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Moment Map and Gelfand Transform for the Enveloping Algebra

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Abstract. Describing the Gelfand construction for the analytic states on an universal enveloping algebra, we characterize pure states and re-find the main result of a preceding work with L. Abdelmoula and J. Ludwig on the separation of unitary irreducible representations of a connected Lie group by their generalized moment sets.

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1. Introduction

There is a direct relationship between representations of the $C^\star$-algebra of a locally compact group G and unitary representations of G. The Gelfand construction is the main tool to describe the cyclic and irreducible representations of a $C^\star$-algebra A, starting with a state or a pure state on A (see [8] for instance). In this paper, we consider a real connected and simply connected Lie group G, and the complex enveloping associative algebra $\mathfrak{A}(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$. We then define the notion of analytic state φ on $\mathfrak{A}(\mathfrak{g})$, and perform the Gelfand construction, starting with φ. This construction was described by K. H. Neeb in [12] for the large class of BCH-Lie algebras. We present it here in the really simpler setting of finite dimensional Lie algebras. The result is a unitary representations of G itself. This representation is irreducible if and only if φ is a pure state.

Conversely, associated to any unitary representation π of G, there is a notion of generalized moment map and set (see [14, 6, 3, 1]): each smooth vector f in the space $\mathcal{H}_\pi$ of the representation π defines naturally a state $\Phi_\pi(f)$ (see Section 2 for definition). It is proved in [1] that, if π is irreducible, the convex hull $J_{\pi,\mathbb{C}}$ of this set of states characterizes the representation π.

In the present paper, we re-find this result, by using the Gelfand construction when f is an analytic vector, and proving that, if π is irreducible, the set of $\Phi_\pi(f)$ (f analytic) is the subset of analytic pure states in $J_{\pi,\mathbb{C}}$ and the disjointness of the sets $\{\Phi_\pi(f):\ f \text{ analytic}\}$ for two non-equivalent irreducible representations of G.

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2. The moment map

In this section, we recall definitions and main results for the moment map of a unitary representation.
Let G be a connected and simply connected real Lie group, and π a unitary representation of G on an Hilbert space $\mathcal{H}_\pi$. Let us denote by $\mathfrak{g}$ the (real) Lie algebra of G and $\mathfrak{A}(\mathfrak{g})$ the complex universal enveloping algebra of $\mathfrak{g}$. The associative algebra $\mathfrak{A}(\mathfrak{g})$ acts naturally on the space $\mathcal{H}_\pi^\infty$ of C^∞ vectors in $\mathcal{H}_\pi$, we denote by $d\pi$ the corresponding representation.

Definition 2.1. Let f be a non vanishing C^∞ vector in $\mathcal{H}_\pi$. The *moment* of π in f is by definition the element $\Psi_\pi(f)$ of the real linear dual of $\mathfrak{A}(\mathfrak{g})$ defined by

$$\Psi_\pi(f)(A) = \mathfrak{Re}\Big(\frac{1}{i}\frac{\langle d\pi(A)f, f\rangle}{\langle f, f\rangle}\Big).$$

The map $\Psi_\pi : \mathcal{H}_\pi^\infty \setminus \{0\} \to (\mathfrak{A}(\mathfrak{g}))^\star_\mathbb{R}$ is the *moment map* of π. The range I_π of the map Ψ_π is the *moment set* of π. We denote by J_π the *convex hull* of I_π. ■

Since each complex linear form φ on $\mathfrak{A}(\mathfrak{g})$ can be written as $\varphi(A) = \psi(iA) + i\psi(A)$, where ψ is the real linear map $\psi(A) = \mathfrak{Im}(\varphi(A))$, we continue with

Definition 2.2. Let f be a non vanishing C^∞ vector in $\mathcal{H}_\pi$. The *complex moment* of π in f is by definition the element $\Phi_\pi(f)$ of the linear dual $(\mathfrak{A}(\mathfrak{g}))^\star$ defined by

$$\Phi_\pi(f)(A) = \frac{\langle d\pi(A)f, f\rangle}{\langle f, f\rangle}.$$

The map $\Phi_\pi : \mathcal{H}_\pi^\infty \setminus \{0\} \to (\mathfrak{A}(\mathfrak{g}))^\star$ is the *complex moment map* of the representation π. The range $I_{\pi,\mathbb{C}}$ of the map Φ_π is the *complex moment set* of π. We denote by $J_{\pi,\mathbb{C}}$ the *convex hull* of $I_{\pi,\mathbb{C}}$.

These objects were studied in [1, 2, 3, 6]. The main result is:

Theorem 2.3. ([1]) *Let π and ρ be two irreducible unitary representations of G, then π and ρ are equivalent if and only if $J_\pi = J_\rho$, if and only if $J_{\pi,\mathbb{C}} = J_{\rho,\mathbb{C}}$.*

Especially, if G is solvable, for each regular, integral coadjoint orbit $\mathcal{O}$ is associated a family of irreducible unitary representations $\pi_{\mathcal{O},\chi}$ (see [5, 7]). In [2], the authors present a direct construction, allowing to re-find $\mathcal{O}$ and χ from $I_{\pi_{\mathcal{O},\chi}}$.
Since $\mathfrak{g} \subset \mathfrak{A}(\mathfrak{g})$, we can restrict the real forms $\Psi_\pi(f)$ to $\mathfrak{g}$. Then $[f] \mapsto \Psi_\pi(f)|_\mathfrak{g}$ is the moment map for the strongly Hamiltonian action of G on the natural symplectic manifold $\mathbb{P}(\mathcal{H}^\infty)$ (see [14, 4]). The denomination 'moment map' for the map Ψ_π is coming from this observation.

