

Metric Symplectic Lie Algebras

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Abstract. It is shown that any metric symplectic Lie algebra has the structure of a quadratic extension. A standard model for such extensions is proposed, and the equivalence classes at the level of the quadratic cohomology sets is given. Further, a classification scheme for metric symplectic Lie algebras is provided, as well as an explicit list of isomorphism classes for some special types.

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1. Introduction

In this paper we concentrate on metric symplectic Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \omega)$. These are symplectic Lie algebras (\mathfrak{g}, ω) which are also metric Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. Of course, an isomorphism of metric symplectic Lie algebras is an isomorphism of the corresponding symplectic Lie algebras which is also an isomorphism of the corresponding metric Lie algebras. We call a metric symplectic Lie algebra decomposable if it is isomorphic to the direct sum of two non-trivial metric symplectic Lie algebras. Here a symplectic Lie algebra (\mathfrak{g}, ω) is a Lie algebra \mathfrak{g} admitting a closed non-degenerate 2-form ω on \mathfrak{g} , which we call symplectic form. Two symplectic Lie algebras $(\mathfrak{g}_i, \omega_i)$, $i = 1, 2$ are isomorphic if there is an isomorphism $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of Lie algebras which preserves the symplectic forms in the sense $\varphi^*\omega_2 = \omega_1$. Symplectic Lie algebras are in one-to-one correspondence with simply connected Lie groups with leftinvariant symplectic forms. Symplectic Lie algebras are also called quasi-Frobenius Lie algebras, since Frobenius Lie Algebras, i. e. Lie algebras admitting an non-degenerate exact 2-form, are examples of symplectic Lie algebras. Moreover, symplectic Lie algebras are examples of Vinberg algebras, since they naturally carry an affine structure.

A metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a Lie algebra \mathfrak{g} with an non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , which is ad-invariant, i. e.

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$$

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for all $X, Y, Z \in \mathfrak{g}$. Although, $\langle \cdot, \cdot \rangle$ is not necessary positive definite, we will call it inner product. In the literature these Lie algebras are also called quadratic Lie algebras. Two metric Lie algebras are isomorphic if there is an isomorphism of the corresponding Lie algebras, which is also an isometry of the inner products.

Metric Lie algebras are in one-to-one correspondence with simply connected Lie groups with biinvariant metrics. They are also used for describing symmetric spaces, since the Lie algebra of the transvection group of a symmetric space admits such a structure [5, Proposition 1.6].

There are several classifications of metric or symplectic Lie algebras in low dimension ([6], [9], [19], [20]). Especially, the nilpotent case ([4], [12], [15], see also [10]) is important for our problem, since every metric symplectic Lie algebra is nilpotent.

If the aim is to give information for arbitrary dimensions, then usually a reduction scheme is used. In this context we mention double extensions ([16], [17]), T^* -extensions [3], symplectic double extensions ([7], [8], [18]) and oxidation [2]. The main idea is, that for every isotropic ideal \mathfrak{j} of a metric or symplectic Lie algebra \mathfrak{g} the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{j}^\perp/\mathfrak{j}$ inherits an inner product or symplectic form respectively from \mathfrak{g} . Conversely, they give a construction scheme taking a metric or symplectic Lie algebra $\bar{\mathfrak{g}}$ and some additional structure and construct a higher dimensional metric respectively symplectic Lie algebra, which can be reduced to $\bar{\mathfrak{g}}$ again. Since the choice of the isotropic ideal is in general not canonical, it is hard to give a general statement on the isomorphy of this Lie algebras with the help of these schemes. For instance, it is possible that extensions of two non-isomorphic low dimensional metric or symplectic Lie algebras are isomorphic. Moreover, the presentation of a metric or symplectic Lie algebra as such an extension is not unique, since it depends on the chosen ideal.

The aim of this paper is to take this choices canonically for metric symplectic Lie algebras such that there is a certain standard model and the possibility to analyse the isomorphy systematically with this standard model.

Until now, there is just a few literature about metric symplectic Lie algebras (for example [1]). These Lie algebras are in one-to-one correspondence to nilpotent metric Lie algebras with bijective skew-symmetric derivations. On every Lie group with Lie algebra \mathfrak{g} there is a flat and torsion-free connection ∇^ω induced by ω and an left-invariant pseudo-Riemannian metric whose Levi-Civita connection equals ∇^ω [18].

A classification scheme for metric Lie algebras from Kath and Olbrich denoted as quadratic extension is very useful for our aim. This scheme was first introduced in [13], where metric Lie algebras were discussed comprehensively. Using the main idea of [13] they introduced a classification scheme in [14], which can be used for metric Lie algebras and metric Lie algebras with additional structure. Especially, metric Lie algebras with semisimple skew-symmetric derivations were treated in that paper in the notion of $(\mathbb{R}, \{e\})$ -equivariant metric Lie algebras. Thus, this is useful for our problem. Our aim is to expand the classification scheme in [14] for metric Lie algebras with semisimple skew-symmetric derivations in such a way that it classifies all metric symplectic Lie algebras.

In the following, we describe the main idea of the present paper. For every metric Lie algebra \mathfrak{g} with bijective skew-symmetric derivation D there is a D -invariant isotropic ideal \mathfrak{i} of \mathfrak{g} in a canonical way such that $\mathfrak{i}^\perp/\mathfrak{i}$ is abelian. This ideal is given by

$$\mathfrak{i} = \mathfrak{i}(\mathfrak{g}) := \sum_{k=0}^{n-1} \mathfrak{g}^{k+1} \cap \mathfrak{g}^{k+1\perp} \tag{1}$$

(compare [12]). Here $\mathfrak{g}^1, \dots, \mathfrak{g}^n$ denotes the descending central sequence of the nilpotent Lie algebra \mathfrak{g} and n the smallest positive integer such that $\mathfrak{g}^n = \{0\}$. For the definition of this ideal in the case of non-nilpotent metric Lie algebras see [13] or [14]. We set $\mathfrak{a} = \mathfrak{i}^\perp/\mathfrak{i}$ and $\mathfrak{l} = \mathfrak{g}/\mathfrak{i}^\perp$ and obtain that \mathfrak{a} is an abelian metric symplectic Lie algebra and inherits the structure of a trivial \mathfrak{l} -module. Since \mathfrak{i} is isomorphic to \mathfrak{l}^* , we can write \mathfrak{g} as two abelian extensions of Lie algebras with bijective derivations

$$0 \rightarrow \mathfrak{l}^* \rightarrow \mathfrak{g} \rightarrow \mathfrak{h} \rightarrow 0, \quad 0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{h} \rightarrow \mathfrak{l} \rightarrow 0 \tag{2}$$

in a canonical way. Here $\mathfrak{h} = \mathfrak{g}/\mathfrak{i}$. Conversely, two abelian extensions given as in (2) define a metric symplectic Lie algebra for every Lie algebra \mathfrak{l} with bijective derivation and \mathfrak{l} -module \mathfrak{a} , unless the abelian extensions satisfy certain compatibility conditions. Then the image of \mathfrak{l}^* in \mathfrak{g} is usually not equal to the canonical isotropic ideal.

We will introduce this construction scheme under the notion of quadratic extensions. In general, not every quadratic extension of \mathfrak{l} by \mathfrak{a} is given by the canonical isotropic ideal $\mathfrak{i}(\mathfrak{g})$. Thus we also introduce balanced quadratic extensions. We call a quadratic extension balanced, if the image of \mathfrak{l}^* in \mathfrak{g} equals the canonical isotropic ideal. Every metric symplectic Lie algebra has the structure of a balanced quadratic extension in a canonical way. There is a natural equivalence relation on the set of quadratic extensions of \mathfrak{l} by \mathfrak{a} . Moreover, we define a non-linear cohomology $H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ of the Lie algebra \mathfrak{l} with coefficients in \mathfrak{a} , which includes the cohomology $H_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ considered in [14]. Then we prove that the equivalence classes of quadratic extensions of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ are in bijection to the second cohomology set $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$. For the proof we cite necessary results for metric Lie algebras with semisimple skew-symmetric derivations and fit this to the case of not necessary semisimple derivations. Moreover, we give the standard model $(\mathfrak{d}_{\alpha,\gamma}, D_{\delta,\epsilon})$ of a metric Lie algebra with skew-symmetric derivation, which defines a quadratic extension of \mathfrak{l} by \mathfrak{a} for every $[\alpha, \gamma, \delta, \epsilon] \in H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.

The equivalence classes of balanced quadratic extensions of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ are described by $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b \subset H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ on the level of the cohomology classes. We also call these cocycles balanced. Since being balanced is a property of a quadratic extension which does only depend on the structure of the corresponding metric Lie algebra, we can use [13] for describing balanced cocycles, or more precisely [12] for nilpotent, metric Lie algebras. We also have the notion of indecomposable balanced cohomology classes, contained in $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0 \subset H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$. The automorphism group $G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ of the pair $((\mathfrak{l}, D_{\mathfrak{l}}), \mathfrak{a})$ acts on $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$. Altogether, the isomorphism classes of metric, symplectic Lie algebras are in one-to-one correspondence to

$$\coprod_{(\mathfrak{l}, D_{\mathfrak{l}}) \in \mathcal{L}} \coprod_{\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}} H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0 / G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}),$$

where \mathcal{L} is a system of representatives of the isomorphism classes of nilpotent Lie algebras with bijective derivations and $\mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}$ a system of representatives of the

isomorphism classes of abelian metric Lie algebras with bijective skew-symmetric derivations (considered as trivial \mathfrak{l} -modules). Using this classification scheme we obtain the following results:

- There is only one non-abelian metric Lie algebra (up to isomorphism) of dimension less than eight, which admits symplectic forms.
- There are no metric symplectic Lie algebras whose inner product has an index of one or two, except for abelian ones.
- We calculate all isomorphism classes of metric symplectic Lie algebras of dimension less than ten and
- we calculate every non-abelian metric symplectic Lie algebra with an index of three up to isomorphism.

The paper is organized as follows. We introduce the notion of (balanced) quadratic extensions in section 3 and show that every metric symplectic Lie algebra has the structure of a (balanced) quadratic extension in a canonical way. Furthermore, we give an equivalence relation on the set of quadratic extensions. Then, we define the quadratic cohomology $H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ in section 4. We also define the isomorphism of pairs and give their action on the quadratic cohomology. In section 5 we define the standard model $(\mathfrak{d}_{\alpha,\gamma}, D_{\delta,\epsilon})$ and show necessary and sufficient conditions on the level of the corresponding cocycles, when $(\mathfrak{d}_{\alpha,\gamma}, D_{\delta,\epsilon})$ has the structure of a (balanced) quadratic extension. We show that every quadratic extension is equivalent to a suitable standard model (section 6) and describe the equivalence of standard models (section 7). In the end of section 7 we obtain a bijection between $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ and the equivalence classes of quadratic extensions of \mathfrak{l} by \mathfrak{a} . In section 8 we describe the isomorphism of standard models on the level of the corresponding cohomology classes and obtain, finally, the classification scheme for metric symplectic Lie algebras. As an application, we calculate in section 9 all non-abelian metric symplectic Lie algebras (up to isomorphisms) whose index of the inner product is less than four. Moreover, we give a system of representatives of all non-abelian metric symplectic Lie algebras of dimension less than ten.

2. Metric symplectic Lie algebras

Definition 2.1. A *metric, symplectic Lie algebra* $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \omega)$ is a Lie algebra \mathfrak{g} with a non-degenerate (not necessarily positive definite) symmetric bilinear form $\langle \cdot, \cdot \rangle: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ and a non-degenerate skew-symmetric bilinear form $\omega: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ satisfying

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle \quad \text{and}$$

$$d\omega(X, Y, Z) = -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Usually, we will call $\langle \cdot, \cdot \rangle$ *inner product* and ω *symplectic form*.

Definition 2.2. An *isomorphism* φ between metric symplectic Lie algebras $(\mathfrak{g}_1, \langle \cdot, \cdot \rangle_1, \omega_1)$ and $(\mathfrak{g}_2, \langle \cdot, \cdot \rangle_2, \omega_2)$ is an isomorphism $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of the corresponding Lie algebras, which is an isometry, i. e. $\varphi^* \langle \cdot, \cdot \rangle_2 = \langle \cdot, \cdot \rangle_1$, and satisfies $\varphi^* \omega_2 = \omega_1$.

3. Quadratic extensions

In this section we introduce the necessary notion of quadratic extensions for the new classification scheme for metric symplectic Lie algebras. Moreover, we show that every metric symplectic Lie algebra has the structure of a quadratic extension in a canonical way.

Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \omega)$ be a metric symplectic Lie algebra. Then

$$\omega = \langle \cdot, D \cdot \rangle \tag{3}$$

defines a bijective skew-symmetric map D on \mathfrak{g} , since $\langle \cdot, \cdot \rangle$ is non-degenerate. Here D is called *skew-symmetric*, if $\langle DX, Y \rangle = -\langle X, DY \rangle$ for all $X, Y \in \mathfrak{g}$. Moreover, D is a derivation on \mathfrak{g} , since ω is closed. Now, the existence of a bijective derivation on \mathfrak{g} implies, that \mathfrak{g} is nilpotent [11, Theorem 3]. Conversely, every bijective skew-symmetric derivation of a nilpotent metric Lie-Algebra \mathfrak{g} defines a symplectic form ω on \mathfrak{g} by equation (3). Thus we obtain the following Lemma.

Lemma 3.1. *Metric symplectic Lie algebras are in one-to-one correspondence with nilpotent metric Lie-Algebras with skew-symmetric bijective derivations.*

From now on, we concentrate on metric Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ with skew-symmetric derivation D and write abbreviatory $(\mathfrak{g}, \langle \cdot, \cdot \rangle, D)$, (\mathfrak{g}, D) or simply \mathfrak{g} unless it is clear from the context that this is a metric Lie algebra with a skew-symmetric derivation.

Definition 3.2. An isomorphism (homomorphism) φ between Lie algebras with derivations (\mathfrak{g}_1, D_1) and (\mathfrak{g}_2, D_2) is an isomorphism (homomorphism) $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of the corresponding Lie algebras, which satisfies $D_2\varphi = \varphi D_1$. An isomorphism between metric Lie algebras with derivations is an isomorphism of Lie algebras with derivations, which is in addition an isometry.

Lemma 3.3. *Let (\mathfrak{g}, D) be a metric Lie algebra with derivation. Then the semisimple part D_s of the Jordan-Chevalley decomposition of the skew-symmetric derivation D is also a skew-symmetric derivation of \mathfrak{g} .*

Proof. Consider the complexification of D and $\langle \cdot, \cdot \rangle$, if D has nonreal eigenvalues. Let v_1 and v_2 denote two generalized eigenvectors of D corresponding to the eigenvalues λ_1 and λ_2 . For a sufficiently large $k \in \mathbb{R}$ we have that

$$(D - (\lambda_1 + \lambda_2) \text{id})^k [v_1, v_2] = \sum_{i=0}^k \binom{k}{i} [(D - \lambda_1 \text{id})^{k-i} v_1, (D - \lambda_2 \text{id})^i v_2]$$

vanishes. Thus $[v_1, v_2]$ is an vector of the generalized eigenspace for $\lambda_1 + \lambda_2$ and we obtain $D_s[v_1, v_2] = (\lambda_1 + \lambda_2)[v_1, v_2]$. Thus, D_s is a derivation, since

$$D_s[v_1, v_2] = (\lambda_1 + \lambda_2)[v_1, v_2] = [\lambda_1 v_1, v_2] + [v_1, \lambda_2 v_2] = [D_s v_1, v_2] + [v_1, D_s v_2].$$

Since D is skew-symmetric, the generalized eigenspaces for λ_1 and λ_2 , $\lambda_1 \neq -\lambda_2$ are orthogonal to each other. These subspaces are invariant under D_s so that we

have $\langle D_s v_1, v_2 \rangle + \langle v_1, D_s v_2 \rangle = 0$. For $\lambda_1 = -\lambda_2$ we obtain

$$\langle D_s v_1, v_2 \rangle + \langle v_1, D_s v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle - \lambda_1 \langle v_1, v_2 \rangle = 0.$$

Hence D_s is skew-symmetric. ■

Definition 3.4. Let \mathfrak{l} be a Lie algebra. The triple $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ is called *orthogonal \mathfrak{l} -module*, if $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ is an abelian metric Lie algebra and $\rho: \mathfrak{l} \rightarrow \text{Hom}(\mathfrak{a})$ a representation of the Lie algebra \mathfrak{l} on \mathfrak{a} , which is skew-symmetric, i. e. $\langle \rho(L)A_1, A_2 \rangle = -\langle A_1, \rho(L)A_2 \rangle$ for all $L \in \mathfrak{l}$ and $A_1, A_2 \in \mathfrak{a}$.

