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Real Forms of Contragredient Lie Superalgebras with Isomorphic Even Parts

Meng-Kiat Chuah and Rita Fioresi

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Abstract. We study how the real forms $\mathfrak{g}$ of contragredient Lie superalgebras are determined by their even parts. We prove that if the even parts of $\mathfrak{g}$ and $\mathfrak{g}'$ are inner isomorphic, then $\mathfrak{g}$ and $\mathfrak{g}'$ are inner isomorphic. Also, if the even parts of $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic.

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1. Introduction

The Lie superalgebras [5] enjoy rapid developments in their structure and representation theories, as well as applications in mathematical physics and supersymmetry [1]. In particular, the real forms of contragredient Lie superalgebras were studied and classified up to isomorphism in [5][7]. This article provides a finer classification under the notion of inner isomorphism.

Let $\mathfrak{g}, \mathfrak{g}'$ be real forms of a contragredient Lie superalgebra L, namely they are real subalgebras of L such that $L = \mathfrak{g} + i\mathfrak{g} = \mathfrak{g}' + i\mathfrak{g}'$ are direct sums of real vector spaces. We write $L = L_{\bar{0}} + L_{\bar{1}}$ and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ for their even and odd parts. It is known that if $\mathfrak{g}_{\bar{0}} = \mathfrak{g}'_{\bar{0}}$, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic [5, Prop. 5.3.2]. In this article, we consider the weaker conditions where $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}'_{\bar{0}}$ are merely isomorphic or inner isomorphic. If $f : \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{g}'_{\bar{0}}$ is an isomorphism, it extends uniquely by $\mathbb{C}$-linearity to $f \in \mathrm{aut}(L_{\bar{0}})$. If f lies in the identity component $\mathrm{int}(L_{\bar{0}})$ of $\mathrm{aut}(L_{\bar{0}})$, we say that $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}'_{\bar{0}}$ are inner isomorphic. We similarly say that $\mathfrak{g}$ and $\mathfrak{g}'$ are inner isomorphic if there exists an isomorphism between them which extends uniquely to $\mathrm{int}(L)$ by $\mathbb{C}$-linearity. We write $\simeq$ for isomorphism, and $\cong$ for inner isomorphism. The following is our main result.

Theorem 1.1. *Let $\mathfrak{g}, \mathfrak{g}'$ be real forms of a contragredient Lie superalgebra L.*
(a) *If $\mathfrak{g}_{\bar{0}} \cong \mathfrak{g}'_{\bar{0}}$, then $\mathfrak{g} \cong \mathfrak{g}'$.*
(b) *If $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{g}'_{\bar{0}}$, then $\mathfrak{g} \simeq \mathfrak{g}'$.*

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Earlier a related work [7] adopted a different approach. It used semi-linear maps on L to represent $\mathfrak{g}$ (so that $\mathfrak{g}$ is its invariant subalgebra), whereas we use linear maps to represent $\mathfrak{g}$ as Cartan automorphism [4]. Also, [7, Lem. 2.2] required the automorphism to be inner.

Theorem 1.1 leads to the following classification of real forms. The classification up to isomorphism has been done in [7], so we focus on inner isomorphism. Let $D(n,m)$ be the Lie superalgebra whose even part is $D_n + C_m$. The real form $\mathfrak{so}^*(2n)$ of D_n is defined in [6, I-8].

Corollary 1.2. *The real forms of a contragredient Lie superalgebra which are isomorphic but not inner isomorphic occur only on $D(n,m)$ for n even. There are two real forms $\mathfrak{g},\mathfrak{g}'$ of $D(n,m)$ such that $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{g}'_{\bar{0}} \simeq \mathfrak{so}^*(2n) + \mathfrak{sp}(p,q)$ where $p+q=m$, and $\mathfrak{g},\mathfrak{g}'$ are not inner isomorphic to each other.*

We shall review some preliminary background materials in Section 2, then prove Theorem 1.1 and Corollary 1.2 in Section 3.

2. Cartan automorphisms and markings

In this section, we review some properties of Cartan automorphisms and markings [4], which will be used to characterize real forms of contragredient Lie superalgebras later.