3. Gelfand construction for $\mathfrak{A}(\mathfrak{g})$

Let $(X_1, \dots, X_d)$ be a basis of the Lie algebra $\mathfrak{g}$. Recall that each unitary representation π of G has a dense subspace $\mathcal{H}_\pi^\omega$ of analytic vectors (see [13]), that are vectors f for which there is $r > 0$, $C > 0$ such that for each n, each $1 \leq i_1, \dots, i_n \leq d$,

$$\|d\pi(X_{i_1} \dots X_{i_n})f\| \leq rC^n n! \tag{1}$$

From now one, we restrict ourselves to the space $\mathcal{H}_{\pi,1}^\omega$ of analytic unit vectors f: $f \in \mathcal{H}^\omega$, $\|f\| = 1$. We thus put

$$\Phi_\pi^\omega = \Phi_\pi|_{\mathcal{H}_{\pi,1}^\omega}, \quad I_{\pi,\mathbb{C}}^\omega = \Phi_\pi^\omega(\mathcal{H}_{\pi,1}^\omega), \text{ and } J_{\pi,\mathbb{C}}^\omega = \mathrm{Conv}(I_{\pi,\mathbb{C}}^\omega).$$

Observe that if π' is a representation unitarily equivalent to π, there is a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ such that $\pi'(x) = U \circ \pi(x) \circ U^{-1}$, and for each $f \in \mathcal{H}_{\pi,1}^\omega$, Uf is in $\mathcal{H}_{\pi',1}^\omega$ and

$$\Phi_{\pi'}^\omega(Uf)(A) = \langle d\pi'(A)Uf, Uf\rangle = \langle Ud\pi(A)f, Uf\rangle = \Phi_\pi^\omega(f)(A).$$

Therefore $\Phi_{\pi'}^\omega(Uf) = \Phi_\pi^\omega(f)$ and $I_{\pi',\mathbb{C}}^\omega = I_{\pi,\mathbb{C}}^\omega$.
Now $\mathfrak{A}(\mathfrak{g})$ is an involutive algebra when we equip it with the involution (called principal anti-isomorphism in [9]):

$$(\alpha Y_1 \dots Y_n)^\star = (-1)^n \overline{\alpha} Y_n \dots Y_1,$$

if $Y_i \in \mathfrak{g}$, $\alpha \in \mathbb{C}$. Therefore we shall now repeat the usual Gelfand construction for the involutive algebra $(\mathfrak{A}(\mathfrak{g}), \star)$ as for $C^\star$-algebras (see for instance [8]).

Definition 3.1.

(1) A linear form $\varphi : \mathfrak{A}(\mathfrak{g}) \to \mathbb{C}$ is *positive* if $\varphi(A^\star A) \geq 0$, for each $A \in \mathfrak{A}(\mathfrak{g})$.

(2) A *state* is a positive linear form φ on $\mathfrak{A}(\mathfrak{g})$, such that $\varphi(1) = 1$. We denote by $S(\mathfrak{A}(\mathfrak{g}))$ the set of states of $\mathfrak{A}(\mathfrak{g})$.

(3) A state φ is *analytic* if there is a basis $(X_1, \dots, X_d)$ of $\mathfrak{g}$, and positive constants r, C such that, for any n, any $1 \leq i_1, \dots, i_n \leq d$,

$$|\varphi(X_{i_1} \dots X_{i_n})| \leq rC^n n! \tag{2}$$

Fix C, the set of states φ for which there is $r = r_\varphi > 0$ such that the relation (2) holds for any n, $i_1, \dots, i_n$ is denoted by S_C^ω. The *set of analytic states* of $\mathfrak{A}(\mathfrak{g})$ is denoted by $S^\omega(\mathfrak{A}(\mathfrak{g}))$: $S^\omega(\mathfrak{A}(\mathfrak{g})) = \cup_{C>0} S_C^\omega$. Clearly, the set $S^\omega(\mathfrak{A}(\mathfrak{g}))$ does not depend of the choice of the basis.

(4) A state φ is *pure* if the only state ψ such that there exists $a > 0$ such that $a\psi(A^\star A) \leq \varphi(A^\star A)$ for each $A \in \mathfrak{A}(\mathfrak{g})$ is φ itself. Denote by $P(\mathfrak{A}(\mathfrak{g}))$ (resp. $P^\omega(\mathfrak{A}(\mathfrak{g}))$) the *set of pure states* (resp. of pure analytic states).

Relation to the Neeb theory

Recall that in [12], K. H. Neeb studies systematically the integrability problem for a functional φ on $\mathfrak{A}(\mathfrak{g})$, where $\mathfrak{g}$ is a BCH-Lie algebra. To solve this problem, he defines an analytic functional as a linear form φ such that $\sum_n \frac{\varphi(X^n)}{n!}$ converges for each X in a 0-neighborhood in $\mathfrak{g}$. If $\mathfrak{g}$ is finite dimensional, a functional φ is analytic if and only if the relation (2) holds.