Now, let $D_{\mathfrak{l}}$ be a derivation on \mathfrak{l} . We call $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\mathfrak{a}})$ *orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module*, if $D_{\mathfrak{a}}$ is a skew-symmetric map on \mathfrak{a} and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ an orthogonal \mathfrak{l} -module satisfying $\rho(D_{\mathfrak{l}}L) = [D_{\mathfrak{a}}, \rho(L)]$ for all $L \in \mathfrak{l}$. We will also write $(\mathfrak{a}, D_{\mathfrak{a}})$ abbreviately for an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module.

Moreover, if ρ is the trivial representation of \mathfrak{l} on \mathfrak{a} , then we will call $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\mathfrak{a}})$ *trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module*.

Lemma 3.5. Let \mathfrak{l} be a Lie algebra, $D_{\mathfrak{l}}$ a derivation on \mathfrak{l} and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\mathfrak{a}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. Then $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\mathfrak{a}_s})$ is an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}_s})$ -module, where $D_{\mathfrak{l}_s}$ and $D_{\mathfrak{a}_s}$ denote the semisimple parts of the Jordan-Chevalley decomposition of $D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$.

Proof. Because of Lemma 3.3 it remains to show that

$$\rho(D_{\mathfrak{l}_s}L) = [D_{\mathfrak{a}_s}, \rho(L)]$$

holds for all $L \in \mathfrak{l}$. Consider the complexification of $D_{\mathfrak{a}}$, $D_{\mathfrak{l}}$ and ρ if $D_{\mathfrak{a}}$ or $D_{\mathfrak{l}}$ have nonreal eigenvectors. Let $L \in \mathfrak{l}$ be a vector in the generalized eigenspace of $D_{\mathfrak{l}}$ for λ and $A \in \mathfrak{a}$ in the generalized eigenspace of $D_{\mathfrak{a}}$ for μ . Similary to the proof of Lemma 3.3 we obtain

$$(D_{\mathfrak{a}} - (\lambda + \mu) \text{id})^k \rho(L)A = \sum_{i=0}^k \binom{k}{i} \rho((D_{\mathfrak{l}} - \lambda \text{id})^{k-i}) (D_{\mathfrak{a}} - \mu \text{id})^i A.$$

Thus $(D_{\mathfrak{a}} - (\lambda + \mu) \text{id})^k \rho(L)A$ vanishes for sufficiently large k . Hence $\rho(L)A$ is a vector in the generalized eigenspace for $\lambda + \mu$. Finally $D_{\mathfrak{a}_s} \rho(L)A = (\lambda + \mu) \rho(L)A$ and

$$D_{\mathfrak{a}_s} \rho(L)A - \rho(L)D_{\mathfrak{a}_s}A = \lambda \rho(L)A = \rho(D_{\mathfrak{l}_s}L)A. \quad \blacksquare$$

Definition 3.6. Let \mathfrak{l} be a Lie algebra, $D_{\mathfrak{l}}$ a derivation of \mathfrak{l} and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle, D_{\mathfrak{a}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. A *quadratic extension* of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ is a quadruple $(\mathfrak{g}, \mathfrak{i}, i, p)$, where

- $\mathfrak{g} = (\mathfrak{g}, \langle \cdot, \cdot \rangle, D)$ is a metric Lie algebra with skew-symmetric derivation,
- $\mathfrak{i} \subset \mathfrak{g}$ is an isotropic D -invariant ideal of \mathfrak{g} ,
- $i: (\mathfrak{a}, D_{\mathfrak{a}}) \rightarrow (\mathfrak{g}/\mathfrak{i}, \bar{D})$ and $p: (\mathfrak{g}/\mathfrak{i}, \bar{D}) \rightarrow (\mathfrak{l}, D_{\mathfrak{l}})$ are homomorphisms of Lie algebras with derivations such that

$$0 \rightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{l} \rightarrow 0$$

is a short exact sequence of Lie algebras. Here \overline{D} denotes the skew-symmetric derivation of $\mathfrak{g}/\mathfrak{i}$ induced by D . Moreover,

$$i(\rho(L)A) = [\tilde{L}, i(A)] \in i(\mathfrak{a}) \tag{4}$$

holds for all $\tilde{L} \in \mathfrak{g}/\mathfrak{i}$ satisfying $p(\tilde{L}) = L$. Furthermore, $\text{im}(i) = \mathfrak{i}^\perp/\mathfrak{i}$ and i is an isometry onto $\mathfrak{i}^\perp/\mathfrak{i}$.

Lemma 3.7. *Let $(\mathfrak{g}, D; \mathfrak{i}, i, p)$ be a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$. Then $(\mathfrak{g}, D_s; \mathfrak{i}, i, p)$ is also a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}_s})$ by $(\mathfrak{a}, D_{\mathfrak{a}_s})$. Here $D_s, D_{\mathfrak{l}_s}$ and $D_{\mathfrak{a}_s}$ denote the semisimple parts of the Jordan-Chevalley decomposition of $D, D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$.*

Proof. It is well known that every D -invariant subspace is also invariant under the semisimple part D_s of the Jordan-Chevalley decomposition of D . So it is not hard to see that $i: (\mathfrak{a}, D_{\mathfrak{a}}) \rightarrow (\mathfrak{g}/\mathfrak{i}, \overline{D})$ and $p: (\mathfrak{g}/\mathfrak{i}, \overline{D}) \rightarrow (\mathfrak{l}, D_{\mathfrak{l}})$ are homomorphisms of Lie algebras with corresponding semisimple derivations. ■

Let \mathfrak{i} be an isotropic D -invariant ideal of a metric Lie algebra \mathfrak{g} with bijective skew-symmetric derivation D such that $\mathfrak{i}^\perp/\mathfrak{i}$ is abelian. Then the short exact sequence

$$0 \rightarrow \mathfrak{i}^\perp/\mathfrak{i} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{g}/\mathfrak{i}^\perp \rightarrow 0 \tag{5}$$

defines a quadratic extension of $\mathfrak{g}/\mathfrak{i}^\perp$ by $\mathfrak{i}^\perp/\mathfrak{i}$ with corresponding induced derivations.

Remark 3.8. This was already proved for metric Lie algebras without additional structure in [13, Page 94]. In [14], this statement was generalized for metric Lie algebras with additional structure, the so called (\mathfrak{h}, K) -equivariant metric Lie algebras. This also includes metric Lie algebras with semisimple skew-symmetric derivations in the notation of $(\mathbb{R}, \{e\})$ -equivariant metric Lie algebras. But it is not necessary that the derivations are semisimple to define a quadratic extension using the short exact sequence (5). ■

We already know that nilpotent metric Lie algebras with bijective skew-symmetric derivations are the main structure to determine the isomorphism classes of metric symplectic Lie algebras. So, let \mathfrak{g} denote a nilpotent metric Lie algebra and D a skew-symmetric, bijective derivation of \mathfrak{g} for the rest of this section.

We are interested in quadratic extensions whose ideal is the canonical isotropic ideal

$$\mathfrak{i}(\mathfrak{g}) := \sum_{k=0}^{n-1} \mathfrak{g}^{k+1} \cap \mathfrak{g}^{k+1}{}^\perp. \tag{6}$$

Here $\mathfrak{g}^1 := \mathfrak{g}, \dots, \mathfrak{g}^k := [\mathfrak{g}, \mathfrak{g}^{k-1}]$ denotes the lower central series of \mathfrak{g} and n the smallest positive integer such that $\mathfrak{g}^n = \{0\}$. This is the definition of the canonical isotropic ideal for nilpotent \mathfrak{g} as it was already used in [12]. The definition of $\mathfrak{i}(\mathfrak{g})$ for a not necessary nilpotent \mathfrak{g} is given in [13, Definition 3.3].

This ideal is isotropic for every nilpotent metric Lie algebra \mathfrak{g} , it is D -invariant for every derivation D of \mathfrak{g} and moreover $\mathfrak{i}^\perp/\mathfrak{i}$ is abelian [13, Lemma 3.4 (d)] [14,

Proposition 2.7]. Thus we will call this ideal $\mathfrak{i}(\mathfrak{g})$ the canonical isotropic ideal. Furthermore, it is not hard to prove that every isometry $F: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of nilpotent metric Lie algebras satisfies $F(\mathfrak{i}(\mathfrak{g}_1)) = \mathfrak{i}(\mathfrak{g}_2)$.

Lemma 3.9. ([12]) *Let \mathfrak{g} be a nilpotent metric Lie algebra and $\mathfrak{i}(\mathfrak{g})$ the canonical isotropic ideal. If $(\mathfrak{g}, \mathfrak{i}(\mathfrak{g}), i, p)$ is a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$, then $(\mathfrak{a}, D_{\mathfrak{a}})$ is a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module, i. e. ρ is trivial.*

Definition 3.10. Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle, D)$ be a nilpotent metric Lie algebra with bijective skew-symmetric derivation D . Moreover, let \mathfrak{l} be a nilpotent Lie algebra, $D_{\mathfrak{l}}$ a bijective derivation of \mathfrak{l} and $(\mathfrak{a}, D_{\mathfrak{a}})$ a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module, where $D_{\mathfrak{a}}$ is bijective. A quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ is called *balanced*, if \mathfrak{i} is the canonical isotropic ideal $\mathfrak{i} = \mathfrak{i}(\mathfrak{g})$.

Theorem 3.11. *Every nilpotent metric Lie algebra \mathfrak{g} with skew-symmetric bijective derivation D has the structure of a balanced quadratic extension. I. e., there is a nilpotent Lie algebra \mathfrak{l} with bijective derivation $D_{\mathfrak{l}}$, an abelian metric Lie algebra \mathfrak{a} with bijective skew-symmetric derivation $D_{\mathfrak{a}}$ considered as a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module and homomorphisms i and p of Lie algebras such that $(\mathfrak{g}, \mathfrak{i}(\mathfrak{g}), i, p)$ is a balanced quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$.*

Proof. We choose $\mathfrak{i} = \mathfrak{i}(\mathfrak{g})$. This ideal is isotropic, D -invariant and $\mathfrak{i}(\mathfrak{g})^{\perp}/\mathfrak{i}(\mathfrak{g})$ is abelian. Thus the sequence (5) defines a quadratic extension with corresponding derivations, which is balanced by definition. ■

It is also possible that $(\mathfrak{g}, \mathfrak{i}, i, p)$ is a balanced quadratic extension of the trivial Lie algebra $\{0\}$ by \mathfrak{a} . This means that $\mathfrak{i}(\mathfrak{g}) = \{0\}$, which is equivalent to \mathfrak{g} is abelian. Thus a non-abelian \mathfrak{g} has the structure of a non-trivial balanced quadratic extension. There is a natural equivalence relation on the set of quadratic extensions.

Definition 3.12. Two quadratic extensions $(\mathfrak{g}_j, D_j; \mathfrak{i}_j, i_j, p_j)$, $j = 1, 2$ of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ are *equivalent*, if there is an isomorphism $F: (\mathfrak{g}_1, D_1) \rightarrow (\mathfrak{g}_2, D_2)$ of the metric Lie algebras with derivations such that $F(\mathfrak{i}_1) = \mathfrak{i}_2$ and the induced isomorphism $\bar{F}: \mathfrak{g}_1/\mathfrak{i}_1 \rightarrow \mathfrak{g}_2/\mathfrak{i}_2$ satisfies $\bar{F} \circ i_1 = i_2$ and $p_2 \circ \bar{F} = p_1$.

We will determine this equivalence relation of quadratic extensions with the help of a certain cohomology class, which we will introduce in section 4.

There is a natural notion of the direct sum of quadratic extensions. That is, if $(\mathfrak{g}_j, \mathfrak{i}_j, i_j, p_j)$, $j = 1, 2$ are quadratic extensions of \mathfrak{l}_j by \mathfrak{a}_j , then

$$(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{i}_1 \oplus \mathfrak{i}_2, i_1 \oplus i_2, p_1 \oplus p_2)$$

is a quadratic extension of $\mathfrak{l}_1 \oplus \mathfrak{l}_2$ by $\mathfrak{a}_1 \oplus \mathfrak{a}_2$.

We call a direct sum *non-trivial*, if both summands are different from the trivial quadratic extension $(\{0\}, \{0\}, 0, 0)$ of $\{0\}$ by $\{0\}$. We call a quadratic extension *decomposable*, if it can be written as a non-trivial direct sum of two quadratic extensions. A quadratic extension, which is equivalent to a decomposable one, is also decomposable. Moreover, if a quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ is decomposable,

then the corresponding metric symplectic Lie algebra \mathfrak{g} is decomposable as a metric symplectic Lie algebra. Conversely, we have the following lemma from [14] (see also [13]), since it does not depend on the semisimplicity of the derivation.

Lemma 3.13. ([14], see also [13]) *Let $(\mathfrak{g}, \mathfrak{i}, i, p)$ be a balanced quadratic extension of \mathfrak{l} by \mathfrak{a} . The quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ is decomposable if and only if \mathfrak{g} is decomposable as a metric symplectic Lie algebra.*

4. Quadratic cohomology

The purpose of this section is to introduce the cocycle, which we will use to describe the quadratic extensions. Afterwards, we define the quadratic cohomology by using a certain group action on the set of cocycles.

Let $\rho: \mathfrak{l} \rightarrow \text{Hom}(\mathfrak{a})$ be a representation of the Lie algebra \mathfrak{l} on the vector space \mathfrak{a} . Let $C^p(\mathfrak{l}, \mathfrak{a}) = \text{Hom}(\bigwedge^p \mathfrak{l}, \mathfrak{a})$ denote the space of alternating p -linear maps of \mathfrak{l} with values in \mathfrak{a} and $(C^*(\mathfrak{l}, \mathfrak{a}), d)$ the standard Lie algebra cochain complex, where $d: C^p(\mathfrak{l}, \mathfrak{a}) \rightarrow C^{p+1}(\mathfrak{l}, \mathfrak{a})$ is defined by

$$\begin{aligned} d\tau(L_1, \dots, L_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(L_i) \tau(L_1, \dots, \hat{L}_i, \dots, L_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \tau([L_i, L_j], L_1, \dots, \hat{L}_i, \dots, \hat{L}_j, \dots, L_{p+1}) \end{aligned}$$

for all $\tau \in C^p(\mathfrak{l}, \mathfrak{a})$. Moreover, let $Z^p(\mathfrak{l}, \mathfrak{a})$ and $B^p(\mathfrak{l}, \mathfrak{a})$ denote the groups of cocycles and coboundaries of $C^p(\mathfrak{l}, \mathfrak{a})$. The cochain complex of \mathfrak{l} with the trivial representation on \mathbb{R} is denoted by $C^*(\mathfrak{l})$.

Definition 4.1. Let $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle)$ be an orthogonal \mathfrak{l} -module. We define a bilinear multiplication $\langle \cdot \wedge \cdot \rangle: C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \rightarrow C^{p+q}(\mathfrak{l}, \mathbb{R})$ by

$$\begin{aligned} \langle \alpha \wedge \tau \rangle(L_1, \dots, L_{p+q}) &= \sum_{[\sigma] \in \mathcal{S}_{p+q}/\mathcal{S}_p \times \mathcal{S}_q} \text{sgn}(\sigma) \langle \alpha(L_{\sigma(1)}, \dots, L_{\sigma(p)}), \tau(L_{\sigma(p+1)}, \dots, L_{\sigma(p+q)}) \rangle. \end{aligned}$$

Here \mathcal{S}_k denotes the symmetric group of k letters.

