

Derivations of Extended Multi-Loop Algebras

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Abstract. We develop the notion of “extended multi-loop algebras” and determine their derivation algebras. Extended multi-loop algebras appear naturally as the core modulo center of locally extended affine Lie algebras; they are in fact an extension of n -step multi-loop algebras where the number of automorphisms are allowed to be possibly infinite and also the coordinate algebras (Laurent polynomials) are allowed to be over an infinite number of variables.

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1. Introduction

Affine Kac-Moody Lie algebras, as infinite dimensional analogues of semisimple Lie algebras, have been developed in the past five decades with great success. Connections to other areas, such as mathematical physics (conformal field theory) and combinatorics, make this class of Lie algebras of great interest. Later, the need seems to have arisen for some “higher nullity” generalizations of affine Kac-Moody Lie algebras. *Extended affine Lie algebras* and *locally extended affine Lie algebras* are natural and higher nullity generalizations of affine Kac-Moody Lie algebras, see [1] and [14] for details. After that, some other generalizations were introduced and studied by others, see e.g. [5], [8] and [18].

An n -step multi-loop algebra [4], is determined by a base algebra \mathcal{A} over a field \mathbb{k} , n commuting finite order algebra automorphisms of \mathcal{A} and a Laurent polynomial in n variables as the *coordinate algebra*. Multi-loop algebras are natural generalizations of the classical loop algebras that are used to realize affine Kac-Moody Lie algebras. Indeed, V. Kac showed that any affine Kac-Moody Lie algebra is built out of a loop algebra (of a finite order automorphism of a finite dimensional simple Lie algebra) by adding a central extension and a space of derivations [12, Chapter 8] and moreover, E. Neher [17] showed how to build an extended affine Lie algebra out of a Lie torus in a way reminiscent of the affine construction. But in [2], the authors showed that almost all centerless Lie tori can be realized using multi-loop algebras, see also [3].

Consequently,

every extended affine Lie algebra can be realized as a multi-loop algebra of a finite number of automorphisms based on a finite-dimensional simple Lie algebra by adding a central extension and a space of derivations. (1)

This fact is of great importance in the theory of affine and extended affine Lie algebras. In particular, in the above realization problem, the space of derivations of a multi-loop algebra plays an important role. To this end, S. Azam [7] showed that the derivation algebra of the tensor product of two algebras can be obtained from the derivation algebras and the centroids of the algebras, and then using this result, he studied the derivation algebra of a multi-loop algebra [6]. Also, A. Pianzola [20] studied the derivation algebra of a class of algebras given by étale descent containing the multi-loop algebras. Moreover, in [19] the authors studied the étale descent of derivations of algebras with values in a module. The algebras under consideration in the latter reference are twisted forms of algebras over rings containing multi-loop algebras.

Motivated by the realization problem (1) for locally extended affine Lie algebras, we introduce here an extension of multi-loop algebras, called *extended multi-loop algebras* (see Definition 5.1), and determine their derivation algebras. An extended multi-loop algebra is determined by a base algebra \mathcal{A} over a field \mathbb{k} , a family of possibly infinitely many commuting finite order admissible automorphisms, and an algebra of Laurent polynomials in possibly infinitely many variables as the coordinate algebra. We expect that the centerless cores of locally extended affine Lie algebras could be in general realized by extended multi-loop algebras whence allowing to establish a realization framework for locally extended affine Lie algebras. Moreover, since the derivations of these algebras are of great importance in “realization problem”, we determine and study the structure of the derivation algebra of an extended multi-loop algebra.

The article is organized as follows. In Section 2, we recall some preliminaries and definitions such as the derivation algebra and centroid of an algebra and some of their properties. In Section 3, we study the derivation algebra of the tensor product of algebras \mathcal{A} and \mathcal{B} over a field and, as the main result of this section, determine the structure of the derivation algebra of the degree-zero space of $\mathcal{A} \otimes \mathcal{B}$ with respect to a G -grading, where G is a certain abelian group, see Theorem 3.4. In Section 4, we apply the previous results to algebras with gradings induced by automorphisms. Finally, in Section 5, we introduce extended multi-loop algebras, see Definition 5.1, and study their algebras of derivations.

2. Basic definitions and facts

In this work we fix a field \mathbb{k} and two algebras \mathcal{A} and \mathcal{B} over \mathbb{k} . All other algebras will be over \mathbb{k} except otherwise stated. Throughout G denotes an abelian group, and every group in this paper is abelian and written additively. The *multiplication* algebra of \mathcal{A} , denoted $\text{Mult}(\mathcal{A})$, is the subalgebra of endomorphisms of \mathcal{A} over \mathbb{k} generated by the identity operator and the left and right multiplication operators of

\mathcal{A} . The *centroid* of \mathcal{A} , denoted $\mathcal{C}(\mathcal{A})$, is by definition the set of all endomorphisms of \mathcal{A} which commute with all elements of $\text{Mult}(\mathcal{A})$, that is

$$\mathcal{C}(\mathcal{A}) = \{\gamma \in \text{End}(\mathcal{A}) \mid \gamma(xy) = \gamma(x)y = x\gamma(y) \text{ for all } x, y \in \mathcal{A}\}.$$

Note that $\mathcal{C}(\mathcal{A})$ is a unital subalgebra of $\text{End}(\mathcal{A})$. Also, we consider \mathcal{A} as a left $\mathcal{C}(\mathcal{A})$ -module by $\gamma \cdot a = \gamma(a)$, for $\gamma \in \mathcal{C}(\mathcal{A})$ and $a \in \mathcal{A}$. \mathcal{A} is called *central* if $\mathcal{C}(\mathcal{A}) = \text{kid}_{\mathcal{A}}$ and it is called *perfect* if $\mathcal{A} = \mathcal{A}^{(1)}$, the \mathbb{k} -span of all products aa' for $a, a' \in \mathcal{A}$. If \mathcal{A} is perfect, then $\mathcal{C}(\mathcal{A})$ is commutative, and if \mathcal{A} is unital, commutative and associative, then $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}$, $\gamma \mapsto \gamma(1)$ is an algebra isomorphism.

