

4-Dimensional Almost-Kähler Lie Algebras of Constant Hermitian Holomorphic Sectional Curvature are Kähler

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Abstract. We prove that any 4-dimensional almost-Kähler Lie algebra of constant Hermitian holomorphic sectional curvature with respect to the canonical Hermitian connection is Kähler.

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1. Introduction

An almost-Hermitian manifold is an even dimensional smooth manifold M equipped with an almost-complex structure J and a compatible Riemannian metric g . When the induced 2-form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ is closed the quadruple (M, ω, J, g) is called *almost-Kähler* and when J is further integrable it is called *Kähler*.

A well-known conjecture of Goldberg [9] states that compact almost-Kähler Einstein manifolds are necessarily Kähler. The conjecture is still open, but some partial results are known. Sekigawa [19] proved that the conjecture is true if the scalar curvature is non-negative and other results have been proved under various conditions specially in dimension 4 (see for instance [3, 4, 13]). Moreover, the conjecture is false if the assumption of being compact is dropped since Nurowski and Przanowski constructed in [12] a non-compact example of almost-Kähler non-Kähler Ricci flat manifold which turns out to be of pointwise positive constant holomorphic sectional curvature. Here we recall that an almost-Kähler manifold is of *pointwise constant holomorphic sectional curvature* if, at any point of M , $g(R_{X, JX}^g X, JX)$ is constant for any vector field X of unit length with respect to g , where R^g denotes the Riemannian curvature of the Levi-Civita connection of g . Also, Sato [16, 17, 18] studied 4-dimensional almost-Kähler manifolds of pointwise constant holomorphic sectional curvature. In particular, he constructed a non-compact example of pointwise negative constant holomorphic sectional curvature [18]. Furthermore, almost-Kähler manifolds of pointwise constant totally-real sectional curvature were examined [6, 7, 2]. In dimension 4, it turns out that such manifolds are self-dual and are conjectured to be Kähler in the compact case [2].

On an almost-Kähler non-Kähler manifold (M, ω, J, g) the Levi-Civita connection does not preserve the almost-complex structure and its role is usually replaced by the canonical Hermitian connection [11, 8] which is the one defined as the unique connection ∇ preserving J and g and having J -anti-invariant torsion. Hence it is natural to study the condition on a 4-dimensional almost-Kähler manifold of having pointwise constant holomorphic sectional curvature with respect to the canonical Hermitian connection instead of the Levi-Civita connection. We refer to these manifolds as *of pointwise constant Hermitian holomorphic sectional curvature*.

The study of such manifolds was initiated in [10] where it is proved that 4-dimensional almost-Kähler manifolds of pointwise constant Hermitian holomorphic sectional curvature are self-dual. It is also shown that a closed 4-dimensional almost-Kähler manifold of globally non-negative constant sectional curvature is Kähler.

In analogy to the Riemannian case, it is rather natural to pursue the study of these manifolds in the non-compact case. In this note we study left-invariant almost-Kähler structures of having pointwise constant Hermitian holomorphic sectional curvature on Lie groups.

Theorem 1.1. *Any left-invariant almost-Kähler structure of pointwise constant Hermitian holomorphic sectional curvature on a 4-dimensional Lie group is Kähler.*

Since we assume the almost-Kähler structure *invariant*, we can work on Lie algebras in an algebraic fashion. In view of [10] we can restrict our attention to 4-dimensional Lie algebras with a self-dual metric. 4-dimensional Lie algebras with a non-conformally flat self-dual metric are completely described in [5] and from the classification it turns out that all the compatible almost-Kähler structures are Kähler. For the conformally flat case we use a classification in [14] and we give a description of all the conformally flat metrics in terms of the existence of a special coframe. From our description and the classification of symplectic Lie algebras [15] it follows every conformally flat almost-Kähler Lie algebra of pointwise constant Hermitian holomorphic sectional curvature is Kähler.

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2. Preliminaries

On an almost-Kähler manifold (M, ω, J, g) the *canonical Hermitian connection* ∇ is defined by the following formula [11, 8]

$$\nabla_X Y = D_X^g Y - \frac{1}{2} J (D_X^g J) Y,$$

where D^g is the Levi-Civita connection of the metric g . We denote by R^∇ and

R^g the curvatures of ∇ and D^g , respectively. We adopt the convention of writing $R_{X,Y}^\nabla = -[\nabla_X, \nabla_Y] + \nabla_{[X,Y]}$, where X, Y are vector fields on M and $[\cdot, \cdot]$ is the commutator. The *Hermitian holomorphic sectional curvature* H is defined by

$$H(X) := \frac{g(R_{X,JX}^\nabla X, JX)}{g(X, X)g(X, X)}.$$

We call the Hermitian holomorphic sectional curvature *pointwise constant* k if

$$g(R_{X,JX}^\nabla X, JX) = k,$$

at any point $p \in M$ and for any vector field X of unit length with respect to g .