Lemma 4.2. ([13, page 90]) *Assume $\alpha \in C^p(\mathfrak{l}, \mathfrak{a})$ and $\tau \in C^q(\mathfrak{l}, \mathfrak{a})$. Then*

$$d\langle \alpha \wedge \tau \rangle = \langle d\alpha \wedge \tau \rangle + (-1)^p \langle \alpha \wedge d\tau \rangle \text{ and} \tag{7}$$

$$\langle \alpha \wedge \tau \rangle = (-1)^{pq} \langle \tau \wedge \alpha \rangle. \tag{8}$$

We consider the pairs $(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ of Lie algebras with derivations $(\mathfrak{l}, D_{\mathfrak{l}})$ and orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -modules $(\mathfrak{a}, D_{\mathfrak{a}})$. This pairs form a category, whose morphisms are pairs (S, U) containing an homomorphism $S: \mathfrak{l}_1 \rightarrow \mathfrak{l}_2$ of Lie algebras and an isometric embedding $U: \mathfrak{a}_2 \rightarrow \mathfrak{a}_1$ satisfying $SD_{\mathfrak{l}_1} = D_{\mathfrak{l}_2}S$, $UD_{\mathfrak{a}_2} = D_{\mathfrak{a}_1}U$ and $U \circ \rho_2(SL) = \rho_1(L) \circ U$ for all $L \in \mathfrak{l}_1$.

We will denote the morphisms of this category by morphisms of pairs.

Remark 4.3. If (S, U) is a morphism from $(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)$ to $(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$, so it is a morphism from $(\mathfrak{l}_1, D_{\mathfrak{l}_{1s}}, \mathfrak{a}_1)$ to $(\mathfrak{l}_2, D_{\mathfrak{l}_{2s}}, \mathfrak{a}_2)$, where $D_{\mathfrak{l}_{is}}$ denotes the semisimple part of the Jordan-Chevalley decomposition of $D_{\mathfrak{l}_i}$ and $\mathfrak{a}_i = (\mathfrak{a}_i, D_{\mathfrak{a}_{is}})$ the orthogonal $(\mathfrak{l}_i, D_{\mathfrak{l}_{is}})$ -module with the semisimple part $D_{\mathfrak{a}_{is}}$ of the Jordan-Chevalley decomposition of $D_{\mathfrak{a}_i}$ for $i = 1, 2$.

Remark 4.4. The category of pairs of Lie algebras and orthogonal modules was already introduced in [13]. The morphisms of that category were also called morphisms of pairs. In [14], morphisms of pairs were build, which also respect additional structure on the pairs of Lie algebras and orthogonal modules. A morphism of pairs from $(\mathfrak{l}_1, D_{\mathfrak{l}_{1s}}, \mathfrak{a}_1)$ to $(\mathfrak{l}_2, D_{\mathfrak{l}_{2s}}, \mathfrak{a}_2)$ in the sense of our definition, where $D_{\mathfrak{l}_{is}}$ is a semisimple derivation of \mathfrak{l}_i and $\mathfrak{a}_i = (\mathfrak{a}_i, D_{\mathfrak{a}_{is}})$ an orthogonal $(\mathfrak{l}_i, D_{\mathfrak{l}_{is}})$ -module with semisimple $D_{\mathfrak{a}_{is}}$ for $i = 1, 2$, is exactly the special case of the definition of morphisms of pairs of $(\mathbb{R}, \{e\})$ -equivariant metric Lie algebras in [14].

Definition 4.5. The *direct sum* of two pairs $(\mathfrak{l}_j, D_{\mathfrak{l}_j}, \mathfrak{a}_j)$, $j = 1, 2$ is defined by

$$(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}) = (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \oplus (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2) := (\mathfrak{l}_1 \oplus \mathfrak{l}_2, D_{\mathfrak{l}_1} \oplus D_{\mathfrak{l}_2}, \mathfrak{a}_1 \oplus \mathfrak{a}_2),$$

where \mathfrak{a}_1 and \mathfrak{a}_2 are orthogonal to each other, $D_{\mathfrak{a}_1} \oplus D_{\mathfrak{a}_2}$ is the derivation on $\mathfrak{a}_1 \oplus \mathfrak{a}_2$ and for $i \neq j$, $i, j = 1, 2$ the Lie algebra \mathfrak{l}_i acts trivially on \mathfrak{a}_j . We call a direct sum *non-trivial*, if both summands are different from the trivial pair $(0, 0, 0)$.

Of course, if $(S, U): (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ is an isomorphism of pairs and $(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ a non-trivial direct sum of pairs, then $(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)$ is also a non-trivial direct sum of pairs.

Let $(S, U): (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ be a morphism of pairs. We define the following pull back maps

$$(S, U)^*: C^p(\mathfrak{l}_2, \mathfrak{a}_2) \rightarrow C^p(\mathfrak{l}_1, \mathfrak{a}_1), \quad (S, U)^*\alpha(L_1, \dots, L_p) := U \circ \alpha(S(L_1), \dots, S(L_p))$$

and $(S, U)^*: C^p(\mathfrak{l}_2) \rightarrow C^p(\mathfrak{l}_1), \quad (S, U)^*\gamma(L_1, \dots, L_p) := \gamma(S(L_1), \dots, S(L_p)).$

Lemma 4.6. ([13, page 92]) *The pull backs $(S, U)^*$ commute with the differential d and we have*

$$(S, U)^*\langle \alpha \wedge \tau \rangle = \langle (S, U)^*\alpha \wedge (S, U)^*\tau \rangle \tag{9}$$

for $\alpha \in C^p(\mathfrak{l}_2, \mathfrak{a}_2)$ and $\tau \in C^q(\mathfrak{l}_2, \mathfrak{a}_2)$.

Let \mathfrak{l} be a Lie algebra, $D_{\mathfrak{l}}$ a derivation of \mathfrak{l} and $(\mathfrak{a}, D_{\mathfrak{a}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. Then $(e^{-tD_{\mathfrak{l}}}, e^{tD_{\mathfrak{a}}})$ is an isomorphism of pairs for every $t \in \mathbb{R}$. For $\alpha \in C^p(\mathfrak{l}, \mathfrak{a})$ denote $D^\circ \alpha = \frac{d}{dt}(e^{-tD_{\mathfrak{l}}}, e^{tD_{\mathfrak{a}}})^* \alpha|_{t=0}$. We obtain

$$D^\circ \alpha(L_1, \dots, L_p) = D_{\mathfrak{a}}(\alpha(L_1, \dots, L_p)) - \sum_{i=1}^p \alpha(L_1, \dots, D_{\mathfrak{l}}L_i, \dots, L_p).$$

For $\gamma \in C^p(\mathfrak{l})$ we get analogous

$$D^\circ \gamma(L_1, \dots, L_p) = - \sum_{i=1}^p \gamma(L_1, \dots, D_{\mathfrak{l}}L_i, \dots, L_p).$$

Lemma 4.7. *Suppose $\alpha \in C^p(\mathfrak{l}, \mathfrak{a})$, $\tau \in C^q(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^p(\mathfrak{l})$. Then we have*

$$D^\circ \langle \alpha \wedge \tau \rangle = \langle D^\circ \alpha \wedge \tau \rangle + \langle \alpha \wedge D^\circ \tau \rangle, \tag{10}$$

$$D^\circ d\gamma = dD^\circ \gamma, \tag{11}$$

$$D^\circ d\alpha = dD^\circ \alpha. \tag{12}$$

Proof. This follows from the properties of $(S, U)^*$. ■

Remark 4.8. The pull back maps and D° commute, i. e. $D_1^\circ(S, U)^* \gamma = (S, U)^* D_2^\circ \gamma$ and $D_1^\circ(S, U)^* \alpha = (S, U)^* D_2^\circ \alpha$.

Now, we give a cohomology, which also respects the (not necessary semisimple) derivations.

Let $D_{\mathfrak{l}s}$ and $D_{\mathfrak{a}s}$ denote the semisimple parts of the Jordan-Chevalley decomposition of the derivations $D_{\mathfrak{l}}$ of \mathfrak{l} and $D_{\mathfrak{a}}$ of \mathfrak{a} for the rest of this work. The nilpotent parts are denoted by $D_{\mathfrak{l}n}$ and $D_{\mathfrak{a}n}$. Moreover, denote $D_s^\circ \alpha = \frac{d}{dt}(e^{-tD_{\mathfrak{l}s}}, e^{tD_{\mathfrak{a}s}})^* \alpha \Big|_{t=0}$.

Definition 4.9. Let p be even. We set (exactly as in [14])

$$Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}) := \left\{ (\alpha, \gamma) \in C^p(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-1}(\mathfrak{l}) \left| \begin{array}{l} d\alpha = 0, \quad d\gamma = \frac{1}{2} \langle \alpha \wedge \alpha \rangle, \\ D_s^\circ \alpha = 0, \quad D_s^\circ \gamma = 0 \end{array} \right. \right\}. \tag{13}$$

it is easy to see that $(\alpha, \gamma) \in Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ is invariant under the morphisms of pairs

$$(e^{tD_{\mathfrak{l}s}}, e^{-tD_{\mathfrak{a}s}}), \quad t \in \mathbb{R}. \tag{14}$$

Let $C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}) \subset C^{p-1}(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-2}(\mathfrak{l})$ denote the set of tuples (τ, σ) , which are invariant under the morphisms of pairs (14). In our notation, this means that (τ, σ) satisfies

$$D_s^\circ \tau = 0 \quad \text{and} \quad D_s^\circ \sigma = 0. \tag{15}$$

Lemma 4.10. ([14, page 13], see also [13, Definition 1.1, Lemma 1.2]) *The set $C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ becomes a group with group multiplication*

$$(\tau_1, \sigma_1)(\tau_2, \sigma_2) := (\tau_1 + \tau_2, \sigma_1 + \sigma_2 + \frac{1}{2} \langle \tau_1 \wedge \tau_2 \rangle). \tag{16}$$

Moreover, suppose $(\alpha, \gamma) \in C^p(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-1}(\mathfrak{l})$ and $(\tau, \sigma) \in C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Then

$$(\alpha, \gamma)(\tau, \sigma) := (\alpha + d\tau, \gamma + d\sigma + \langle (\alpha + \frac{1}{2}d\tau) \wedge \tau \rangle) \tag{17}$$

defines a right action of the group $C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ on $C^p(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-1}(\mathfrak{l})$, which leaves $Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ invariant.

We set the p -th cohomology set $H_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ of \mathfrak{l} with coefficients in \mathfrak{a} as

$$H_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}) := Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}) / C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}).$$

For $(\alpha, \gamma) \in Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ let $[\alpha, \gamma]$ denote the corresponding cohomology class.

Remark 4.11. The cohomology set $H_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ was already studied in [14]. It was introduced to classify the equivalence classes of quadratic extensions of (\mathfrak{h}, K) -equivariant metric Lie algebras. Here this cohomology set is a special case and describes the equivalence classes of metric Lie algebras with semisimple skew-symmetric derivations. We shall discuss this later in more detail.

Lemma 4.12. *Let p be even and $(\delta, \epsilon), (\tau, \sigma) \in C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Then*

$$(\delta, \epsilon)(\tau, \sigma) := \left(\delta + D^\circ \tau, \epsilon + D^\circ \sigma + \left\langle \left(\delta + \frac{1}{2} D^\circ \tau \right) \wedge \tau \right\rangle \right) \quad (18)$$

defines a right action of $C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ on $C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$.

Proof. Because of equation (10) we have

$$(\delta, \epsilon)((\tau_1, \sigma_1)(\tau_2, \sigma_2)) = ((\delta, \epsilon)(\tau_1, \sigma_1))(\tau_2, \sigma_2).$$

Since $D_s^\circ D^\circ = D^\circ D_s^\circ$, we obtain $D_s^\circ(\delta + D^\circ \tau) = 0$ and

$$D_s^\circ(\epsilon + D^\circ \sigma + \langle (\delta + \frac{1}{2} D^\circ \tau) \wedge \tau \rangle) = 0$$

for $(\delta, \epsilon), (\tau, \sigma) \in C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. ■

Definition 4.13. Let $Z_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ denote the set of all $(\alpha, \gamma, \delta, \epsilon) \in Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}) \oplus C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ satisfying $d\delta = D^\circ \alpha$ and $d\epsilon = D^\circ \gamma - \langle \alpha \wedge \delta \rangle$.

Lemma 4.14. *Let p be even, $(\alpha, \gamma, \delta, \epsilon) \in Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}) \oplus C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ and $(\tau, \sigma) \in C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Then*

$$\begin{aligned} (\alpha, \gamma, \delta, \epsilon)(\tau, \sigma) &:= ((\alpha, \gamma)(\tau, \sigma), (\delta, \epsilon)(\tau, \sigma)) \\ &= \left(\alpha + d\tau, \gamma + d\sigma + \left\langle \left(\alpha + \frac{1}{2} d\tau \right) \wedge \tau \right\rangle, \delta + D^\circ \tau, \epsilon + D^\circ \sigma + \left\langle \left(\delta + \frac{1}{2} D^\circ \tau \right) \wedge \tau \right\rangle \right) \end{aligned} \quad (19)$$

defines a right action of $C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ on $Z_{Q+}^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}) \oplus C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$, which leaves the set $Z_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ invariant.

Proof. Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ and $(\tau, \sigma) \in C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Since the group action of $C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ leaves the cocycles $Z_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ invariant, it remains to show that the equations

$$\begin{aligned} d(\delta + D^\circ \tau) - D^\circ(\alpha + d\tau) &= 0 \text{ and} \\ \langle (\alpha + d\tau) \wedge (\delta + D^\circ \tau) \rangle + d(\epsilon + D^\circ \sigma + \langle \delta \wedge \tau \rangle) + \frac{1}{2} \langle D^\circ \tau \wedge \tau \rangle \\ &\quad - D^\circ(\gamma + d\sigma + \langle \alpha \wedge \tau \rangle) + \frac{1}{2} \langle \tau \wedge d\tau \rangle = 0 \end{aligned}$$

hold. The first equation follows directly from (12). Because of (7), (10) and the commutativity of d and D° the second equation is equivalent to

$$\langle \alpha \wedge \delta \rangle + d\epsilon - D^\circ \gamma + \langle (d\delta - D^\circ \alpha) \wedge \tau \rangle = 0.$$

Since $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$, this equation is satisfied. ■

We set the p -th quadratic cohomology $H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ of \mathfrak{l} with coefficients in \mathfrak{a} as

$$H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}) := Z_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})/C_Q^{p-1}(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}).$$

In addition, we denote the cohomology class of $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ by $[\alpha, \gamma, \delta, \epsilon]$. Let $(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}) = (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \oplus (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ be the direct sum of two pairs. Let $j_i: \mathfrak{a}_i \rightarrow \mathfrak{a}$ and $q_i: \mathfrak{l} \rightarrow \mathfrak{l}_i$ denote the canonical embeddings and projections for $i = 1, 2$. The addition in $C^p(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-1}(\mathfrak{l}) \oplus C^{p-1}(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-2}(\mathfrak{l})$ defines a map

$$+: ((q_1, j_1)^* Z_{Q+}^p(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)) \times ((q_2, j_2)^* Z_{Q+}^p(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)) \rightarrow Z_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}).$$

Since the addition respects the group action, we have a natural injective map

$$+: ((q_1, j_1)^* H_{Q+}^p(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)) \times ((q_2, j_2)^* H_{Q+}^p(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)) \rightarrow H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}).$$

We call a cohomology class $[\alpha, \gamma, \delta, \epsilon] \in H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ decomposable if there is a decomposition $(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}) = (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \oplus (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ into a non-trivial direct sum of pairs such that

$$[\alpha, \gamma, \delta, \epsilon] \in (q_1, j_1)^* H_{Q+}^p(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) + (q_2, j_2)^* H_{Q+}^p(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2).$$

Otherwise, the cohomology class is called indecomposable. We denote the set of all indecomposable cohomology classes by $H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_i$.

