus \bigoplus_{j=k}^{n-2} \big(\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Proof of the claim by descending induction on k. By hypothesis we have

$$H^0(s_{k+1} \cdots s_n u, \alpha_{n-1}) = \tag{4}$$
$$= \bigoplus_{j=a_r}^{k} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})}\big) \oplus \tag{5}$$
$$\oplus \bigoplus_{j=k+1}^{n-2} \big(\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \tag{6}$$

Let $V = \bigoplus_{j=a_r}^{k-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})}\big)$

and $V' = \bigoplus_{j=k+2}^{n-2} \big(\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}$.

Then the roots

$$\{(\beta_j+\alpha_n), (\beta_j+2\alpha_n), (\beta_j+2\alpha_n+\beta_{n-1}), \ldots, (\beta_j+2\alpha_n+\beta_{k+2}) : a_r \leq j \leq k-1\},$$
$$\{(\beta_j+2\alpha_n), (\beta_j+2\alpha_n+\beta_{n-1}), \ldots, (\beta_j+2\alpha_n+\beta_{j+1}) : k+2 \leq j \leq n-2\}$$

and $-(\beta_{n-1}+2\alpha_n)$ are orthogonal to α_k. Therefore V, V' are direct sums of irreducible $\tilde{L}_{\alpha_k}$-modules. Lemma 2.3(2) implies

$$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, V) = V, \quad H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, V') = V', \quad \text{and}$$
$$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}) = \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Further, the remaining roots of (4), (5), and (6) are

$$\{-(\beta_k+\alpha_n), -(\beta_k+2\alpha_n), -(\beta_k+2\alpha_n+\beta_{n-1}), ..., -(\beta_k+2\alpha_n+\beta_{k+2}),$$
$$-(\beta_k+2\alpha_n+\beta_{k+1})\}, \{-(\beta_{k+1}+2\alpha_n), -(\beta_{k+1}+2\alpha_n+\beta_{n-1}), ..., -(\beta_k+2\alpha_n+\beta_{k+2})\}$$

and $\{-(\beta_j+2\alpha_n+\beta_{k+1}) : a_r \leq j \leq k\}$. Since $\langle -(\beta_k+\alpha_n), \alpha_k \rangle = -1$, by Lemma 2.3(4) we have

$$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, \mathbb{C}_{-(\beta_k+\alpha_n)}) = 0.$$

Since $\mathbb{C}_{-(\beta_k+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_{k+1}+2\alpha_n)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_k}$-module, by Lemma 2.3(2) we have

$$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, \mathbb{C}_{-(\beta_k+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_{k+1}+2\alpha_n)}) = \mathbb{C}_{-(\beta_k+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_{k+1}+2\alpha_n)}.$$

Similarly for each $k+2 \leq j \leq n-1$, $\mathbb{C}_{-(\beta_k+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_{k+1}+2\alpha_n+\beta_j)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_k}$-module. Therefore by Lemma 2.3(2) we have

$$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, \mathbb{C}_{-(\beta_k+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_{k+1}+2\alpha_n+\beta_j)}) = \mathbb{C}_{-(\beta_k+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_{k+1}+2\alpha_n+\beta_j)}$$

for each $k+2 \leq j \leq n-1$. Moreover, $\langle -(\beta_j + 2\alpha_n + \beta_{k+1}), \alpha_k \rangle = 1$ for all $a_r \leq j \leq k-1$. Therefore by Lemma 2.3(2) we have

$$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})}) = \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_k)}$$

for all $a_r \leq j \leq k-1$. Since $\langle -(\beta_k + 2\alpha_n + \beta_{k+1}), \alpha_k \rangle = 0$, Lemma 2.3(2) implies

$$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, \mathbb{C}_{-(\beta_k+2\alpha_n+\beta_{k+1})}) = \mathbb{C}_{-(\beta_k+2\alpha_n+\beta_{k+1})}.$$

From the above discussion, we have $H^0(s_k s_{k+1} \cdots s_n u, \alpha_{n-1}) =$

$$= \bigoplus_{j=a_r}^{k-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_k)}\big) \oplus$$
$$\oplus \bigoplus_{j=k}^{n-2} \big(\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Therefore the claim follows. The claim implies

$$H^0(v, \alpha_{n-1}) = H^0(s_{a_{r-1}} s_{a_r+1} \cdots s_n v'_r, \alpha_{n-1}) =$$
$$= \bigoplus_{j=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus$$
$$\oplus \bigoplus_{j=a_{r-1}}^{n-2} \big(\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Proof of (2): By (1) we have

$$H^0(s_2 s_3 \cdots s_n s_1 s_2 \cdots s_{n-1}, \alpha_{n-1}) = \tag{7}$$
$$= \mathbb{C}_{-(\beta_1+\alpha_n)} \oplus \mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_2)} \oplus \tag{8}$$
$$\oplus \bigoplus_{j=2}^{n-2} \big(\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \tag{9}$$

The roots of (7), (8), and (9), which are not orthogonal to α_1 are

$$\{-(\beta_1+\alpha_n),\ -(\beta_1+2\alpha_n),\ -(\beta_1+2\alpha_n+\beta_{n-1}),\ \ldots,\ -(\beta_1+2\alpha_n+\beta_3)\},$$
$$\{-(\beta_2+2\alpha_n),\ -(\beta_2+2\alpha_n+\beta_{n-1}),\ \ldots,\ -(\beta_2+2\alpha_n+\beta_3)\}$$

Since $-(\beta_{n-1}+2\alpha_n)$ is orthogonal to α_1, Lemma 2.3(2) implies

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}) = \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Since $\langle -(\beta_1+\alpha_n), \alpha_1 \rangle = -1$, by Lemma 2.3(4) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+\alpha_n)}) = 0.$$

Since $\mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_1}$-module, by Lemma 2.3(2) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n)}) = \mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n)}.$$

Similarly, for each $3 \le j \le n-1$, $\mathbb{C}_{-(\beta_1+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n+\beta_j)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_1}$-module. Therefore by Lemma 2.3(2) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n+\beta_j)}) = \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n+\beta_j)})$$

for each $3 \le j \le n-1$. Since $\langle -(\beta_1 + 2\alpha_n + \beta_2), \alpha_1 \rangle = 0$, Lemma 2.3(2) implies

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_2)}) = \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_2)}.$$

From the above discussion, we obtain

$$H^0(v', \alpha_{n-1}) = \\ = \bigoplus_{i=1}^{n-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \quad \blacksquare$$

Lemma 4.2. *Let $3 \le r \le k$ and let $v = s_{a_{r-1}} s_{a_{r-1}+1} \cdots s_n s_{a_r} s_{a_r+1} \cdots s_{n-1}$. Then*

(1) $H^0(s_{a_{r-2}} \cdots s_n v, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus$
$\oplus \bigoplus_{i=a_{r-1}}^{a_{r-2}-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big)$.

(2) $H^0(w_r, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus$
$\oplus \bigoplus_{i=a_{r-1}}^{a_{r-2}-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big)$.

(3) *Let $u_1 = (s_{a_1} \cdots s_n)(s_{a_2} \cdots s_n) \cdots (s_{a_{k-1}} \cdots s_n) v'$, where v' is defined as in Lemma 4.1. Then we have*

$$H^0(u_1, \alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{k-1}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big).$$

Proof. (1): Since $r \ge 3$, we have $a_{r-1} < n$. By Lemma 4.1(1), we have

$$H^0(v, \alpha_{n-1}) = \\ = \bigoplus_{i=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus \\ \oplus \bigoplus_{i=a_{r-1}}^{n-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Since $\{-(\beta_i+2\alpha_n+\beta_{n-1}), \dots, -(\beta_i+2\alpha_n+\beta_{a_{r-1}}) : a_r \le i \le a_{r-1}-1\}$ are orthogonal to α_n, by Lemma 2.3(2), we have for all $a_r \le i \le a_{r-1}-1$ and $a_{r-1} \le t \le n-1$

$$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_t)}) = \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_t)}.$$

Since $\{-(\beta_i+2\alpha_n+\beta_{n-1}), \dots, -(\beta_i+2\alpha_n+\beta_{i+1}) : a_{r-1} \le i \le n-2\}$ are orthogonal to α_n, by Lemma 2.3(2) we have

$$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}) = \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}$$

for all $i+1 \le l \le n-1$, where $a_{r-1} \le i \le n-2$. Since $\langle -(\beta_i + 2\alpha_n), \alpha_n \rangle = -2$ for all $a_r \le i \le n-1$, by Lemma 2.3(3) we have $H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}) = 0$ for all $a_{r-1} \le i \le n-1$.

Moreover, for each $a_r \le i \le a_{r-1}-1$, $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ is an indecomposable two dimensional B_{α_n}-module. Therefore by Lemma 2.4, we have $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} = V_i \otimes \mathbb{C}_{-\omega_n}$, where V_i is the irreducible two dimensional representation of $\tilde{L}_{\alpha_n}$. By Lemma 2.3(4) we have

$$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}) = 0$$

for each $a_r \le i \le a_{r-1}-1$. From the above discussion, we have

$$H^0(s_n v_r, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus$$
$$\oplus \bigoplus_{i=a_{r-1}}^{n-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big).$$

Since $\langle -(\beta_i + 2\alpha_n + \beta_{n-1}), \alpha_{n-1} \rangle = -1$ for each $a_r \le i \le n-2$, by Lemma 2.3(4) we have $H^0(\tilde{L}_{\alpha_{n-1}}/\tilde{B}_{\alpha_{n-1}}, \mathbb{C}_{-(\beta_i+\alpha_n+\beta_{n-1})}) = 0$ for each $a_r \le i \le n-2$.
Moreover, $\{-(\beta_i + 2\alpha_n + \beta_{n-2}), \ldots, -(\beta_i + 2\alpha_n + \beta_{a_{r-1}}) : a_r \le i \le a_{r-1}-1\}$, $\{-(\beta_i + 2\alpha_n + \beta_{n-2}), \ldots, -(\beta_i + 2\alpha_n + \beta_{i+1}) : a_{r-1} \le i \le n-3\}$, and $\beta_{n-1} + 2\alpha_n$ are orthogonal to α_{n-1}. Therefore we have

$$H^0(s_{n-1}s_n v, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-2})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus$$
$$\oplus \bigoplus_{i=a_{r-1}}^{n-3} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-2})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big).$$