Let L be a contragredient Lie superalgebra. It has an invariant non-degenerate bilinear form B, which is unique up to multiplication by non-zero scalars, and is symmetric on $L_{\bar{0}}$ and skew-symmetric on $L_{\bar{1}}$ [5, §2.5]. Let $\mathrm{aut}_{2,4}(L)$ denote L-automorphisms of order 2 on $L_{\bar{0}}$, and order 4 on $L_{\bar{1}}$. We say that $\theta \in \mathrm{aut}_{2,4}(L)$ is a *Cartan automorphism* of a real form $\mathfrak{g}$ of L if θ stabilizes $\mathfrak{g}$, and $-B(\cdot,\theta\cdot)$ is an inner product on $\mathfrak{g}$. Each $\theta \in \mathrm{aut}_{2,4}(L)$ is a Cartan automorphism of a real form, and conversely each real form has a Cartan automorphism [4, Thm.1.1]. This generalizes the notion of Cartan involutions of real semisimple Lie algebras.

Let D be an extended Dynkin diagram of L, namely its vertices represent a simple system of L together with its lowest root, and its edges are drawn according to the method in [5, p. 54-55]. We write $\mathrm{D} = \mathrm{D}_{\bar{0}} \cup \mathrm{D}_{\bar{1}}$, where $\mathrm{D}_{\bar{0}}$ (resp. $\mathrm{D}_{\bar{1}}$) contains even (resp. odd) roots. The vertices of $\mathrm{D}_{\bar{0}}$ have white color, and the vertices of $\mathrm{D}_{\bar{1}}$ have black or grey color (the latter denoted by $\otimes$). There are many choices of extended Dynkin diagrams, and we always let D denote the preferred one where [2, Thm.1.1]

$$\begin{aligned} &\mathrm{D}_{\bar{0}} \text{ is the Dynkin diagram of } [L_{\bar{0}}, L_{\bar{0}}], \\ &\mathrm{D}_{\bar{1}} \text{ are the lowest weights of } L_{\bar{1}}. \end{aligned} \tag{1}$$

In the last condition of (1), the lowest weights refer to the adjoint $L_{\bar{0}}$-representation on $L_{\bar{1}}$.

If H is a Cartan subalgebra of L (and hence of $L_{\bar{0}}$), we write $L = H + \sum_\Delta L_\alpha$ for the root space decomposition, where $\Delta \subset H^*$ are the roots and L_α are the root spaces. If $\Pi \subset \Delta$ is a simple system, we let φ denote its lowest root. We can choose $\Pi \cup \{\varphi\}$ such that it is represented by D of (1). There are canonical positive integers $\{a_\alpha\}_{\mathrm{D}}$ without nontrivial common factor such that

$$\sum_{\mathrm{D}} a_\alpha \alpha = 0, \tag{2}$$

known as the labels. A list of all extended Dynkin diagrams D, along with labels $\{a_\alpha\}_{\mathrm{D}}$, is given in [2, Fig. 1].

We say that L is of type 1 if $L_{\bar{0}}$ has a 1-dimensional center and its adjoint representation on $L_{\bar{1}}$ has two irreducible factors. In this case $\mathrm{D}_{\bar{1}} = \{\beta, \gamma\}$, where $a_\beta = a_\gamma = 1$. We say that L is of type 2 if $L_{\bar{0}}$ is semisimple and its adjoint representation on $L_{\bar{1}}$ is irreducible. In this case $\mathrm{D}_{\bar{1}} = \{\gamma\}$, where $a_\gamma = 2$. Thus $A(m,n)$ and $C(n)$ are of type 1, while the remaining contragredient Lie superalgebras are of type 2.

Let S^1 be the unit circle in $\mathbb{C}$. A *marking* on D is a pair (c,d) such that

$$c : \mathrm{D} \longrightarrow S^1 \ , \ d \in \mathrm{aut}(\mathrm{D}).$$

Here $c_\alpha \in S^1$ for all vertices $\alpha \in \mathrm{D}$, and let $\mathrm{im}(c) = \{c_\alpha \ ; \ \alpha \in \mathrm{D}\}$ be its image. Also, d is required to preserve vertex colors. We write the marking as $(c,1)$ if d is the trivial diagram automorphism, and as $(1,d)$ if c assigns 1 to every vertex.