Lemma 4.15. ([14, page 13], see also [13, page 92]) *Let $(S, U): (\mathfrak{l}_1, D_{\mathfrak{l}_{1,s}}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, D_{\mathfrak{l}_{2,s}}, \mathfrak{a}_2)$ be a morphism of pairs, where $D_{\mathfrak{l}_{i,s}}$ is semisimple and $(\mathfrak{a}_i, D_{\mathfrak{a}_{i,s}})$ orthogonal $(\mathfrak{l}_i, D_{\mathfrak{l}_{i,s}})$ -modules with semisimple derivations $D_{\mathfrak{a}_{i,s}}$ for $i = 1, 2$. Then*

$$(S, U)^*(\alpha_2, \gamma_2) := ((S, U)^*\alpha_2, (S, U)^*\gamma_2)$$

defines a pull back map from $Z_Q^p(\mathfrak{l}_2, \phi_{\mathfrak{l}_2}, \mathfrak{a}_2)$ to $Z_Q^p(\mathfrak{l}_1, \phi_{\mathfrak{l}_1}, \mathfrak{a}_1)$.

Lemma 4.16. *Let $(S, U): (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ be a morphism of pairs. Then*

$$(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2) := ((S, U)^*\alpha_2, (S, U)^*\gamma_2, (S, U)^*\delta_2, (S, U)^*\epsilon_2) \quad (20)$$

defines a pull back map from $Z_{Q+}^p(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ to $Z_{Q+}^p(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)$.

Proof. From Lemma 4.15 we know that equation (20) defines a pull back from $Z_Q^p(\mathfrak{l}_2, \phi_{\mathfrak{l}_2}, \mathfrak{a}_2) \oplus C_Q^{p-1}(\mathfrak{l}_2, \phi_{\mathfrak{l}_2}, \mathfrak{a}_2)$ to $Z_Q^p(\mathfrak{l}_1, \phi_{\mathfrak{l}_1}, \mathfrak{a}_1) \oplus C_Q^{p-1}(\mathfrak{l}_1, \phi_{\mathfrak{l}_1}, \mathfrak{a}_1)$. Because of equation (9), (11),(12) and remark 4.8, we get

$$d(S, U)^*\delta_2 - D^\circ(S, U)^*\alpha_2 = 0, \quad \text{and}$$

$$\langle (S, U)^*\alpha_2 \wedge (S, U)^*\delta_2 \rangle + d(S, U)^*\epsilon_2 - D^\circ(S, U)^*\gamma_2 = 0.$$

Thus $(S, U)^*$ defines a pull back map from $Z_{Q+}^p(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ to $Z_{Q+}^p(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)$. ■

Definition 4.17. Since the pull back map respects the group action in the sense

$$(S, U)^*((\alpha_2, \gamma_2, \delta_2, \epsilon_2)(\tau, \sigma)) = ((S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2))((S, U)^*\tau, (S, U)^*\sigma),$$

we also have a pull back map from $H_{Q+}^p(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ to $H_{Q+}^p(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)$ given by

$$(S, U)^*[\alpha_2, \gamma_2, \delta_2, \epsilon_2] := [(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2)].$$

Of course, the pull back map of isomorphisms of pairs also maps decomposable cohomology classes to decomposable ones.

5. The standard model

In this section we define the canonical example $(\mathfrak{d}_{\alpha,\gamma}, D_{\delta,\epsilon})$ of a metric Lie algebra with skew-symmetric derivation for every $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$. Moreover, we show that this standard model is also a standard example of a quadratic extension and we describe this balanced quadratic extensions on the level of the quadratic cocycles. Therefore, we define the subset $Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ of $Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.

Example 5.1. Let $(\mathfrak{l}, [\cdot, \cdot]_{\mathfrak{l}})$ be a Lie algebra, $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ an orthogonal \mathfrak{l} -module and $(\alpha, \gamma) \in C^2(\mathfrak{l}, \mathfrak{a}) \oplus C^3(\mathfrak{l})$. We define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$ by

$$\langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2 \rangle = Z_1(L_2) + \langle A_1, A_2 \rangle_{\mathfrak{a}} + Z_2(L_1)$$

for all $Z_1, Z_2 \in \mathfrak{l}^*$, $A_1, A_2 \in \mathfrak{a}$ and $L_1, L_2 \in \mathfrak{l}$. Moreover, we consider a skew-symmetric bilinear map $[\cdot, \cdot] : \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \times \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$, which is given by

$$\begin{aligned} [\mathfrak{l}^*, \mathfrak{l}^* \oplus \mathfrak{a}] &= 0, \\ [L_1, L_2] &= \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}}, \\ [L, A] &= -\langle A, \alpha(L, \cdot) \rangle + \rho(L)A, \\ [A_1, A_2] &= \langle \rho(\cdot)A_1, A_2 \rangle, \\ [L, Z] &= \text{ad}^*(L)(Z) = -Z([L, \cdot]_{\mathfrak{l}}) \end{aligned}$$

for all $Z \in \mathfrak{l}^*$, $A, A_1, A_2 \in \mathfrak{a}$ and $L, L_1, L_2 \in \mathfrak{l}$.

This definition is exactly the definition of the standard model in [13].

Lemma 5.2. ([14, page 9]) *Let $(\mathfrak{l}, D_{\mathfrak{l}_s})$ be a Lie algebra with semisimple derivation $D_{\mathfrak{l}_s}$ and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle, D_{\mathfrak{a}_s})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}_s})$ -module, where $D_{\mathfrak{a}_s}$ is semisimple. Then $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}) := (\mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is a metric Lie algebra with skew-symmetric derivation $D_{0,0}(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a})) := -D_{\mathfrak{l}_s}^* \oplus D_{\mathfrak{a}_s} \oplus D_{\mathfrak{l}_s}$ if and only if $(\alpha, \gamma) \in Z_{Q^+}^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$.*

If \mathfrak{l} and \mathfrak{a} are clear from the context, we simply write $\mathfrak{d}_{\alpha,\gamma}$ for $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a})$.

Example 5.3. Let $D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$ be derivations of \mathfrak{l} and \mathfrak{a} . We consider $(\delta, \epsilon) \in C^1(\mathfrak{l}, \mathfrak{a}) \oplus C^2(\mathfrak{l})$ and define

$$D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}) = \begin{pmatrix} -D_{\mathfrak{l}}^* & -\delta^* & \bar{\epsilon} \\ 0 & D_{\mathfrak{a}} & \delta \\ 0 & 0 & D_{\mathfrak{l}} \end{pmatrix}.$$

Here $\bar{\epsilon}$ is the uniquely determined linear map from \mathfrak{l} to \mathfrak{l}^* given by $\epsilon(L_1, L_2) = \langle \bar{\epsilon}(L_1), L_2 \rangle = (\bar{\epsilon}(L_1))(L_2)$ for $L_1, L_2 \in \mathfrak{l}$ and $\delta^* : \mathfrak{a} \rightarrow \mathfrak{l}^*$ is the dual map of δ given by $\langle \delta^* A, L \rangle = \langle A, \delta L \rangle$ for $L \in \mathfrak{l}$ and $A \in \mathfrak{a}$. If $D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$ are clear from the context, we simply write $D_{\delta,\epsilon}$. \blacksquare

Let $i : \mathfrak{a} \rightarrow \mathfrak{a} \oplus \mathfrak{l}$ denote the canonical embedding and $p : \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{l}$ the canonical projection. If $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is a metric Lie algebra with skew-symmetric derivation, then $((\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})), \mathfrak{l}^*, i, p)$ is a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$, where we identify $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a})/\mathfrak{l}^*$ and $\mathfrak{a} \oplus \mathfrak{l}$.

Let $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ also denote the quadratic extension of the standard model.

Theorem 5.4. *Let $(\mathfrak{l}, D_{\mathfrak{l}})$ be a Lie algebra with derivation $D_{\mathfrak{l}}$ and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle, D_{\mathfrak{a}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. Then $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is a metric Lie algebra with skew-symmetric derivation, whose semisimple part of the Jordan-Chevalley decomposition of $D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$ is equal to $D_{0,0}(D_{\mathfrak{l}_s}, D_{\mathfrak{a}_s})$, if and only if $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.*

Proof. Because of Lemma 5.2 the standard model $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}_s}, D_{\mathfrak{a}_s}))$ is a metric Lie algebra with skew-symmetric derivation, if and only if (α, γ) is an element in $Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Thus it remains to show that $D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$ is a skew-symmetric derivation of $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a})$ if and only if the two conditions

$$d\delta = D^\circ\alpha, \tag{21}$$

$$d\epsilon = D^\circ\gamma - \langle \alpha \wedge \delta \rangle \tag{22}$$

hold and that $D_{0,0}(D_{\mathfrak{l}_s}, D_{\mathfrak{a}_s})$ is exactly the semisimple part of $D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$, if and only if $D_s^\circ\delta = 0$ and $D_s^\circ\epsilon = 0$ is satisfied.

So, let $D_{\delta,\epsilon}$ be a skew-symmetric derivation of $\mathfrak{d}_{\alpha,\gamma}$. It holds $D[L_1, L_2] = [DL_1, L_2] + [L_1, DL_2]$ for $L_1, L_2 \in \mathfrak{l}$ if and only if

$$\begin{aligned} D_{\mathfrak{l}}[L_1, L_2]_{\mathfrak{l}} &= [D_{\mathfrak{l}}L_1, L_2]_{\mathfrak{l}} + [L_1, D_{\mathfrak{l}}L_2]_{\mathfrak{l}}, \\ D_{\mathfrak{a}}\alpha(L_1, L_2) &= -\delta[L_1, L_2]_{\mathfrak{l}} + \alpha(D_{\mathfrak{l}}L_1, L_2) - \rho(L_2)(\delta L_1) + \alpha(L_1, D_{\mathfrak{l}}L_2) + \rho(L_1)(\delta L_2), \\ \bar{\epsilon}[L_1, L_2]_{\mathfrak{l}} &= D_{\mathfrak{l}}^*\gamma(L_1, L_2, \cdot) + \gamma(D_{\mathfrak{l}}L_1, L_2, \cdot) + \langle \delta L_1, \alpha(L_2, \cdot) \rangle - \text{ad}^*(L_2)(\bar{\epsilon}L_1) + \\ &\quad + \delta^*\alpha(L_1, L_2) + \gamma(L_1, D_{\mathfrak{l}}L_2, \cdot) - \langle \delta L_2, \alpha(L_1, \cdot) \rangle + \text{ad}^*(L_1)(\bar{\epsilon}L_2) \end{aligned}$$

is satisfied for all $L_1, L_2 \in \mathfrak{l}$. These equations are equivalent to (21) and (22). On the other side it is easy to prove for $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ that $D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$ is a skew-symmetric derivation of $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a})$, because of condition (21) and (22). Finally, $D_{0,0}(D_{\mathfrak{l}_s}, D_{\mathfrak{a}_s})$ is the semisimple part of $D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$, if and only if

$$D_{0,0}(D_{\mathfrak{l}_s}, D_{\mathfrak{a}_s})D_{\delta,\epsilon}(D_{\mathfrak{l}_m}, D_{\mathfrak{a}_m}) = D_{\delta,\epsilon}(D_{\mathfrak{l}_m}, D_{\mathfrak{a}_m})D_{0,0}(D_{\mathfrak{l}_s}, D_{\mathfrak{a}_s})$$

holds, which is equivalent to $D_s^\circ\delta = 0$ and $D_s^\circ\epsilon = 0$. ■

Again, we remark that nilpotent metric symplectic Lie algebra with bijective skew-symmetric derivations are our objects of interest to understand the metric, symplectic Lie algebras. So, we already defined balanced quadratic extensions only for this kind of metric Lie algebras with derivations in section 3 and now we again limit our observations for the rest of this section.

Definition 5.5. Let \mathfrak{l} be a nilpotent Lie algebra with bijective derivation $D_{\mathfrak{l}}$ and $(\mathfrak{a}, D_{\mathfrak{a}})$ a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module with bijective $D_{\mathfrak{a}}$. Let m denote the smallest positive integer such that $\mathfrak{l}^{m+2} = 0$. We define the set of all balanced cocycles $Z_{Q^+}^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$ as the set of all cocycles $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$, which satisfy the following conditions for every $k = 0, \dots, m$:

(A_k) If there is an $A_0 \in \mathfrak{a}$ and $Z_0 \in (\mathfrak{l}^{k+1})^*$ for a given $L_0 \in \mathfrak{z}(\mathfrak{l}) \cap \mathfrak{l}^{k+1}$ such that (i) $\alpha(L, L_0) = 0$, and (ii) $\gamma(L, L_0, \cdot) = -\langle A_0, \alpha(L, \cdot) \rangle_{\mathfrak{a}} + \langle Z_0, [L, \cdot]_{\mathfrak{l}} \rangle$ as an element of $(\mathfrak{l}^{k+1})^*$, is satisfied for all $L \in \mathfrak{l}$, then $L_0 = 0$.

(B_k) The subspace $\alpha(\ker[\cdot, \cdot]_{\mathfrak{l} \otimes \mathfrak{l}^{k+1}}) \subset \mathfrak{a}$ is non-degenerate with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$.

Moreover, let $Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ denote the set of cocycles $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$, where $(\alpha, \gamma) \in Z_{Q+}^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$.

The set $Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$ is used to describe on the level of quadratic cocycles when standard models of nilpotent metric Lie algebras define balanced quadratic extensions. Here we have:

Lemma 5.6. ([14], see also [12]) *Let $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}))$ be a nilpotent, metric Lie algebra with bijective derivation. Then the quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}))$ is balanced if and only if $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$.*

The property that a quadratic extension is balanced is a property of the corresponding metric Lie algebra and does not depend on the derivation. Thus, we obtain the following lemma:

Lemma 5.7. *Let $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ be a nilpotent, metric Lie algebra with bijective derivation and let $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ be given. The quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ of a nilpotent Lie algebra \mathfrak{l} with bijective derivation $D_{\mathfrak{l}}$ by a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module $(\mathfrak{a}, D_{\mathfrak{a}})$ with bijective $D_{\mathfrak{a}}$ is balanced if and only if $(\alpha, \gamma, \delta, \epsilon)$ is an element in $Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$.*

6. Equivalence to the standard model

The notation standard model for $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ comes from the fact that every quadratic extension of metric Lie algebras with skew-symmetric derivations is equivalent to $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ for a suitable $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ as we will prove in this section.