The proof of the following lemma can be found in [4], [9] and [7].

Lemma 2.1. *The map*

$$\psi: \mathcal{C}(\mathcal{A}) \otimes \mathcal{B} \longrightarrow \mathcal{C}(\mathcal{A} \otimes \mathcal{B}); \quad \gamma \otimes b \mapsto \gamma \otimes L_b \quad (2)$$

where L_b , $b \in \mathcal{B}$, denotes the left multiplication by b , is an algebra isomorphism if \mathcal{A} is perfect and \mathcal{B} is unital, commutative and associative and one of the following holds:

- (a) \mathcal{B} is finite dimensional,
- (b) \mathcal{A} is unital,
- (c) \mathcal{A} is finitely generated over \mathbb{k} ,
- (d) \mathcal{A} is finitely generated as a module over $\mathcal{C}(\mathcal{A})$,
- (e) \mathcal{A} is finitely generated as a module over $\text{Mult}(\mathcal{A})$,
- (f) \mathcal{A} is central.

Let (M, \cdot) be an \mathcal{A} -bimodule. A *derivation* of \mathcal{A} with values in M is a \mathbb{k} -linear map $d: \mathcal{A} \rightarrow M$ satisfying $d(aa') = d(a) \cdot a' + a \cdot d(a')$ for all $a, a' \in \mathcal{A}$. The vector space of all such derivations is denoted $\mathcal{D}(\mathcal{A}, M)$. An example of a derivation is the so-called *inner derivation* ∂_m determined by $m \in M$ and defined by $\partial_m(a) = a \cdot m$, $a \in \mathcal{A}$. The inner derivations form a subspace of $\mathcal{D}(\mathcal{A}, M)$ denoted $\mathcal{ID}(\mathcal{A}, M)$. If $M = \mathcal{A}$, we write $\mathcal{D}(\mathcal{A})$ and $\mathcal{ID}(\mathcal{A})$ instead of $\mathcal{D}(\mathcal{A}, \mathcal{A})$ and $\mathcal{ID}(\mathcal{A}, \mathcal{A})$, respectively.

Let \mathfrak{g} be a finite dimensional semisimple Lie algebra. If M is a finite dimensional \mathfrak{g} -module, Whitehead's Lemma says that every derivation $\mathfrak{g} \rightarrow M$ is inner, see [13, III.7, Lemma 3]. This immediately extends to the locally finite case, i.e., $\mathcal{ID}(\mathfrak{g}, M) = \mathcal{D}(\mathfrak{g}, M)$ if M is a locally finite \mathfrak{g} -module (see [19, Lemma 3.1]). One knows in general that $\mathcal{ID}(\mathfrak{g})$ is an ideal of $\mathcal{D}(\mathfrak{g})$.

2.1. Centroids and derivations of graded algebras

The algebra \mathcal{A} is said to be *G-graded*, or *graded* if the grading group G is clear from the context, if $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ is a direct sum of subspaces \mathcal{A}_g such that $\mathcal{A}_g \mathcal{A}_{g'} \subseteq \mathcal{A}_{g+g'}$, for all $g, g' \in G$. In this case, $\text{supp}(\mathcal{A}) = \{g \in G \mid \mathcal{A}_g \neq \{0\}\}$ is called the *support* of \mathcal{A} and the elements of \mathcal{A}_g are said to be *homogeneous* (of degree g). A subalgebra (or ideal) \mathcal{A}' of \mathcal{A} is graded if $\mathcal{A}' = \bigoplus_{g \in G} (\mathcal{A}' \cap \mathcal{A}_g)$. A map $f \in \text{End}(\mathcal{A})$ is of *degree* $g \in G$ if $f(\mathcal{A}_{g'}) \subseteq \mathcal{A}_{g'+g}$, for all $g' \in G$. Denote

by $\text{End}(\mathcal{A})_g$ the subspace containing all $f \in \text{End}(\mathcal{A})$ of degree $g \in G$. Then, the (internal) sum of the subspaces $\text{End}(\mathcal{A})_g$ is direct and we set

$$\text{grEnd}(\mathcal{A}) := \bigoplus_{g \in G} \text{End}(\mathcal{A})_g. \tag{3}$$

This is a graded associative subalgebra and, in general, a proper subalgebra of $\text{End}(\mathcal{A})$. However, if \mathcal{A} is finite dimensional or if the support of \mathcal{A} is finite, then $\text{grEnd}(\mathcal{A}) = \text{End}(\mathcal{A})$ and thus the space $\text{End}(\mathcal{A})$ becomes graded.