Now we focus on dimension 4. In this case the bundle of 2-forms $\Lambda^2 M$ admits, under the action of the (Riemannian) Hodge operator, the g -orthogonal splitting

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,$$

where $\Lambda^+ M$ and $\Lambda^- M$ correspond to the bundles of self-dual and anti-self-dual 2-forms, respectively. The Riemannian curvature R^g , viewed as a symmetric linear operator of 2-forms $\Omega^2(M) = \Omega^+(M) \oplus \Omega^-(M)$, decomposes as

$$R^g = \begin{pmatrix} W^+ + \frac{s^g}{12} \text{Id}|_{\Omega^+(M)} & \tilde{r}_0|_{\Omega^-(M)} \\ \tilde{r}_0|_{\Omega^+(M)} & W^- + \frac{s^g}{12} \text{Id}|_{\Omega^-(M)} \end{pmatrix}, \tag{1}$$

where W^+ and W^- are called the self-dual Weyl tensor and the anti-self-dual Weyl tensor, respectively. Both W^+ and W^- are trace-free. Here s^g denotes the Riemannian scalar curvature and \tilde{r}_0 is the Kulkarni–Nomizu extension of the trace-free part $r_0 = r^g - \frac{s^g}{4}g$ of the Riemannian Ricci tensor r^g . The manifold (M, g) is said to be self-dual (anti-self-dual) if $W^- = 0$ ($W^+ = 0$). (M, g) is conformally flat if $W^+ = W^- = 0$. The Riemannian curvature R^g can also be viewed as a $(4, 0)$ -tensor via the metric, as well as the Weyl tensor $W = W^+ + W^-$. Note also the decomposition of W^+ in the almost-Kähler case (see [1] or [3] for more details)

$$W^+ = \begin{pmatrix} \|N_J\|^2 + \frac{s^g}{6} & W_\omega^+ \\ W_\omega^+ & W_{00}^+ - \frac{1}{2} (\|N_J\|^2 + \frac{s^g}{6}) \text{Id}|_{\Omega^{J,-}(M)} \end{pmatrix}, \tag{2}$$

where $\Omega^{J,-}(M)$ are the J -anti-invariant 2-forms on M and N_J is the Nijenhuis tensor of J . It turns out that if a 4-dimensional almost-Kähler manifold (M, ω, J, g) is of pointwise constant Hermitian holomorphic sectional curvature, then $W^- = 0$ [10].

Theorem 2.1. [10] *Let (M, ω, J, g) be an almost-Kähler manifold of dimension 4. If M is of pointwise constant Hermitian holomorphic sectional curvature then (M, g) is self-dual.*

Remark that the compactness assumption is not needed in the above Theorem. Furthermore, M. Upmeyer and the first named author [10] proved in the compact case that if the sectional curvature is globally constant and if k is non-negative then (M, ω, J, g) is Kähler.

3. Lie algebras admitting non-conformally flat self-dual metrics

In this section we study 4-dimensional almost-Kähler self-dual Lie algebras.

In [5], De Smedt and Salamon proved the a Lie algebra \mathfrak{g} of a 4-dimensional Lie group admitting a non-conformally flat left-invariant metric g such that $W^+ = 0$ has a coframe $\{e^1, e^2, e^3, e^4\}$ satisfying the following structure equations

$$de^1 = 0, \quad de^2 = -e^{12} - \lambda e^{13}, \quad de^3 = \lambda e^{12} - e^{13}, \quad de^4 = -2e^{14} + e^{23},$$

for some $\lambda \geq 0$, where $e^{12} = e^1 \wedge e^2$, etc. Furthermore, the metric g has the (oriented) orthonormal basis $\{\frac{1}{k}e_1, e_2, e_3, e_4\}$ for $k = 1$ or $k = 2$, where $\{e_i\}$ is the dual basis of $\{e^i\}$. We remark that all the Lie algebras of this kind are not unimodular and, consequently, their Lie groups do not admit compact quotients. To get non-conformally left-invariant metrics such that $W^- = 0$, we only have to change the orientation and consider the opposite orientation given by the volume form

$$-|k|e^1 \wedge e^2 \wedge e^3 \wedge e^4. \quad (3)$$

Now, we will find the almost-Kähler structures on \mathfrak{g} with g as a Riemannian metric and compatible with the opposite orientation (3).