An important result of Neeb is the following: let φ be a functional on $\mathfrak{A}(\mathfrak{g})$, p a sub-multiplicative semi-norm on $\mathfrak{g}$ $(p([X,Y]) \leq p(X)p(Y))$. Put:

$$\|\varphi_n^s\|_p = \sup\Big\{|\frac{1}{n!}\sum_{\sigma\in S_n}\varphi(X_{\sigma(1)}\dots X_{\sigma(n)})| : \quad p(X_i)\leq 1 \quad (1\leq i\leq n)\Big\}.$$

Now Theorem 6.10 in [12] says that if G is a BCH-Lie group with Lie algebra $\mathfrak{g}$, if φ is an analytic functional which is positive and such that there is a sub-multiplicative semi-norm p on $\mathfrak{g}$ for which $\sum_n \frac{\|\varphi_n^s\|_p t^n}{n!}$ converges for some $t > 0$, then there is a unique unitary representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of G and a vector f in $\mathcal{H}_\varphi$ such that

$$\varphi(A) = \langle d\pi_\varphi(A)f, f\rangle.$$

It is not difficult to prove that if G is a finite dimensional Lie group, the above condition holds for any analytic state φ in the sense of Definition 3.1, therefore the next theorem is a direct consequence of Theorem 6.10 in [12]. However, to be complete, we present here a direct proof of this theorem.

Theorem 3.2. *If π is a unitary representation of G and f a analytic unit vector for π, then: $\Phi_\pi^\omega(f) : A \mapsto \langle d\pi(A)f, f\rangle$ is an analytic state for $\mathfrak{A}(\mathfrak{g})$.*

Conversely, let φ be an analytic state on $\mathfrak{A}(\mathfrak{g})$, then there is a unitary representation τ of G and an analytic vector $f \in \mathcal{H}_{\tau,1}^\omega$ such that $\varphi = \Phi_\tau^\omega(f)$.

Proof. Let π be a unitary representation of G and $f \in \mathcal{H}_{\pi,1}^\omega$. By definition, $\Phi_\pi^\omega(f)$ is a positive form: $\Phi_\pi^\omega(f)(A^\star A) = \langle d\pi(A^\star A)f, f\rangle = \|d\pi(A)f\|^2 \geq 0$, and a state since $\Phi_\pi^\omega(f)(1) = \langle f, f\rangle = 1$. Now since f is analytic, Formula (1) says that there is $r > 0$, $C > 0$ such that:

$$|\Phi_\pi^\omega(f)(X_{i_1}\dots X_{i_n})| = |\langle d\pi(X_{i_1}\dots X_{i_n})f, f\rangle| \leq \|d\pi(X_{i_1}\dots X_{i_n})f\| \leq rC^n n!$$

Thus $\Phi_\pi^\omega(f) \in S^\omega(\mathfrak{A}(\mathfrak{g}))$.

Conversely, if $\varphi \in S^\omega(\mathfrak{A}(\mathfrak{g}))$, put $\mathrm{Ann}_\varphi = \{A \in \mathfrak{A}(\mathfrak{g}) : \ \varphi(A^\star A) = 0\}$. Since φ is positive, the relation $b(A,B) = \varphi(B^\star A)$ defines a positive form b, linear in the argument A, anti-linear in the argument B. By the Cauchy-Schwarz inequality:

$$|b(A,B)| \leq (b(A,A))^{\frac{1}{2}}(b(B,B))^{\frac{1}{2}}.$$

Therefore, Ann_φ is the kernel of b: $\mathrm{Ann}_\varphi = \{A \in \mathfrak{A}(\mathfrak{g}) : \ \varphi(B^\star A) = 0 \ \ (B \in \mathfrak{A}(\mathfrak{g}))\}$, it is a left ideal. Put $V = \mathfrak{A}(\mathfrak{g})/\mathrm{Ann}_\varphi$, and let $d\tau$ be the natural $\mathfrak{A}(\mathfrak{g})$-action on V:

$$d\tau(A)[B] = [AB].$$

The form b defines a scalar product $([A]|[B]) = b(A,B) = \varphi(B^\star A)$ on V. Completing V, we get an Hilbert space $\mathcal{H} = \overline{V}$. Let $A = X_{j_1} \dots X_{j_a}$. For any n, any $1 \le i_1 \dots i_n \le d$, the square of the norm of the vector $d\tau(X_{i_1} \dots X_{i_n})[A]$ is

$$\|[X_{i_1} \dots X_{i_n} A]\|^2 = |\varphi(A^\star X_{i_n} \dots X_{i_1} X_{i_1} \dots X_{i_n} A)| \le r C^{2n+2a}(2n+2a)!$$

Using the Stirling formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, if $C' > C$, we get:

$$\frac{C^{2n+2a}(2n+2a)!}{((C')^n n!)^2} \sim \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{n}} \frac{C^{2a}}{e^{2a}} \left[\frac{2n+2a}{2n}\right]^{2n} (2n+2a)^{2a} \left[\frac{C}{C'}\right]^{2n} 2^{2n}.$$

Since $\left[\frac{2n+2a}{2n}\right]^{2n} \to e^{2a}$ and $(2n+2a)^{2a}\left[\frac{C}{C'}\right]^{2n} \to 0$, there is $r_A > 0$ such that:

$$\|[X_{i_1} \dots X_{i_n} A]\|^2 \le r_A \big((2C')^n n!\big)^2. \tag{3}$$

This implies that the vectors in V are analytic for the representation of $\mathfrak{g}$, thus Theorem 1 in [10] proves there is a unique unitary representation τ of G on $\mathcal{H}$ such that $V \subset \mathcal{H}^\infty$, and $d\tau(X)[A] = [XA]$: on V, $d\tau$ is the representation of $\mathfrak{A}(\mathfrak{g})$ associated to τ by differentiation.