For $g \in G$, set $\mathcal{C}(\mathcal{A})_g := \mathcal{C}(\mathcal{A}) \cap \text{End}(\mathcal{A})_g$ and $\mathcal{D}(\mathcal{A})_g := \mathcal{D}(\mathcal{A}) \cap \text{End}(\mathcal{A})_g$, and

$$\text{gr } \mathcal{C}(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{C}(\mathcal{A})_g \text{ and } \text{gr } \mathcal{D}(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{D}(\mathcal{A})_g.$$

If \mathcal{A} is finitely generated $\text{Mult}(\mathcal{A})$ -module, then $\text{gr } \mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{A})$ [9, §2], and $\text{gr } \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A})$ if \mathcal{A} is finitely generated as an algebra over \mathbb{k} [11, Prop. 1].

3. Derivations of tensor product algebras

In this section, we study the derivations of the tensor product $\mathcal{A} \otimes \mathcal{B}$ over \mathbb{k} . Recall that $\mathcal{A} \otimes \mathcal{B}$ is an algebra under the product $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$, for $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. For $f \in \text{End}(\mathcal{A})$ and $g \in \text{End}(\mathcal{B})$ we consider the map $f \tilde{\otimes} g \in \text{End}(\mathcal{A} \otimes \mathcal{B})$ defined by $f \tilde{\otimes} g(a \otimes b) = f(a) \otimes g(b)$, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Definition 3.1. Let R be a commutative associative ring and M be an R -module. A family $\{g_j\}_{j \in J} \subset \text{End}_R(M)$ is called *summable* if for each $m \in M$, $g_j(m) = 0$ for all but finitely many j . In this case, for each family $\{r_j\}_{j \in J} \subseteq R$, $\sum_{j \in J} r_j g_j$ defined by $(\sum_{j \in J} r_j g_j)(m) = \sum_{j \in J} r_j g_j(m)$ is a well-defined endomorphism of M , moreover the set of all endomorphisms of this form is a submodule of $\text{End}_R(M)$. If $\{g_i \mid i \in I\}$ is an R -linearly independent subset of $\text{End}_R(M)$, then the family $\{g_i\}_{i \in I}$ is called a *summable R -basis* for this submodule.

Suppose that $E_{\mathcal{A}}$ and $E_{\mathcal{B}}$ are two arbitrary subspaces of $\text{End}(\mathcal{A})$ and $\text{End}(\mathcal{B})$ respectively. If $\{g_j\}_{j \in J} \subseteq E_{\mathcal{B}}$ is summable and $\{f_j\}_{j \in J} \subseteq E_{\mathcal{A}}$, then $\{f_j \tilde{\otimes} g_j\}_{j \in J} \subseteq \text{End}(\mathcal{A} \otimes \mathcal{B})$ is also summable. Then we define $E_{\mathcal{A}} \overrightarrow{\otimes} E_{\mathcal{B}}$ to be the set of all $\sum_{j \in J} f_j \tilde{\otimes} g_j$, for all summable families $\{g_j\}_{j \in J} \subseteq E_{\mathcal{B}}$ and all subsets $\{f_j\}_{j \in J} \subseteq E_{\mathcal{A}}$. In a similar manner, we can define $E_{\mathcal{A}} \overleftarrow{\otimes} E_{\mathcal{B}}$. An element $F \in \text{End}(\mathcal{A} \otimes \mathcal{B})$ is called of *finite \mathcal{A} -image* if for each $b \in \mathcal{B}$, $F(\mathcal{A} \otimes b) \subseteq \sum_{j \in T_b} \mathcal{A} \otimes b_j$ for some $b_j \in \mathcal{B}$ and a finite set T_b . Clearly, in this definition b_j 's can be considered as elements of a basis of \mathcal{B} . Here is a description of $\text{End}(\mathcal{A}) \overrightarrow{\otimes} \text{End}(\mathcal{B})$ in terms of finite \mathcal{A} -image maps.

Lemma 3.2. $\text{End}(\mathcal{A}) \overrightarrow{\otimes} \text{End}(\mathcal{B}) = \{F \in \text{End}(\mathcal{A} \otimes \mathcal{B}) \mid F \text{ is of finite } \mathcal{A}\text{-image}\}.$

Proof. We fix a basis $\{a_k\}_{k \in K}$ of \mathcal{A} and a basis $\{b_j\}_{j \in J}$ of \mathcal{B} . Suppose first that F is an element of $\text{End}(\mathcal{A}) \overrightarrow{\otimes} \text{End}(\mathcal{B})$. Then there exists a summable family $\{g_i\}$ of $\text{End}(\mathcal{B})$ and a family $\{f_i\}$ of $\text{End}(\mathcal{A})$ such that $F := \sum_{i \in I} f_i \tilde{\otimes} g_i$. Since $\{g_i\}$ is summable, for each $j \in J$, there exists a finite set I_j such that $g_i(b_j) \neq 0$ if and

only if $i \in I_j$. Then for each $a \in \mathcal{A}$ and $j \in J$, $F(a \otimes b_j) = \sum_{i \in I_j} f_i(a) \otimes g_i(b_j) \in \sum_{i \in I_j} \mathcal{A} \otimes g_i(b_j)$ and so F is of finite \mathcal{A} -image. To complete the proof, suppose that F is of finite \mathcal{A} -image; we must show that $F \in \text{End}(\mathcal{A}) \overleftrightarrow{\otimes} \text{End}(\mathcal{B})$. Now for each $k \in K$ and $j \in J$, we have $F(a_k \otimes b_j) = \sum_{t \in J} a_{jk}^t \otimes b_t$, for some unique $a_{jk}^t \in \mathcal{A}$ where $a_{jk}^t = 0$ for all but finitely many t . Since F is of finite \mathcal{A} -image, for each j , $J_j = \{t \in J \mid a_{jk}^t \neq 0 \text{ for some } k\}$ is finite. Let \bar{J}_j be a copy of J_j such that $\bar{J}_j \cap \bar{J}_i = \emptyset$ for $j \neq i$. Set $T := \cup_{j \in J} \bar{J}_j$ and, for $t \in T$, define $g_t \in \text{End}(\mathcal{B})$ by

$$g_t(b_i) = \begin{cases} b_t & \text{if } t \in \bar{J}_i \\ 0 & \text{otherwise.} \end{cases}$$