Proceeding recursively, we have

$$H^0(s_{a_{r-2}} s_{a_{r-2}+1} \cdots s_n v, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus$$
$$\oplus \bigoplus_{i=a_{r-1}}^{a_{r-2}-2} (\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Proof of (2): Since $\{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1}), \ldots, -(\beta_i+2\alpha_n+\beta_{a_{r-1}}) : a_r \le i \le a_{r-1}-1\}$, $\{-(\beta_i + 2\alpha_n + \beta_{a_{r-2}-1}), \ldots, -(\beta_i + 2\alpha_n + \beta_{i+1}) : a_{r-1} \le i \le a_{r-2}-2\}$ are orthogonal to α_j for all $a_{r-3} \le j \le n$, by Lemma 2.3(2) we have

$$H^0(s_{a_{r-3}} \cdots s_n s_{a_{r-2}} \cdots s_n v, \alpha_{n-1}) = H^0(s_{a_{r-2}} \cdots s_n v, \alpha_{n-1}).$$

Proceeding recursively we have

$$H^0(w_r, \alpha_{n-1}) = H^0(s_{a_{r-2}} \cdots s_n v, \alpha_{n-1}) =$$
$$= \bigoplus_{i=a_r}^{a_{r-1}-1} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big) \oplus$$
$$\oplus \bigoplus_{i=a_{r-1}}^{a_{r-2}-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big).$$

Proof of (3): By Lemma 4.1(2) we have

$$H^0(v', \alpha_{n-1}) =$$
$$= \bigoplus_{i=1}^{n-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Since $\{-(\beta_i + 2\alpha_n + \beta_{n-1}), \ldots, -(\beta_i + 2\alpha_n + \beta_{i+1}) : 1 \leq i \leq n-2\}$ are orthogonal to α_n, by Lemma 2.3(2) we have for all $i+1 \leq l \leq n-1$, where $1 \leq i \leq n-2$,

$$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}) = \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}.$$

Since $\langle -(\beta_i + 2\alpha_n), \alpha_n \rangle = -2$ for all $1 \leq i \leq n-1$, by Lemma 2.3(3) we have $H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}) = 0$ for all $1 \leq i \leq n-1$. Consequently

$$H^0(s_n v', \alpha_{n-1}) = \bigoplus_{i=1}^{n-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big).$$

Since $\langle -(\beta_i + 2\alpha_n + \beta_{n-1}), \alpha_{n-1} \rangle = -1$ for each $1 \leq i \leq n-2$, by Lemma 2.3(4) we have $H^0(\tilde{L}_{\alpha_{n-1}}/\tilde{B}_{\alpha_{n-1}}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})}) = 0$ for each $1 \leq i \leq n-2$. Moreover, $\{-(\beta_i + 2\alpha_n + \beta_{n-2}), \ldots, -(\beta_i + 2\alpha_n + \beta_{i+1}) : 1 \leq i \leq n-3\}$, and $\beta_{n-1} + 2\alpha_n$ are orthogonal to α_{n-1}. Therefore we have

$$H^0(s_{n-1}s_n v', \alpha_{n-1}) = \bigoplus_{i=1}^{n-3} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-2})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big).$$

Proceeding recursively we have

$$H^0(s_{a_{k-1}}s_{a_{k-1}+1} \cdots s_n v', \alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{k-1}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big).$$

Since $\{-(\beta_i+2\alpha_n+\beta_{a_{k-2}-1}), \ldots, -(\beta_i+2\alpha_n+\beta_{i+1}) : 1 \leq i \leq a_{k-1}-2\}$ are orthogonal to α_j for all $a_{k-2} \leq j \leq n$, we have

$$H^0(s_{a_{k-2}} \cdots s_n s_{a_{k-1}} \cdots s_n v', \alpha_{n-1}) = H^0(s_{a_{k-1}} \cdots s_n v', \alpha_{n-1}).$$

Proceeding recursively we have

$$H^0(u_1, \alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-2} \big(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{k-1}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big). \qquad \blacksquare$$

Lemma 4.3. *Let $3 \leq r \leq k$. Then $H^0(w_{r-2}s_n s_{a_{r-1}} s_{a_{r-1}+1} \cdots s_{n+2-r}, \alpha_{n+2-r})_\mu \neq 0$ if μ is of the form $\mu = -(\beta_j + \alpha_n)$ for some $a_{r-1} \leq j \leq a_{r-2} - 1$.*

Proof. By applying SES repeatedly, it is easy to see that

$$H^0(s_n s_{a_{r-1}} s_{a_{r-1}+1} \cdots s_{n+2-r}, \alpha_{n+2-r}) = \mathbb{C}h(\alpha_{n+2-r}) \oplus \Big(\bigoplus_{j=a_{r-1}}^{n+2-r} \mathbb{C}_{-\gamma_{j,n+2-r}}\Big),$$

where $\gamma_{j,j'} = (\alpha_j + \cdots + \alpha_{j'})$ for $j' \geq j$. Let $V_1 = H^0(s_n s_{a_{r-1}} s_{a_{r-1}+1} \cdots s_{n+2-r}, \alpha_{n+2-r})$. We next calculate the space $H^0(s_{a_{r-2}} \cdots s_{n-1}, V_1)$. Since $n \geq a_1 > a_2 > \cdots > a_k = 1$,

we have $a_i \leq n+1-i$ for all $1 \leq i \leq k$. Assume $l \geq n+4-r$, then $\langle \gamma_{j,n+2-r}, \alpha_l \rangle = 0$ for all $a_{r-1} \leq j \leq n+2-r$. By Lemma 2.3(2) we have $H^0(\tilde{L}_{\alpha_l}/\tilde{B}_{\alpha_l}, V_1) = V_1$ for all $l \geq n+4-r$. Therefore, $H^0(s_{a_{r-2}} \cdots s_{n-1}, V_1) = H^0(s_{a_{r-2}} \cdots s_{n+2-r} s_{n+3-r}, V_1)$.
Note that, since $\langle -\gamma_{j,n+2-r}, \alpha_{n+3-r} \rangle = 1$ for all $a_{r-1} \leq j \leq n+2-r$, by Lemma 2.3(2) we have

$$H^0(s_{n+3-r}, V_1) = \mathbb{C}h(\alpha_{n+2-r}) \oplus \Big(\bigoplus_{j=a_{r-1}}^{n+2-r} \big(\mathbb{C}_{-\gamma_{j,n+2-r}} \oplus \mathbb{C}_{-\gamma_{j,n+3-r}} \big) \Big).$$

Since $\mathbb{C}h(\alpha_{n+2-r}) \oplus \mathbb{C}_{-\gamma_{n+2-r,n+2-r}}$ is an indecomposable two dimensional $B_{\alpha_{n+2-r}}$-module, by Lemma 2.4, $\mathbb{C}h(\alpha_{n+2-r}) \oplus \mathbb{C}_{-\gamma_{n+2-r,n+2-r}} = V \otimes \mathbb{C}_{-\omega_{n+2-r}}$, where V is the irreducible two dimensional $\tilde{L}_{\alpha_{n+2-r}}$-module.
Since $\langle \gamma_{j,n+2-r}, \alpha_{n+2-r} \rangle = -1$, for all $a_{r-1} \leq j \leq n+1-r$, by Lemma 2.3(4) we have $H^0(\tilde{L}_{\alpha_{n+2-r}}/\tilde{B}_{\alpha_{n+2-r}}, \mathbb{C}_{\gamma_{j,n+2-r}}) = 0$ for all $a_{r-1} \leq j \leq n+1-r$. Hence

$$H^0(s_{n+2-r}, H^0(s_{n+3-r}, V_1)) = \bigoplus_{j=a_{r-1}}^{n+1-r} \mathbb{C}_{-\gamma_{j,n+3-r}}.$$

Since $\langle \gamma_{n+1-r,n+3-r}, \alpha_{n+1-r} \rangle = -1$, by Lemma 2.3(4), we have

$$H^0(\tilde{L}_{\alpha_{n+1-r}}/\tilde{B}_{\alpha_{n+1-r}}, \mathbb{C}_{\gamma_{n+1-r,n+3-r}}) = 0.$$

Moreover, $\langle \gamma_{j,n+3-r}, \alpha_{n+1-r} \rangle = 0$ for all $a_{r-1} \leq j \leq n-r$. Therefore we have

$$H^0(s_{n+1-r}, H^0(s_{n+2-r}, H^0(s_{n+3-r}, V_1))) = \bigoplus_{j=a_{r-1}}^{n-r} \mathbb{C}_{-\gamma_{j,n+3-r}}.$$

Proceeding recursively we have

$$H^0(s_{a_{r-2}} ... s_{n-1}, V_1) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+3-r}}, \quad \text{and} \quad H^0(s_n s_{a_{r-2}} ... s_{n-1}, V_1) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+3-r}}.$$

Let $V_2 = H^0(s_n s_{a_{r-2}} \cdots s_{n-1}, V_1)$. Similarly, we have

$$H^0(s_{a_{r-3}} ... s_{n-1}, V_2) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+4-r}} \quad \text{and} \quad H^0(s_n s_{a_{r-3}} ... s_{n-1}, V_2) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+4-r}}.$$

Proceeding recursively we have

$$V_{r-2} = H^0(s_{a_2} \cdots s_{n-1}, V_{r-3}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n-1}},$$

where $V_{r-3} = H^0((s_n s_3 \cdots s_n) \cdots (s_{a_{r-2}} \cdots s_{n+2-r}), \alpha_{n+2-r})$. Note that $\gamma_{j,n-1} = \beta_j$. Since $\langle -\alpha_{n-1}, \alpha_n \rangle = 2$, by Lemma 2.3(2) and Lemma 2.4, we have

$$H^0(s_n, V_{r-2}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \big(\mathbb{C}_{-\beta_j} \oplus \mathbb{C}_{-(\beta_j + \alpha_n)} \oplus \mathbb{C}_{-(\beta_j + 2\alpha_n)} \big).$$

Moreover, $\langle -\beta_j, \alpha_{n-1}\rangle = -1$, $\langle -(\beta_j+\alpha_n), \alpha_{n-1}\rangle = 0$ and $\langle -(\beta_j+2\alpha_n), \alpha_{n-1}\rangle = 1$ for all $a_{r-1} \le j \le a_{r-2}-1$. Therefore by Lemma 2.3(2), Lemma 2.3(4) and Lemma 2.4 we have

$$H^0(s_{n-1}, H^0(s_n, V_{r-2})) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\big).$$