Finally $[1]$ is an analytic vector for τ (take $A = 1$ in the preceding computation), and by definition:

$$\Phi^\omega_\tau([1])(A) = \langle d\tau(A)[1], [1]\rangle = ([A]|[1]) = \varphi(A),$$

for each $A \in \mathfrak{A}(\mathfrak{g})$, this finishes the proof. ■

The relation between pure states and irreducible representations of G is:

Proposition 3.3. (1) *Let π be an irreducible unitary representation of G, and $f \in \mathcal{H}^\omega_{\pi,1}$ an analytic unit vector for π, then the state $\Phi^\omega_\pi(f)$ is pure.*

(2) *Conversely, if φ is a pure state, then the representation τ defined in Theorem 3.2 is irreducible.*

Proof. **(1)** Suppose $a > 0$ and φ is a state such that $a\varphi(A^\star A) \le \Phi^\omega_\pi(f)$. Since $d\pi(X^n)f \in d\pi(\mathfrak{A}(\mathfrak{g}))f$, for any n, if g is a vector in $(d\pi(\mathfrak{A}(\mathfrak{g}))f)^\perp$, then $g \in (\pi(G)f)^\perp$, thus g is orthogonal to the closure $\mathcal{K}$ of the vector space generated by $\pi(G)f$. But $\mathcal{K}$ is a closed vector subspace of $\mathcal{H}_\pi$, invariant under the action of $\pi(G)$, and containing $f \neq 0$, therefore $\mathcal{K} = \mathcal{H}_\pi$, $g = 0$, and $W = d\pi(\mathfrak{A}(\mathfrak{g}))f$ is a dense subspace in $\mathcal{H}_\pi$.

Now since $\|d\pi(A)f\|^2 = \Phi^\omega_\pi(f)(A^\star A) \ge a\varphi(A^\star A)$,

$$\mathrm{Ann}_f = \{A \in \mathfrak{A}(\mathfrak{g}) : \; d\pi(A)f = 0\} = \mathrm{Ann}_{\Phi^\omega_\pi(f)} \subset \mathrm{Ann}_\varphi.$$

Thus the surjective map $A \mapsto d\pi(A)f$ induces a bijective map $[A] \mapsto d\pi(A)f$ between the two spaces $V = \mathfrak{A}(\mathfrak{g})/\mathrm{Ann}_f$ and W. Moreover, the Hermitian form $b([A],[B]) = \varphi(B^\star A)$ is well defined on V and:

$$a^2\,|b(A,B)|^2 \le a^2 b(A,A)b(B,B) \le \|d\pi(A)f\|^2 \|d\pi(B)f\|^2.$$

By passing to the completion of these spaces, there is a positive self-adjoint operator T on $\mathcal{H}$ bounded by 1, such that, for any A and B in $\mathfrak{A}(\mathfrak{g})$,

$$\varphi(B^\star A) = \langle Td\pi(A)f, d\pi(B)f\rangle.$$

But T commutes with $d\pi$: indeed for each A, B and C in $\mathfrak{A}(\mathfrak{g})$,

$$\begin{aligned}\langle Td\pi(AB)f, d\pi(C)f\rangle &= \varphi(C^\star AB) = \varphi((A^\star C)^\star B)\\ &= \langle Td\pi(B)f, d\pi(A^\star C)f\rangle\\ &= \langle d\pi(A)Td\pi(B)f, d\pi(C)f\rangle.\end{aligned}$$

thus by the density of W, $Td\pi(A) = d\pi(A)T$ on W, by T-continuity, for each x in G, $T\pi(x) = \pi(x)T$, on W. This implies T commutes with $\pi(x)$ on $\mathcal{H}_\pi$ for each x. By the Schur Lemma $T = \lambda\mathrm{Id}$, but $1 = \varphi(1) = \langle Tf, f\rangle = \lambda$, $\varphi = \Phi^\omega_\pi(f)$, $\Phi^\omega_\pi(f)$ is a pure state.

(2) Suppose now that φ is a pure analytic state, perform the Gelfand construction for φ, as in Theorem 3.2. With the notation of the Theorem, the vector $f = [1]$ is analytic in $\mathcal{H}_\tau$, and $\varphi = \Phi^\omega_\tau(f)$.

Let $\mathcal{K}$ be an invariant closed subspace of $\mathcal{H}_\tau$, let us decompose f on the direct sum $\mathcal{K} \oplus \mathcal{K}^\perp$ into $f = f_1 + f_2$. If $f_1 = 0$, then $\mathcal{K}$ being the closure of the projection of $d\tau(\mathfrak{A}(\mathfrak{g}))f$ is vanishing. Suppose now $f_1 \neq 0$. Since $\mathcal{K}$ and $\mathcal{K}^\perp$ are invariant, for any n, any $1 \leq i_1, \dots, i_n \leq d$,

$$\|d\tau(X_{i_1}\dots X_{i_n})f\|^2 = \|d\tau(X_{i_1}\dots X_{i_n})f_1\|^2 + \|d\tau(X_{i_1}\dots X_{i_n})f_2\|^2 \leq rC^n n!.$$

Thus f_1 is an analytic vector, and

$$\Phi^\omega_\tau(f_1)(A^\star A) = \langle d\tau(A^\star A)f_1, f_1\rangle = \|d\tau(A)f_1\|^2 \leq \|d\tau(A)f\|^2 = \varphi(A^\star A).$$