Also we define $f_t \in \text{End}(\mathcal{A})$ as follows. If $t \in \bar{J}_i$, then we require $f_t(a_k) = a_{ik}^t$. Note that f_t 's are well-defined since whenever $i \neq j$, we have $\bar{J}_i \cap \bar{J}_j = \emptyset$, for all $i, j \in J$. Here in the definitions of f_t and g_t we have identified $t \in \bar{J}_i$ with its corresponding element in J_i . Since J_j is finite for each j , $\{g_t\}_{t \in T}$ is summable. So $F' := \sum_{t \in T} f_t \overleftrightarrow{\otimes} g_t \in \text{End}(\mathcal{A}) \overleftrightarrow{\otimes} \text{End}(\mathcal{B})$. Moreover,

$$F'(a_k \otimes b_j) = \sum_{i \in J} \sum_{t \in \bar{J}_i} f_t(a_k) \otimes g_t(b_j) = \sum_{t \in \bar{J}_j} f_t(a_k) \otimes b_t = \sum_{t \in T} a_{jk}^t \otimes b_t = F(a_k \otimes b_j),$$

so $F = F'$ as required. ■

Let \mathcal{A} be perfect and \mathcal{B} be unital, commutative and associative. Also, assume that the map ψ in (2) is an isomorphism. Therefore, by [7, Theorem 2.9], we have

$$\mathcal{D}(\mathcal{A} \otimes \mathcal{B}) = (\mathcal{D}(\mathcal{A}) \overleftarrow{\otimes} \mathcal{B}) \oplus (\mathcal{C}(\mathcal{A}) \overrightarrow{\otimes} \mathcal{D}(\mathcal{B})). \tag{4}$$

Let \mathcal{A} and \mathcal{B} be G -graded. Therefore, $\mathcal{A} \otimes \mathcal{B} = \bigoplus_{g \in G} (\mathcal{A} \otimes \mathcal{B})_g$ is a G -graded algebra where $(\mathcal{A} \otimes \mathcal{B})_g = \sum_{g' \in G} \mathcal{A}_{g'} \otimes \mathcal{B}_{g-g'}$, for all $g \in G$.

Proposition 3.3. *Assume that \mathcal{A} is perfect and \mathcal{B} is unital, commutative and associative. Also, let \mathcal{A} and \mathcal{B} be G -graded and the map ψ in (2) be an isomorphism.*

Then
$$\text{gr } \mathcal{D}(\mathcal{A} \otimes \mathcal{B}) = \sum_{g \in G} \mathcal{D}(\mathcal{A} \otimes \mathcal{B})_g = \sum_{g \in G} (\mathcal{D}(\mathcal{A}) \overleftarrow{\otimes} \mathcal{B})_g \oplus (\mathcal{C}(\mathcal{A}) \overrightarrow{\otimes} \mathcal{D}(\mathcal{B}))_g$$

such that

$$(\mathcal{D}(\mathcal{A}) \overleftarrow{\otimes} \mathcal{B})_g = \sum_{g' \in G} \mathcal{D}(\mathcal{A})_{g'} \overleftarrow{\otimes} \mathcal{B}_{g-g'} \text{ and } (\mathcal{C}(\mathcal{A}) \overrightarrow{\otimes} \mathcal{D}(\mathcal{B}))_g = \sum_{g' \in G} \mathcal{C}(\mathcal{A})_{g'} \overrightarrow{\otimes} \mathcal{D}(\mathcal{B})_{g-g'}.$$

Proof. The proof is similar to the proof of [7, Lemma 3.1]. ■

For a direct sum $G = \bigoplus_{i \in I} G_i$ of groups, we let $\iota_i : G_i \rightarrow G$ and $\pi_i : G \rightarrow G_i$ be the canonical injection and projection maps, respectively. Then each $x \in G$, setting $\pi_i(x) := x_i$, for all i , can be represented as $x = \sum_{i \in I} x_i$. For an abelian group H and a positive integer m , we call a map $\phi : H \rightarrow \mathbb{Z}$, m -additive if for $x, y \in H$, $\phi(x + y) = \phi(x) + \phi(y) - \epsilon m$ for $\epsilon \in \{0, 1\}$. Clearly the composition of an m -additive map with the canonical projection $\mathbb{Z} \rightarrow \mathbb{Z}_m$ is a group homomorphism.

