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On the Component Factor Group G/G_0 of a Pro-Lie Group G

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Abstract. A pro-Lie group G is a topological group such that G is isomorphic to the projective limit of all quotient groups G/N (modulo closed normal subgroups N) such that G/N is a finite dimensional real Lie group. A topological group is almost connected if the totally disconnected factor group $G_t \stackrel{\rm def}{=} G/G_0$ of G modulo the identity component G_0 is compact. In this case it is straightforward that each Lie group quotient G/N of G has finitely many components. However, in spite of a comprehensive literature on pro-Lie groups, the following theorem, proved here, was not available until now:

Theorem. *A pro-Lie group G is almost connected if each of its Lie group quotients G/N has finitely many connected components.*

The difficulty of the proof is the verification of the completeness of G_t.

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1. Projective Limits of Almost Connected Lie Groups

A notorious problem in the structure theory of pro-Lie groups is the completeness of quotient groups, notably that of the group G/G_0 of connected components. In one of the sources on pro-Lie groups, [2], the section following Definition 4.24 on pp.195ff. exhibits some of the characteristic difficulties involving the completeness of quotients of a pro-Lie group G in general and the quotient $G_t = G/G_0$ in particular. In their entirety, these difficulties remain unresolved today. We shall settle the completeness issue of G_t here for any pro-Lie group whose Lie group quotients have finitely many components.

Existing literature (see [4], Corollary 8.4) provides the following conclusion, which reinforces the independent interest in the result of this note:

An almost connected pro-Lie group G contains a maximal compact subgroup C and a closed subspace V homeomorphic to $\mathbb{R}^J$ for a set J such that

$$(c,x) \mapsto cx \colon C \times V \to G$$

is a homeomorphism.

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So, let G denote a topological group and $\mathcal{N}(G)$ the set of all closed normal subgroups of G for which G/N is a Lie group. With these conventions we formulate a theorem, to be proved subsequently. The proof requires some effort. It is based on information from [2].

Theorem 1.1. *For a pro-Lie group G, the following statements are equivalent:*

(1) *G_t is compact,*

(2) *There is a compact totally disconnected subspace $D \subseteq G$ being mapped homeomorphically onto G_t by the quotient map $q_t \colon G \to G_t$.*

(3) *$(G/N)_t$ is finite for all $N \in \mathcal{N}(G)$.*

The proof of the theorem will require the proof of some new lemmas and some references to existing literature. The first one is cited from [4], Main Theorem 8.1, Corollary 8.3.

Lemma 1.2. *Let G be an almost connected pro-Lie group. Then the following conclusions hold:*

(i) *G contains a maximal compact subgroup C, and any compact subgroup of G has a conjugate inside C.*

(ii) *$G = G_0C$.*

(iii) *G contains a profinite subgroup D such that $G = G_0D$.*

For every compact group K there is a compact totally disconnected subspace $D \subseteq K$ such that $(k, d) \mapsto kd : K_0 \times D \to K$ is a homeomorphism (see [3], Corollary 10.38, p. 573). From Lemma 1.2 we know that G is almost connected iff there is a compact subgroup $K \subseteq G$ such that $G = G_0K$. Write $K = K_0D$ with the topological direct factor $D \subseteq K$ as we just pointed out. Then $G = G_0K_0D = G_0D$ and so $(g, d) \mapsto gd : G_0 \times D \to G$ is readily seen to be a homeomorphism. Thus, in Theorem 1.1, Condition (1) implies (2), and for (2)⇒(1) there is nothing to prove.

Let us establish that (1)⇒(3):

Assume G/N to be a Lie group quotient of G. Then $(G/N)_0 = G_0N/N \cong G_0/(G_0 \cap N)$ (cf. [2], Lemma 3.29, p.152). Let K be a compact subgroup of G such that $G = G_0K$, and let $L = G/N$. Then $L = (G_0N/N)(KN/N) = L_0C$ for the compact Lie group $C = KN/N$. Thus $L_t = L_0C/L_0 \cong C/(C \cap L_0)$ is a compact totally disconnected Lie group and is therefore finite. This proves (3).

There remains a proof of the implication (3)⇒(1):

For the moment let us assume that the following hypothesis is satisfied

$$\textbf{(H)}\quad G_t \textit{ is a complete topological group.}$$

By [2], Corollary 3.31, hypothesis (H) implies that G_t is prodiscrete, that is, $G_t = \lim_{N\in\mathcal{N}(G_t)} G_t/N$ where G_t/N is discrete. Now G_t/N is a Lie group quotient of G and so is finite by (3). Hence G_t is profinite and thus compact. This proves Condition (1).

It therefore remains to prove (H). For this purpose we shall invoke results from [2], pp. 195ff.

Firstly, we define $\mathcal{M}(G)$ to be the subset of all $M \in \mathcal{N}(G)$ with the additional property that each open subgroup $N \subseteq M$ from $\mathcal{N}(G)$ has finite index in M. We shall then use

Lemma 1.3. *If G is a pro-Lie group such that*

(*) *each Lie group quotient G/N, $N \in \mathcal{N}(G)$ is almost connected,*

then $\mathcal{M}(G)$ is cofinal in $\mathcal{N}(G)$ and thus is a filter basis.
Moreover, G is the strict projective limit of the G/M, $M \in \mathcal{M}(G)$.

Proof. For the proof see Lemma 4.25 in [2], pp. 195 and 196. ∎

We note that Lemma 4.25 in [2] states as hypothesis that G is almost connected which implies **(*)**. But the hypothesis **(*)** is all that is used in the proof of Lemma 4.25.