Since φ is pure, this implies $\Phi^\omega_\tau(f_1) = \varphi$, and choosing $A = 1$, $f_2 = 0$. Thus the only closed invariant subspaces of $\mathcal{H}_\tau$ are $\{0\}$ and $\mathcal{H}_\tau$: τ is irreducible. ∎

Proposition 3.3 means that there is a canonical map $F : P^\omega(\mathfrak{A}(\mathfrak{g})) \to \widehat{G}$, defined by $F(\varphi) = [\tau]$ where τ is defined in Theorem 3.2. Since $\varphi = \Phi_\tau([1])$, this map is onto. Now let $[\pi]$ be in $\widehat{G}$, by definition, we then get $F^{-1}([\pi]) = \{\Phi_\pi(f) : f \in \mathcal{H}^\omega_{\pi,1}\}$, so that

$$F^{-1}([\pi]) = I^\omega_{\pi,\mathbb{C}}.$$

Corollary 3.4. *Let π and ρ be two unitary irreducible representations of G.*

If π and ρ are equivalent, then: $I^\omega_{\pi,\mathbb{C}} = I^\omega_{\rho,\mathbb{C}}$ and $I^\omega_\pi = I^\omega_\rho$.

If π and ρ are not equivalent, then: $I^\omega_{\pi,\mathbb{C}} \cap I^\omega_{\rho,\mathbb{C}} = \emptyset$ and $I^\omega_\pi \cap I^\omega_\rho = \emptyset$.

4. Convex hulls

Let X be a convex subset in a real vector space. Recall that the extremal points in X are points v such that $v = av_1 + (1-a)v_2$, with $0 \leq a \leq 1$ and $v_i \in X$, $v_1 \neq v_2$, implies $a = 0$ or $a = 1$. We denote by $\mathrm{Ext}(X)$ the set of extremal points in X.

Proposition 4.1. *The subsets* S_C^ω, $S^\omega(\mathfrak{A}(\mathfrak{g}))$ *(*$C>0$*) of* $(\mathfrak{A}(\mathfrak{g}))^\star$ *are convex, and:*

$$\operatorname{Ext}(S^\omega(\mathfrak{A}(\mathfrak{g}))) = P^\omega(\mathfrak{A}(\mathfrak{g})).$$

Proof. Since $S_C^\omega \subset S_{C'}^\omega$ if $C \leq C'$, and $S^\omega(\mathfrak{A}(\mathfrak{g})) = \cup_{C>0} S_C^\omega$, the convexity of $S^\omega(\mathfrak{A}(\mathfrak{g}))$ is a consequence of the convexity of each S_C^ω. Let now $0 \leq a \leq 1$, φ, ψ in S_C^ω, then $\theta = a\varphi + (1-a)\psi$ is clearly a state and

$$|\theta(X_{i_1}\dots X_{i_n})| \leq a\,|\varphi(X_{i_1}\dots X_{i_n})| + (1-a)\,|\psi(X_{i_1}\dots X_{i_n})| \leq (ar_\varphi + (1-a)r_\psi)C^n n!$$

Therefore S_C^ω is convex.

Let θ bean analytic state. Suppose first θ is not extremal: there is $0<a<1$ and $\varphi \neq \psi$ in $S^\omega(\mathfrak{A}(\mathfrak{g}))$ such that $\theta = a\varphi + (1-a)\psi$. Thus $a\varphi(A^\star A) \leq \theta(A^\star A)$, for any A. This means that θ is not a pure state.

Conversely, suppose that θ is a non pure analytic state, more precisely, suppose $\theta \in S_C^\omega$. Let π be the unitary representation of G in $\mathcal{H}_\pi = \overline{\mathfrak{A}(\mathfrak{g})/\mathrm{Ann}_\theta}$ and $f \in \mathcal{H}_\pi^\omega$ as in Theorem 3.2, such that $\theta = \Phi_\pi^\omega(f)$. Since θ is not pure, π is not irreducible (Proposition 3.3): there is a non trivial invariant closed subspace $\mathcal{K}$ in $\mathcal{H}_\pi$. Decompose f on $\mathcal{H}_\pi = \mathcal{K} \oplus \mathcal{K}^\perp$ into $f = f_1 + f_2$. Then $f_i \neq 0$. Indeed if $f_2 = 0$, then $d\pi(\mathfrak{A}(\mathfrak{g}))f = d\pi(\mathfrak{A}(\mathfrak{g}))f_1 \subset \mathcal{K}$, taking the closure, we get: $\mathcal{H}_\pi = \mathcal{K}$, a contradiction. Let p be the orthogonal projection onto $\mathcal{K}$, since $\mathcal{K}$ and $\mathcal{K}^\perp$ are invariant, p commutes with $\pi(x)$ for each x. Therefore $p(\mathcal{H}_\pi^\omega) \subset \mathcal{K}_{\pi|_\mathcal{K}}^\omega$. Thus f_1, f_2 are analytic vectors in $\mathcal{H}_\pi$. Put $\varphi = \Phi_\pi^\omega(\frac{f_1}{\|f_1\|})$, $\psi = \Phi_\pi^\omega(\frac{f_2}{\|f_2\|})$, and $a = \|f_1\|^2$. Then:

$$\theta = \Phi_\pi(\|f_1\|\frac{f_1}{\|f_1\|} + \|f_2\|\frac{f_2}{\|f_2\|}) = a\varphi + (1-a)\psi.$$