Theorem 3.4. *Suppose that \mathcal{A} and \mathcal{B} are G -graded algebras where $G = \bigoplus_{i \in I} G_i$ is a direct sum of abelian groups G_i . Assume that \mathcal{A} is perfect and \mathcal{B} is unital, commutative and associative, and for each i , m_i is a positive integer such that m_i^{-1} makes sense in \mathbb{k} . Assume further that*

- (a) *the map ψ in (2) is an isomorphism,*
- (b) *for each i , there exists an m_i -additive map $\phi_i : G_i \rightarrow \mathbb{Z}$,*
- (c) *for each i , there exists a unit u_i in \mathcal{B} such that $u_i^{m_i} \in \mathcal{B}_0$ and for all $r_i \in G_i$, $u_i^{\phi_i(r_i)} \in \mathcal{B}_{\iota_i(r_i)}$.*

If $(\mathcal{A} \otimes \mathcal{B})_0$ and $\mathcal{D}(\mathcal{A} \otimes \mathcal{B})_0$ respectively denote the degree-zero homogeneous subspaces of $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{D}(\mathcal{A} \otimes \mathcal{B})$ with respect to the G -grading, then the restriction map

$$\pi : \mathcal{D}(\mathcal{A} \otimes \mathcal{B})_0 \longrightarrow \mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0); \quad D \mapsto D|_{(\mathcal{A} \otimes \mathcal{B})_0}$$

is an isomorphism. In particular,

$$\mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0) \cong \sum_{r \in G} (\mathcal{D}(\mathcal{A})_r \overleftarrow{\otimes} \mathcal{B}_{-r}) \oplus (\mathcal{C}(\mathcal{A})_r \overrightarrow{\otimes} \mathcal{D}(\mathcal{B})_{-r}).$$

Under the assumptions of Theorem 3.4 and assuming $n = \sum_{i \in I} n_i \in \bigoplus_{i \in I} \mathbb{Z}$, we set $u^n := \prod_{i \in I} u_i^{n_i}$. Since $\{\phi_i \pi_i\}$ is summable, $\phi : G \rightarrow \bigoplus_{i \in I} \mathbb{Z}$ defined by $\phi(r) = \sum_{i \in I} \phi_i(r_i)$ is a well-defined group homomorphism. Moreover we have $u^{\phi(r)} = \prod_{i \in I} u_i^{\phi_i(r_i)} \in \mathcal{B}_r$ where $r = \sum_{i \in I} \iota_i(r_i) \in G$.

Before we prove the above theorem, we need to prove two lemmas. For this we need to fix elements $r, r', s, s' \in G$, also fix $a := a_r \in \mathcal{A}_r$, $a' := a_{r'} \in \mathcal{A}_{r'}$, $b := b_{-r+s} \in \mathcal{B}_{-r+s}$ and $b' := b_{-r'+s'} \in \mathcal{B}_{-r'+s'}$. Since ϕ_i 's are m_i -additive maps, for each i , we have $\phi_i(s_i + s'_i) = \phi_i(s_i) + \phi_i(s'_i) - \epsilon_i m_i$ and $\phi_i(r_i + r'_i) = \phi_i(r_i) + \phi_i(r'_i) - \epsilon'_i m_i$, for $\epsilon_i, \epsilon'_i \in \{0, 1\}$. We set $m(s, s') := \sum_{i \in I} \epsilon_i m_i$ and $m(r, r') := \sum_{i \in I} \epsilon'_i m_i$ in $\bigoplus_{i \in I} \mathbb{Z}$. Therefore, we then have $\phi(s + s') = \phi(s) + \phi(s') - m(s, s')$ and also $\phi(r + r') = \phi(r) + \phi(r') - m(r, r')$.

Lemma 3.5. *Under the assumptions of Theorem 3.4, let $d \in \mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0)$. Then for a unit $b_0 \in \mathcal{B}_0$,*

$$\begin{aligned} & u^{\phi(r+r')} \left(b_0^{-1} d(aa' \otimes u^{-\phi(r+r')} b_0) - d(aa' \otimes u^{-\phi(r+r')} b_0) \right) \\ &= u^{\phi(r)+\phi(r')} \left(b_0^{-1} d(aa' \otimes u^{-\phi(r)-\phi(r')} b_0) - d(aa' \otimes u^{-\phi(r)-\phi(r')} b_0) \right). \end{aligned}$$

Proof. Since $\phi(r + r') = \phi(r) + \phi(r') - m(r, r')$, it is enough to show that

$$\begin{aligned} & u^{-m(r,r')} \left(b_0^{-1} d(aa' \otimes u^{m(r,r')-\phi(r)-\phi(r')} b_0) - d(aa' \otimes u^{m(r,r')-\phi(r)-\phi(r')} b_0) \right) \\ &= b_0^{-1} d(aa' \otimes u^{-\phi(r)-\phi(r')} b_0) - d(aa' \otimes u^{-\phi(r)-\phi(r')} b_0). \end{aligned}$$

But we have

$$u^{-m(r,r')} b_0^{-1} d(aa' \otimes u^{m(r,r')-\phi(r)-\phi(r')} b_0) - u^{-m(r,r')} d(aa' \otimes u^{m(r,r')-\phi(r)-\phi(r')} b_0)$$