Any set $\mathcal{Z}$ of subsets of a set G may be considered as a subbasis of closed sets for a topology. If G is a topological group, and $\mathcal{Z}$ is the set of all cosets gM with $g \in G$ and $M \in \mathcal{M}(G)$, then $\mathcal{Z}$ generates the set of closed sets of a topology on G, called the Z-*topology.*

Lemma 1.4. *The* Z-*topology on a pro-Lie group G satisfying* Condition **(*)** *of* Lemma 1.3 *is a compact T_1-topology.*

Proof. See Proposition 4.27 of [2], pp. 197–201. ∎

Again we note that Proposition 4.27 in [2] assumes the hypothesis that G is almost connected, but the proof of the conclusion of Lemma 1.4 only uses Hypothesis **(*)** of Lemma 1.3.

We now adjust the proof of Theorem 4.28 on p. 202 of [2] for our purposes.

Lemma 1.5. *Let G be a pro-Lie group satisfying hypothesis* **(*)** *of* Lemma 1.3*. Then G_t is complete.*

We note right away that Lemma 1.5 will prove hypothesis (H) and therefore complete the proof of Theorem 1.1.

Proof of Lemma 1.5. We let $f\colon G \to G_t = G/G_0$ be the quotient morphism and consider a Cauchy filter $\mathcal{C}$ on G_t. We have to show that $\mathcal{C}$ converges. By Lemma 1.3, $\mathcal{M}(G)$ is cofinal in $\mathcal{N}(G)$. For each $N \in \mathcal{M}(G)$ let $N^* = \overline{f(N)}$ and $p_{N^*}\colon G_t \to G_t/N^*$ be the quotient morphism. Then the image $p_{N^*}(\mathcal{C})$ is a Cauchy filter in the Lie group G_t/N^* and thus has a limit g_N. Then $(g_N)_{N\in\mathcal{M}(G)} \in \prod_{N\in\mathcal{M}(G)} G_t/N^*$ is an element of $\lim_{N\in\mathcal{M}(G)} G_t/N^*$; indeed $\mathcal{C}$ has to converge to a point in the completion of G_t.

Now let $F_N = (p_{N^*} \circ f)^{-1}(g_N)$. Then $\{F_N : N \in \mathcal{M}(G)\}$ is a filter basis consisting of cosets modulo $N_* \stackrel{\text{def}}{=} \overline{G_0 N}$ of G. We claim that $N_* \in \mathcal{M}(G)$. Indeed we have $N_* \in \mathcal{N}(G)$. Now we let M be an open subgroup of $N_* \supseteq G_0$. Then $G_0 \subseteq M$, so $G_0N \subseteq MN$ and MN is open-closed in $N_* = \overline{G_0N}$. Thus $N_* = MN$. So $MN/M \cong N/(N \cap M)$ is discrete and then $M \cap N$ is open in N. But $N \in \mathcal{M}(G)$ then implies that $N_*/M \cong N/(M \cap N)$ is finite.

This shows that $N_* \in \mathcal{M}(G)$ as claimed. Since G is Z-compact by Lemma 1.4, we find an element $g \in \bigcap_{N \in \mathcal{M}(G)} F_N$. But then $p_{N^*}(f(g)) = g_N$ for all $N \in \mathcal{M}(G)$ which implies that $f(g) = \lim \mathcal{C}$. Thus every Cauchy filter in G_t converges showing that G_t is complete. ∎

This completes the proof of Theorem 1.1.

An inspection of [2] shows that the following questions appear to be unsettled:

Question 1. For which pro-Lie groups G is $\mathcal{M}(G)$ cofinal in $\mathcal{N}(G)$?

For each of these groups G we would know that G is (isomorphic to) the strict projective limit $\lim_{M \in \mathcal{M}(G)} G/M$. In [2] this is proved of all almost connected pro-Lie groups.

Test examples are the nondiscrete pro-discrete groups $\mathbb{Z}^{(\mathbb{N})}$ and $\mathbb{Z}^{\mathbb{N}}$ (see e.g. [2], Example 4.4ff., Proposition 5.2).

Question 2. For which pro-Lie groups is the Z-topology compact?

In [2], Proposition 4.27, this is shown for almost connected pro-Lie groups, and here we have proved it for those pro-Lie groups G all of whose Lie group quotients are known to be almost connected.

Theorem 1.1 suggests the following rather general question:

Question 3. When is the projective limit of a projective system of almost connected topological groups almost connected?

Theorem 1.1 says that within the category of pro-Lie groups we have an affirmative answer for the projective system of all Lie group quotients. See also some background information in [2] in and around Theorem 1.27, p. 88.

One remark is in order in the context of the Z-topology discussed in [2], pp. 197–203: In Exercise E4.2(i), p. 202, it is pointed out that $\mathcal{M}(\mathbb{Z}) = \{\{0\}, \mathbb{Z}\}$ and that therefore the Z-topology on $\mathbb{Z}$, being the cofinite topology, is compact. Whereas the topology generated by the set of cosets $z + N$, $N \in \mathcal{N}(\mathbb{Z})$, the set of all subgroups of $\mathbb{Z}$, fails to be compact.

Theorem 1.1 will play a significant role in the authors' study [1] of weakly complete real or complex topological algebras with identity, which will explore in detail their relation to pro-Lie theory and aims for a systematic treatment of weakly complete group algebras of topological groups and their representation and duality theories.

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