This finishes the proof of the proposition. ■

Let us now come back to the complex analytic moment set $I_{\pi,\mathbb{C}}^\omega$ of a unitary irreducible representation π and its convex hull, $J_{\pi,\mathbb{C}}^\omega$. We have:

Proposition 4.2. *The following holds:*

$$J_{\pi,\mathbb{C}}^\omega \cap P^\omega(\mathfrak{A}(\mathfrak{g})) = J_{\pi,\mathbb{C}}^\omega \cap \operatorname{Ext}(S^\omega(\mathfrak{A}(\mathfrak{g}))) = I_{\pi,\mathbb{C}}^\omega.$$

Proof. Let θ be in $J_{\pi,\mathbb{C}}^\omega \cap P^\omega(\mathfrak{A}(\mathfrak{g}))$, thus there are k, $\varphi_i \in I_{\pi,\mathbb{C}}^\omega$, $a_i > 0$ $(1 \leq i \leq k)$, such that $\sum_i a_i = 1$ and:

$$\theta = \sum_i a_i \varphi_i.$$

Especially, φ_1 is a state such that $a_1\varphi_1(A^\star A) \leq \theta(A^\star A)$. Since θ is pure, this implies $\varphi_1 = \theta$, $\theta \in I_{\pi,\mathbb{C}}^\omega$.

Conversely, since π is irreducible, each state $\Phi_\pi(f)$ in $I_{\pi,\mathbb{C}}^\omega$ is pure and in $J_{\pi,\mathbb{C}}^\omega$. This proves the proposition. ■

Observe that Theorem 2.3 (see [1]) is now a direct consequence of Proposition 4.2. Recall the notation $J_\pi = \operatorname{Conv}(I_\pi)$, $J_{\pi,\mathbb{C}} = \operatorname{Conv}(I_{\pi,\mathbb{C}})$.

Corollary 4.3. *Let π and ρ be two unitary irreducible representations of a connected Lie group G, then π and ρ are equivalent if and only if $J_\pi = J_\rho$.*

Proof. Let $\widetilde{G}$ be the universal covering group of G and $\widetilde{\pi}$, $\widetilde{\rho}$ the representations of $\widetilde{G}$ corresponding to π and ρ. Then $d\pi = d\widetilde{\pi}$, $I_{\pi,\mathbb{C}} = I_{\widetilde{\pi},\mathbb{C}}$, and $J_{\pi,\mathbb{C}} = J_{\widetilde{\pi},\mathbb{C}}$.
Let $f \in \mathcal{H}^\omega_{\widetilde{\pi}} = \mathcal{H}^\omega_\pi$, then

$$\Phi_{\widetilde{\pi}}(f) \in J^\omega_{\widetilde{\pi},\mathbb{C}} \cap P^\omega(\mathfrak{A}(\mathfrak{g})) = I^\omega_{\widetilde{\pi},\mathbb{C}}.$$

On the other hand, if $J_\pi = J_\rho$, then this analytic state is in $J_{\widetilde{\rho},\mathbb{C}}$, thus we also have:

$$\Phi_{\widetilde{\pi}}(f) \in J^\omega_{\widetilde{\rho},\mathbb{C}} \cap P^\omega(\mathfrak{A}(\mathfrak{g})) = I^\omega_{\widetilde{\rho},\mathbb{C}}.$$

Therefore $\widetilde{\pi} \sim \widetilde{\rho}$, and $\pi \sim \rho$.
The converse being evident, this proves the corollary. ∎

5. Complete convex sums

In [1], the complete moment set $\widetilde{J}_{\pi,\mathbb{C}}$ of a unitary representation π is defined as

$$\widetilde{J}_{\pi,\mathbb{C}} = \Big\{ \sum_{n=0}^{\infty} \Phi_\pi(f_n) : f_n \in \mathcal{H}^\infty_\pi,\ \sum_n \|f_n\|^2 = 1,\ \sum_n \|d\pi(A) f_n\|^2 < \infty \text{ for all } A \in \mathfrak{A}(\mathfrak{g}) \Big\}.$$

Similarly, let us now define the complete states ψ:

Definition 5.1. An analytic state ψ is the *sum of a convex series* of states if there exists a sequence (φ_n) of states and a sequence (a_n) of positive real numbers such that:

$$\psi = \sum_{n=0}^{\infty} a_n \varphi_n.$$

Lemma 5.2. *If $\psi = \sum_n a_n \varphi_n$ is the sum of a convex series of states, then for each n, $a_n \neq 0$ implies that φ_n is an analytic state.*

Proof. Fix a basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{g}$, since ψ is analytic, there are $r > 0$ and $C > 0$ such that, for any k, and any $1 \leq i_1, \ldots, i_k \leq d$,

$$\begin{aligned} |\psi(X_{i_k} \ldots X_{i_1} X_{i_1} \ldots X_{i_k})| &= (-1)^k \psi(X_{i_k} \ldots X_{i_1} X_{i_1} \ldots X_{i_k}) \\ &= \sum_n a_n (-1)^k \varphi_n(X_{i_k} \ldots X_{i_1} X_{i_1} \ldots X_{i_k}) \leq r C^{2k} (2k)!\,. \end{aligned}$$

Thus if $a_n > 0$, $$|\varphi_n(X_{i_k} \ldots X_{i_1} X_{i_1} \ldots X_{i_k})| \leq \frac{r}{a_n} C^{2k} (2k)!\,.$$