$$\begin{aligned}
 & -b_0^{-1}d(aa' \otimes u^{-\phi(r)-\phi(r')}b_0) = u^{-m(r,r')}b_0^{-1}\left(d(a \otimes u^{-\phi(r)}b_0)(a' \otimes u^{m(r,r')-\phi(r')})\right. \\
 & \left. + (a \otimes u^{-\phi(r)}b_0)d(a' \otimes u^{m(r,r')-\phi(r')})\right) \\
 & - u^{-m(r,r')}\left(d(a \otimes u^{-\phi(r)})(a' \otimes u^{m(r,r')-\phi(r')}) + (a \otimes u^{-\phi(r)})d(a' \otimes u^{m(r,r')-\phi(r')})\right) \\
 & - b_0^{-1}\left(d(a \otimes u^{-\phi(r)}b_0)(a' \otimes u^{-\phi(r')}) + (a \otimes u^{-\phi(r)}b_0)d(a' \otimes u^{-\phi(r')})\right) \\
 & = d(a \otimes u^{-\phi(r)}b_0)(a' \otimes u^{-\phi(r')}b_0^{-1}) + (a \otimes u^{-m(r,r')-\phi(r)})d(a' \otimes u^{m(r,r')-\phi(r')}) \\
 & - d(a \otimes u^{-\phi(r)})(a' \otimes u^{-\phi(r')}) - (a \otimes u^{-m(r,r')-\phi(r)})d(a' \otimes u^{m(r,r')-\phi(r')}) \\
 & - d(a \otimes u^{-\phi(r)}b_0)(a' \otimes u^{-\phi(r')}b_0^{-1}) - (a \otimes u^{-\phi(r)})d(a' \otimes u^{-\phi(r')}) \\
 & = -d(aa' \otimes u^{-\phi(r)-\phi(r')}),
 \end{aligned}$$

which completes the proof. ■

Lemma 3.6. *Under the assumptions of Theorem 3.4, for $d \in \mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0)$ and $n = \sum_{i \in I} n_i \in \bigoplus_{i \in I} \mathbb{Z}$ with $u_i^{n_i} \in \mathcal{B}_0$, for all $i \in I$, we have*

$$\begin{aligned}
 & (a \otimes b)\left(d(a' \otimes u^{n-\phi(s')}b')u^{\phi(s')-n} - d(a' \otimes u^{-\phi(s')}b')u^{\phi(s')}\right) \\
 & = \sum_{i \in I} \left(d(a \otimes u_i^{n_i}u^{-\phi(s)}b)u_i^{-n_i}u^{\phi(s)} - d(a \otimes u^{-\phi(s)}b)u^{\phi(s)}\right)(a' \otimes b').
 \end{aligned} \tag{5}$$

In particular,

$$\begin{aligned}
 & (a \otimes 1)\left(d(a' \otimes u^{n-\phi(s')}b')u^{\phi(s')-n} - d(a' \otimes u^{-\phi(s')}b')u^{\phi(s')}\right) \\
 & = \sum_{i \in I} \left(d(a \otimes u_i^{n_i}u^{-\phi(r)}b)u_i^{-n_i}u^{\phi(r)} - d(a \otimes u^{-\phi(r)}b)u^{\phi(r)}\right)(a' \otimes b')
 \end{aligned} \tag{6}$$

and, for each $i \in I$,

$$\begin{aligned}
 & (a \otimes 1)u^{\phi(r')}\left(d(a' \otimes u_i^{n_i}u^{-\phi(r')}b')u_i^{-n_i} - d(a' \otimes u^{-\phi(r')}b')\right) \\
 & = \left(d(a \otimes u_i^{n_i}u^{-\phi(r)}b)u_i^{-n_i} - d(a \otimes u^{-\phi(r)}b)u^{\phi(r)}\right)u^{\phi(r)}(a' \otimes 1).
 \end{aligned} \tag{7}$$

Proof. Set $I_0 := \{i \in I \mid n_i \neq 0\}$. Clearly $|I_0| < \infty$ and thus the sum in (5) is finite. Let $|I_0| = \ell$ and we use induction on ℓ . For $\ell = 0$ there is nothing to prove. Now, let (5) hold for $\ell = k$ and we show that it also holds for $\ell = k + 1$. To do this, consider the fixed element $i_0 \in I_0$ and let $n' \in \bigoplus_{i \in I} \mathbb{Z}$ be such that $n'_i = n_i$ for all $i \in I \setminus \{i_0\}$ and $n'_{i_0} = 0$. Now, we compute

$$M := d(aa' \otimes u^{n-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')-n} - d(aa' \otimes u^{-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')}$$

in the following two ways:

$$\begin{aligned}
 M & = d(a \otimes u^{-\phi(s)}b)u^{\phi(s)}(a' \otimes b') + (a \otimes b)d(a' \otimes u^{n-\phi(s')}b')u^{\phi(s')-n} \\
 & \quad - d(a \otimes u^{-\phi(s)}b)u^{\phi(s)}(a' \otimes b') - (a \otimes b)d(a' \otimes u^{-\phi(s')}b')u^{\phi(s')} \\
 & = (a \otimes b)\left(d(a' \otimes u^{n-\phi(s')}b')u^{\phi(s')-n} - d(a' \otimes u^{-\phi(s')}b')u^{\phi(s')}\right)
 \end{aligned}$$

and, by the induction hypothesis,

$$\begin{aligned}
 M &= d(a \otimes u_{i_0}^{n_{i_0}} u^{-\phi(s)} b) u_{i_0}^{-n_{i_0}} u^{\phi(s)} (a' \otimes b') + (a \otimes b) d(a' \otimes u^{n' - \phi(s')} b') u^{\phi(s') - n'} \\
 &\quad - d(a \otimes u^{-\phi(s)} b) u^{\phi(s)} (a' \otimes b') - (a \otimes b) d(a' \otimes u^{-\phi(s')} b') u^{\phi(s')} \\
 &= \left(d(a \otimes u_{i_0}^{n_{i_0}} u^{-\phi(s)} b) u_{i_0}^{-n_{i_0}} u^{\phi(s)} - d(a \otimes u^{-\phi(s)} b) u^{\phi(s)} \right) (a' \otimes b') \\
 &\quad + (a \otimes b) \left(d(a' \otimes u^{n' - \phi(s')} b') u^{\phi(s') - n'} - d(a' \otimes u^{-\phi(s')} b') u^{\phi(s')} \right) \\
 &= \left(d(a \otimes u_{i_0}^{n_{i_0}} u^{-\phi(s)} b) u_{i_0}^{-n_{i_0}} u^{\phi(s)} - d(a \otimes u^{-\phi(s)} b) u^{\phi(s)} \right) (a' \otimes b') \\
 &\quad + \sum_{i \in I_0 \setminus \{i_0\}} \left(d(a \otimes u_i^{n_i} u^{-\phi(s)} b) u_i^{-n_i} u^{\phi(s)} - d(a \otimes u^{-\phi(s)} b) u^{\phi(s)} \right) (a' \otimes b').
 \end{aligned}$$