We shall relate some markings (c,d) on D to L-automorphisms. To do this, we first let $\epsilon \in \{\pm 1\}$, which depends on d. In fact $\epsilon = 1$ in almost all cases (including $d = 1$) except when d switches two adjacent grey vertices or is given by [2, Fig. 2] (interested readers may read [2, §1]). We say that (c,d) is *admissible* if [2, (1.4)]

$$\epsilon \prod_{\mathrm{D}} c_\alpha^{a_\alpha} = 1. \tag{3}$$

We say that (c,d) represents an L-automorphism θ if D represents some $\Pi \cup \{\varphi\} \subset \Delta$, and there exist vector space automorphisms θ_c, θ_d on $\sum_D L_\alpha$ such that [2, Def.1.2]

$$\begin{array}{ll} \text{(a)} & \theta_c \text{ acts as multiplication by } c_\alpha \text{ on } L_\alpha, \\ \text{(b)} & \text{there exist root vectors } \{X_\alpha \in L_\alpha\}_{\alpha\in\mathrm{D}} \text{ such that } \theta_d X_\alpha = X_{d\alpha}, \\ \text{(c)} & \theta = \theta_c\theta_d \text{ on } \sum_{\mathrm{D}} L_\alpha. \end{array} \tag{4}$$

Let $U \subset S^1$ denote the roots of unity, namely all $z \in S^1$ such that $z^m = 1$ for some $m \in \mathbb{N}$. The definition of markings (c,d) in [2, §1] requires $\mathrm{im}(c) \subset U$. In that case if (c,d) represents θ, then θ has finite order.

Theorem 2.1. *Every admissible marking on* D *represents an* L*-automorphism. Conversely, every finite order* L*-automorphism is represented by an admissible marking on* D.

Proof. Most of this theorem is handled in [2, Thm.1.3], which says that the admissible markings (c,d) with $\mathrm{im}(c) \subset U$ correspond to finite order L-automorphisms. If (c,d) is an admissible marking without the condition $\mathrm{im}(c) \subset U$, then the same arguments in [2, Thm.1.3] show that it also represents an (infinite order) L-automorphism. ∎

In Theorem 2.1, the finite order condition ensures that the L-automorphism stabilizes a Cartan subalgebra of L and a simple system of $[L_{\bar{0}}, L_{\bar{0}}]$, which is necessary and sufficient for it to be represented by a marking on D. An infinite order L-automorphism may or may not stabilize a Cartan subalgebra and simple system. An arbitrary assignment of eigenvalues to the vertices of D may not represent an L-automorphism, because these vertices represent linearly dependent vectors $\Pi \cup \{\varphi\}$. This is why all the markings which represent automorphisms need to satisfy an additional condition (3), namely admissibility.

If a marking represents some $\theta \in \mathrm{aut}_{2,4}(L)$, then θ is a Cartan automorphism of a real form $\mathfrak{g}$ of L [4, Thm.1.1]. In that case we also say that the marking represents $\mathfrak{g}$. Suppose that L is of type 2, so that D has a unique grey vertex γ with $a_\gamma = 2$. If $(c,1)$ represents some $\theta \in \mathrm{aut}_{2,4}(L)$, then $c_\gamma = \pm i$ so that $\theta|_{L_\gamma}$ has order 4. Then $c_\gamma^{a_\gamma} = (\pm i)^2 = -1$, so (3) implies that $\prod_{\mathrm{D}_{\bar{0}}} c_\alpha^{a_\alpha} = -1$. Hence:

Theorem 2.2. [4, Def. 1.2(c),Thm. 1.3] *For L of type 2, a marking $(c,1)$ represents a real form of L if and only if $\prod_{\mathrm{D}_{\bar{0}}} c_\alpha^{a_\alpha} = -1$.*

This theorem puts some restrictions on the real forms of $L_{\bar{0}}$ which are even parts of real forms of L. For example we cannot choose a compact real form of $L_{\bar{0}}$ when L is of type 2. We write $\theta = \theta_{\bar{0}} + \theta_{\bar{1}}$ to denote its restrictions to $L_{\bar{0}}$ and $L_{\bar{1}}$.