Now since the form $(A, B) \mapsto \varphi_n(B^\star A)$ on $\mathfrak{A}(\mathfrak{g})$ is positive, the Cauchy-Schwarz inequality $|\varphi_n(B^\star A)|^2 \leq \varphi_n(B^\star B) \varphi_n(A^\star A)$ holds and:

$$|\varphi_n(X_{i_1} \ldots X_{i_k})| \leq \sqrt{\frac{r}{a_n}} C^k \sqrt{(2k)!}\,.$$

Now, the Stirling formula implies:

$$(2k)! \sim \sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k} \sim 2\sqrt{\pi k}\ 2^{2k}\left(\frac{k!}{\sqrt{2\pi k}}\right)^2 = \frac{2^{2k}}{\sqrt{\pi k}}(k!)^2,$$

or $\sqrt{(2k)!} \sim (\pi k)^{-1/4}\ 2^k k!$. This proves that φ_n is an analytic state. ∎

From now on, we suppose that if $a_n = 0$, then $\varphi_n(X_{i_1}\dots X_{i_k}) = 0$ for any $k > 0$ (φ_n is the map called augmentation in [9]).

Observe that, if τ_n is the unitary representation such that $\varphi_n = \Phi^\omega_{\tau_n}(f_n)$ defined in Theorem 3.2, then putting $f'_n = \sqrt{a_n}\ f_n$, and denoting $\oplus\tau_n$ the direct sum of the τ_n, we get:

$$\|\oplus_n f'_n\|^2 = \sum_n \|f'_n\|^2 = \sum_n a_n = \psi(1) = 1.$$

Moreover, for any A in $\mathfrak{A}(\mathfrak{g})$,

$$\begin{aligned}\|\oplus_n d\tau_n(A)f'_n\|^2 &= \sum_n \|d\tau_n(A)f'_n\|^2 = \sum_n a_n\|d\tau_n(A)f_n\|^2\\ &= \sum_n a_n\Phi^\omega_{\tau_n}(f_n)(A^\star A) = \psi(A^\star A) < \infty\end{aligned}$$

Therefore $\psi = \Phi^\omega_{\oplus\tau_n}(\oplus f'_n)$ belongs to $I^\omega_{\oplus\tau_n,\mathbb{C}}$.

Proposition 5.3. *Let $\psi = \sum_n a_n\varphi_n$ be the sum of a convex series of analytic states and π (resp. τ_n) the representation associated to ψ (resp. to φ_n) by Theorem 3.2. Then π is unitarily equivalent to a subrepresentation of $\oplus_n\tau_n$.*

Proof. Recall that $\mathcal{H}_\pi$ is the completion of the pre-Hilbert space $V = \mathfrak{A}(\mathfrak{g})/Ann_\psi$, equipped with the bilinear form:

$$([A]|[B]) = \psi(B^\star A).$$

As above, denote $\varphi_n = \Phi_{\tau_n}(f_n)$, and $f'_n = \sqrt{a_n} f_n$. Define thus the map $U : V \to \oplus_n\mathcal{H}_{\tau_n}$ by

$$U([A]) = \sum_n d\tau_n(A)f'_n.$$

Observe that U is well defined because, if A belongs to Ann_ψ,

$$0 = \psi(A^\star A) = \sum_n a_n\langle d\tau_n(A^\star A)f_n, f_n\rangle = \sum_n a_n\|d\tau_n(A)f_n\|^2.$$

Thus $U([A]) = 0$. Moreover U is linear and an isometry, since

$$\|U([A])\|^2 = \sum_n a_n\|d\tau_n(A)f_n\|^2 = \psi(A^\star A) = \|[A]\|^2.$$

We can thus extend U to the space $\mathcal{H}_\pi$.

We saw that if C is such that $\psi \in S^\omega_C$, then each φ_n is in S^ω_{2C}. Fix the basis $\{X_1,\dots,X_d\}$ of $\mathfrak{g}$ and define the norm on $\mathfrak{g}$ by $\|\sum_i x_iX_i\| = \sup|x_i|$. Let X be in $\mathfrak{g}$ such that $\|X\| < \frac{1}{4Cd}$, then we claim that the following holds:

$$U(\pi(\exp\ X)[A]) = \sum_k \frac{1}{k!}U(d\pi(X^k)[A]) = \sum_k\sum_n \frac{1}{k!}d\tau_n(X^kA)f'_n$$
$$= \sum_n\sum_k \frac{1}{k!}d\tau_n(X^kA)f'_n = \sum_n \tau_n(\exp X)d\tau_n(A)f'_n = (\oplus_n\tau_n)(\exp X)U([A]).$$

Indeed, thanks to the inequality (3) for any C' such that $2C > C' > C$, we saw there is $r_A > 0$ such that:

$$\|d\tau_n(X^kA)f'_n\| \le \sum_{i_1,\dots i_k} |x_{i_1}\dots x_{i_k}|\|d\tau_n(X_{i_1}\dots X_{i_k}A)f'_n\|$$
$$\le r_A(2C')^k k!\frac{d^k}{(4Cd)^k} \le r_A k!\left(\frac{C'}{2C}\right)^k.$$

Therefore by Lebesgue's theorem of dominated convergence, we can exchange the sums over k and n in the above computation, this proves our claim. Since G is connected and simply connected, this implies that for each x in G, each $f \in \mathcal{H}_\pi$,

$$U(\pi(x)f) = (\oplus_n\tau_n)(x)(Uf).$$

This proves the proposition. ■

As in [1], if π is a unitary representation of G, we denote by $\widetilde{\pi}$ the representation $\aleph_0\pi$ sum of a countably many representations unitarily equivalent to π. Moreover the proof of Proposition 3.2 in [1] says that $\widetilde{J_{\pi,\mathbb{C}}} = I_{\widetilde{\pi},\mathbb{C}}$. Therefore Theorem 1.2 of [1] is a direct corollary of Proposition 5.3. Recall that two unitary representations π and ρ are called quasi equivalent if the representations $\widetilde{\pi}$ and $\widetilde{\rho}$ are unitarily equivalent.