Proceeding recursively we have

$$H^0(s_{a_1}, H^0(s_{a_1+1}\cdots s_n, V_{r-2})) =$$
$$= \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{a_1})}\big).$$

Hence the proof of the lemma follows. ∎

Lemma 4.4. *Let $3 \le r \le k$. Then $H^0(w_{r-1}s_n, \alpha_n)_\mu \neq 0$ if and only if μ is of the form $\mu = -(\beta_j+\alpha_n)$, for some $a_{r-1} \le j \le a_{r-2}-1$.*

Proof. By applying SES repeatedly, it is easy to see that

$$H^0(s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{n-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Let $V_1 = H^0(s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n)$. Since $\langle -(\beta_j+\alpha_n), \alpha_n\rangle = 0$, we get $H^0(s_n, V_1) = V_1$. Since $\langle -(\beta_{n-1}+\alpha_n), \alpha_{n-1}\rangle = -1$ and $\langle -(\beta_j+\alpha_n, \alpha_{n-1})\rangle = 0$, for all $a_{r-1} \le j \le n-2$, by Lemma 2.3(2), Lemma 2.3(4) and Lemma 2.4 we have

$$H^0(s_{n-1}, V_1) = \bigoplus_{j=a_{r-1}}^{n-2} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^0(s_{a_{r-2}}\cdots s_{n-1}s_n, V_1) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $n \ge a_1 > a_2 > \cdots > a_k = 1$, we see that $\langle -(\beta_j+\alpha_n), \alpha_t\rangle = 0$, for all $a_{r-1} \le j \le a_{r-2}-1$ and for all $a_{r-3} \le t \le n$, therefore by Lemma 2.3(2) and Lemma 2.4, we have

$$H^0(s_{a_{r-3}}\cdots s_n s_{a_{r-2}}\cdots s_n s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $n \ge a_1 > a_2 > \cdots > a_k = 1$, we see that $\langle -(\beta_j+\alpha_n), \alpha_t\rangle = 0$ for all $a_{r-1} \le j \le a_{r-2}-1$ and for all $a_{r-4} \le t \le n$. By Lemma 2.3(2) and Lemma 2.4, we have

$$H^0(s_{a_{r-4}}\cdots s_n s_{a_{r-3}}\cdots s_n s_{a_{r-2}}\cdots s_n s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^0(w_{r-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Hence the lemma follows. ∎

Lemma 4.5. *If μ is of the form $\mu = -(\beta_j + \alpha_n)$ for some $1 \leq j \leq a_{k-1} - 1$, then we have $H^0(w_{k-1}s_n s_1 s_2 \cdots s_{n+1-k}, \alpha_{n+1-k})_\mu \neq 0$.*

Proof. By applying SES repeatedly, it is easy to see that

$$H^0(s_n s_{a_1} s_2 \cdots s_{n+1-k}, \alpha_{n+1-k}) = \mathbb{C}h(\alpha_{n+1-k}) \oplus \Big(\bigoplus_{j=1}^{n+1-k} \mathbb{C}_{-\gamma_{j,n+1-k}} \Big),$$

where $\gamma_{j,j'} = (\alpha_j + \cdots + \alpha_{j'})$ for $j' \geq j$. Let $V_1 = H^0(s_n s_1 s_2 \cdots s_{n+1-k}, \alpha_{n+1-k})$. We next calculate $H^0(s_{a_{k-1}} \cdots s_{n-1}, V_1)$. Since $n \geq a_1 > a_2 > \cdots > a_k = 1$, we have $a_i \leq n+1-i$ for all $1 \leq i \leq k$. Moreover, $\langle \gamma_{j,n+1-k}, \alpha_l \rangle = 0$ for all $1 \leq j \leq n+1-k$ and for all $l \geq n+3-k$. Therefore by using Lemma 2.3(2) and Lemma 2.4, we have

$$H^0(s_{a_{k-1}} \cdots s_{n-1}, V_1) = H^0(s_{a_{k-1}} \cdots s_{n+1-k} s_{n+2-k}, V_1).$$

Since $\langle -\gamma_{j,n+1-k}, \alpha_{n+2-k} \rangle = 1$ for all $1 \leq j \leq n+1-k$, by using Lemma 2.3(2) and Lemma 2.4, we have

$$H^0(s_{n+2-k}, V_1) = \mathbb{C}h(\alpha_{n+1-k}) \oplus \Big(\bigoplus_{j=1}^{n+1-k} (\mathbb{C}_{-\gamma_{j,n+1-k}} \oplus \mathbb{C}_{-\gamma_{j,n+2-k}}) \Big).$$

Since $\mathbb{C}h(\alpha_{n+1-k}) \oplus \mathbb{C}_{-\gamma_{n+1-k,n+1-k}}$ is an indecomposable two dimensional $B_{\alpha_{n+1-k}}$-module, by Lemma 2.4 we have $\mathbb{C}h(\alpha_{n+1-k}) \oplus \mathbb{C}_{-\gamma_{n+1-k,n+1-k}} = V \otimes \mathbb{C}_{-\omega_{n+1-k}}$, where V is the irreducible two dimensional $\tilde{L}_{\alpha_{n+1-k}}$-module. By Lemma 2.3(4) we have

$$H^0(\tilde{L}_{\alpha_{n+1-k}}/\tilde{B}_{\alpha_{n+1-k}}, \mathbb{C}h(\alpha_{n+1-k}) \oplus \mathbb{C}_{-\gamma_{n+1-k,n+1-k}}) = 0.$$

Since $\langle \gamma_{j,n+1-k}, \alpha_{n+1-k} \rangle = -1$ for all $1 \leq j \leq n-k$, by Lemma 2.3(4) we have $H^0(\tilde{L}_{\alpha_{n+1-k}}/\tilde{B}_{\alpha_{n+1-k}}, \mathbb{C}_{\gamma_{j,n+1-k}}) = 0$ for all $1 \leq j \leq n-k$. Therefore

$$H^0(s_{n+1-k}, H^0(s_{n+2-k}, V_1)) = \bigoplus_{j=1}^{n-k} \mathbb{C}_{-\gamma_{j,n+2-k}}.$$

Since $\langle \gamma_{j,n+2-k}, \alpha_{n-k} \rangle = 0$ for all $1 \leq j \leq n-k-1$, and $\langle \gamma_{n-k,n+2-k}, \alpha_{n-k} \rangle = -1$, by using Lemma 2.3(2), Lemma 2.3(2)(4) and Lemma 2.4, we have

$$H^0(s_{n-k}, H^0(s_{n+1-k}, H^0(s_{n+2-k}, V_1))) = \bigoplus_{j=1}^{n-k-1} \mathbb{C}_{-\gamma_{j,n+2-k}}.$$

Proceeding recursively we have

$$H^0(s_n s_{a_{k-1}} \cdots s_{n-1}, V_1) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-\gamma_{j,n+2-k}}.$$

Let $V_2 = H^0(s_n s_{a_{k-1}} \cdots s_{n-1}, V_1)$. Then similarly, we have

$$H^0(s_{a_{k-2}} \cdots s_{n-1}, V_2) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-\gamma_{j,n+3-k}}.$$

Proceeding recursively we have

$$V_{k-1} = H^0(s_{a_2}\cdots s_{n-1}, V_{k-2}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-\gamma_{j,n-1}},$$

where $V_{k-2} = H^0((s_n s_3 \cdots s_n)\cdots(s_{a_{k-1}}\cdots s_{n+1-k}), \alpha_{n+1-k})$. Note that $\gamma_{j,n-1} = \beta_j$. Since $\langle -\alpha_{n-1}, \alpha_n\rangle = 2$, by using Lemma 2.3(2) and Lemma 2.4 we have

$$H^0(s_n, V_{k-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \big(\mathbb{C}_{-\beta_j} \oplus \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)}\big).$$

Moreover, $\langle -\beta_j, \alpha_{n-1}\rangle = -1$, $\langle -(\beta_j+\alpha_n), \alpha_{n-1}\rangle = 0$ and $\langle -(\beta_j+2\alpha_n), \alpha_{n-1}\rangle = 1$, for all $1 \le j \le a_{k-1}-1$. Therefore by using Lemma 2.3(2), Lemma 2.3(4) and Lemma 2.4 we have

$$H^0(s_{n-1}, H^0(s_n, V_{k-1})) = \bigoplus_{j=1}^{a_{k-1}-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\big).$$

Proceeding recursively we have

$$H^0(s_{a_1}, H^0(s_{a_1+1}\cdots s_n, V_{k-1})) =$$
$$= \bigoplus_{j=1}^{a_{k-1}-1} \big(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{a_1})}\big).$$

Hence the proof of the lemma follows. ■

Lemma 4.6. *$H^0(w_k s_n, \alpha_n)_\mu \ne 0$ if and only if μ is of the form $\mu = -(\beta_j+\alpha_n)$ for some $1 \le j \le a_{k-1}-1$.*

Proof. By applying SES repeatedly it is easy to see that

$$H^0(s_1\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=1}^{n-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Let $V_1 = H^0(s_1\cdots s_{n-1}s_n, \alpha_n)$. Since $\langle -(\beta_j+\alpha_n), \alpha_n\rangle = 0$, for all $1 \le j \le n$, by Lemma 2.3(2) and Lemma 2.4 we have $H^0(s_n, V_1) = V_1$.
Moreover, $\langle -(\beta_{n-1}+\alpha_n), \alpha_{n-1}\rangle = -1$ and $\langle -(\beta_j+\alpha_n, \alpha_{n-1})\rangle = 0$ for all $1 \le j \le n-2$. Therefore by using Lemma 2.3(4), Lemma 2.3(2) and Lemma 2.4, we have

$$H^0(s_{n-1}, V_1) = \bigoplus_{j=1}^{n-2} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^0(s_{a_{k-1}}\cdots s_{n-1}s_n, V_1) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $n \ge a_1 > a_2 > \cdots > a_k = 1$, we have $\langle -(\beta_j+\alpha_n), \alpha_t\rangle = 0$ for all $1 \le j \le a_{k-1}-1$ and for all $a_{k-2} \le t \le n$. By using Lemma 2.3(2), Lemma 2.4 we have

$$H^0(s_{a_{k-2}}\cdots s_n s_{a_{k-1}}\cdots s_n s_1\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Similarly, since $n \geq a_1 > a_2 > \cdots > a_k = 1$, we have $\langle -(\beta_j + \alpha_n), \alpha_t \rangle = 0$ for all $1 \leq j \leq a_{k-1} - 1$ and for all $a_{k-3} \leq t \leq n$, therefore by using Lemma 2.3(2) and Lemma 2.4 we have

$$H^0(s_{a_{k-3}} \cdots s_n s_{a_{k-2}} \cdots s_n s_{a_{k-1}} \cdots s_n s_1 \cdots s_{n-1} s_n, \alpha_n) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have $H^0(w_k s_n, \alpha_n) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}$.