Proposition 2.3. *Let $\theta, \theta' \in \mathrm{aut}_{2,4}(L)$. If $\theta_{\bar{0}} = \theta'_{\bar{0}}$, then $\theta_{\bar{1}} = \pm\theta'_{\bar{1}}$.*

Proof. Let (c,d) and (c',d') be markings on D which represent θ and θ' respectively. We may assume that their restrictions to $\mathrm{D}_{\bar{0}}$ are equal, namely $(c,d)_{\bar{0}} = (c',d')_{\bar{0}}$. We divide the arguments into two cases, for L of types 1 and 2. Suppose first that L is of type 2, so that $\mathrm{D}_{\bar{1}} = \{\gamma\}$. The condition $\theta, \theta' \in \mathrm{aut}_{2,4}(L)$ implies

$$c_\gamma, c'_\gamma \in \{\pm i\},$$

so that $\theta|_{L_\gamma}$ and $\theta'|_{L_\gamma}$ have order 4. Recall that γ is the lowest root of $L_{\bar{1}}$. If $c_\gamma = c'_\gamma$, then $\theta_{\bar{1}} = \theta'_{\bar{1}}$. If $c_\gamma = -c'_\gamma$, then $\theta_{\bar{1}} = -\theta'_{\bar{1}}$. This solves the type 2 case.

Suppose now that L is of type 1, so that $\mathrm{D}_{\bar{1}} = \{\beta, \gamma\}$, where $a_\beta = a_\gamma = 1$. Since θ and θ' have order 4 on $L_\beta + L_\gamma$, by (4),

$$\begin{array}{l} d \text{ fixes } \beta,\gamma \implies c_\beta, c_\gamma, c'_\beta, c'_\gamma \in \{\pm i\}, \\ d \text{ interchanges } \beta,\gamma \implies \{c_\beta, c_\gamma\} = \{c'_\beta, c'_\gamma\} = \{1,-1\}. \end{array} \tag{5}$$

In the second case of (5), if d interchanges β, γ, then each of θ and θ' acts as $X_\beta \mapsto X_\gamma \mapsto -X_\beta$ on some root vectors X_β and X_γ.

By Theorem 2.1, $\prod_{\mathrm{D}} c_\alpha^{a_\alpha} = \prod_{\mathrm{D}} (c'_\alpha)^{a_\alpha}$. The condition $(c,d)_{\bar{0}} = (c',d')_{\bar{0}}$ then implies that

$$c_\beta c_\gamma = c'_\beta c'_\gamma. \tag{6}$$

By (5) and (6), one of the following holds,

$$\{c_\beta = c'_\beta \text{ and } c_\gamma = c'_\gamma\} \text{ or } \{c_\beta = -c'_\beta \text{ and } c_\gamma = -c'_\gamma\}. \tag{7}$$

Since β and γ are the lowest roots of the two irreducible factors of $L_{\bar{1}}$, the first case of (7) implies that $\theta_{\bar{1}} = \theta'_{\bar{1}}$, and the second case of (7) implies that $\theta_{\bar{1}} = -\theta'_{\bar{1}}$. This proves the proposition. ∎

Corollary 2.4. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be real forms of L. If $\mathfrak{g}_{\bar 0} = \mathfrak{g}'_{\bar 0}$, then either $\mathfrak{g}_{\bar 1} = \mathfrak{g}'_{\bar 1}$ or $\mathfrak{g}_{\bar 1} = i\mathfrak{g}'_{\bar 1}$.*

Proof. Let $\mathfrak{g} = \mathfrak{g}_{\bar 0} + \mathfrak{g}_{\bar 1}$ be a real form of L. Let $\mathfrak{g}' = \mathfrak{g}_{\bar 0} + i\mathfrak{g}_{\bar 1}$. It is easy to check that $\mathfrak{g}'$ is also a real form of L, and we shall show that these are the only two real forms whose even part is $\mathfrak{g}_{\bar 0}$.
Let $\theta \in \mathrm{aut}_{2,4}(L)$ be a Cartan automorphism of $\mathfrak{g}$, namely θ stabilizes $\mathfrak{g}$, and $-B(\cdot, \theta\cdot)$ is an inner product of $\mathfrak{g}$. Let $\theta' = \theta_{\bar 0} - \theta_{\bar 1}$. Then θ' stabilizes $\mathfrak{g}'$, and $-B(\cdot, \theta'\cdot)$ is an inner product of $\mathfrak{g}'$. So θ' is a Cartan automorphism of $\mathfrak{g}'$. By Proposition 2.3, θ and θ' are the only members of $\mathrm{aut}_{2,4}(L)$ whose even parts are $\theta_{\bar 0}$. The members of $\mathrm{aut}_{2,4}(L)$ correspond to real forms of L [4, Thm.1.1], so $\mathfrak{g}$ and $\mathfrak{g}'$ are the only real forms of $\mathfrak{g}$ with even part $\mathfrak{g}_{\bar 0}$. ■