Let $(\mathfrak{g}, D_s; \mathfrak{i}, i, p)$ be a quadratic extension of the metric Lie algebra with skew-symmetric semisimple derivation $(\mathfrak{l}, D_{\mathfrak{l}s})$ by an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}s})$ -module $(\mathfrak{a}, D_{\mathfrak{a}s})$, where $D_{\mathfrak{a}s}$ is semisimple. The subspace \mathfrak{i} is invariant under D_s , thus \mathfrak{i}^{\perp} is also invariant. Since D_s is semisimple and \mathfrak{i} isotropic, we can choose an isotropic complement of \mathfrak{i}^{\perp} , which is invariant under D_s . Let $s: \mathfrak{l} \rightarrow V_{\mathfrak{l}}$ be a section with $\tilde{p} \circ s = \text{id}$. Here $\tilde{p}: \mathfrak{g} \rightarrow \mathfrak{l}$ is the composition of the natural projection π from \mathfrak{g} to $\mathfrak{g}/\mathfrak{i}$ and p . This section satisfies $D_s \circ s = s \circ D_{\mathfrak{l}s}$ because of

$$\tilde{p} \circ D_s \circ s = p \circ \pi \circ D_s \circ s = p \circ \overline{D}_s \circ \pi \circ s = D_{\mathfrak{l}s} \circ \tilde{p} \circ s = \tilde{p} \circ s \circ D_{\mathfrak{l}s}.$$

Here we used that \tilde{p} is a bijection between \mathfrak{l} and $V_{\mathfrak{l}}$ and that p is a homomorphism from $(\mathfrak{g}/\mathfrak{i}, \overline{D}_s)$ to $(\mathfrak{l}, D_{\mathfrak{l}s})$. Let $V_{\mathfrak{a}}$ be an orthogonal complement of $\mathfrak{i} \oplus s(\mathfrak{l})$ in \mathfrak{g} . We define $t: \mathfrak{a} \rightarrow V_{\mathfrak{a}}$ by

$$i(A) = t(A) + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}.$$

This map t is an isometry, since $i: \mathfrak{a} \rightarrow \mathfrak{i}^\perp/\mathfrak{i}$ is one. Moreover, we define an isomorphism $p^*: \mathfrak{l}^* \rightarrow \mathfrak{i}$ by

$$\langle p^*(Z), s(L) \rangle := \langle Z, (\tilde{p} \circ s)(L) \rangle = Z(L)$$

for all $Z \in \mathfrak{l}^*$ and $L \in \mathfrak{l}$. Then we define $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^3(\mathfrak{l})$ by

$$i(\alpha(L_1, L_2)) := [sL_1, sL_2] - s[L_1, L_2] + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}, \tag{23}$$

$$\gamma(L_1, L_2, L_3) := \langle [s(L_1), s(L_2)], s(L_3) \rangle. \tag{24}$$

Proposition 6.1. ([14], see also [13, Lemma 2.8 and proof]) *It holds $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Moreover, the map $\varphi := p^* + t + s: \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{g}$ is an equivalence of the quadratic extensions $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}))$ and $(\mathfrak{g}, D_s; \mathfrak{i}, i, p)$.*

Proposition 6.2. *Let $(\mathfrak{g}, D; \mathfrak{i}, i, p)$ be a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$. Then there is an element $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ such that the quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon})$ is equivalent to $(\mathfrak{g}, D; \mathfrak{i}, i, p)$. In addition $(\alpha, \gamma, \delta, \epsilon)$ is an element in $Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ for an balanced quadratic extension.*

Proof. We choose $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ by equations (23) and (24) with respect to the quadratic extension $(\mathfrak{g}, D_s; \mathfrak{i}, i, p)$ of $(\mathfrak{l}, D_{\mathfrak{l}s})$ by $(\mathfrak{a}, D_{\mathfrak{a}s})$. Here D_s denotes the semisimple part of the Jordan-Chevalley decomposition of the derivation D (analogous to $D_{\mathfrak{l}s}$ and $D_{\mathfrak{a}s}$ for $D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$). Proposition 6.1 says that the mapping $\varphi := p^* + t + s: \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{g}$ is an equivalence of quadratic extensions with semi-simple parts of the derivations of $(\mathfrak{d}_{\alpha, \gamma}, D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}), \mathfrak{l}^*, i, p)$ to $(\mathfrak{g}, D_s, \mathfrak{i}, i, p)$. Set

$$\langle A, \delta(L) \rangle_{\mathfrak{a}} := \langle t(A), D(s(L)) \rangle, \tag{25}$$

$$\epsilon(L_1, L_2) := \langle D(s(L_1)), s(L_2) \rangle. \tag{26}$$

Since φ is an isometry and D and $D_{\delta, \epsilon}$ are skew-symmetric, we get that $\varphi \circ D_{\delta, \epsilon} = D \circ \varphi$ holds if and only if

$$\langle (D \circ \varphi)(L), X \rangle = \langle (\varphi \circ D_{\delta, \epsilon})(L), X \rangle, \tag{27}$$

$$\langle (D \circ \varphi)(A_1), t(A_2) \rangle = \langle (\varphi \circ D_{\delta, \epsilon})(A_1), t(A_2) \rangle \tag{28}$$

is satisfied for all $L \in \mathfrak{l}$, $X \in \mathfrak{g}$ and $A_1, A_2 \in \mathfrak{a}$. Because of

$$(\tilde{p} \circ D \circ t)(A_1) = (p \circ \pi \circ D \circ t)(A_1) = (p \circ \overline{D} \circ i)(A_1) = (D_{\mathfrak{l}} \circ p \circ i)(A_1) = 0$$

the element $(D \circ t)(A_1)$ lies in $V_{\mathfrak{a}} \oplus \mathfrak{i} = \mathfrak{i}^\perp$ for all $A_1 \in \mathfrak{a}$ and we obtain

$$\begin{aligned} \langle (D \circ \varphi)(A_1), t(A_2) \rangle &= \langle (D \circ t)(A_1), t(A_2) \rangle = \langle (\pi \circ D \circ t)(A_1), i(A_2) \rangle \\ &= \langle (\overline{D} \circ i)(A_1), i(A_2) \rangle_{\mathfrak{g}/\mathfrak{i}} = \langle (i \circ D_{\mathfrak{a}})(A_1), i(A_2) \rangle_{\mathfrak{g}/\mathfrak{i}} \\ &= \langle (t \circ D_{\mathfrak{a}})(A_1), t(A_2) \rangle = \langle (\varphi \circ D_{\delta, \epsilon})(A_1), t(A_2) \rangle. \end{aligned}$$

Here we used that i is a homomorphism of Lie algebras with derivations from $(\mathfrak{a}, D_{\mathfrak{a}})$ to $(\mathfrak{g}/\mathfrak{i}, \overline{D})$. Using

$$(\tilde{p} \circ D \circ s)(L_1) = (p \circ \overline{D} \circ \pi \circ s)(L_1) = (D_{\mathfrak{l}} \circ \tilde{p} \circ s)(L_1) = D_{\mathfrak{l}}(L_1)$$

for all $L_1 \in \mathfrak{l}$, we get furthermore

$$\begin{aligned}
& \langle (\varphi \circ D_{\delta, \epsilon})(L_1), p^*(Z_2) + t(A_2) + s(L_2) \rangle \\
&= \langle p^* \bar{\epsilon}(L_1), s(L_2) \rangle + \langle (t \circ \delta)(L_1), t(A_2) \rangle + \langle (s \circ D_l)(L_1), p^*(Z_2) \rangle \\
&= \epsilon(L_1, L_2) + \langle \delta(L_1), A_2 \rangle_{\mathfrak{a}} + Z_2(D_l(L_1)) \\
&= \langle (D \circ s)(L_1), p^*(Z_2) + t(A_2) + s(L_2) \rangle
\end{aligned}$$

for all $L_2 \in \mathfrak{l}$, $A_2 \in \mathfrak{a}$ and $Z_2 \in \mathfrak{l}^*$.

Finally, φ is an isomorphism from $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_l, D_{\mathfrak{a}}))$ to (\mathfrak{g}, D) , which maps $D_{0,0}(D_{l_s}, D_{\mathfrak{a}_s})$ to the semisimple part of D . Thus $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon}(D_l, D_{\mathfrak{a}}))$ is a skew-symmetric derivation, whose semisimple part equals $D_{0,0}(D_{l_s}, D_{\mathfrak{a}_s})$. So $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{l}, D_l, \mathfrak{a})$ and φ an equivalence of quadratic extensions, because of Theorem 5.4. If $(\mathfrak{g}, \mathfrak{i}, i, p)$ is balanced, then $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ is balanced, because of $\varphi(\mathfrak{l}^*) = \mathfrak{i}$. Using Lemma 5.7, we obtain that the cocycle $(\alpha, \gamma, \delta, \epsilon)$ is balanced. ■

7. Equivalence classes of quadratic extensions

The task of this section is to prove the bijection between $H_{Q^+}^2(\mathfrak{l}, D_l, \mathfrak{a})$ and the equivalence classes of quadratic extensions of (\mathfrak{l}, D_l) by \mathfrak{a} .

Lemma 7.1. ([14], compare with the prove of Lemma 2.9 in [13]) *Let (\mathfrak{l}, D_{l_s}) be a Lie algebra with semisimple derivation and $(\mathfrak{a}, D_{\mathfrak{a}_s})$ an orthogonal (\mathfrak{l}, D_{l_s}) -module, where D_{l_s} is semisimple. Assume $(\alpha_i, \gamma_i) \in Z_{Q^+}^2(\mathfrak{l}, \phi_l, \mathfrak{a})$ for $i = 1, 2$. An isomorphism $F: \mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}, \mathfrak{a}) \rightarrow \mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}, \mathfrak{a})$ of Lie algebras is an equivalence of the quadratic extensions $(\mathfrak{d}_{\alpha_i, \gamma_i}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{l_s}, D_{\mathfrak{a}_s}))$ if and only if*

$$F = \begin{pmatrix} \text{id} & -\tau^* & \bar{\sigma} - \frac{1}{2}\tau^*\tau \\ & \text{id} & \tau \\ & & \text{id} \end{pmatrix}$$

for a $(\tau, \sigma) \in C^1(\mathfrak{l}, \phi_l, \mathfrak{a})$ satisfying $\sigma(\cdot, \cdot) = \langle \bar{\sigma}(\cdot), \cdot \rangle$ and $(\alpha_1, \gamma_1) = (\alpha_2, \gamma_2)(\tau, \sigma)$. Here $\tau^*: \mathfrak{a} \rightarrow \mathfrak{l}^*$ denotes the dual map of τ defined by $\langle \tau^*A, L \rangle = \langle A, \tau L \rangle$ for $A \in \mathfrak{a}$ and $L \in \mathfrak{l}$.

Lemma 7.2. *Let us suppose that $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z_{Q^+}^2(\mathfrak{l}, D_l, \mathfrak{a})$ for $i = 1, 2$. The quadratic extensions $(\mathfrak{d}_{\alpha_i, \gamma_i}, D_{\delta_i, \epsilon_i})$ are equivalent if and only if*

$$[\alpha_1, \gamma_1, \delta_1, \epsilon_1] = [\alpha_2, \gamma_2, \delta_2, \epsilon_2] \in H_{Q^+}^2(\mathfrak{l}, D_l, \mathfrak{a}).$$

Proof. Suppose $(\alpha_1, \gamma_1, \delta_1, \epsilon_1), (\alpha_2, \gamma_2, \delta_2, \epsilon_2) \in Z_{Q^+}^2(\mathfrak{l}, D_l, \mathfrak{a})$. Then $(\mathfrak{d}_{\alpha_i, \gamma_i}, D_{\delta_i, \epsilon_i})$ is a quadratic extension and $D_{0,0}(D_{l_s}, D_{\mathfrak{a}_s})$ the semisimple part of the Jordan-Chevalley decomposition of the derivation $D_{\delta_i, \epsilon_i}(D_{l_s}, D_{\mathfrak{a}_s})$.

For a bijection F from $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$ to $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{\delta_2, \epsilon_2})$ the linear map $F \circ D_{0,0}(D_{l_s}, D_{\mathfrak{a}_s}) \circ F^{-1}$ is the semisimple part of the Jordan-Chevalley decomposition of D_{δ_2, ϵ_2} . Thus the map F is an equivalence of $(\mathfrak{d}_{\alpha_i, \gamma_i}, D_{\delta_i, \epsilon_i})$, $i = 1, 2$, if and only if it is an equivalence of the corresponding quadratic extensions from $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{0,0}(D_{l_s}, D_{\mathfrak{a}_s}))$ to $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{0,0}(D_{l_s}, D_{\mathfrak{a}_s}))$ and $F \circ D_{\delta_1, \epsilon_1} = D_{\delta_2, \epsilon_2} \circ F$ holds.

From Lemma 7.1 we obtain that the first equation is satisfied if and only if

$$F = \begin{pmatrix} \text{id} & -\tau^* & \bar{\sigma} - \frac{1}{2}\tau^*\tau \\ & \text{id} & \tau \\ & & \text{id} \end{pmatrix}$$

for a $(\tau, \sigma) \in C^1(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$, where $\sigma(\cdot, \cdot) = \langle \bar{\sigma}(\cdot), \cdot \rangle$ and $(\alpha_1, \gamma_1) = (\alpha_2, \gamma_2)(\tau, \sigma)$. Moreover, the condition $D_{\delta_2, \epsilon_2} \circ F = F \circ D_{\delta_1, \epsilon_1}$ holds, if

$$\langle D_{\delta_2, \epsilon_2} \circ FL, A \rangle = \langle F \circ D_{\delta_1, \epsilon_1} L, A \rangle \tag{29}$$

$$\langle D_{\delta_2, \epsilon_2} \circ FL_1, L_2 \rangle = \langle F \circ D_{\delta_1, \epsilon_1} L_1, L_2 \rangle \tag{30}$$

is satisfied for all $L, L_1, L_2 \in \mathfrak{l}$ and $A \in \mathfrak{a}$. Here equation (29) is equivalent to

$$\delta_1 = \delta_2 + D_{\mathfrak{a}} \circ \tau - \tau \circ D_{\mathfrak{l}}. \tag{31}$$

Equation (30) is equivalent to

$$\begin{aligned} \epsilon_1(L_1, L_2) &= \epsilon_2(L_1, L_2) + D^\circ \sigma(L_1, L_2) + \langle \delta_1 L_1, \tau L_2 \rangle \\ &\quad - \langle \tau L_1, \delta_2 L_2 \rangle + \frac{1}{2} \langle \tau \circ D_{\mathfrak{l}} L_1, \tau L_2 \rangle + \frac{1}{2} \langle \tau L_1, \tau \circ D_{\mathfrak{l}} L_2 \rangle, \end{aligned}$$

which leads to $(\delta_1, \epsilon_1) = (\delta_2, \epsilon_2)(\tau, \sigma)$ with the use of equation (31). ■

Remark 7.3. Using Lemma 5.7 and Lemma 7.2 we get that the group action of $C^1_Q(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ on $Z^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ leaves the set of balanced cocycles $Z^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ invariant.

For a nilpotent Lie algebra \mathfrak{l} with bijective derivation $D_{\mathfrak{l}}$ and a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module $(\mathfrak{a}, D_{\mathfrak{a}})$, where $D_{\mathfrak{a}}$ is bijective, we define

$$H^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b := Z^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b / C^1_Q(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a}). \tag{32}$$

Theorem 7.4. *The equivalence classes of quadratic extensions of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ are in one-to-one correspondence with $H^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$. Moreover, the equivalence classes of balanced quadratic extensions are in bijection with $H^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$.*

Proof. Let $(\mathfrak{g}, D; \mathbf{i}, i, p)$ be a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$. Consider $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$, which are given by the equations (23), (24), (25) and (26) with respect to two sections $s_i: \mathfrak{l} \rightarrow \mathfrak{g}$, $i = 1, 2$. Proposition 6.2 says that $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$ and $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{\delta_2, \epsilon_2})$ are equivalent, since both are equivalent to $(\mathfrak{g}, D; \mathbf{i}, i, p)$. Thus $[\alpha_1, \gamma_1, \delta_1, \epsilon_1] = [\alpha_2, \gamma_2, \delta_2, \epsilon_2]$, because of Lemma 7.2. This shows that the cohomology group of $(\alpha, \gamma, \delta, \epsilon) \in Z^2_{Q+}(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ defined by Proposition 6.2 does not depend on the choice of the section s . Using Theorem 5.4, Proposition 6.2 and Lemma 7.2 we obtain the assertion. ■

8. Isomorphism classes of metric, symplectic Lie algebras

Until now, we know that we can represent the isomorphism classes of metric, symplectic Lie algebras by a standard model. Now, we determine when two standard models are isomorphic as metric, symplectic Lie algebras and, finally, give a classification scheme for metric, symplectic Lie algebras.