A 2-form $\omega = \sum_{1 \leq i < j \leq 4} \alpha_{ij} e^{ij}$ on \mathfrak{g} is closed if

$$\begin{aligned} d\omega &= \alpha_{14} (-e^1 \wedge e^{23}) + \alpha_{23} (-e^{12} \wedge e^3) + \alpha_{23} (-e^2 \wedge -e^{13}) \\ &\quad + \alpha_{24} ((-e^{12} - \lambda e^{13}) \wedge e^4) + \alpha_{24} (-e^2 \wedge -2e^{14}) \\ &\quad + \alpha_{34} ((\lambda e^{12} - e^{13}) \wedge e^4) + \alpha_{34} (-e^3 \wedge -2e^{14}) \\ &= e^{123}(-\alpha_{14} - 2\alpha_{23}) + e^{124}(-3\alpha_{24} + \lambda\alpha_{34}) + e^{134}(-\lambda\alpha_{24} - 3\alpha_{34}) = 0. \end{aligned}$$

We deduce that $-2\alpha_{23} = \alpha_{14}$, $\lambda\alpha_{34} = 3\alpha_{24}$, $\lambda\alpha_{24} = -3\alpha_{34}$.

We conclude that ω is closed if and only $\alpha_{34} = \alpha_{24} = 0$ and $\alpha_{14} = -2\alpha_{23}$. Hence, ω is closed and non-degenerate if it is of the form

$$\omega = ae^{12} + be^{13} - 2ce^{14} + ce^{23}$$

for some $a, b, c \in \mathbb{R}$ with $c \neq 0$. Now, one can easily check that ω is compatible with the metric g if and only if $a = b = 0, c^2 = 1$ and $k^2 = 4$. As a result, we get

Corollary 3.1. *Any 4-dimensional almost-Kähler Lie algebra with a self-dual non-conformally flat metric is Kähler.*

Proof. From the above discussion, we obtain that, for such almost-Kähler structures (ω, J, g) , the 2-form ω takes the following expression

$$\omega = \mp 2e^{14} \pm e^{23}.$$

It is also compatible with the opposite orientation (3). Moreover, since g is the standard metric with respect to basis $\{e_i\}$, the almost-complex structure J satisfies

$$Je_1 = -2e_4, \quad Je_2 = e_3,$$

or

$$Je_1 = 2e_4, \quad Je_2 = -e_3.$$

Then a direct computation of the Nijenhuis tensor yields that J is integrable and that (ω, J, g) is a Kähler structure, as required. \blacksquare

4. Lie algebras with conformally flat left-invariant metrics

In this section we study conformally flat almost-Kähler structures on 4-dimensional Lie algebras.

The spectrum of the Riemannian curvature operators on 4-dimensional Lie algebras with a conformally flat metrics was studied in [14, Theorem 2]. It turns out that there are only eight models of 4-dimensional Lie algebras admitting a conformally flat metric. Moreover, by using the classification of 4-dimensional symplectic Lie algebras in [15], we get that the only three conformally flat Lie algebras admitting symplectic structures are the following

- the abelian Lie algebra;
- $\mathfrak{rr}_{3,0}$: $[e_1, e_3] = -e_2, [e_2, e_3] = e_1$;
- \mathfrak{r}'_2 : $[e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, [e_2, e_4] = -e_3$.

The abelian Lie algebra and $\mathfrak{rr}_{3,0}$ are unimodular and the spectrum of the Riemannian curvature operator (1) of their conformally flat metrics is $\{0, 0, 0, 0, 0, 0\}$. In particular in these algebras any conformally flat metric has vanishing scalar curvature.