3. Inner automorphisms

In this section, we prove Theorem 1.1 and Corollary 1.2. As discussed in Section 1, we write $\mathfrak{g} \simeq \mathfrak{g}'$ for isomorphism, and $\mathfrak{g} \cong \mathfrak{g}'$ for inner isomorphism.

Proposition 3.1. *Let $\mathfrak{g}, \mathfrak{g}'$ be real forms of a contragredient Lie superalgebra L. If $\mathfrak{g}_{\bar 0} \cong \mathfrak{g}'_{\bar 0}$, then $\mathfrak{g} \cong \mathfrak{g}'$.*

Proof. Let $\mathfrak{g}, \mathfrak{g}'$ be real forms of a contragredient Lie superalgebra L. We first prove the special case

$$\mathfrak{g}_{\bar 0} = \mathfrak{g}'_{\bar 0} \implies \mathfrak{g} \cong \mathfrak{g}'. \tag{8}$$

Suppose that $\mathfrak{g}_{\bar 0} = \mathfrak{g}'_{\bar 0}$. By Corollary 2.4, we may assume that $\mathfrak{g}' = \mathfrak{g}_{\bar 0} + i\mathfrak{g}_{\bar 1}$. Recall that the preferred extended Dynkin diagram D of (1) is equipped with the labels $\{a_\alpha\}_{\mathrm{D}}$ as in (2). Notice that either $\mathrm{D}_{\bar 1}$ consists of two vertices with label 1, or of one vertex with label 2. Fix an even vertex $\beta \in \mathrm{D}_{\bar 0}$ with $a_\beta = 1$. Let $(c, 1)$ be the marking given by

$$c_\alpha = \begin{cases} 1 & \text{for all } \alpha \in \mathrm{D}_{\bar 0} \backslash \{\beta\}, \\ -1 & \text{for } \alpha = \beta, \\ i & \text{for all } \alpha \in \mathrm{D}_{\bar 1}. \end{cases} \tag{9}$$

Then $\prod_{\mathrm{D}} c_\alpha^{a_\alpha} = (\prod_{\mathrm{D}_{\bar 0}\backslash\{\beta\}} c_\alpha^{a_\alpha}) c_\beta^{a_\beta} (\prod_{\mathrm{D}_{\bar 1}} c_\alpha^{a_\alpha}) = 1 \cdot (-1) \cdot i^2 = 1$, so by Theorem 2.1, it represents some $f \in \mathrm{aut}(L)$. The construction (9) shows that f acts as ± 1 on each even root space, and acts as $\pm i$ on each odd root space. Hence $f(\mathfrak{g}) = \mathfrak{g}'$.
We claim that $f \in \mathrm{int}(L)$. Let $S^1 \subset \mathbb{C}$ be the unit circle, and let σ be a continuous path given by

$$\sigma : [0, 1] \longrightarrow S^1 \ ; \ \sigma(0) = 1 \ , \ \sigma(1) = i.$$

For each $t \in [0, 1]$, we define a marking $(c(t), 1)$ by

$$c(t)_\alpha = \begin{cases} 1 & \text{for all } \alpha \in \mathrm{D}_{\bar 0} \backslash \{\beta\}, \\ \sigma(t)^{-2} & \text{for } \alpha = \beta, \\ \sigma(t) & \text{for all } \alpha \in \mathrm{D}_{\bar 1}. \end{cases}$$

Then each $(c(t), 1)$ is admissible and hence represents some $f_t \in \mathrm{aut}(L)$. Furthermore, $f_0 = 1$ and $f_1 = f$. We have proved that $f \in \mathrm{int}(L)$ as claimed, hence (8).