Proposition 8.1. ([14], see also the proof of Lemma 4.1 in [13]) *Let $(\mathfrak{l}_i, D_{\mathfrak{l}_i, s})$ be Lie algebras with semisimple derivations $D_{\mathfrak{l}_i, s}$ and $(\mathfrak{a}_i, D_{\mathfrak{a}_i, s})$ trivial $(\mathfrak{l}_i, D_{\mathfrak{l}_i, s})$ -modules with semisimple $D_{\mathfrak{a}_i, s}$. Suppose $(\alpha_i, \gamma_i) \in Z_{Q^2}^2(\mathfrak{l}_i, \phi_{\mathfrak{l}_i}, \mathfrak{a}_i)_b$ for $i = 1, 2$. Let*

$$F: (\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \mathfrak{a}_1), D_{\delta_1, \epsilon_1}) \rightarrow (\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \mathfrak{a}_2), D_{\delta_2, \epsilon_2})$$

be an isomorphism and $\overline{F}: \mathfrak{a}_1 \oplus \mathfrak{l}_1 \rightarrow \mathfrak{a}_2 \oplus \mathfrak{l}_2$ the corresponding isomorphism on the quotient induced by F . We set

$$S(L) := (p \circ \overline{F})(L), \quad U(A) := \overline{F}^{-1}(A)$$

for $L \in \mathfrak{l}_1$, $A \in \mathfrak{a}_2$. Then (S, U) is an isomorphism of pairs of $(\mathfrak{l}_1, D_{\mathfrak{l}_1, s}, \mathfrak{a}_1)$ and $(\mathfrak{l}_2, D_{\mathfrak{l}_2, s}, \mathfrak{a}_2)$.

Theorem 8.2. *Suppose $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z_{Q^+}^2(\mathfrak{l}_i, D_{\mathfrak{l}_i}, \mathfrak{a}_i)_b$ for $i = 1, 2$. The metric Lie algebras with bijective skew-symmetric derivations $(\mathfrak{d}_{\alpha_i, \gamma_i}(\mathfrak{l}_i, \mathfrak{a}_i), D_{\delta_i, \epsilon_i})$ are isomorphic if and only if there is an isomorphism of pairs $(S, U): (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ such that*

$$(S, U)^*[\alpha_2, \gamma_2, \delta_2, \epsilon_2] = [\alpha_1, \gamma_1, \delta_1, \epsilon_1] \in H_{Q^+}^2(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)_b.$$

Proof. Let (S, U) be an isomorphism of pairs such that $(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2)$ lies in the cohomology set of $(\alpha_1, \gamma_1, \delta_1, \epsilon_1)$. The map

$$\varphi = \text{diag}(S^{*-1}, U^{-1}, S): \mathfrak{l}_1^* \oplus \mathfrak{a}_1 \oplus \mathfrak{l}_1 \rightarrow \mathfrak{l}_2^* \oplus \mathfrak{a}_2 \oplus \mathfrak{l}_2 \quad (33)$$

defines an isomorphism of the Lie algebra $\mathfrak{d}_{(S, U)^*\alpha_2, (S, U)^*\gamma_2}(\mathfrak{l}_1, \mathfrak{a}_1)$ with corresponding derivation $D_{(S, U)^*\delta_2, (S, U)^*\epsilon_2}(D_{\mathfrak{l}_1}, D_{\mathfrak{a}_1})$ to the Lie algebra $\mathfrak{d}_{\alpha_2, \gamma_2}$ with derivation D_{δ_2, ϵ_2} . Since $(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2)$ and $(\alpha_1, \gamma_1, \delta_1, \epsilon_1)$ lie in the same cohomology set, the corresponding quadratic extensions are equivalent. In particular, the pair $(\mathfrak{d}_{(S, U)^*\alpha_2, (S, U)^*\gamma_2}, D_{(S, U)^*\delta_2, (S, U)^*\epsilon_2})$ is isomorphic to $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$. Thus the pairs $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$ and $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{\delta_2, \epsilon_2})$ are isomorphic.

Now, suppose $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z_{Q^+}^2(\mathfrak{l}_i, D_{\mathfrak{l}_i}, \mathfrak{a}_i)_b$, $i = 1, 2$. Let

$$F: (\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \mathfrak{a}_1), D_{\delta_1, \epsilon_1}) \rightarrow (\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \mathfrak{a}_2), D_{\delta_2, \epsilon_2})$$

be an isomorphism of the metric Lie algebras. Since the cocycles are balanced, we have $F(\mathfrak{l}_1^*) = \mathfrak{l}_2^*$ and F induces an isomorphism $\overline{F}: \mathfrak{a}_1 \oplus \mathfrak{l}_1 \rightarrow \mathfrak{a}_2 \oplus \mathfrak{l}_2$. Now, we set

$$S(L) := (p \circ \overline{F})(L), \quad U(A) := \overline{F}^{-1}(A)$$

for $L \in \mathfrak{l}_1$, $A \in \mathfrak{a}_2$. Because of $D_{\delta_1, \epsilon_1} \circ F = F \circ D_{\delta_2, \epsilon_2}$ and Proposition 8.1 we obtain that (S, U) is an isomorphism of the pairs $(\mathfrak{l}_i, D_{\mathfrak{l}_i}, \mathfrak{a}_i)$. Thus, φ given by equation (33)

defines an isomorphism from the metric Lie algebra with skew-symmetric bijective derivation $(\mathfrak{d}_{(S,U)^*\alpha_2, (S,U)^*\gamma_2}(\mathfrak{l}_1, \mathfrak{a}_1), D_{(S,U)^*\delta_2, (S,U)^*\epsilon_2})$ to $(\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \mathfrak{a}_2), D_{\delta_2, \epsilon_2})$. Thus the corresponding quadratic extensions $(\mathfrak{d}_{(S,U)^*\alpha_2, (S,U)^*\gamma_2}, D_{(S,U)^*\delta_2, (S,U)^*\epsilon_2}, \mathfrak{l}_1^*, i, p)$ and $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1}, \mathfrak{l}_1^*, i, p)$ are equivalent. Using Lemma 7.2 yields the assertion. ■

Remark 8.3. For an isomorphism of pairs (S, U) the map φ defined by equation (33) is an isomorphism of metric Lie algebras with derivations and it holds $\varphi(\mathfrak{l}_1^*) = \mathfrak{l}_2^*$. Thus, using Lemma 5.7 we obtain that the isomorphisms of pairs define pull back maps, which map the balanced cocycles of $Z_{Q^+}^2(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ to balanced cocycles of $Z_{Q^+}^2(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)$.

Lemma 8.4. Assume $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$. The quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is decomposable if and only if $[\alpha, \gamma, \delta, \epsilon]$ is decomposable.

Proof. In general, if a quadratic extension is decomposable, then the corresponding cohomology class $[\alpha, \gamma, \delta, \epsilon]$ given by equations (23), (24), (25) and (26) equals

$$(q_1, j_1)^*[\alpha^1, \gamma_1, \delta_1, \epsilon_1] + (q_2, j_2)^*[\alpha^2, \gamma_2, \delta_2, \epsilon_2],$$

where $[\alpha_i, \gamma_i, \delta_i, \epsilon_i]$ are the corresponding cohomology classes of the direct summands of the quadratic extension. This immediately implies that $[\alpha, \gamma, \delta, \epsilon] \in H^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ is decomposable for an decomposable quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$. Conversely, if $[\alpha, \gamma, \delta, \epsilon] \in H^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ is decomposable, then

$$[\alpha, \gamma, \delta, \epsilon] = (q_1, j_1)^*[\alpha_1, \gamma_1, \delta_1, \epsilon_1] + (q_2, j_2)^*[\alpha_2, \gamma_2, \delta_2, \epsilon_2]$$

for certain $[\alpha_i, \gamma_i, \delta_i, \epsilon_i] \in H_{Q^+}^2(\mathfrak{l}_i, D_{\mathfrak{l}_i}, \mathfrak{a}_i)$. Then the observation in the beginning of the proof and Lemma 7.2 yield that $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is equivalent to the direct sum $(\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \mathfrak{a}_1), D_{\delta_1, \epsilon_1}(D_{\mathfrak{l}_1}, D_{\mathfrak{a}_1})) \oplus (\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \mathfrak{a}_2), D_{\delta_2, \epsilon_2}(D_{\mathfrak{l}_2}, D_{\mathfrak{a}_2}))$ and hence is decomposable. ■

Let $G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ denote the set of all morphisms of pairs (S, U) constituting of automorphisms S of \mathfrak{l} and isometries U of \mathfrak{a} , which satisfy $SD_{\mathfrak{l}} = D_{\mathfrak{l}}S$ and $UD_{\mathfrak{a}} = D_{\mathfrak{a}}U$. We simply write G , if $(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ is clear from the context. Let $H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$ denote the set of all balanced and indecomposable cohomology classes in $H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.

The final result of this section is the following classification scheme. It follows directly from Lemma 3.1, Theorem 3.11, Proposition 6.2 and Theorem 8.2.

Theorem 8.5. Let \mathfrak{l} be a (nilpotent) Lie algebra, $D_{\mathfrak{l}}$ a bijective derivation of \mathfrak{l} and $(\mathfrak{a}, D_{\mathfrak{a}})$ a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module with skew-symmetric bijective derivation $D_{\mathfrak{a}}$ of \mathfrak{a} . The set of isomorphism classes of indecomposable metric symplectic Lie algebras \mathfrak{g} with the following properties

- $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ together with the induced bijective skew-symmetric derivation is isomorphic to $(\mathfrak{a}, D_{\mathfrak{a}})$ as a metric Lie algebra with derivation and
- $\mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp$ together with the induced bijective derivation is isomorphic to $(\mathfrak{l}, D_{\mathfrak{l}})$ as a Lie algebra with derivation

is in one-to-one correspondence to the $G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ -orbits of $H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$.

The set of isomorphism classes of indecomposable metric symplectic Lie algebras is in bijection to the set

$$\coprod_{(\mathfrak{l}, D_{\mathfrak{l}}) \in \mathcal{L}} \coprod_{\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}} H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0 / G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}),$$

where \mathcal{L} is a system of representatives of the isomorphism classes of nilpotent Lie algebras with bijective derivations and $\mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}$ a system of representatives of the isomorphism classes of abelian metric Lie algebras with bijective skew-symmetric derivations (considered as trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -modules).

Remark 8.6. Here, the union was taken over a system of representatives \mathcal{L} of the isomorphism classes of all Lie algebras \mathfrak{l} with bijective derivations $D_{\mathfrak{l}}$ of \mathfrak{l} . But often $(\mathfrak{l}, D_{\mathfrak{l}})$ does not have a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module \mathfrak{a} such that $H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$ is not empty. We call a Lie algebra \mathfrak{l} with bijective $D_{\mathfrak{l}}$ admissible if there is such a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. The set of isomorphism classes of indecomposable metric symplectic Lie algebras is in one-to-one correspondence to the set

$$\coprod_{(\mathfrak{l}, D_{\mathfrak{l}}) \in \mathcal{L}_{ad}} \coprod_{\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}} H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0 / G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}),$$

where \mathcal{L}_{ad} is a system of representatives of the isomorphism classes of admissible Lie algebras with bijective derivations and $\mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}$ a system of representatives of isomorphism classes of abelian metric Lie algebras with bijective skew-symmetric derivations (considered as trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -modules).

9. Metric symplectic Lie algebras in special cases

In this section we determine metric symplectic Lie algebras in special cases up to isomorphism with the help of the classification scheme in section 8.

We call a basis $\{a_1, \dots, a_p, a_{p+1}, \dots, a_{p+q}\}$ of \mathfrak{a} *orthonormal*, if $a_i \perp a_j$ for $i \neq j$ and $\langle a_i, a_i \rangle_{\mathfrak{a}} = -1$ for $i = 1, \dots, p$ and $\langle a_j, a_j \rangle_{\mathfrak{a}} = 1$ for $j = p + 1, \dots, p + q$. In this case (p, q) is called signature and p the index of $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$. Let $\langle \cdot, \cdot \rangle_{p,q}$ denote the non-degenerate symmetric bilinear form with signature (p, q) on \mathbb{R}^{p+q} , which satisfies that the standard basis of the \mathbb{R}^{p+q} is an orthonormal basis. Then we call $\mathbb{R}^{p,q} := (\mathbb{R}^{p+q}, \langle \cdot, \cdot \rangle_{p,q})$ the *standard pseudo-euclidian space*. Of course we will denote $\mathbb{R}^{0,n}$ as \mathbb{R}^n . Often, we choose a Witt basis for $\mathbb{R}^{1,1}$. This is a basis $\{a_1, a_2\}$ of \mathbb{R}^2 of isotropic vectors, which satisfy $\langle a_1, a_2 \rangle_{1,1} = 1$. For $z = a + ib \in \mathbb{C}$ we set

$$D_z := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Moreover, denote $D_{z_1, \dots, z_n} = \text{diag}(D_{z_1}, \dots, D_{z_n})$ for $z_1, \dots, z_n \in \mathbb{C}$ the block diagonal matrix with the matrices D_{z_1} till D_{z_n} on the diagonal. Denote for $b \in \mathbb{R}$

$$N_b = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}.$$

For a basis $\{\sigma^1, \dots, \sigma^n\}$ of \mathfrak{l}^* we write shortly $\sigma^{ij} := \sigma^i \wedge \sigma^j$ and $\sigma^{ijk} := \sigma^i \wedge \sigma^j \wedge \sigma^k$.

Theorem 9.1. *The only non-abelian, metric Lie algebra of dimension smaller than eight having symplectic forms is $\mathfrak{d}_{0,\sigma^{123}}(\mathbb{R}^3, 0)$. Its symplectic forms are given (up to isomorphism) by exactly one of the following derivations:*

$$D_{0,0}(\text{diag}(a, b, c), 0), \quad D_{0,0}(\text{diag}(N_b, -2b), 0), \quad D_{0,0}(\text{diag}(D_{b+id}, -2b), 0)$$

mit $a \leq b \leq c$, $a, b, c \neq 0$, $d > 0$ und $a + b + c = 0$.

Theorem 9.2. *The set of $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta,\epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$, where*

- (1) $\mathfrak{l} = \mathbb{R}^3$, $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$, $D_{\mathfrak{a}} = D_{is}$ and
 - $D_{\mathfrak{l}} = \text{diag}(D_{b+is}, -b)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0)$,
- (2) $\mathfrak{l} = \mathbb{R}^3$, $\mathfrak{a} = \mathbb{R}^{1,1}$, $D_{\mathfrak{a}} = \text{diag}(s, -s)$ and
 - $D_{\mathfrak{l}} = \text{diag}(s - e, e, -s - e)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0)$,
 - $D_{\mathfrak{l}} = \text{diag}(-s, 2s, -3s)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, \sigma^1 \otimes a_2, 0)$,
 - $D_{\mathfrak{l}} = \text{diag}(3s, -2s, s)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, \sigma^3 \otimes a_1, 0)$,
- (3) $\mathfrak{l} = \mathbb{R}^3$, $\mathfrak{a} = \mathbb{R}^{1,1}$, $D_{\mathfrak{a}} = \text{diag}(\pm s, \mp s)$ and
 - $D_{\mathfrak{l}} = \text{diag}(N_{\pm s/2}, \mp \frac{3}{2}s)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0)$,
 - $D_{\mathfrak{l}} \in \left\{ \text{diag}(\pm s, e, \mp s - e), \text{diag}(N_{\pm s}, \mp 2s), \text{diag}(\pm s, N_{\mp s/2}), \text{diag}(\pm s, D_{\mp s+id}) \right\}$,
 $(\alpha, \gamma, \delta, \epsilon) = (0, \sigma^{123}, \sigma^1 \otimes a_1, 0)$

with $a \leq b \leq c$, $a, b, c \neq 0$, $a + b + c = 0$, $d, s > 0$, $e \notin \{0, s, -s\}$ forms a system of representatives of the isomorphism classes of indecomposable non-abelian metric Lie algebras of dimension 8 with bijective skew-symmetric derivations. Here $\{a_1, a_2\}$ is a Witt basis for $\mathfrak{a} = \mathbb{R}^{1,1}$ and an orthonormal basis for $\mathfrak{a} = \mathbb{R}^2$ or $\mathbb{R}^{2,0}$. Moreover, denote by $\{\sigma^1, \sigma^2, \sigma^3\}$ the dual basis of the standard basis of \mathbb{R}^3 .