Hence the proof of the lemma is complete. ■

5. Cohomology module H^1 of the relative tangent bundle

In this section, we describe the weights of H^1 of the relative tangent bundle. Let $n \geq a_1 > a_2 > \ldots > a_{k-1} > a_k = 1$ be a decreasing sequence of integers such that $k \geq 3$. Fix $3 \leq r \leq k$.

Lemma 5.1. *Let $v_r = s_n s_{a_r} \ldots s_{n-2}$, $v_{r-1} = s_{a_{r-1}} \ldots s_{n-1}$, and $v_{r-2} = s_{a_{r-2}} \ldots s_{n-1} s_n$. Then we have*

(1) $H^1(v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = 0$.

(2) *Assume $w = v_{r-2} v_{r-1} v_r s_{n-1}$. $H^1(w, \alpha_{n-1})_\mu \neq 0$ if and only if μ is of the form $\mu = -(\beta_t + \alpha_n)$ for some $a_{r-1} \leq t \leq a_{r-2} - 1$. In such a case,* $\dim H^1(w, \alpha_{n-1})_\mu = 1$.

Proof. Proof of (1): Note that

$$H^0(s_{n-1}, \alpha_{n-1}) = \mathbb{C}_{-\alpha_{n-1}} \oplus \mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{\alpha_{n-1}} \tag{10}$$

(see [5, Corollary 2.5]). Since $\langle \alpha_{n-1}, \alpha_{n-1} \rangle = 2$, we have $H^1(s_{n-1}, \alpha_{n-1}) = 0$. Since $\langle \alpha_{n-1}, \alpha_{n-2} \rangle = -1$, we have $H^1(s_{n-2}, H^0(s_{n-1}, \alpha_{n-1})) = 0$. Therefore by using SES we have

$$H^1(s_{n-2} s_{n-1}, \alpha_{n-1}) = 0. \tag{11}$$

Since $\langle -\alpha_{n-1}, \alpha_{n-2} \rangle = 1$, by using (10) we have

$$H^0(s_{n-2} s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}} \oplus \mathbb{C}_{-\beta_{n-2}}.$$

Since $\langle -\alpha_{n-1}, \alpha_{n-3} \rangle = 0$ and $\langle -\beta_{n-2}, \alpha_{n-3} \rangle = 1$ we have

$$H^1(s_{n-3}, H^0(s_{n-2} s_{n-1}, \alpha_{n-1})) = 0. \tag{12}$$

Therefore by using SES together with (11), (12) we have

$$H^1(s_{n-3} s_{n-2} s_{n-1}, \alpha_{n-1}) = 0. \tag{13}$$

Proceeding in this way we have $$H^1(s_{a_r} \cdots s_{n-2} s_{n-1}, \alpha_{n-1}) = 0 \tag{14}$$

and $$H^0(s_{a_r} \cdots s_{n-2} s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \bigoplus_{j=a_r}^{n-1} \mathbb{C}_{-\beta_j}. \tag{15}$$

Since $\langle -\beta_j, \alpha_n\rangle > 0$ for all $a_r \leq j \leq n-1$, by using (15) we have

$$H^1(s_n, H^0(s_{a_r}\cdots s_{n-2}s_{n-1}, \alpha_{n-1})) = 0. \tag{16}$$

Therefore by using SES, (14) and (16) together we have

$$H^1(v_r s_{n-1}, \alpha_{n-1}) = 0. \tag{17}$$

In the proof of Lemma 4.1(1) we notice that

$$H^0(v_r s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \Big(\bigoplus_{j=a_r}^{n-1}(\mathbb{C}_{\beta_j}\oplus \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)})\Big). \tag{18}$$

By (18) we have $$H^1(s_{n-1}, H^0(v_r s_{n-1}, \alpha_{n-1})) = 0, \tag{19}$$

(see lines from (1) to (3)). Proceeding similarly and using (17), (19) we have

$$H^1(v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = 0.$$

Proof of (2): From the Lemma 4.1(1) we have

$$\begin{aligned} &H^0(v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = \\ &= \bigoplus_{i=a_r}^{a_{r-1}-1}\big(\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})}\big)\oplus \\ &\oplus \bigoplus_{i=a_{r-1}}^{n-2}\big(\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big) \oplus \mathbb{C}_{-(\alpha_{n-1}+2\alpha_n)}. \end{aligned}$$

Notice that for each $a_r \leq i \leq a_{r-1}-1$, $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ forms an indecomposable two dimensional B_{α_n}-module. Since $\langle -(\beta_i+2\alpha_n), \alpha_n\rangle = -2$, by Lemma 2.4 we have

$$\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} = V_i \otimes \mathbb{C}_{-\omega_n},$$

where V_i is the irreducible two dimensional $\tilde{L}_{\alpha_n}$-module. By Lemma 2.3(4) we have $H^j(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}) = 0$ for all $j \geq 0$ and for all $a_r \leq i \leq a_{r-1}-1$. Since $\langle -(\beta_i+2\alpha_n+\beta_t), \alpha_n\rangle = 0$ for each $a_r \leq i \leq a_{r-1}-1$, and $a_{r-1} \leq t \leq n-1$, we have $H^1(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_t)}) = 0$ for all $a_r \leq i \leq a_{r-1}-1$ and $a_{r-1} \leq t \leq n-1$. Moreover, we have $\langle -(\beta_i+2\alpha_n+\beta_{i+1}), \alpha_n\rangle = 0$ for each $a_{r-1} \leq i \leq n-2$ and $\langle -(\beta_i+2\alpha_n), \alpha_n\rangle = -2$ for each $a_{r-1} \leq i \leq n-1$. Therefore we have $H^1(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}) = 0$ for all $a_{r-1} \leq i \leq n-2$. By Lemma 2.3(3) we have, for all $a_{r-1} \leq i \leq n-1$,

$$H^1(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n)}) = H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, s_{n-1}\cdot -(\beta_i+2\alpha_n)) = \mathbb{C}_{-(\beta_i+\alpha_n)}.$$

From the above discussion, we have

$$H^1(s_n, H^0(v_{r-1}v_r s_{n-1}, \alpha_{n-1})) = \bigoplus_{i=a_{r-1}}^{n-1}\mathbb{C}_{-(\beta_i+\alpha_n)}. \tag{20}$$

By (1) and using SES we have

$$H^1(s_n, H^0(v_{r-1}v_r s_{n-1}, \alpha_{n-1})) = H^1(s_n v_{r-1}v_r s_{n-1}, \alpha_{n-1}). \tag{21}$$

From (20) and (21) we have

$$H^1(s_n v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{i=a_{r-1}}^{n-1} \mathbb{C}_{-(\beta_i+\alpha_n)}. \tag{22}$$

By Lemma 3.1 we have

$$H^1(s_{n-1}, H^0(s_n v_{r-1} v_r s_{n-1}, \alpha_{n-1})) = 0. \tag{23}$$

Let $v = v_{r-1} v_r s_{n-1}$. Therefore by using SES and (23) we have

$$H^1(s_{n-1} s_n v, \alpha_{n-1}) = H^0(s_{n-1}, H^1(s_n v, \alpha_{n-1})).$$

Since $\langle -(\beta_{n-1}+\alpha_n), \alpha_{n-1}\rangle = -1$ and $\langle -(\beta_i+\alpha_n), \alpha_{n-1}\rangle = 0$ for all $a_{r-1} \leq i \leq n-2$, by using (22) we have

$$H^1(s_{n-1} s_n v, \alpha_{n-1}) = \bigoplus_{i=a_{r-1}}^{n-2} \mathbb{C}_{-(\beta_i+\alpha_n)}.$$

Proceeding recursively we have

$$H^1(v_{r-2} v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = H^1(s_{a_{r-2}} \cdots s_{n-1} s_n v, \alpha_{n-1}) = \bigoplus_{i=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_i+\alpha_n)}. \qquad \blacksquare$$

Recall that $w_r = [a_1, n][a_2, n] \cdots [a_{r-1}, n][a_r, n-1]$, where $1 \leq r \leq k$ and $n \geq a_1 > a_2 > \ldots > a_{k-1} > a_k = 1$.

Lemma 5.2. (1) $H^1(w_1, \alpha_{n-1}) = 0$.

(2) *If* $a_2 \neq n-1$, *then* $H^1(w_2, \alpha_{n-1}) = 0$.

(3) *Let* $3 \leq r \leq k$. *Then,* $H^1(w_r, \alpha_{n-1})_\mu \neq 0$ *if and only if* $\mu = -(\beta_j + \alpha_n)$ *for some* j *such that* $a_{r-1} \leq j \leq a_{r-2} - 1$. *In such case* $\dim H^1(w_r, \alpha_{n-1})_\mu = 1$.

Proof. Proof of (1): Follows from proof of Lemma 3.1 and using SES.
Proof of (2): By proof of (1), we have

$$H^1(s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = 0. \tag{24}$$

Since $a_2 \neq n-1$, $H^0(s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \Big(\bigoplus_{j=a_2}^{n-1} \mathbb{C}_{-\beta_j}\Big)$.

Since $\langle -\beta_j, \alpha_n\rangle \geq 1$ for all $a_2 \leq j \leq n-1$, we have

$$H^1(s_n, H^0(s_{a_2} \cdots s_{n-1}, \alpha_{n-1})) = 0. \tag{25}$$

By SES, (24), (25) we have $H^1(s_n s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = 0$.
By using Lemma 3.1 and SES repeatedly, we see that

$$H^1(w_2, \alpha_{n-1}) = H^1(s_{a_1} \cdots s_n s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = 0.$$

Proof of (3): Let $v_r = s_n s_{a_r} \cdots s_{n-2}$, $v_{r-1} = s_{a_{r-1}} \cdots s_{n-1}$ and $v_{r-2} = s_{a_{r-2}} \cdots s_{n-1} s_n$.
Then by Lemma 5.1(2) we have $H^1(v_{r-2} v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}$.