Proposition 4.1. *Any conformally flat almost-Kähler structure on $\mathfrak{rr}_{3,0}$ is Kähler.*

Proof. Let (ω, J, g) be a conformally flat almost-Kähler structure on $\mathfrak{rr}_{3,0}$. From the above discussion the Riemannian scalar curvature s^g is vanishing. Now, from the decomposition (2), we have

$$\|W^+\|^2 = 2\|W_\omega^+\|^2 + \|W_{00}^+\|^2 + \frac{3}{2} \left(\|N_J\|^2 + \frac{s^g}{6} \right)^2.$$

Since g is conformally flat, $\|N_J\|^2 + \frac{s^g}{6} = 0$. So we deduce that $s^g = \|N_J\|^2 = 0$ and that J is integrable. ■

Now, we study the remaining Lie algebra \mathfrak{r}'_2 . This algebra is not unimodular and from the spectrum of the Riemannian curvature operator we can deduce that every conformally flat metric on \mathfrak{r}'_2 has negative scalar curvature. In terms of 1-forms the structure equations of \mathfrak{r}'_2 take the following expression

$$de^1 = 0, \quad de^2 = 0, \quad de^3 = -e^{13} + e^{24}, \quad de^4 = -e^{23} - e^{14},$$

where $\{e^i\}$ is a basis of $(\mathfrak{r}'_2)^*$.

We give a description of all the possible conformally flat metrics on \mathfrak{r}'_2 . Let g be an arbitrary metric on \mathfrak{r}'_2 . By applying the Gram-Schmidt algorithm we construct from the basis $\{e^i\}$ a g -orthonormal basis $\{f^i\}$ of $(\mathfrak{r}'_2)^*$ such that

$$\begin{aligned} f^1 &= a_1 e^1, & f^2 &= a_2 f^1 + a_3 e^2, & f^3 &= a_4 f^1 + a_5 f^2 + a_6 e^3, \\ f^4 &= a_7 f^1 + a_8 f^2 + a_9 f^3 + a_{10} e^4, \end{aligned}$$

for some $a_i \in \mathbb{R}$ with $a_1, a_3, a_6, a_{10} > 0$. By differentiating the coframe $\{f^i\}$ we get

$$df^1 = 0, \quad df^2 = 0, \quad df^3 = -a_6e^{13} + a_6e^{24} \quad df^4 = a_9(-a_6e^{13} + a_6e^{24}) - a_{10}e^{23} - a_{10}e^{14}.$$

And so we deduce the following structure equations with respect to the coframe $\{f^i\}$:

$$df^1 = 0, \quad df^2 = 0,$$

$$df^3 = \frac{a_1a_2a_6a_8 + a_1a_6a_7 + a_{10}a_3a_5}{a_1a_3a_{10}}f^{12} + \frac{a_1a_2a_6a_9 - a_{10}a_3}{a_1a_3a_{10}}f^{13} - \frac{a_6a_2}{a_3a_{10}}f^{14} \\ - \frac{a_6a_9}{a_3a_{10}}f^{23} + \frac{a_6}{a_3a_{10}}f^{24}$$

$$df^4 = \frac{a_1a_2a_6^2a_8a_9 - a_1a_{10}^2a_2a_5 + a_1a_6^2a_7a_9 + a_{10}a_3a_5a_6a_9 - a_1a_{10}^2a_4 + a_{10}a_3a_6a_8}{a_1a_3a_6a_{10}}f^{12} \\ + \frac{a_2a_6^2a_9^2 + a_{10}^2}{a_3a_{10}a_6}f^{13} - \frac{a_1a_2a_6a_9 + a_{10}a_3}{a_1a_3a_{10}}f^{14} - \frac{a_6^2a_9^2 + a_{10}^2}{a_3a_{10}a_6}f^{23} + \frac{a_9a_6}{a_3a_{10}}f^{24}.$$

We would like to find all possible solutions for the vanishing of the Weyl tensor $W = W^+ + W^-$. A direct computation gives

$$W(f_1, f_3, f_2, f_3) = \frac{a_2((a_9^2 - 1)a_6^2 + a_{10}^2)((a_9^2 + 1)a_6^2 + a_{10}^2)a_1 - 2a_{10}a_3a_6^3a_9}{a_1a_3^2a_{10}^2a_6^2}$$

and so

$$W = 0 \implies a_2((a_9^2 - 1)a_6^2 + a_{10}^2)((a_9^2 + 1)a_6^2 + a_{10}^2)a_1 - 2a_{10}a_3a_6^3a_9 = 0.$$

Notice that since $a_1, a_3, a_6, a_{10} > 0$, $(a_9^2 + 1)a_6^2 + a_{10}^2 \neq 0$. Assume g conformally flat (i.e $W = 0$). If we assume $(a_9^2 - 1)a_6^2 + a_{10}^2 \neq 0$, then we get

$$a_2 = \frac{2a_{10}a_3a_6^3a_9}{((a_9^2 - 1)a_6^2 + a_{10}^2)((a_9^2 + 1)a_6^2 + a_{10}^2)a_1}, \quad (4)$$

and we get, assuming (4)

$$W(f_1, f_3, f_2, f_4) = \frac{(a_6^2a_9^2 + a_{10}^2 - 2a_{10}a_6 + a_6^2)(a_6^2a_9^2 + a_{10}^2 + 2a_{10}a_6 + a_6^2)}{-2a_3a_{10}(a_6^2a_9^2 + a_{10}^2 - a_6^2)a_6a_1} = 0$$