Theorem 9.3. *There are no non-abelian metric symplectic Lie-Algebras of index smaller than three.*

Theorem 9.4. *The only indecomposable non-abelian metric symplectic Lie algebras with index 3 (up to isomorphism) are given as the six-dimensional metric Lie algebra $\mathfrak{d}_{0,\sigma^{123}}(\mathbb{R}^3, 0)$ with derivations $D_{0,0}(D_{\mathfrak{l}}, 0)$, where*

$$D_{\mathfrak{l}} \in \{\text{diag}(a, b, c), \text{diag}(N_b, -2b), \text{diag}(D_{b+is}, -2b)\},$$

or the eight-dimensional metric Lie algebra $\mathfrak{d}_{\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0}(\mathbb{R}^3, \mathbb{R}^2)$ with derivations $D_{0,0}(\text{diag}(D_{b+is}, -b), D_{is})$ for $a \leq b \leq c$, $a, b, c \neq 0$, $a + b + c = 0$ and $s > 0$. Here $\{a_1, a_2\}$ is an orthonormal basis of $\mathfrak{a} = \mathbb{R}^2$ and $\{\sigma^1, \sigma^2, \sigma^3\}$ the dual basis of the standard basis of \mathbb{R}^3 .

Proof of all theorems. The variety of computations is so modest in size that we don't have to restrict the computations to indecomposable metric symplectic Lie algebras. We simply compute all non-abelian metric symplectic Lie algebras and can easily decide afterwards which are indecomposable.

For every 4-dimensional non-abelian nilpotent Lie algebra \mathfrak{l} there is an $m \in \mathbb{N}$ such that \mathfrak{l}^{m+1} is one-dimensional. Thus $Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, 0)_b$ is empty for every derivation $D_{\mathfrak{l}}$ of

\mathfrak{l} , since (A_m) is not satisfied. For $\mathfrak{l} = \mathbb{R}^4$ and $\mathfrak{a} = \{0\}$ we have for every $\gamma \in C^3(\mathfrak{l})$ an isomorphism of pairs (S, U) such that $(S, U)^*(0, \gamma) = (0, 0)$ or $(S, U)^*(0, \gamma) = (0, \sigma^2 \wedge \sigma^3 \wedge \sigma^4)$. Since both cocycles $(0, 0)$ and $(0, \sigma^2 \wedge \sigma^3 \wedge \sigma^4)$ do not satisfy (A_0) , the set $H_{Q^+}^2(\mathbb{R}^4, D_{\mathfrak{l}}, 0)_b$ is empty.

Lemma 9.5. *The set of balanced cocycles $Z_{Q^+}^2(\mathbb{R}, D_{\mathfrak{l}}, \mathfrak{a})_b$ and $Z_{Q^+}^2(\mathbb{R}^2, D_{\mathfrak{l}}, \mathfrak{a})_b$ does not contain any elements for every bijective derivation $D_{\mathfrak{l}}$ of \mathbb{R}^2 and for every abelian metric Lie algebra \mathfrak{a} with bijective skew-symmetric derivation $D_{\mathfrak{a}}$.*

Proof. Since $H_{Q^+}^2(\mathbb{R}, \phi_{\mathfrak{l}}, \mathfrak{a})$ contains only the element $[0, 0]$ for every metric vector space \mathfrak{a} and therefor does not have any balanced elements, the set $Z_{Q^+}^2(\mathbb{R}, D_{\mathfrak{l}}, \mathfrak{a})_b$ is also empty. Now, suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathbb{R}^2, D_{\mathfrak{l}}, \mathfrak{a})$ and let $D_{\mathfrak{l}_s}$ be diagonalizable over \mathbb{R} at the moment. I. e. there is a diagonal basis $\{L_1, L_2\}$ of \mathbb{R}^2 for $D_{\mathfrak{l}_s}$ such that $D_{\mathfrak{l}_s}L_1 = \lambda_1 L_1$ and $D_{\mathfrak{l}_s}L_2 = \lambda_2 L_2$. Because of $D_s^\circ \alpha = 0$ we have

$$\begin{aligned} 0 &= \langle (D_s^\circ \alpha)(L_1, L_2), \alpha(L_1, L_2) \rangle \\ &= \langle D_{\mathfrak{a}s}(\alpha(L_1, L_2)) - \alpha(D_{\mathfrak{l}_s}L_1, L_2) - \alpha(L_1, D_{\mathfrak{l}_s}L_2), \alpha(L_1, L_2) \rangle \\ &= \langle -\alpha(D_{\mathfrak{l}_s}L_1, L_2) - \alpha(L_1, D_{\mathfrak{l}_s}L_2), \alpha(L_1, L_2) \rangle \\ &= -(\lambda_1 + \lambda_2) \langle \alpha(L_1, L_2), \alpha(L_1, L_2) \rangle. \end{aligned}$$

So, on the one hand $\alpha = 0$ (which is in addition the case for $\lambda_1 + \lambda_2 = 0$, since $D^\circ \alpha = 0$ and $D_{\mathfrak{a}}$ is bijective) or on the other hand $\alpha(\mathbb{R}^2, \mathbb{R}^2)$ is an one-dimensional isotropic subspace of \mathfrak{a} .

Now, suppose $D_{\mathfrak{l}_s}$ is not diagonalizable over \mathbb{R} . This means that there is a basis $\{L_1, L_2\}$ of \mathfrak{l} such that $D_{\mathfrak{l}_s}L_1 = aL_1 + bL_2$ and $D_{\mathfrak{l}_s}L_2 = -bL_1 + aL_2$ for $a, b \in \mathbb{R}$, $b \neq 0$. Then analogously

$$0 = \langle (D_s^\circ \alpha)(L_1, L_2), \alpha(L_1, L_2) \rangle = -2a \langle \alpha(L_1, L_2), \alpha(L_1, L_2) \rangle.$$

For $\alpha \neq 0$ we obtain that $2a \neq 0$ is an eigenvalue of $D_{\mathfrak{a}s}$ because of $D_s^\circ \alpha(L_1, L_2) = 0$. Then $\alpha(\mathfrak{l}, \mathfrak{l})$ is an one-dimensional isotropic subspace. Moreover, the 3-form γ on \mathbb{R}^2 is trivial. Thus, for every of these cases the condition (A_0) or (B_0) is not satisfied and hence $(\alpha, \gamma, \delta, \epsilon) \notin Z_{Q^+}^2(\mathbb{R}^2, D_{\mathfrak{l}}, \mathfrak{a})_b$. ■

Since the dimension of \mathfrak{l} is limited by the index, we obtain directly Theorem 9.3 from Lemma 9.5. In addition Lemma 9.5 shows that there is no symplectic metric Lie algebra of dimension less than 6, except for abelian ones.

Denote by \mathfrak{h}_3 the three-dimensional Heisenberg algebra given by $[e_1, e_2] = e_3$.

Lemma 9.6. *Given a bijective derivation $D_{\mathfrak{l}}$ on the Heisenberg Lie algebra \mathfrak{h}_3 and an abelian metric Lie algebra \mathfrak{a} with skew-symmetric bijective derivation, the set of cocycles $Z_{Q^+}^2(\mathfrak{h}_3, D_{\mathfrak{l}}, \mathfrak{a})$ is free from balanced elements.*

Proof. Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{h}_3, D_{\mathfrak{l}}, \mathfrak{a})$. If $\alpha(\mathfrak{h}_3, \mathfrak{z}(\mathfrak{h}_3)) = 0$, then condition (A_1) is not satisfied. Thus $(\alpha, \gamma, \delta, \epsilon) \notin Z_{Q^+}^2(\mathfrak{h}_3, D_{\mathfrak{l}}, \mathfrak{a})_b$.

Now, suppose $\alpha(\mathfrak{h}_3, \mathfrak{z}(\mathfrak{h}_3)) \neq 0$ and assume that $D_{\mathfrak{l}_s}$ is diagonalizable over \mathbb{R} . Then there is a diagonal basis $\{L_1, L_2, L_3\}$ for $D_{\mathfrak{l}_s}$ satisfying $[L_1, L_2] = L_3$ and $\alpha(L_1, L_3) \neq 0$. This means that

$$D_{\mathfrak{l}_s}L_1 = \lambda_1 L_1, \quad D_{\mathfrak{l}_s}L_2 = \lambda_2 L_2, \quad D_{\mathfrak{l}_s}L_3 = (\lambda_1 + \lambda_2)L_3,$$

where $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \neq 0$. Because of

$$D_{\mathfrak{a}_s}\alpha(L_1, L_3) = D_s^\circ\alpha(L_1, L_3) + \alpha(D_{\mathfrak{l}_s}L_1, L_3) + \alpha(L_1, D_{\mathfrak{l}_s}L_3) = (2\lambda_1 + \lambda_2)\alpha(L_1, L_3)$$

we obtain that $2\lambda_1 + \lambda_2 \neq 0$ is an eigenvalue of $D_{\mathfrak{a}_s}$. Thus $\alpha(L_1, L_3) \neq 0$ is isotropic as an eigenvector for $2\lambda_1 + \lambda_2$, since $D_{\mathfrak{a}_s}$ is skew-symmetric. Moreover,

$$\begin{aligned} 0 &= \langle D_{\mathfrak{a}_s}\alpha(L_1, L_3), \alpha(L_2, L_3) \rangle + \langle \alpha(L_1, L_3), D_{\mathfrak{a}_s}\alpha(L_2, L_3) \rangle \\ &= (2\lambda_1 + \lambda_2)\langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle + (\lambda_1 + 2\lambda_2)\langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle \\ &= 3(\lambda_1 + \lambda_2)\langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle. \end{aligned}$$

Since $D_{\mathfrak{l}}$ is bijective, it holds that $\lambda_1 + \lambda_2 \neq 0$ and hence $\langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle = 0$. Now, assume that $D_{\mathfrak{l}_s}$ is not diagonalizable over \mathbb{R} . Then there is a basis $\{L_1, L_2, L_3\}$ of \mathfrak{l} , which satisfies $D_{\mathfrak{l}_s}L_1 = aL_1 + bL_2$, $D_{\mathfrak{l}_s}L_2 = -bL_1 + aL_2$, $D_{\mathfrak{l}_s}L_3 = 2aL_3$ for $a, b \neq 0$ and $[L_1, L_2] = L_3$, since the center of \mathfrak{h}_3 is invariant under $D_{\mathfrak{l}}$. For $i, j = 1, 2$ we consider

$$0 = \langle D_s^\circ\alpha(L_i, L_3), \alpha(L_j, L_3) \rangle$$

and obtain analogously that

$$\langle \alpha(L_1, L_3), \alpha(L_1, L_3) \rangle = \langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle = 0.$$

So, $\alpha(L_1, L_3)$ is orthogonal to every element in $\alpha(\mathfrak{l}, \mathfrak{z}(\mathfrak{l}))$ and (B_1) is not satisfied. Thus $(\alpha, \gamma, \delta, \epsilon) \notin Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$. \blacksquare

Since $H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, 0)_b$ is empty for every 4-dimensional Lie algebra \mathfrak{l} and the Lie algebras \mathbb{R} , \mathbb{R}^2 and also \mathfrak{h}_3 are not admissible, we obtain Theorem 9.1 by determining $H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, 0)_b/G$. For Theorem 9.2 it remains to determine $H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b/G$ with $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ and Theorem 9.4 follows by calculating the G -orbits of $H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathbb{R}^{2n})_b$ for $n \in \mathbb{N}$. For $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$ we have

$$\begin{aligned} 0 &= -\langle (D_s^\circ\alpha)(L_1, L_2), \alpha(L_3, L_4) \rangle - \langle \alpha(L_1, L_2), (D_s^\circ\alpha)(L_3, L_4) \rangle \quad (34) \\ &= \langle \alpha(D_{\mathfrak{l}_s}L_1, L_2) + \alpha(L_1, D_{\mathfrak{l}_s}L_2), \alpha(L_3, L_4) \rangle \\ &\quad + \langle \alpha(L_1, L_2), \alpha(D_{\mathfrak{l}_s}L_3, L_4) + \alpha(L_3, D_{\mathfrak{l}_s}L_4) \rangle \end{aligned}$$

for vectors $L_1, \dots, L_4 \in \mathfrak{l}$ because of $D_s^\circ\alpha = 0$ and the skewsymmetry of $D_{\mathfrak{a}_s}$. Moreover, we have

$$0 = -\langle (D_s^\circ\alpha)(L_1, L_2), \alpha(L_1, L_2) \rangle = \langle \alpha(D_{\mathfrak{l}_s}L_1, L_2) + \alpha(L_1, D_{\mathfrak{l}_s}L_2), \alpha(L_1, L_2) \rangle. \quad (35)$$

Lemma 9.7. For $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$ the following conditions are equivalent:

- (i) $\alpha = 0$,
- (ii) $\gamma \neq 0$,
- (iii) $\text{tr}(D_{\mathfrak{l}}) = 0$.

Proof. Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$. Let $D_{\mathfrak{l}}$ be a bijective zero-trace matrix. At first, let $D_{\mathfrak{l}}$ be diagonalizable over \mathbb{R} . Denote $\{L_1, L_2, L_3\}$ the basis of

eigenvectors of $D_{\mathfrak{l}_s}$ for the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3 \neq 0$. Consider

$$\begin{aligned} 0 &= \langle \alpha(D_{\mathfrak{l}_s}L_i, L_j) + \alpha(L_i, D_{\mathfrak{l}_s}L_j), \alpha(L_p, L_k) \rangle \\ &\quad + \langle \alpha(L_i, L_j), \alpha(D_{\mathfrak{l}_s}L_p, L_k) + \alpha(L_p, D_{\mathfrak{l}_s}L_k) \rangle \\ &= (\lambda_i + \lambda_j + \lambda_p + \lambda_k) \langle \alpha(L_i, L_j), \alpha(L_p, L_k) \rangle \end{aligned}$$

(compare to equation (34)). Because of $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we have $\langle \alpha(\mathfrak{l}, \mathfrak{l}), \alpha(\mathfrak{l}, \mathfrak{l}) \rangle = 0$. Now (B_0) implies that $\alpha(\mathfrak{l}, \mathfrak{l})$ is non-degenerate with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ and thus $\alpha = 0$. If the zero-trace matrix $D_{\mathfrak{l}_s}$ is not diagonalizable over \mathbb{R} , then there is a basis $\{L_1, L_2, L_3\}$ of \mathfrak{l} such that $D_{\mathfrak{l}_s}L_1 = aL_1 + bL_2$, $D_{\mathfrak{l}_s}L_2 = -bL_1 + aL_2$ and $D_{\mathfrak{l}_s}L_3 = -2aL_3$ for $a, b \neq 0$. As above, we can consider the equation

$$0 = \langle \alpha(D_{\mathfrak{l}_s}L_i, L_j) + \alpha(L_i, D_{\mathfrak{l}_s}L_j), \alpha(L_p, L_k) \rangle + \langle \alpha(L_i, L_j), \alpha(D_{\mathfrak{l}_s}L_p, L_k) + \alpha(L_p, D_{\mathfrak{l}_s}L_k) \rangle$$

for $i, j, k, l \in \{1, 2, 3\}$ and obtain $\langle \alpha(\mathfrak{l}, \mathfrak{l}), \alpha(\mathfrak{l}, \mathfrak{l}) \rangle = 0$. Thus $\alpha = 0$ because of (B_0) . Now, suppose $\alpha = 0$. Using (A_0) we obtain $\gamma \neq 0$. Finally from $\gamma \neq 0$ follows directly that $D_{\mathfrak{l}}$ has zero trace by using

$$0 = D_s^\circ \gamma = -\text{tr}(D_{\mathfrak{l}})\gamma. \quad \blacksquare$$

Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$. If $D_{\mathfrak{l}_s}$ is not diagonalizable over \mathbb{R} , then we consider the complexification of the derivations and differential forms. Denote $\{L_1, L_2, L_3\}$ a basis of eigenvectors of $D_{\mathfrak{l}_s}$ for the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3 \neq 0$. We consider

$$0 = (D_s^\circ \epsilon)(L_i, L_j) = -\epsilon(D_{\mathfrak{l}_s}L_i, L_j) - \epsilon(L_i, D_{\mathfrak{l}_s}L_j) = -(\lambda_i + \lambda_j)\epsilon(L_i, L_j). \quad (36)$$