By Lemma 4.2(1) if $H^0(v_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1})_\mu \neq 0$ then $\langle \mu, \alpha_n\rangle \geq 0$. Therefore we have

$$H^1(s_n, H^0(v_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1})) = 0. \tag{26}$$

By using SES and (26) we have

$$H^1(s_nv_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1}) = H^0(s_n, H^1(v_{r-2}v_{r-1}v_r,\alpha_{n-1})). \tag{27}$$

Since $\langle -(\beta_j+\alpha_n), \alpha_n\rangle = 0$ for all $a_{r-1} \leq j \leq a_{r-2}-1$, we have from (27)

$$H^1(s_nv_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}$$

and by Lemma 3.1 $\quad H^1(s_{n-1}, H^0(s_nv_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1})) = 0. \quad (28)$

Therefore by using SES together with (28) we have

$$H^1(s_{n-1}s_nv_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1}) = H^0(s_{n-1}, H^1(s_nv_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1})). \tag{29}$$

Since $\langle -(\beta_j+\alpha_n), \alpha_{n-1}\rangle = 0$ for all $a_{r-1} \leq j \leq a_{r-2}-1$, using (29) we have

$$H^1(s_{n-1}s_nv_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^1(s_{a_{r-3}}\cdots s_{n-1}s_nv_{r-2}v_{r-1}v_rs_{n-1},\alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $\langle -(\beta_j+\alpha_n), \alpha_t\rangle$ for all $a_{r-1} \leq j \leq a_{r-2}-1$, and $a_{r-4} \leq t \leq n$, using similar arguments as above we have

$$H^1(w_r,\alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$ ■

Corollary 5.3. *Let $3 \leq r \leq k$. If $H^1(w_r,\alpha_{n-1})_\mu \neq 0$, then*

$$H^0(w_{r-2}s_ns_{a_{r-1}}\cdots s_{n+2-r},\alpha_{n+2-r})_\mu \neq 0.$$

Proof. Follows from Lemma 4.3 and Lemma 5.2(2). ■

Corollary 5.4. *Let $3 \leq r \leq k$. If $H^1(w_r,\alpha_{n-1})_\mu \neq 0$, then $H^0(w_{r-1}s_n,\alpha_n)_\mu \neq 0$.*

Proof. Follows from Lemma 4.4 and Lemma 5.2(2). ■

Lemma 5.5. *Let $u_1 = w_ks_n[a_k, n-1]$. Then $H^1(u_1,\alpha_{n-1})_\mu \neq 0$ if and only if μ is of the form $\mu = -(\beta_j+\alpha_n)$ for some $1 \leq j \leq a_{k-1}-1$.*

Proof. Let $v_{k+1} = s_ns_1\cdots s_{n-2}$, $v_k = s_1\cdots s_{n-1}$ and $v_{k-1} = s_{a_{k-1}}\cdots s_{n-1}s_n$.

Step 1: $H^1(v_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}$.

Proof of Step 1: By Lemma 4.1(2), we have

$$H^0(s_1s_2\cdots s_ns_1\cdots s_{n-1},\alpha_{n-1}) =$$
$$= \bigoplus_{i=1}^{n-2}\big(\mathbb{C}_{-(\beta_i+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}\big)\oplus\mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

By Lemma 5.2(1), $H^1(s_2\cdots s_ns_1\cdots s_{n-1},\alpha_{n-1})=0$. Therefore by SES, we have

$$H^1(v_kv_{k+1}s_{n-1},\alpha_{n-1}) = H^1(s_1, H^0(s_2\cdots s_ns_1\cdots s_{n-1},\alpha_{n-1})) = 0 \tag{30}$$

(by Lemma 3.1). By SES and (30) we have

$$H^1(s_nv_kv_{k+1}s_{n-1},\alpha_{n-1}) = H^1(s_n, H^0(v_kv_{k+1}s_{n-1},\alpha_{n-1})) = \bigoplus_{i=1}^{n-1}\mathbb{C}_{-(\beta_i+\alpha_n)}.$$

By Lemma 3.1, we have $H^1(s_{n-1}, H^0(s_nv_kv_{k+1}s_{n-1},\alpha_{n-1}))=0$.
Therefore by SES, we have

$$H^1(s_{n-1}s_nv_kv_{k+1}s_{n-1},\alpha_{n-1}) = H^0(s_{n-1}, H^1(s_nv_kv_{k+1}s_{n-1},\alpha_{n-1})).$$

Since $\langle -(\beta_{n-1}+\alpha_n),\alpha_{n-1}\rangle = -1$ and $\langle -(\beta_i+\alpha_n),\alpha_{n-1}\rangle = 0$ for all $1\le i\le n-2$, we have

$$H^1(s_{n-1}s_nv_kv_{k+1}s_{n-1},\alpha_{n-1}) = \bigoplus_{i=1}^{n-2}\mathbb{C}_{-(\beta_i+\alpha_n)}.$$

Proceeding in this way recursively, we see that

$$H^1(v_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-1}\mathbb{C}_{-(\beta_i+\alpha_n)}.$$

Hence Step 1 follows.

Step 2: By Lemma 4.2, we have

$$H^1(s_n, H^0(v_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1})) = 0. \tag{31}$$

Therefore by using SES and (31) we have

$$H^1(s_nv_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1}) = H^0(s_n, H^1(v_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1})). \tag{32}$$

Since $\langle -(\beta_j+\alpha_n),\alpha_n\rangle = 0$ for all $1\le j\le a_{k-1}-1$, by using (32) we have

$$H^1(s_nv_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1}\mathbb{C}_{-(\beta_j+\alpha_n)}.$$

By Lemma 3.1, we have $$H^1(s_{n-1}, H^0(s_nv_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1})) = 0. \tag{33}$$

Therefore by using SES and (33) we have

$$H^1(s_{n-1}s_nv_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1}) = H^0(s_{n-1}, H^1(s_nv_{k-1}v_kv_{k+1}s_{n-1},\alpha_{n-1})). \tag{34}$$

Since $\langle -(\beta_j + \alpha_n), \alpha_{n-1} \rangle = 0$ for all $1 \le j \le a_{k-1} - 1$, by using (34) we have

$$H^1(s_{n-1}s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^1(s_{a_{k-2}} \cdots s_{n-1}s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Using similar arguments as above we have

$$H^1(u_1, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}. \qquad \blacksquare$$

Corollary 5.6. *If $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, then $H^0(w_{k-1}s_ns_1s_2 \cdots s_{n+1-k}, \alpha_{n+1-k})_\mu \neq 0$.*

Proof. Corollary follows from Lemma 4.5 and Lemma 5.5. ■

Corollary 5.7. *If $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, then $H^0(w_ks_n, \alpha_n)_\mu \neq 0$.*

Proof. Corollary follows from Lemma 4.6 and Lemma 5.5. ■

Let $3 \le r \le k$. We define $M_r := \{\mu \in X(T) : H^1(w_r, \alpha_{n-1})_\mu \neq 0\}$ and furthermore $M_0 := \{\mu \in X(T) : H^1(u_1, \alpha_{n-1})_\mu \neq 0\}$. Then we have

Corollary 5.8. (1) *$M_r \cap M_{r'} = \emptyset$ whenever $r \neq r'$.*

(2) *$M_0 \cap M_r = \emptyset$ for every $1 \le r \le k$.*

Proof. Proofs of (1) and (2) follow from Lemma 5.2 and Lemma 5.5. ■

Lemma 5.9. *Let $c = s_1s_2 \cdots s_n$. Let $T_r = c^{r-1}s_1s_2 \cdots s_{n-1}$, for all $2 \le r \le n$. Then, $T_r(\alpha_j) < 0$ for all $n+1-r \le j \le n-1$.*

Proof. Assume $r = 2$. Then we see that $T_r(\alpha_{n-1}) = -\alpha_0$. We assume that for $2 < l < n$, we have $T_l(\alpha_j) < 0$ for $n+1-l \le j \le n-1$. Note that $T_{l+1} = T_ls_ns_1 \cdots s_{n-1}$. Then for all $n-l \le i \le n-3$, we have $s_1 \cdots s_{n-1}(\alpha_i) = \alpha_{i+1}$. Since $n-(i+1) \ge 2$ and $n+1-l \le i+1 \le n-1$, we have $T_{l+1}(\alpha_i) = T_l(\alpha_{i+1}) < 0$ (by assumption). Since $s_1 \cdots s_{n-1}(\alpha_{n-2}) = \alpha_{n-1}$, we have $T_{l+1}(\alpha_{n-2}) = T_l(s_n(-\alpha_{n-1}))$. Since $s_n(\alpha_{n-1}) = \alpha_{n-1} + 2\alpha_n$, we have $T_ls_n(\alpha_{n-1}) = T_l(\alpha_{n-1} + 2\alpha_n)$. Since $s_1 \cdots s_{n-1}(\alpha_{n-1}+2\alpha_n) = \beta_1 + 2\alpha_n$, we get $T_l(\alpha_{n-1}+2\alpha_n) = T_{l-1}s_n(\beta_1+2\alpha_n)$.