Corollary 5.4. *Two unitary representations π and ρ of a connected Lie group G are quasi equivalent if and only if $\widetilde{J_{\pi,\mathbb{C}}} = \widetilde{J_{\rho,\mathbb{C}}}$, if and only if their complete moment sets $I_{\widetilde{\pi}}$ and $I_{\widetilde{\rho}}$ coincide.*

Proof. By considering the extensions of the representations π and ρ to the universal covering group of G, we can suppose that G is simply connected.
If π and ρ are quasi-equivalent, then $\widetilde{\pi}$ and $\widetilde{\rho}$ are equivalent and $I_{\widetilde{\pi}} = I_{\widetilde{\rho}}$.
Conversely, if $I_{\widetilde{\pi}} = I_{\widetilde{\rho}}$, then $I^\omega_{\widetilde{\pi},\mathbb{C}} = I^\omega_{\widetilde{\rho},\mathbb{C}}$. Pick $f \in \mathcal{H}^\omega_{\pi,1}$, then $\Phi^\omega_\pi(f)$ is in $I^\omega_{\pi,\mathbb{C}} \subset I^\omega_{\widetilde{\pi},\mathbb{C}} = I^\omega_{\widetilde{\rho},\mathbb{C}}$. This means there is a sequence (f_n) of vectors in $\mathcal{H}^\omega_{\rho,1}$, a sequence (a_n) of positive numbers, such that:

$$\psi = \Phi^\omega_\pi(f) = \sum_n a_n\Phi^\omega_\rho(f_n) = \sum a_n\varphi_n.$$

For each n define $\mathcal{H}_{\tau_n}$ as the closure of the subspace $d\rho(\mathfrak{A}(\mathfrak{g}))f_n$ of $\mathcal{H}_\rho$, and τ_n as the restriction of ρ to $\mathcal{H}_{\tau_n}$. Thus τ_n is the representation coming from φ_n by Theorem 3.2. Now Proposition 5.3 implies that π is a subrepresentation of $\oplus_n\tau_n$, thus a subrepresentation of $\widetilde{\rho}$.

Therefore $\widetilde{\pi}$ is a subrepresentation of $\widetilde{\widetilde{\rho}} = \widetilde{\rho}$. Interchanging the roles of π and ρ, we obtain that $\widetilde{\rho}$ is a subrepresentation of $\widetilde{\pi}$, an usual argument proves that $\widetilde{\pi}$ and $\widetilde{\rho}$ are equivalent (see [8] or [1]). ■

References

[1] L. Abdelmoula, D. Arnal, J. Ludwig, M. Selmi: *Separation of unitary representations of connected Lie groups by their moment sets*, J. Funct. Anal. 228 (2005) 189–206.

[2] L. Abdelmoula, A. Baklouti, Y. Bouaziz: *On the generalized moment separability teorem for type 1 solvable Lie groups*, Adv. Pure Appl. Math. 9 (2018) 247–277.

[3] D. Arnal, A. Baklouti, J. Ludwig, M. Selmi: *Separation of unitary representations of exponential Lie groups*, J. Lie Theory 10 (2000) 399–410.

[4] D. Arnal, M. Selmi, A. Zergane: *Universal overgroups*, J. Geom. Phys. 61 (2011) 217–229.

[5] L. Auslander, B. Kostant: *Polarizations and unitary representations of solvable Lie groups*, Inventiones Math. 14 (1971) 255–354.

[6] A. Baklouti, J. Ludwig, M. Selmi: *Séparation des représentations unitaires des groupes de Lie nilpotents*, in: *Lie Theory and its Applications in Physics II*, Proc. Workshop held at Clausthal, Germany, August 1997, H.-D. Doebner, J. Hilgert and V. K. Dobrev (eds), World Scientific, Singapore (1999) 75–91.

[7] P. Bernat, N. Conze, M. Duflo, M. Lévy-Nahas, M. Rais, P. Renouard, M. Vergne: *Représentations des Groupes de Lie Résolubles*, Monographies de la S.M.F., Dunod, Paris (1972).

[8] J. Dixmier: *Les $C^\star$-Algèbres et leurs Représentations*, Cahiers Scientifiques, Gauthier-Villars, Paris (1957).

[9] J. Dixmier: *Algèbres Enveloppantes*, Cahiers Scientifiques, Gauthier-Villars, Paris (1974).

[10] M. Flato, J. Simon, H. Snellman, D. Sternheimer: *Simple facts about analytic vectors and integrability*, Ann. Sci. École Norm. Sup. 4 (1972) 423–434.

[11] R. Goodman: *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. 143 (1969) 55–76.

[12] K. H. Neeb: *On analytic vectors for unitary representations of infinite dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) 61 (2011) 1839–1874.

[13] E. Nelson: *Analytic vectors*, Ann. of Math. II 70 (1959) 572–615.

[14] N. J. Wildberger: *Convexity and unitary representations of nilpotent Lie groups*, Invent. Math. 98 (1989) 281–292.

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