By comparing the two parts, we get (5). To get (6), substitute $s = 0$ and $b = u^{-\phi(r)}$, and to get (7), let $|I_0| = 1$ and substitute $s = 0$, $b = u^{-\phi(r)}$, $s' = 0$ and $b' = u^{-\phi(r')}$ in (5). ■

Now, we are ready to prove Theorem 3.4:

Proof of Theorem 3.4. Assume that $d \in \mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0)$ is extended to an element $D \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})_0$. Then, for nonzero integers n_i such that n_i^{-1} makes sense in \mathbb{k} , $i \in I$, we claim that

$$\begin{aligned}
 D(a \otimes b) &= d(a \otimes u^{-\phi(s)} b) u^{\phi(s)} \\
 &\quad + \sum_{i \in I} \phi_i(s_i) (n_i m_i)^{-1} u^{\phi(r)} \left(u_i^{-n_i m_i} d(a \otimes u^{-\phi(r)} u_i^{n_i m_i}) - d(a \otimes u^{-\phi(r)}) \right) b. \tag{8}
 \end{aligned}$$

Note that the above sum is finite since $\phi_i(s_i) = 0$ except for finitely many i . To prove the claim, using Proposition 3.3, we have $D = D_1 + D_2$ where

$$D_1 \in \sum_{x \in G} \mathcal{D}(\mathcal{A})_x \overleftarrow{\otimes} \mathcal{B}_{-x} \text{ and } D_2 \in \sum_{x \in G} \mathcal{C}(\mathcal{A})_x \overrightarrow{\otimes} \mathcal{D}(\mathcal{B})_{-x}.$$

Therefore, $D_1 = \sum_{t \in T} d_t \otimes b_t$ and $D_2 = \sum_{j \in J} \gamma_j \otimes d'_j$ where $\{b_t\}_{t \in T}$ is a basis of \mathcal{B} , $\{d_t\}_{t \in T}$ is a summable family in $\mathcal{D}(\mathcal{A})$, $\{\gamma_j\}_{j \in J}$ is a basis of $\mathcal{C}(\mathcal{A})$ and $\{d'_j\}_{j \in J}$ is a summable family in $\mathcal{D}(\mathcal{B})$. Then, one checks that $x_{-r} := u^{-\phi(s)} b \in \mathcal{B}_{-r}$ and

$$\begin{aligned}
 D(a \otimes b) &= D(a \otimes u^{\phi(s)} x_{-r}) = D_1(a \otimes u^{\phi(s)} x_{-r}) + D_2(a \otimes u^{\phi(s)} x_{-r}) \\
 &= D_1(a \otimes x_{-r}) u^{\phi(s)} + D_2(a \otimes u^{\phi(s)} x_{-r}) + D_2(a \otimes x_{-r}) u^{\phi(s)} \\
 &= D(a \otimes x_{-r}) u^{\phi(s)} + D_2(a \otimes u^{\phi(s)} x_{-r}) \\
 &= d(a \otimes x_{-r}) u^{\phi(s)} + D_2(a \otimes u^{\phi(s)} x_{-r}). \tag{9}
 \end{aligned}$$

Also, for nonzero integers n_i such that n_i^{-1} makes sense in \mathbb{k} , $i \in I$, and for $t = \sum_{i \in I} t_i \in \bigoplus_{i \in I} \mathbb{Z}$, one checks that

$$D_2(a \otimes u^t) = \sum_{i \in I} t_i u^t u_i^{-1} D_2(a \otimes u_i) = \sum_{i \in I} t_i u^t n_i^{-1} u_i^{-n_i} D_2(a \otimes u_i^{n_i}) \tag{10}$$

and, using (9), for $i \in I$ we have

$$\begin{aligned} d(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) &= D(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) \\ &= D(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) + D_2(a \otimes u_i^{n_i m_i})u^{-\phi(r)} \\ &= d(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) + D_2(a \otimes u_i^{n_i m_i})u^{-\phi(r)}. \end{aligned}$$

Therefore,

$$D_2(a \otimes u_i^{n_i m_i}) = \left(d(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) - d(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) \right) u^{\phi(r)}. \tag{11}$$

Thus, using (10) and (11), we get that

$$\begin{aligned} D_2(a \otimes u^{\phi(s)}) &= \sum_{i \in I} \phi_i(s_i)u^{\phi(s)}(n_i m_i)^{-1}u_i^{-n_i m_i} D_2(a \otimes u_i^{n_i m_i}) \\ &= \sum_{i \in I} \phi_i(s_i)u^{\phi(s)}(n_i m_i)^{-1}u_i^{-n_i m_i} \left(d(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) - d(a \otimes u^{-\phi(r)}u_i^{n_i m_i}) \right) u^{\phi(r)}. \end{aligned}$$

Therefore, replacing this in (9) and also replacing x_{-r} with $u^{-\phi(s)}b$, we complete the proof of the claim. Now, let $D \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B})_0$ and $d := \pi(D) = 0$. Therefore, using (8), $D = 0$ is the unique extension of d , whence implies that π is injective.