Suppose that $\mathfrak{g}, \mathfrak{g}'$ are real forms of L such that $\mathfrak{g}_{\bar{0}} \cong \mathfrak{g}'_{\bar{0}}$. There exists $f_{\bar{0}} \in \mathrm{int}(L_{\bar{0}})$ given by $f_{\bar{0}} = \prod_j \exp \mathrm{ad}_{X_j}$ for some $X_j \in L_{\bar{0}}$, such that $f_{\bar{0}}(\mathfrak{g}_{\bar{0}}) = \mathfrak{g}'_{\bar{0}}$. Since L is an $L_{\bar{0}}$-module via the adjoint action, we have $\mathrm{ad}_{X_j} : L \longrightarrow L$, so that $f_{\bar{0}}$ extends naturally to $f \in \mathrm{int}(L)$.
Define $\mathfrak{g}'' = f(\mathfrak{g})$. Then $\mathfrak{g}'' \cong \mathfrak{g}$. We have $\mathfrak{g}''_{\bar{0}} = \mathfrak{g}'_{\bar{0}}$, so by (8), $\mathfrak{g}'' \cong \mathfrak{g}'$. Hence $\mathfrak{g} \cong \mathfrak{g}'$. This proves the proposition. ■

Recall that if a marking (c, d) represents $\theta \in \mathrm{aut}_{2,4}(L)$, then it represents a real form of L. In particular if $d = 1$, then c_α is the eigenvalue of $\theta|_{L_\alpha}$. We shall study real forms of $L_{\bar{0}}$ which are isomorphic but not inner isomorphic. The next theorem provides all the cases where such a situation may occur on real forms of complex simple Lie algebras.

Figure 1: Real forms of D_n ($n > 4$ even) that are not inner isomorphic.

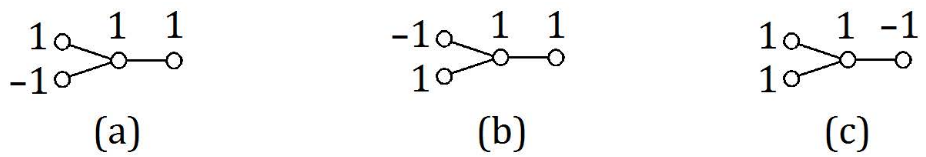

Figure 2: Real forms of D_4 that are not inner isomorphic.

Theorem 3.2. [3, Fig.4, §10] *The real forms of a complex simple Lie algebra which are isomorphic but not inner isomorphic occur only on D_n for n even, and are represented by the markings in Figures 1 and 2. In all other cases, isomorphic real forms are inner isomorphic.*

Proposition 3.3. *Let $\mathfrak{g}, \mathfrak{g}'$ be real forms of a contragredient Lie superalgebra L. If $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{g}'_{\bar{0}}$, then $\mathfrak{g} \simeq \mathfrak{g}'$.*

Proof. By Proposition 3.1, it suffices to consider real forms $\mathfrak{g}, \mathfrak{g}'$ such that $\mathfrak{g}_{\bar{0}}, \mathfrak{g}'_{\bar{0}}$ are isomorphic but not inner isomorphic. Let $f : \mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{g}'_{\bar{0}}$ be an isomorphism. It extends uniquely to $\mathrm{aut}(L_{\bar{0}})$ by $\mathbb{C}$-linearity. We have the semi-direct product $\mathrm{aut}(L_{\bar{0}}) = \mathrm{aut}(\mathrm{D}_{\bar{0}}) \ltimes \mathrm{int}(L_{\bar{0}})$ [6, Thm.7.8], and we may assume that

$$f(\mathfrak{g}_{\bar{0}}) = \mathfrak{g}'_{\bar{0}} \ , \ 1 \neq f \in \mathrm{aut}(\mathrm{D}_{\bar{0}}) \subset \mathrm{aut}(L_{\bar{0}}). \tag{10}$$