Of course, $\alpha = 0$ and $\delta = 0$ for $\mathfrak{a} = 0$. So $D_{\mathfrak{l}}$ is a zero-trace matrix and $\gamma = k\sigma^{123} \neq 0$, where $\{\sigma^1, \sigma^2, \sigma^3\}$ is the dual of the standard basis of \mathbb{R}^3 . Since the sum of two eigenvalues of $D_{\mathfrak{l}_s}$ is not zero, we have $\epsilon = 0$. Thus $(\alpha, \gamma, \delta, \epsilon) = (0, k\sigma^{123}, 0, 0)$ with $k \neq 0$. Finally $(k^{-\frac{1}{3}} \text{id}, \text{id}) \in G(\mathbb{R}^3, D_{\mathfrak{l}}, 0)$ and we get

$$(k^{-\frac{1}{3}} \text{id}, \text{id})^*(0, k\sigma^{123}, 0, 0) = (0, \sigma^{123}, 0, 0). \quad (37)$$

Now assume $\mathfrak{a} \neq \{0\}$. At first, let us assume that $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$ for a bijective zero-trace-matrix $D_{\mathfrak{l}}$. Then, again, $\alpha = 0$, $\gamma = k\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \neq 0$ and $\epsilon = 0$ because of equation (36). Now, consider

$$0 = (D_s^\circ \delta)(L_i) = D_{\mathfrak{a}s}(\delta(L_i)) - \delta(D_{\mathfrak{l}_s}L_i) = D_{\mathfrak{a}s}\delta(L_i) - \lambda_i\delta(L_i). \quad (38)$$

The eigenvalues of $D_{\mathfrak{a}}$ are purely imaginary for $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$. But the eigenvalues of the zero-trace matrix $D_{\mathfrak{l}_s}$ can't be purely imaginary, so we have $\delta = 0$. Because of $(k^{-\frac{1}{3}} \text{id}, \text{id}) \in G(\mathbb{R}^3, D_{\mathfrak{l}}, 0)$ and equation (37) the cohomology class

$$H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b / G(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})$$

consists of only one element for bijective zero-trace matrices $D_{\mathfrak{l}}$ and $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$, which is represented by $[0, \sigma^{123}, 0, 0]$. The same holds for an arbitrary euclidian vector space \mathfrak{a} of even dimension.

Now, assume $\mathfrak{a} = \mathbb{R}^{1,1}$. If there is no real number, which is an eigenvalue of $D_{\mathfrak{a}s} = D_{\mathfrak{a}}$ and $D_{\mathfrak{t}s}$ at the same time, then $\delta = 0$ because of equation (38). If there is such a real number, then this number is unique, since both eigenvalues $\pm s \neq 0$ of $D_{\mathfrak{a}}$ can't be eigenvalues of the bijective zero-trace matrix $D_{\mathfrak{t}s}$ at the same time. Thus $\delta(\mathfrak{l})$ is either $\{0\}$ or it spans a one-dimensional eigenspace of $D_{\mathfrak{a}}$. Assume $\delta \neq 0$. If there is no eigenvector \hat{v} of $D_{\mathfrak{l}}$ such that $\delta(\hat{v}) \neq 0$, then there is a vector v in the generalized eigenspace $\pm s$ of $D_{\mathfrak{l}}$ satisfying $\delta(v) \neq 0$ because of equation (38). Especially the generalized eigenspace for $\pm s$ is two-dimensional and the generalized eigenspace for $\mp 2s$ is one-dimensional. The vector v is no eigenvector of $D_{\mathfrak{l}}$. Thus there is an eigenvector \tilde{v} of $D_{\mathfrak{l}}$ with $D_{\mathfrak{l}}v = \pm s + \tilde{v}$. We define $\tau \in C^1(\mathbb{R}^3, \phi_{\mathfrak{l}}, \mathbb{R}^{1,1})$ by $\tau(\tilde{v}) = \delta(v)$, $\tau(v) = 0$ and $\tau(w) = 0$ for an eigenvector w for the eigenvalue $\mp 2s$. Then $D_s^\circ \tau = 0$ because of $D_s^\circ \delta = 0$ and

$$D^\circ \tau(v) = D_s^\circ \tau(v) - \tau(\tilde{v}) = -\delta(v).$$

Hence $(0, \gamma, \delta, 0)(\tau, 0) = (0, \gamma, 0, 0)$ and thus $(0, \gamma, \delta, 0)$ and $(0, \gamma, 0, 0)$ are equivalent. Using the morphism of pairs $(S, U) = (k^{-\frac{1}{3}} \text{id}, \text{id})$ we obtain

$$(S, U)^*(0, \gamma, 0, 0) = (0, \sigma^{123}, 0, 0).$$

Now, let there be a real eigenvector $D_{\mathfrak{l}}$, which spans the one-dimensional subspace $\delta(\mathfrak{l})$. This case can't be reduced to the previous case, since this property is invariant under the G -action. Moreover, it is invariant under equivalence, since

$$\begin{aligned} (\delta + D^\circ \tau)(v) &= \delta(v) + D_{\mathfrak{a}}\tau(v) - \tau(D_{\mathfrak{l}}(v)) = \delta(v) + D_{\mathfrak{a}s}\tau(v) - \tau(D_{\mathfrak{t}s}(v)) \\ &= (\delta + D_s^\circ \tau)(v) = \delta(v) \end{aligned}$$

holds for an eigenvector v of $D_{\mathfrak{l}}$. We choose an eigenvector $v_1 \in \mathfrak{l}$ of $D_{\mathfrak{l}}$, which satisfies $\delta(v_1) \neq 0$. Then we extend v_1 to a real Jordan or orthonormal basis $\{v_1, v_2, v_3\}$ of $D_{\mathfrak{l}}$ satisfying $\delta(v_2) = \delta(v_3) = 0$. We then obtain $\gamma = k\sigma^{123} \neq 0$. Furthermore, we choose a vector a_2 in \mathfrak{a} such that $a_1 := \delta(v_1)$ and a_2 form a Witt basis of $\mathfrak{a} = \mathbb{R}^{1,1}$. We set

$$(S, U) = (k^{-\frac{1}{3}} \text{id}, \text{diag}(k^{\frac{1}{3}}, k^{-\frac{1}{3}})) \in G(\mathbb{R}^3, D_{\mathfrak{l}}, \mathbb{R}^{1,1})$$

and obtain

$$(S, U)^*(0, \gamma, \delta, 0) = (0, \sigma^{123}, \delta, 0).$$

So this shows that every $(0, \gamma, \delta, 0)$ with an one-dimensional subspace $\delta(\mathfrak{l})$, which is spanned by an eigenvector of $D_{\mathfrak{l}}$, lives in the same G -orbit. Finally we have shown that for a bijective zero-trace matrix $D_{\mathfrak{l}}$

$$H_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathbb{R}^{1,1})_b / G(\mathbb{R}^3, D_{\mathfrak{l}}, \mathbb{R}^{1,1})$$

consists of only one element, if no eigenvalue of $D_{\mathfrak{t}s}$ is one of $D_{\mathfrak{a}}$. If there is an eigenvalue of $D_{\mathfrak{t}s}$, which is one of $D_{\mathfrak{a}}$, then the cohomology class has two elements. Assume $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$ for bijective $D_{\mathfrak{l}}$ with $\text{tr}(D_{\mathfrak{l}}) \neq 0$. Then $\gamma = 0$ and thus $\alpha \neq 0$.

At first, we assume $\mathfrak{a} = \mathbb{R}^2, \mathbb{R}^{2,0}$. Using $D_s^\circ \alpha = 0$ we see that $D_{\mathfrak{l}}$ has the eigenvalues $a + is$, $a - is$ and $-a$ for an $a \neq 0$, where $\pm is$ are the eigenvalues of $D_{\mathfrak{a}}$ with $s > 0$. Since $D_{\mathfrak{l}}$ has no purely imaginary eigenvalues and, furthermore, the sum of two eigenvalues is not equal to zero, we get $\delta = 0$ and $\epsilon = 0$ (also compare to equation (36) and (38)).

Now, we consider $H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b/G$, where

$$D_{\mathfrak{l}} = \begin{pmatrix} a & -s & \\ s & a & \\ & & -a \end{pmatrix}, \quad D_{\mathfrak{a}} = \begin{pmatrix} & -s & \\ s & & \end{pmatrix}$$

with $s > 0$ and $a \neq 0$. We have $\alpha(e_1, e_2) = 0$, since $D_s^\circ \alpha = 0$. It also holds that $\alpha(e_1, e_3) \neq 0$ and $\alpha(e_2, e_3) \neq 0$ are linearly independent because of (A_0) . Moreover, $\alpha(e_1, e_3)$ and $\alpha(e_2, e_3)$ are orthogonal to each other, since

$$0 = \langle \alpha(D_{\mathfrak{l}s}e_1, e_3) + \alpha(e_1, D_{\mathfrak{l}s}e_3), \alpha(e_1, e_3) \rangle = -s \langle \alpha(e_2, e_3), \alpha(e_1, e_3) \rangle,$$

and they have the same length because of

$$\begin{aligned} 0 &= \langle \alpha(D_{\mathfrak{l}s}e_1, e_3) + \alpha(e_1, D_{\mathfrak{l}s}e_3), \alpha(e_2, e_3) \rangle + 0 \\ &= \langle \alpha(e_1, e_3), \alpha(D_{\mathfrak{l}s}e_2, e_3) + \alpha(e_2, D_{\mathfrak{l}s}e_3) \rangle \\ &= s \langle \alpha(e_2, e_3), \alpha(e_2, e_3) \rangle - s \langle \alpha(e_1, e_3), \alpha(e_1, e_3) \rangle. \end{aligned}$$

We normalize the vectors $\{\alpha(e_1, e_3), \alpha(e_2, e_3)\}$ to an orthonormal basis $\{v_1, v_2\}$ of \mathfrak{a} . Then $\alpha = \mu(\sigma^{13} \otimes v_1 + \sigma^{23} \otimes v_2)$ with $\mu \neq 0$ because of $D_s^\circ \alpha = 0$ and using

$$(S, U) = (\text{diag}(1, 1, \mu^{-1}), \text{id}) \in G$$

yields $(S, U)^*(\alpha, 0, 0, 0) = (\sigma^{13} \otimes v_1 + \sigma^{23} \otimes v_2, 0, 0, 0)$.

Hence, we can represent $H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b/G$ by $[\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0]$. Here $\{a_1, a_2\}$ denotes the standard basis of \mathfrak{a} . For an arbitrary euclidian vectorspace \mathfrak{a} of even dimension with orthonormal basis $\{a_1, \dots, a_{2n}\}$ we have analogous

$$H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b/G = \{[\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0]\}.$$

Now assume $\mathfrak{a} = \mathbb{R}^{1,1}$. Because of $D_s^\circ \alpha = 0$ and (B_0) we know that the sum of two eigenvalues of $D_{\mathfrak{l}s}$ equals s and another sum equals $-s$, where $\pm s$ denote the eigenvalues of $D_{\mathfrak{a}}$ with $s > 0$. This especially implies that the eigenvalues of $D_{\mathfrak{l}s}$ are real numbers. Since the sum of two eigenvalues is not zero, we obtain $\epsilon \equiv 0$.

Let $D_{\mathfrak{l}}$ be semisimple. Since the eigenvalues of $D_{\mathfrak{l}}$ are real numbers, there is a basis of eigenvectors $\{L_1, L_2, L_3\}$. This basis is w. l. o. g. given such that, the sum of the eigenvalues of L_1 and L_2 equals s and the sum of the eigenvalues of L_2 and L_3 equals $-s$. Since $\text{tr } D_{\mathfrak{l}} \neq 0$, we have $\gamma = 0$ and $\alpha(\mathfrak{l}, \mathfrak{l}) \neq 0$ is non-degenerate. Thus $\alpha(L_1, L_2)$ is an eigenvector of $D_{\mathfrak{a}}$ for the eigenvalue s and $\alpha(L_2, L_3)$ an eigenvector of $D_{\mathfrak{a}}$ for $-s$. Moreover, $\text{tr } D_{\mathfrak{l}} \neq 0$ implies that the sum of the eigenvalues of L_1 and

L_3 is not equal to $\pm s$ and hence $\alpha(L_1, L_3) = 0$. W. l. o. g. we choose $\{L_1, L_2, L_3\}$ such that $\{\alpha(L_1, L_2), \alpha(L_2, L_3)\}$ is a Witt basis of \mathfrak{a} .

If no eigenvalue of $D_{\mathfrak{l}}$ is an eigenvalue of $D_{\mathfrak{a}}$, then $\delta = 0$, because of (38). If there is such an eigenvalue, then we have on the one hand $\delta = k\sigma^1 \otimes a_2$ for $-s$ being an eigenvalue of $D_{\mathfrak{l}}$ or on the other hand $\delta = k\sigma^3 \otimes a_1$ for s being an eigenvalue of $D_{\mathfrak{l}}$, since $D^\circ\delta = 0$. Here $k \in \mathbb{R}$. For $\delta \neq 0$ the cocycles $(\alpha, 0, 0, 0)$ and $(\alpha, 0, \delta, 0)$ are not equivalent, nor they lie in the same G -orbit. Moreover, $(S, U)^*(\alpha, 0, \delta, 0) = (\alpha, 0, k^{-1}\delta, 0)$ for $(S, U) = (\text{diag}(k^{-1}, k, k^{-1}), \text{id}) \in G$. For a semisimple $D_{\mathfrak{l}}$, whose trace doesn't vanish, the cohomology class $H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b/G$ consists of

- no elements, if one eigenvalue of $D_{\mathfrak{a}}$ is not the sum of two eigenvalues of $D_{\mathfrak{l}}$,
- exactly one element, if both eigenvalue of $D_{\mathfrak{a}}$ are the sum of two eigenvalues of $D_{\mathfrak{l}}$, but there is no eigenvalue of $D_{\mathfrak{l}}$ which is also an eigenvalue of $D_{\mathfrak{a}}$ and
- exactly two elements, otherwise.

For a $D_{\mathfrak{l}}$, which is not semisimple, we have

$$0 = (d\delta - D^\circ\alpha)(L_2, L_3) = D_s^\circ\alpha(L_2, L_3) - \alpha(L_1, L_3) = -\alpha(L_1, L_3).$$

Here $\{L_1, L_2, L_3\}$ denotes a Jordan basis of $D_{\mathfrak{l}}$, where $D_{\mathfrak{l}}L_1 = \pm\frac{s}{2}L_1$, $D_{\mathfrak{l}}L_2 = \pm\frac{s}{2}L_2 + L_1$ and $D_{\mathfrak{l}}L_3 = \mp\frac{3}{2}sL_3$. It is easy to see that there is a Jordan basis such that $\alpha(L_1, L_2)$ is an eigenvector for $\pm s$, $\alpha(L_2, L_3)$ is one for $\mp s$ and $\{\alpha(L_1, L_2), \alpha(L_2, L_3)\}$ is a Witt basis of $\mathbb{R}^{1,1}$. Since $\pm s$ is no eigenvalue of $D_{\mathfrak{l}}$, we obtain $\delta = 0$ using equation (38). Thus, in this case, $H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathbb{R}^{1,1})_b/G$ consists of exactly one element.

At this moment we determined the G -orbits of $H_{Q^+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$ for every derivation $D_{\mathfrak{l}}$ of $\mathfrak{l} = \mathbb{R}^3$ and every trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -modul $(\mathfrak{a}, D_{\mathfrak{a}})$, where $D_{\mathfrak{a}}$ is bijective on $\mathfrak{a} \in \{0, \mathbb{R}^2, \mathbb{R}^{1,1}, \mathbb{R}^{2,0}, \mathbb{R}^{2n}\}$. Choosing representatives of the conjugation classes of the derivations of \mathfrak{l} and \mathfrak{a} yield the theorems. ■

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