Since $a_6^2a_9^2 + a_{10}^2 + 2a_{10}a_6 + a_6^2 \neq 0$ we deduce

$$a_6^2a_9^2 + a_{10}^2 - 2a_{10}a_6 + a_6^2 = a_6^2a_9^2 + (a_{10} - a_6)^2 = 0.$$

Hence $a_{10} = a_6$ and $a_9 = 0$. This is in contradiction to our hypothesis above, namely $(a_9^2 - 1)a_6^2 + a_{10}^2 \neq 0$, and we conclude that $(a_9^2 - 1)a_6^2 + a_{10}^2 = 0$. Thus,

$$a_9 = \pm \sqrt{1 - \frac{a_{10}^2}{a_6^2}}. \quad (5)$$

We get, assuming (5), $W(f_1, f_3, f_2, f_3) = \pm \frac{a_6 \sqrt{1 - \frac{a_{10}^2}{a_6^2}}}{a_1a_3a_{10}} = 0$.

We deduce that (recall that $a_6, a_{10} > 0$): $a_{10} = a_6$. (6)

We obtain, assuming (5) and (6),

$$W(f_1, f_2, f_2, f_3) = \frac{a_1 a_2 a_5 + a_1 a_4 - a_3 a_8}{-4a_3^2 a_1} = 0.$$

Hence, $a_8 = \frac{a_1 a_2 a_5 + a_1 a_4}{a_3}$. (7)

Assuming (5), (6) and (7), we get

$$W(f_2, f_4, f_3, f_4) = \frac{a_1^2 a_2^2 a_5 + a_1^2 a_2 a_4 + a_1 a_3 a_7 + a_3^2 a_5}{-4a_3^2 a_1^2} = 0.$$

We deduce that $a_7 = \frac{-a_1^2 a_2^2 a_5 - a_1^2 a_2 a_4 - a_3^2 a_5}{a_1 a_3}$. (8)

Under the conditions (5), (6), (7) and (8), one can verify that the Weyl tensor vanishes and so we get

Proposition 4.2. *Conformally flat metrics on \mathfrak{r}'_2 are described by the orthonormal basis $\{f^i\}$ of $(\mathfrak{r}'_2)^*$*

$$f^1 = a_1 e^1, \quad f^2 = a_2 f^1 + a_3 e^2, \quad f^3 = a_4 f^1 + a_5 f^2 + a_6 e^3, \quad f^4 = a_7 f^1 + a_8 f^2 + a_9 f^3 + a_{10} e^4,$$

($a_1, a_3, a_6, a_{10} > 0$) subject to the relations

$$(1) \quad a_6 = a_{10}, \quad (2) \quad a_9 = 0, \quad (3) \quad a_8 = \frac{a_1 a_2 a_5 + a_1 a_4}{a_3},$$

$$(4) \quad a_7 = \frac{-a_1^2 a_2^2 a_5 - a_1^2 a_2 a_4 - a_3^2 a_5}{a_1 a_3}.$$

Now, we consider a conformally flat almost-Kähler structures (ω, J, g) on \mathfrak{r}'_2 . The symplectic form can be written as

$$\omega = b_1 f^{12} + b_2 f^{13} + b_3 f^{14} + b_4 f^{23} + b_5 f^{24} + b_6 f^{34},$$

where $b_i \in \mathbb{R}$ and the basis $\{f^i\}$ is a g -orthonormal basis of $(\mathfrak{r}'_2)^*$ as described in Proposition 4.2. In order for ω to be closed it is easy to check that (keeping in mind that $a_1, a_3, a_6 > 0$)

$$(1) \quad b_6 = 0, \quad (2) \quad b_5 := \frac{-a_1 a_2 a_6 b_4 - a_1 a_6 b_2}{a_3 a_6}, \quad (3) \quad b_4 = \frac{-a_1^2 a_2 b_2 + a_1 a_3 b_3}{a_1^2 a_2^2 + a_3^2},$$

and the 2-form ω has the form

$$\omega = b_1 f^{12} + b_2 f^{13} + b_3 f^{14} - \frac{a_1(a_1 a_2 b_2 - a_3 b_3)}{a_1^2 a_2^2 + a_3^2} f^{23} - \frac{a_1(a_1 a_2 b_3 + a_3 b_2)}{a_1^2 a_2^2 + a_3^2} f^{24}.$$