Here $\mathfrak{g}_{\bar{0}}, \mathfrak{g}'_{\bar{0}}$ contain ideals $\mathfrak{s}, \mathfrak{s}'$ which are not inner isomorphic, and $f(\mathfrak{s}) = \mathfrak{s}'$. By Theorem 3.2, $\mathfrak{s}, \mathfrak{s}'$ are represented by Figure 1 or 2. This happens only on $L = D(n, m)$ and $L_{\bar{0}} = D_n + C_m$, where n is even. Write $\mathrm{D}_{\bar{0}} = \mathrm{D}_{\bar{0}}(D_n) \cup \mathrm{D}_{\bar{0}}(C_m)$ for the Dynkin diagrams of the two simple ideals. Since $\mathrm{D}_{\bar{0}}(C_m)$ has no nontrivial diagram symmetry, we have $f \in \mathrm{aut}(\mathrm{D}_{\bar{0}}(D_n)) = \mathrm{aut}(\mathrm{D}_{\bar{0}})$.
If $\mathfrak{s}$ and $\mathfrak{s}'$ are represented by Figure 1, then f is given by Figure 3(a), and it extends naturally to $f \in \mathrm{aut}(\mathrm{D})$. The marking $(1, f)$ on D is admissible, so by Theorem

2.1, it represents a member of $\mathrm{aut}(L)$. Let $\mathfrak{g}'' = f(\mathfrak{g})$. Then $\mathfrak{g}''$ is a real form of L such that $\mathfrak{g}''_{\bar 0} = \mathfrak{g}'_{\bar 0}$, so $\mathfrak{g}'' \cong \mathfrak{g}'$ by Proposition 3.1. Hence $\mathfrak{g} \simeq \mathfrak{g}'$.

Next we consider Figure 2. The three endpoints of $\mathrm{D}_{\bar 0}(D_4)$ play symmetric roles within $\mathrm{D}_{\bar 0}(D_4)$. However, they play different roles as vertices of D because only one of them is adjacent to the grey vertex. We orientate the vertices in Figure 2 so that the rightmost endpoint x is adjacent to the grey vertex, as appears in Figure 3(b). If y is one of the other two endpoints, then the labels (2) satisfy

$$a_x = 2 \ , \ a_y = 1. \tag{11}$$

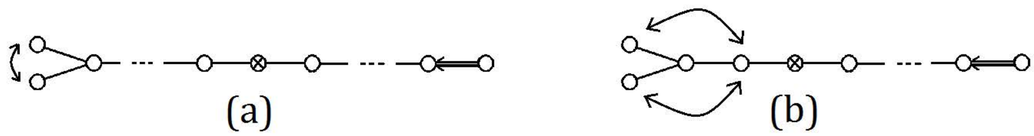

Figure 3: $f \in \mathrm{aut}(\mathrm{D}_{\bar 0})$.

Suppose that $\mathfrak{s}$ and $\mathfrak{s}'$ are represented by Figure 2. There are two cases,

$$\begin{array}{ll} \text{(a)} & \mathfrak{s} \text{ and } \mathfrak{s}' \text{ are represented by Figure 2(a,b),} \\ \text{(b)} & \text{one of } \mathfrak{s}, \mathfrak{s}' \text{ is represented by Figure 2(c).} \end{array} \tag{12}$$

If (12)(a) occurs, then f is again given by Figure 3(a), and the above discussion on D_n implies that $\mathfrak{g} \simeq \mathfrak{g}'$.

We now show that (12)(b) cannot occur. Suppose that (12)(b) occurs. Then $f \in \mathrm{aut}(\mathrm{D}_{\bar 0})$ is given by one of the two arrows in Figure 3(b). Let $(c, 1)$ and $(c', 1)$ be markings which represent $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. We may let c' be given by $c'_\alpha = c_{f(\alpha)}$ for all $\alpha \in \mathrm{D}_{\bar 0}$. By (11), $\prod_{\mathrm{D}_{\bar 0}(D_n)} c_\alpha^{a_\alpha}$ and $\prod_{\mathrm{D}_{\bar 0}(D_n)} (c'_\alpha)^{a_\alpha}$ have opposite signs, namely

$$\{ \prod_{\mathrm{D}_{\bar 0}(D_n)} c_\alpha^{a_\alpha}, \prod_{\mathrm{D}_{\bar 0}(D_n)} (c'_\alpha)^{a_\alpha} \} = \{1, -1\}. \tag{13}$$

Since c and c' are equal on $\mathrm{D}_{\bar 0}(C_m)$, (13) implies that

$$\{\prod_{\mathrm{D}_{\bar 0}} c_\alpha^{a_\alpha}, \prod_{\mathrm{D}_{\bar 0}} (c'_\alpha)^{a_\alpha}\} = \{1, -1\}.$$

This contradicts Theorem 2.2, because both $(c, 1)$ and $(c', 1)$ represent real forms of L. Hence (12)(b) cannot occur.