Since $s_ns_1 \cdots s_{n-1}s_n(\beta_1+2\alpha_n) = -\alpha_1$, we have $T_{l-1}s_n(\beta_1+2\alpha_n) = T_{l-2}(-\alpha_1)$. Since $s_ns_1 \cdots s_{n-1}(-\alpha_1) = -\alpha_2$, we have $T_{l-2}(-\alpha_1) = T_{l-3}(-\alpha_2)$. Therefore by recursively we have $T_{l-3}(-\alpha_2) = -\alpha_{l-1}$. Hence $T_{l+1}(\alpha_{n-2}) = -\alpha_{l-1} < 0$. Also it is clear that $T_{l+1}(\alpha_{n-1}) < 0$. Therefore we have $T_{l+1}(\alpha_j) < 0$ for all $n-l \le j \le n-1$. Hence the result follows. ■

Lemma 5.10. *Let $u, v \in W$, let $v := \left(\prod_{j=1}^{n} s_j\right)^{l-1} s_1 s_2 \cdots s_{n-1}$ for some positive integer $l \le n$, such that $l(uv) = l(u) + l(v)$. Let $w = uv$. If $l \ge 3$, then $H^i(w, \alpha_{n-1}) = 0$ for all $i \ge 0$.*

Proof. We note that by SES, we have $H^0(w, \alpha_{n-1}) = H^0(u, H^0(v, \alpha_{n-1}))$. We show that $H^0(v, \alpha_{n-1}) = 0$. By Lemma 5.9 we have for each $1 \le r \le n-1$, $c^{r-1} s_1 \cdots s_{n-1}(\alpha_j) < 0$ for all $n+1-r \le j \le n-1$. In particular, we have $l(vs_{n-2}) = l(v) - 1$. Therefore, by Lemma 2.2(4) and using SES, we have

$$H^0(v, \alpha_{n-1}) = H^0(vs_{n-2}, H^0(s_{n-2}, \alpha_{n-1})) = 0. \tag{35}$$

Next we show that $H^1(w, \alpha_{n-1}) = H^0(u, H^1(v, \alpha_{n-1}))$. We will prove by induction on $l(u)$. If $l(u) = 0$ then it follows trivially. Next suppose that $l(u) > 1$. Then there exists a simple root γ such that $l(s_\gamma u) = l(u) - 1$. By using SES and (35) we have $H^0(s_\gamma uv, \alpha_{n-1}) = 0$. Again, by induction hypotheses $H^1(s_\gamma uv, \alpha_{n-1}) = H^0(s_\gamma u, H^1(v, \alpha_{n-1}))$. Therefore by SES, we have

$$\begin{aligned} H^1(w, \alpha_{n-1}) &= H^0(s_\gamma, H^1(s_\gamma uv, \alpha_{n-1})) = \\ &= H^0(s_\gamma, H^0(s_\gamma u, H^1(v, \alpha_{n-1}))) = H^0(u, H^1(v, \alpha_{n-1})). \end{aligned}$$

$H^1(v, \alpha_{n-1}) = 0$ follows from the fact that $l(vs_{n-2}) = l(v) - 1$ and Lemma 2.2(4). Thus, we have $H^1(w, \alpha_{n-1}) = 0$. Therefore by [14, Corollary 6.4, p. 780] we have $H^i(w, \alpha_{n-1}) = 0$ for all $i \ge 0$. ■

6. Cohomology modules of the tangent bundle of $Z(w, \underline{i})$

Let $w \in W$ and let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $\tau = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Recall the following long exact sequence of B-modules from [5] (see [5, Proposition 3.1, p. 673]):

$$\begin{aligned} 0 &\longrightarrow H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow \\ &\longrightarrow H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow H^2(w, \alpha_{i_r}) \longrightarrow \\ &\longrightarrow H^2(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^2(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow H^3(w, \alpha_{i_r}) \longrightarrow \cdots \end{aligned}$$

By [14, Corollary 6.4, p. 780], we have $H^j(w, \alpha_{i_r}) = 0$ for every $j \ge 2$. Thus we have the following exact sequence of B-modules:

$$\begin{aligned} 0 &\longrightarrow H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow \\ &\longrightarrow H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i}'), T_{(\tau,\underline{i}')}) \longrightarrow 0 \end{aligned}$$

Now onwards we call this exact sequence by LES.
Let $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ be a reduced expression of w_0 such that $\underline{i} = (j_1, j_2, \ldots, j_r)$ and let and $\underline{j} = (j_1, j_2, \ldots, j_N)$.

Lemma 6.1. *The natural homomorphism*

$$f \colon H^1(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of B-modules is surjective.

Proof. See [4, Lemma 7.1, p. 459]. ■

Lemma 6.2. *Let $J = S \setminus \{\alpha_{n-1}\}$. Let $v \in W_J$ and $u \in W$ be such that $l(uv) = l(u) + l(v)$. Let $u = s_{i_1} \cdots s_{i_r}$ and $v = s_{i_{r+1}} \cdots s_{i_t}$ be reduced expressions of u and v respectively. Let $\underline{i} = (i_1, i_2, \ldots, i_r)$ and $\underline{j} = (i_1, i_2, \ldots, i_r, i_{r+1}, \ldots, i_t)$. Then:*

(1) *The natural homomorphism $H^0(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^0(Z(u, \underline{i}), T_{(u,\underline{i})})$ of B-modules is surjective.*

(2) *The natural homomorphism $H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u,\underline{i})})$ of B-modules is an isomorphism.*

Proof. Let $r + 1 \leq l \leq t$. Let $v_l = us_{i_{r+1}} \cdots s_{i_l}$ and $\underline{i}_l = (\underline{i}, i_{r+1}, \ldots, i_l)$.

Proof of (1): By Lemma 3.3 we have $H^1(v_t, \alpha_{i_t}) = 0$. Therefore, using LES the natural homomorphism $H^0(Z(v_t, \underline{i}_t), T_{(v_t,\underline{i}_t)}) \longrightarrow H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})})$ is surjective. By induction on $l(v)$, the natural homomorphism

$$H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow H^0(Z(u, \underline{i}), T_{(u,\underline{i})})$$

is surjective. Hence we conclude that the natural homomorphism

$$H^0(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^0(Z(u, \underline{i}), T_{(u,\underline{i})})$$

is surjective.

Proof of (2): We will prove by induction on $l(v)$. By LES, we have the following exact sequence of B-modules:

$$0 \longrightarrow H^0(uv, \alpha_{i_t}) \longrightarrow H^0(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow$$
$$\longrightarrow H^1(uv, \alpha_{i_t}) \longrightarrow H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow 0.$$

By induction on $l(v)$, the natural homomorphism

$$H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u,\underline{i})})$$

is an isomorphism. By Lemma 3.3, $H^1(uv, \alpha_{i_t}) = 0$. Therefore, by the above exact sequence we have that $H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})})$ is an isomorphism. Hence we conclude that the homomorphism

$$H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u,\underline{i})})$$

of B-modules is an isomorphism. ■

Recall that by Lemma 2.6(1) and Lemma 2.8(2) we have that

$$w_0 = \left(\prod_{l_1=1}^{k-1} [a_{l_1}, n]\right) \left([a_k, n]^{n+1-k}\right) \left(\prod_{l_2=1}^{k-1} [a_k, a_{l_2} - 1]\right)$$

is a reduced expression for w_0. Let $\underline{i}$ be the tuple corresponding to this reduced of w_0. Let $u_1 = w_k s_n [a_k, n-1]$ and $\underline{i}_1$ be the tuple corresponding to the reduced expression $\prod_{l_1=1}^{k} [a_{l_1}, n]([a_k, n-1])$. Note that $a_k = 1$. With this notation, we have

Lemma 6.3. (1) *The natural homomorphism*

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism.

(2) *The natural homomorphism $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$ of B-modules is an isomorphism.*

Proof. For $1 \leq n-k$, let $u_j = w_k s_n [a_k, n]^{j-1} s_1 s_2 \cdots s_{n-1}$ and $\underline{i}_j$ be the tuple corresponding to the reduced expression $u_j = (\prod_{l_1=1}^{k}[a_{l_1}, n])([a_k, n])^{j-1}[a_k, n-1]$ (see Lemma 2.8(2)). Note that $w_0 = u_{n-k} s_n (\prod_{l_2=1}^{k-1}[a_k, a_{l_2} - 1])$.

Case 1: $a_1 \neq n$. In this case, we have $s_n(\prod_{l_2=1}^{k-1}[a_k, a_{l_2-1}]) \in W_J$, where $J = S \setminus \{\alpha_{n-1}\}$. Then by Lemma 6.2, the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_{n-k}, \underline{i}_{n-k}), T_{(u_{n-k},\underline{i}_{n-k})}) \tag{36}$$

is surjective and the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_{n-k}, \underline{i}_{n-k}), T_{(u_{n-k},\underline{i}_{n-k})}) \tag{37}$$

of B-modules is an isomorphism.

If $j \geq 2$, then by Lemma 5.10, we have $H^1(u_j, \alpha_{n-1}) = 0$. Let $u'_j = u_j s_{n-1}$ and let $\underline{i'}_j$ be the partial subsequence of u'_j such that $\underline{i}_j = (\underline{i'}_j, n-1)$. Hence by LES, we observe that the natural homomorphism

$$H^0(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^0(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \tag{38}$$

is surjective and $$H^1(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^1(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \tag{39}$$

is an isomorphism. Therefore by Lemma 6.2 we have the natural homomorphism

$$H^0(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \longrightarrow H^0(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})}) \tag{40}$$

is surjective and $$H^1(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \longrightarrow H^1(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})}) \tag{41}$$

is an isomorphism. Therefore, by combining the equations (38), (40) with the equations (39), (41) we see that the homomorphism

$$H^0(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^0(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})})$$

is surjective and $$H^1(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^1(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})})$$

is an isomorphism. Proceeding recursively and using (36), (37) we get that the homomorphism $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$ of B-modules is surjective and that the homomorphism $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$ of B-modules is an isomorphism. Since $u_1^{-1}(\alpha_0) < 0$, by [5, Lemma 6.2, p. 667], we have $H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})_{-\alpha_0} \neq 0$.

By [5, Theorem 7.1], $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$ and hence there is a unique B-stable line in $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})})$, namely $\mathfrak{g}_{-\alpha_0}$. Therefore we conclude that the natural homomorphism

$$H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1,\underline{i}_1),T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism.

Case 2 : $a_1 = n$ Then by Lemma 6.2, the natural morphism

$$H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_{n+1-k},\underline{i}_{n+1-k}),T_{(u_{n+1-k},\underline{i}_{n+1-k})})$$

is surjective and the natural homomorphism

$$H^1(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_{n+1-k},\underline{i}_{n+1-k}),T_{(u_{n+1-k},\underline{i}_{n+1-k})})$$

of B-modules is an isomorphism.
If $j \geq 2$, then by Lemma 5.10, we have $H^1(u_j,\alpha_{n-1}) = 0$. Hence by LES, for each $2 \leq j \leq n+1-k$, we observe that the natural homomorphism

$$H^0(Z(u_j,\underline{i}_j),T_{(u_j,\underline{i}_j)}) \longrightarrow H^0(Z(u_{j-1},\underline{i}_{j-1}),T_{(u_{j-1},\underline{i}_{j-1})})$$

is surjective and $H^1(Z(u_j,\underline{i}_j),T_{(u_j,\underline{i}_j)}) \longrightarrow H^1(Z(u_{j-1},\underline{i}_{j-1}),T_{(u_{j-1},\underline{i}_{j-1})})$ is an isomorphism. Therefore, the homomorphism $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1,\underline{i}_1),T_{(u_1,\underline{i}_1)})$ of B-modules is surjective and the homomorphism

$$H^1(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1,\underline{i}_1),T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism.
Since $u_1^{-1}(\alpha_0) < 0$, by [5, Lemma 6.2, p. 667], we have $H^0(Z(u_1,\underline{i}_1),T_{(u_1,\underline{i}_1)})_{-\alpha_0} \neq 0$. By [5, Theorem 7.1], $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$ and hence there is a unique B-stable line in $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})})$, namely $\mathfrak{g}_{-\alpha_0}$. Therefore we conclude that the natural homomorphism

$$H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1,\underline{i}_1),T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism. ■

The following is a useful corollary.