The matrix

$$J = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & -\frac{a_1(a_1a_2b_2 - a_3b_3)}{a_1^2a_2^2 + a_3^2} & -\frac{a_1(a_1a_2b_3 + a_3b_2)}{a_1^2a_2^2 + a_3^2} \\ -b_2 & \frac{a_1(a_1a_2b_2 - a_3b_3)}{a_1^2a_2^2 + a_3^2} & 0 & 0 \\ -b_3 & \frac{a_1(a_1a_2b_3 + a_3b_2)}{a_1^2a_2^2 + a_3^2} & 0 & 0 \end{bmatrix}$$

has to satisfy $J^2 = -\text{Id}$, in order for (ω, J, g) to be an almost-Kähler structure. From $J^2 = -\text{Id}$ we see that $b_1b_2 = b_1b_3 = 0$. If we suppose that $b_1 \neq 0$ then $b_2 = b_3 = 0$ but then $\omega \wedge \omega = 0$. Hence, we deduce that

$$b_1 = 0. \quad (9)$$

Assuming (9), we get, from $J^2 = -\text{Id}$, that $b_2^2 + b_3^2 = 1$. We also obtain

$$\frac{b_2a_1(a_1a_2b_2 - a_3b_3)}{a_1^2a_2^2 + a_3^2} + \frac{b_3a_1(a_1a_2b_3 + a_3b_2)}{a_1^2a_2^2 + a_3^2} = 0.$$

This implies $a_2a_1^2(b_2^2 + b_3^2) = a_2a_1^2 = 0$. We deduce that $a_2 = 0$ and get $a_3 = a_1$.

Proposition 4.3. *Conformally flat almost-Kähler structures (ω, J, g) on \mathfrak{v}'_2 are described by*

$$g = f^1 \otimes f^1 + f^2 \otimes f^2 + f^3 \otimes f^3 + f^4 \otimes f^4,$$

$$\omega = b_2f^{13} + b_3f^{14} - \frac{a_1(a_1a_2b_2 - a_3b_3)}{a_1^2a_2^2 + a_3^2}f^{23} - \frac{a_1(a_1a_2b_3 + a_3b_2)}{a_1^2a_2^2 + a_3^2}f^{24}.$$

The $\{f^i\}$ are described in Proposition 4.2. The constants $a_1, a_2, a_3, a_4, a_5, a_6, b_2, b_3$ are subject to the relations (1) $a_3 = a_1$, (2) $b_2^2 + b_3^2 = 1$, (3) $a_2 = 0$.

All the conformally flat almost-Kähler structures given in Proposition 4.3 are non-Kähler. Indeed if we compute the Nijenhuis tensor we get for instance

$$N(f_1, f_2) = \frac{b_2^2 + b_3^2}{2a_1}f_2 = \frac{1}{2a_1}f_2 \neq 0.$$

We now compute the holomorphic Hermitian sectional curvature of the unit vectors f_i of the conformally flat almost-Kähler structures described in Prop. 4.3. We get

$$(1) H(f_1) = \frac{-1}{a_1^2}, \quad (2) H(f_2) = \frac{-1}{2a_1^2}, \quad (3) H(f_3) = -\frac{1 + b_2^2}{2a_1^2}, \quad (4) H(f_4) = -\frac{1 + b_3^2}{2a_1^2}.$$

We conclude that none of those almost-Kähler structures is of pointwise constant Hermitian holomorphic sectional curvature and we get the following

Proposition 4.4. *The Lie algebra \mathfrak{v}'_2 has no conformally flat almost-Kähler structures of pointwise constant Hermitian holomorphic sectional curvature.*

We can now prove our main result by combining all the previous results.

Proof of Theorem 1.1. Suppose that \mathfrak{g} is a 4-dimensional Lie algebra with an almost-Kähler structure (ω, J, g) of pointwise constant Hermitian holomorphic sectional curvature. It follows from Theorem 2.1 that $W^- = 0$. If the metric is not conformally flat then it is given in [5] and it follows from Corollary 3.1 that it is Kähler. If it is conformally flat then it is described in [14, Theorem 2] and it follows from Corollary 4.1 and Proposition 4.4 that it is Kähler. ■

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