We conclude that when $\mathfrak{g}$ and $\mathfrak{g}'$ are real forms such that $f(\mathfrak{g}_{\bar 0}) = \mathfrak{g}'_{\bar 0}$ satisfies (10), then f is given by Figure 3(a), which implies that $\mathfrak{g} \simeq \mathfrak{g}'$. ■

Proof of Theorem 1.1 and Corollary 1.2.

Theorem 1.1 follows from Propositions 3.1 and 3.3. Next we prove Corollary 1.2. By Theorem 1.1, we start with real forms $\mathfrak{s}, \mathfrak{s}'$ of complex simple Lie algebras such that $\mathfrak{s} \simeq \mathfrak{s}'$ and $\mathfrak{s} \not\cong \mathfrak{s}'$. By Theorem 3.2, this occurs only when $\mathfrak{s}, \mathfrak{s}'$ are represented by the markings in Figure 1 or 2.

We first consider Figure 1, where the markings represent the real forms

$$\mathfrak{s} \simeq \mathfrak{s}' \simeq \mathfrak{so}^*(2n) \tag{14}$$

of D_n with $n > 4$ even (see Chapter I-8 and Appendix C-3 of [6] for $\mathfrak{so}^*(2n)$). The list of real forms of contragredient Lie superalgebras L [7][4] shows that (14) extends to real forms $\mathfrak{g} \simeq \mathfrak{g}'$ of L only when $L = D(n,m)$, where

$$\mathfrak{g}_{\bar{0}} \simeq \mathfrak{g}'_{\bar{0}} \simeq \mathfrak{so}^*(2n) + \mathfrak{sp}(p,q) \tag{15}$$

for $p + q = m$. Here $\mathfrak{g}_{\bar{0}}, \mathfrak{g}'_{\bar{0}}$ are isomorphic but not inner isomorphic. By Theorem 1.1, $\mathfrak{g} \simeq \mathfrak{g}'$. It also implies that $\mathfrak{g} \not\cong \mathfrak{g}'$. This is because if $f(\mathfrak{g}) = \mathfrak{g}'$ and $\{f_t \in \operatorname{aut}(L)\ ;\ t \in [0,1]\}$ satisfy $f_0 = 1$ and $f_1 = f$, then the restrictions of f_t to $L_{\bar{0}}$ imply that $\mathfrak{g}_{\bar{0}} \cong \mathfrak{g}'_{\bar{0}}$, a contradiction. This provides the real forms indicated in this corollary.

Let Figure 2(a,b,c) respectively represent three mutually non-inner isomorphic real forms of D_4: $\mathfrak{s} \simeq \mathfrak{s}' \simeq \mathfrak{s}'' \simeq \mathfrak{so}^*(8)$.

They extend to real forms $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}''$ of $D(4,m)$. Similar to (15), we have $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{g}'_{\bar{0}} \simeq \mathfrak{so}^*(8) + \mathfrak{sp}(p,q)$, so by Theorem 1.1, $\mathfrak{g}, \mathfrak{g}'$ are isomorphic but not inner isomorphic. However, due to the parities of labels (see Theorem 2.2 and (11)), we have $\mathfrak{g}''_{\bar{0}} \simeq \mathfrak{so}^*(8) + \mathfrak{sp}(m,\mathbb{R})$. So $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{g}'_{\bar{0}} \not\simeq \mathfrak{g}''_{\bar{0}}$, and hence $\mathfrak{g} \simeq \mathfrak{g}' \not\simeq \mathfrak{g}''$. The additional case of Figure 2(c) that is not covered by Figure 1 does not contribute to extra isomorphic real form. This proves Corollary 1.2.

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Meng-Kiat Chuah
Department of Mathematics
National Tsing Hua University
Hsinchu 300, Taiwan.
chuah@math.nthu.edu.tw

Rita Fioresi
Dipartimento di Matematica
University of Bologna
Piazza Porta San Donato 5
40126 Bologna, Italy.
rita.fioresi@unibo.it

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