Corollary 6.4. *If* $\mu \in X(T)\setminus\{0\}$, *then we have* $\dim H^0(Z(u_1,\underline{i}_1),T_{(u_1,\underline{i}_1)})_\mu \leq 1$.

Proof. By [5, Theorem 7.1], $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$. By Lemma 6.3(1) we have $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) \simeq H^0(Z(u_1,\underline{i_1}),T_{(u_1,\underline{i_1})})$ (as B-modules). Hence for any $\mu \in X(T)\setminus\{0\}$, we have $\dim H^0(Z(u_1,\underline{i}_1),T_{(u_1,\underline{i}_1)})_\mu \leq 1$. ■

Let $u_1' = w_k s_n[a_k, n-2]$ and let $\underline{i}'_1$ be the tuple corresponding to the reduced expression $\prod_{l_1=1}^{k}[a_{l_1}, n][a_k, n-2]$.

Let $3 \leq r \leq k$, and let $\underline{j}_r = (a_1,\ldots,n; a_2,\ldots,n;\ldots;a_{r-1},\ldots,n;a_r,\ldots,n-1)$ and $\underline{j}'_r = (a_1,\ldots,n;a_2,\ldots,n;a_r,\ldots,n-2)$.

We now prove

Lemma 6.5. *Let* $\mu \in X(T) \setminus \{0\}$.

(1) *If* $H^1(u_1, \alpha_{n-1})_\mu = 0$, *then* $\dim H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu \leq 1$.

(2) *If* $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, *then* $\dim H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu = 2$, *and the natural homomorphism* $H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu \longrightarrow H^1(u_1, \alpha_{n-1})_\mu$ *is surjective.*

Proof. By LES, we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(u_1, \alpha_{n-1}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)}) \longrightarrow \\ \longrightarrow H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)}) \longrightarrow H^1(u_1, \alpha_{n-1}) \longrightarrow \cdots \quad (42)$$

Proof of (1): Assume that $H^1(u_1, \alpha_{n-1})_\mu = 0$ and $\mu \in X(T) \setminus \{0\}$, then by the above exact sequence the natural homomorphism

$$H^0(Z(u_1, \underline{i}_1), T_{(u'_1, \underline{i}_1)})_\mu \longrightarrow H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu$$

is surjective. By Corollary 6.4, we have $\dim H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_\mu \leq 1$.

Proof of (2): Assume that $H^1(u_1, \alpha_{n-1})_\mu \neq 0$. Then by Lemma 5.5, μ is of the form $-(\beta_j + \alpha_n)$ for some $1 \leq j \leq a_{k-1} - 1$ and $\dim H^1(u_1, \alpha_{n-1})_\mu = 1$. Hence by using Corollary 6.4, we see that if $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, then

$$\dim H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu \leq 2. \quad (43)$$

Let $\underline{j''}_k = (\underline{j}_k, n)$ be the tuple corresponding to the reduced expression $w_k s_n = \prod_{l_1=1}^{k} [a_{l_1}, n]$. Then by Lemma 6.2 the natural homomorphism

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)}) \longrightarrow H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})$$

is surjective. By (43), we have

$$\dim H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu \leq 2. \quad (44)$$

Again by Lemma 6.2 the natural homomorphism

$$H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)}) \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)}) \quad (45)$$

is surjective. Recall that $\tau_k = w_k s_{n-1}$. By LES we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w_k, \alpha_{n-1}) \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)}) \longrightarrow \\ \longrightarrow H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k, \underline{j'}_k)}) \longrightarrow H^1(w_k, \alpha_{n-1}) \longrightarrow \cdots$$

Since $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, by Corollary 5.8 we have $H^1(w_k, \alpha_{n-1})_\mu = 0$. Therefore we have an exact sequence

$$0 \longrightarrow H^0(w_k, \alpha_{n-1})_\mu \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow \\ \longrightarrow H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k, \underline{j'}_k)})_\mu \longrightarrow 0.$$

Let $\sigma_k = w_{k-1}s_n s_1 \cdots s_{n+1-k}$ and $\underline{j^*}_k = (\underline{j}_{k-1}, n, 1, 2, \ldots, n+1-k)$ be the tuple corresponding to this reduced expression of σ_k. Then by using Lemma 6.2, the natural homomorphism

$$H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k, \underline{j'}_k)}) \longrightarrow H^0(Z(\sigma_k, \underline{j^*}_k), T_{(\sigma_k, \underline{j^*}_k)}) \tag{46}$$

is surjective. Therefore the natural map

$$H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow H^0(Z(\sigma_k, \underline{j^*}_k), T_{(\sigma_k, \underline{j^*}_k)})_\mu \tag{47}$$

is surjective. Since $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, by Corollary 5.6 we have $H^0(\sigma_k, \alpha_{n+1-k})_\mu \neq 0$. Therefore, $H^0(Z(\sigma_k, \underline{j^*}_k), T_{(\sigma_k, \underline{j^*}_k)})_\mu \neq 0$. Thus from (47) we have

$$H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \neq 0.$$

By LES and (45) we have an exact sequence of T-modules

$$0 \longrightarrow H^0(w_k s_n, \alpha_n)_\mu \longrightarrow H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu \longrightarrow$$
$$\longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow 0$$

Since $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, by Corollary 5.7 we have $H^0(w_k s_n, \alpha_n)_\mu \neq 0$. Therefore $\dim H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu \geq 2$. On the other hand, by (44), we have $\dim H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu \leq 2$. Hence we have

$$\dim H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu = 2.$$

Since the natural homomorphism

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)}) \longrightarrow H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})$$

is surjective, we have $\quad \dim H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)_\mu}) = 2. \qquad (48)$

It is clear from (42), (48), and Corollary 6.4, that

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu \longrightarrow H^1(u_1, \alpha_{n-1})_\mu$$

is surjective. ∎

Corollary 6.6. *The natural homomorphism* $H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)}) \longrightarrow H^1(u_1, \alpha_{n-1})$ *is surjective.*

Proof. Note that by Lemma 5.5, if $H^1(u_1, \alpha_{n-1})_\mu \neq 0$ then $\mu \in X(T) \setminus \{0\}$. Now the proof follows from Lemma 6.5. ∎

Lemma 6.7. (1) *If* $H^1(w_m, \alpha_{n-1})_\mu = 0$ *for all* $r \leq m \leq k$ *and* $H^1(u_1, \alpha_{n-1})_\mu = 0$, *then* $\dim H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu \leq 1$.

(2) *Let* $3 \leq r \leq k$. *If* $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, *then* $\dim H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu = 2$, *and the natural homomorphism* $H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu \longrightarrow H^1(w_r, \alpha_{n-1})_\mu$ *is surjective.*

Proof. Proof of (1): If $H^1(u_1, \alpha_{n-1})_\mu = 0$, then we get by Lemma 6.5 the inequality $\dim H^0(Z(u_1', \underline{i'}_1), T_{(u_1', \underline{i'}_1)})_\mu \leq 1$. By Lemma 6.2, the natural homomorphism

$$H^0(Z(u_1', \underline{i'}_1), T_{(u_1', \underline{i'}_1)})_\mu \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu$$

is surjective. If $H^1(w_m, \alpha_{n-1})_\mu = 0$ for all $r \leq m \leq k$, by using LES, we see that the natural homomorphism $H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu$ is surjective. Therefore, we have $\dim H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu \leq 1$.

Proof of (2): Assume that $H^1(w_r, \alpha_{n-1})_\mu \neq 0$. Then by Corollary 5.8, we have $H^1(w_m, \alpha_{n-1})_\mu = 0$ for all $r+1 \leq m \leq k$ and $H^1(u_1, \alpha_{n-1})_\mu = 0$. Then by (1), we have $\dim H^0(Z(\tau_{r+1}, \underline{j'}_{r+1}), T_{(\tau_{r+1}, \underline{j'}_{r+1})})_\mu \leq 1$. By Lemma 6.2, the natural homomorphism $H^0(Z(\tau_{r+1}, \underline{j'}_{r+1}), T_{(\tau_{r+1}, \underline{j'}_{r+1})})_\mu \longrightarrow H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)})_\mu$ is surjective. Hence, we have

$$\dim H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)})_\mu \leq 1. \tag{49}$$

By LES we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w_r, \alpha_{n-1}) \longrightarrow H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)}) \longrightarrow$$
$$\longrightarrow H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)}) \longrightarrow H^1(w_r, \alpha_{n-1}) \longrightarrow \cdots$$

Since $\dim H^0(Z(w_r, \underline{j}_r), T_{(w_r, \underline{j}_r)})_\mu \leq 1$ and $\dim H^1(w_r, \alpha_{n-1})_\mu = 1$, we get

$$\dim H^0(Z(\tau_r, \underline{j'}_r))_\mu \leq 2. \tag{50}$$

Let $\underline{j''}_{r-1} = (\underline{j}_{r-1}, n)$ be the tuple corresponding to the reduced expression $w_{r-1}s_n = \prod_{l_1=1}^{r-1}[a_{l_1}, n]$. Then by Lemma 6.2 the natural homomorphism

$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)}) \longrightarrow H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})})$$

is surjective. By (50), we have

$$\dim H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})_\mu}) \leq 2. \tag{51}$$

Again by Lemma 6.2 the natural homomorphism

$$H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})}) \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})}) \tag{52}$$

is surjective. By LES we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w_{r-1}, \alpha_{n-1}) \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})}) \longrightarrow$$
$$\longrightarrow H^0(Z(\tau_{r-1}, \underline{j'}_{r-1}), T_{(\tau_{r-1}, \underline{j'}_{r-1})}) \longrightarrow H^1(w_{r-1}, \alpha_{n-1}) \longrightarrow \cdots$$

Since $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, then by Corollary 5.8 we have $H^1(w_{r-1}, \alpha_{n-1})_\mu = 0$. Therefore we have an exact sequence

$$0 \longrightarrow H^0(w_{r-1}, \alpha_{n-1})_\mu \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \longrightarrow$$
$$\longrightarrow H^0(Z(\tau_{r-1}, \underline{j'}_{r-1}), T_{(\tau_{r-1}, \underline{j'}_{r-1})})_\mu \longrightarrow 0. \tag{53}$$

Let $\sigma_{r-1} = w_{r-2}s_n s_{a_{r-1}} \dots s_{n+2-r}$, and $\underline{j^*}_{r-1} = (\underline{j}_{r-2}, n, a_{r-1}, a_{r-1}+1, \dots, n+2-r)$ be the reduced expression of σ_{r-1}. Then by using Lemma 6.2 the natural homomorphism

$$H^0(Z(\tau_{r-1}, \underline{j'}_{r-1}), T_{(\tau_{r-1}, \underline{j'}_{r-1})}) \longrightarrow H^0(Z(\sigma_{r-1}, \underline{j^*}_{r-1}), T_{(\sigma_{r-1}, \underline{j^*}_{r-1})}) \tag{54}$$

is surjective. Therefore, by (53) and (54) we have

$$H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \longrightarrow H^0(Z(\sigma_{r-1}, \underline{j^*}_{r-1}), T_{(\sigma_{r-1}, \underline{j^*}_{r-1})})_\mu \tag{55}$$

is surjective. Since $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, by Corollary 5.3 we get $H^0(\sigma_{r-1}, \alpha_{n+2-r})_\mu \neq 0$. Therefore, $H^0(Z(\sigma_{r-1}, \underline{j^*}_{r-1}), T_{\sigma_{r-1}, \underline{j^*_{r-1}}}))_\mu \neq 0$.

Thus from (55) we have $H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \neq 0$. Then by using (52) we have an exact sequence of T-modules

$$\begin{aligned}0 \longrightarrow H^0(w_{r-1}s_n, \alpha_n)_\mu \longrightarrow H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})})_\mu \longrightarrow \\ \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \longrightarrow 0\end{aligned}$$

Since $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, by Corollary 5.4 we have $H^0(w_{r-1}s_n, \alpha_n)_\mu \neq 0$. Therefore $\dim H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})_\mu}) \geq 2$. Hence, by (51) we have

$$\dim H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})})_\mu = 2.$$

Since the natural homomorphism

$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)}) \longrightarrow H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})})$$

is surjective, we have $\dim H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu = 2$. Therefore by (49),

$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu \longrightarrow H^1(w_r, \alpha_{n-1})_\mu$$

is surjective. ■

7. Main theorem

In this section we prove the main theorem.

Recall that $G = PSO(2n+1, \mathbb{C})(n \geq 3)$, and let c be a Coxeter element of W. Then there exists a decreasing sequence $n \geq a_1 > a_2 > \cdots > a_k = 1$ of positive integers such that $c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j]$ for $i \leq j$ denotes $s_i s_{i+1} \cdots s_j$.

Let $\underline{i} = (\underline{i}^1, \underline{i}^2, \dots, \underline{i}^n)$ be a sequence corresponding to a reduced expression of w_0, where $\underline{i}^r$ $(1 \leq r \leq n)$ is a sequence of reduced expressions of c (see Lemma 2.8).

Theorem 7.1. $H^j(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ *for all* $j \geq 1$ *if and only if* $a_2 \neq n-1$.

Proof. From [5, Proposition 3.1, p. 673], we have $H^j(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $j \geq 2$. It is enough to prove the following: $H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ if and only if c is of the form $[a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_2 \neq n-1$.

Proof of ($\Longrightarrow$): If $a_2 = n-1$, then $a_1 = n$ and $c = s_n s_{n-1} v$, where $v \in W_J$ and $J = S \setminus \{\alpha_{n-1}, \alpha_n\}$. Let $u = s_n s_{n-1}$. Then $c = uv$. Let $\underline{j} = (n, n-1)$ be the sequence corresponding to u. Then using LES, we have:

$$0 \longrightarrow H^0(u, \alpha_{n-1}) \longrightarrow H^0(Z(u, \underline{j}), T_{(u,\underline{j})}) \longrightarrow H^0(s_n, \alpha_n) \longrightarrow$$
$$\longrightarrow H^1(u, \alpha_{n-1}) \xrightarrow{f} H^1(Z(u, \underline{j}), T_{(u,\underline{j})}) \longrightarrow 0.$$

We see that $H^1(s_n s_{n-1}, \alpha_{n-1}) = \mathbb{C}_{\alpha_n + \alpha_{n-1}}$ and $H^0(s_n, \alpha_n)_{\alpha_n + \alpha_{n-1}} = 0$. Hence f is non zero homomorphism. Hence $H^1(Z(u, \underline{j}), T_{(u,\underline{j})})) \neq 0$. By Lemma 6.1, the natural homomorphism $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u, \underline{j}), T_{(u,\underline{j})})$ is surjective. Hence we have $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \neq 0$.

Proof of ($\Longleftarrow$) : Assume that $a_2 \neq n-1$. Recall that

$$w_k = [a_1, n] \cdots [a_{k-1}, n][a_k, n-1], \qquad u_1 = w_k s_n [a_k, n-1].$$

By Lemma 6.3(2), the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \tag{56}$$

of B-modules is an isomorphism. By using LES, we have the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \longrightarrow H^0(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \xrightarrow{h_1} H^1(u_1, \alpha_{n-1}) \longrightarrow$$
$$\longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \longrightarrow H^1(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \longrightarrow 0.$$

By Corollary 6.6, we see that the natural homomorphism

$$h_1 : H^0(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \longrightarrow H^1(u_1, \alpha_{n-1})$$

is surjective. Therefore, the natural homomorphism

$$H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \longrightarrow H^1(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \tag{57}$$

is an isomorphism. By Lemma 6.2(2), the natural homomorphism

$$H^1(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \tag{58}$$

is an isomorphism. Therefore, by (56), (57) and (58) the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \tag{59}$$

is an isomorphism. Recall that $\tau_k = [a_1, n][a_2, n] \cdots [a_{k-1}, n][a_k, n-2]$. By using LES, we have the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \longrightarrow H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \xrightarrow{h_2} H^1(w_k, \alpha_{n-1}) \longrightarrow$$
$$\longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \xrightarrow{h_3} H^1(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \longrightarrow 0.$$

By Lemma 6.7(2), we see that the map

$$h_2 : H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \longrightarrow H^1(w_k, \alpha_{n-1})$$

is surjective. Therefore, the map

$$h_3 : H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \longrightarrow H^1(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \tag{60}$$

is an isomorphism. By using Lemma 6.2(2) we see that the natural map

$$H^1(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \longrightarrow H^1(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1},\underline{j}_{k-1})}) \tag{61}$$

is an isomorphism. Using LES, we get the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1},\underline{j}_{k-1})}) \longrightarrow H^0(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1},\underline{j'}_{k-1})}) \longrightarrow$$
$$\longrightarrow H^1(w_{k-1}, \alpha_{n-1}) \longrightarrow H^1(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1},\underline{j}_{k-1})}) \longrightarrow$$
$$\longrightarrow H^1(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1},\underline{j'}_{k-1})}) \longrightarrow 0.$$

By Lemma 6.7(2), we see that the natural map

$$H^0(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1},\underline{j'}_{k-1})}) \longrightarrow H^1(w_{k-1}, \alpha_{n-1})$$

is surjective. Therefore, the natural map

$$H^1(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1},\underline{j}_{k-1})}) \longrightarrow H^1(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1},\underline{j'}_{k-1})}) \tag{62}$$

is an isomorphism. By using Lemma 6.2(2) and 6.7(2) repeatedly, we see that the natural map

$$H^1(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1},\underline{j'}_{k-1})}) \longrightarrow H^1(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)}) \tag{63}$$

is an isomorphism for all $3 \leq r \leq k-2$. Therefore by (59), (60), (61), (62), (63) we have the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(\tau_3, \underline{j'}_3), T_{(\tau_3,\underline{j'}_3)}) \tag{64}$$

is an isomorphism. Again from Lemma 6.2(2), we see that the natural map

$$H^1(Z(\tau_3, \underline{j'}_3), T_{(\tau_3,\underline{j'}_3)}) \longrightarrow H^1(Z(w_2, \underline{j}_2), T_{(w_2,\underline{j}_2)})$$

is an isomorphism. Since $a_2 \neq n-1$, by Lemma 5.2(2) we have $H^1(w_2, \alpha_{n-1}) = 0$. Note that by Lemma 5.2(1) we have $H^1(w_1, \alpha_{n-1}) = 0$.

By using Lemma 3.3 and using LES, we have $H^1(Z(w_2, \underline{j}_2), T_{(w_2,\underline{j}_2)}) = 0$. Hence, by (64) we conclude that $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$. This completes the proof of the theorem. ∎

Corollary 7.2. *Let c be a Coxeter element such that c is of the form*

$$[a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$$

with $a_2 \neq n-1$ and $a_k = 1$. Let $(w_0, \underline{i})$ be a reduced expression of w_0 in terms of c as in Theorem 7.1. Then, $Z(w_0, \underline{i})$ has no deformations.

Proof. By Theorem 7.1 and by [5, Proposition 3.1, p. 673], we have

$$H^i(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$$

for all $i > 0$. Hence, by [12, Proposition 6.2.10, p. 272], we see that $Z(w_0, \underline{i})$ has no deformations. ■

Remark 7.3. Theorem 7.1 does not hold for $PSO(5, \mathbb{C})$.

Proof. We take $c = s_1 s_2$. Here $a_2 \neq 1$. Further, we have $w_0 = c^2 = s_1 s_2 s_1 s_2$. Let $\underline{i} = (1, 2, 1, 2)$. It is easy to see by using SES repeatedly that

$$H^1(s_1 s_2 s_1, \alpha_1) = \mathbb{C}_{\alpha_1+\alpha_2} \oplus \mathbb{C}_{\alpha_2}.$$

Further, note that $H^0(Z(s_1 s_2, (1, 2)), T_{(s_1 s_2,(1,2))})_{\alpha_1+\alpha_2} = 0$ (see [5, Proposition 6.3(1), p. 688]). Hence by using LES we have

$$H^1(Z(s_1 s_2 s_1, (1, 2, 1)), T_{(s_1 s_2 s_1,(1,2,1))}) \neq 0.$$

Therefore by using Lemma 6.1 we have $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \neq 0$. ■

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