Next, it remains to show that π is surjective. Let $d \in \mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0)$. Define $D \in \text{End}(\mathcal{A} \otimes \mathcal{B})_0$ by

$$\begin{aligned} D(a \otimes b) &= d(a \otimes u^{-\phi(s)}b)u^{\phi(s)} \\ &\quad + \sum_{i \in I} \phi_i(s_i)m_i^{-1}u^{\phi(r)} \left(u_i^{-m_i}d(a \otimes u^{-\phi(r)}u_i^{m_i}) - d(a \otimes u^{-\phi(r)}) \right) b. \end{aligned} \tag{12}$$

It is enough to show that D is a derivation of $\mathcal{A} \otimes \mathcal{B}$. Using (12) and Lemma 3.5, we get that

$$\begin{aligned} D(aa' \otimes bb') &= d(aa' \otimes u^{-\phi(s+s')}bb')u^{\phi(s+s')} + \sum_{i \in I} \phi_i(s_i + s'_i)m_i^{-1}u^{\phi(r+r')} \\ &\quad \times \left(u_i^{-m_i}d(aa' \otimes u^{-\phi(r+r')}u_i^{m_i}) - d(aa' \otimes u^{-\phi(r+r')}) \right) bb' \\ &= d(aa' \otimes u^{m(s,s')-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')-m(s,s')} \\ &\quad + \sum_{i \in I} (\phi_i(s_i) + \phi_i(s'_i) - \epsilon_i m_i)m_i^{-1}u^{\phi(r)+\phi(r')} \\ &\quad \times \left[u_i^{-m_i} \left(d(a \otimes u^{-\phi(r)})(a' \otimes u^{-\phi(r')}u_i^{m_i}) + (a \otimes u^{-\phi(r)})d(a' \otimes u^{-\phi(r')}u_i^{m_i}) \right) \right. \\ &\quad \left. - d(a \otimes u^{-\phi(r)})(a' \otimes u^{-\phi(r')}) - (a \otimes u^{-\phi(r)})d(a' \otimes u^{-\phi(r')}) \right] bb' \\ &= d(aa' \otimes u^{m(s,s')-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')-m(s,s')} \\ &\quad + \sum_{i \in I} \phi_i(s_i)m_i^{-1}(a \otimes b)u^{\phi(r')} \left(u_i^{-m_i}d(a' \otimes u^{-\phi(r')}u_i^{m_i}) - d(a' \otimes u^{-\phi(r')}) \right) b' \\ &\quad + \sum_{i \in I} \phi_i(s'_i)m_i^{-1}(a \otimes b)u^{\phi(r')} \left(u_i^{-m_i}d(a' \otimes u^{-\phi(r')}u_i^{m_i}) - d(a' \otimes u^{-\phi(r')}) \right) b' \\ &\quad - \sum_{i \in I_0} u^{\phi(r)+\phi(r')} \left(u_i^{-m_i}d(aa' \otimes u^{-\phi(r)-\phi(r')}u_i^{m_i}) - d(aa' \otimes u^{-\phi(r)-\phi(r')}) \right) bb', \end{aligned} \tag{13}$$

where $I_0 := \{i \in I \mid \epsilon_i = 1\}$. On the other hand, we have

$$\begin{aligned} D(a \otimes b)(a' \otimes b') + (a \otimes b)D(a' \otimes b') &= \left[d(a \otimes u^{-\phi(s)}b)u^{\phi(s)} \right. \\ &+ \sum_{i \in I} \phi_i(s_i)m_i^{-1}u^{\phi(r)} \left(u_i^{-m_i}d(a \otimes u^{-\phi(r)}u_i^{m_i}) - d(a \otimes u^{-\phi(r)}) \right) b \left. \right] (a' \otimes b') \\ &+ (a \otimes b) \left[d(a' \otimes u^{-\phi(s')}b')u^{\phi(s')} \right. \\ &+ \sum_{i \in I} \phi_i(s'_i)m_i^{-1}u^{\phi(r')} \left(u_i^{-m_i}d(a' \otimes u^{-\phi(r')}u_i^{m_i}) - d(a' \otimes u^{-\phi(r')}) \right) b' \left. \right] \\ &= d(aa' \otimes u^{-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')} \\ &+ \sum_{i \in I} \phi_i(s_i)m_i^{-1}u^{\phi(r)} \left(u_i^{-m_i}d(a \otimes u^{-\phi(r)}u_i^{m_i}) - d(a \otimes u^{-\phi(r)}) \right) b(a' \otimes b') \\ &+ \sum_{i \in I} \phi_i(s'_i)m_i^{-1}(a \otimes b)u^{\phi(r')} \left(u_i^{-m_i}d(a' \otimes u^{-\phi(r')}u_i^{m_i}) - d(a' \otimes u^{-\phi(r')}) \right) b'. \end{aligned}$$

But, multiplying both sides of the formula (7) of Lemma 3.6 by bb' , we conclude that

$$\begin{aligned} &\sum_{i \in I} \phi_i(s_i)m_i^{-1}(a \otimes b)u^{\phi(r')} \left(u_i^{-m_i}d(a' \otimes u^{-\phi(r')}u_i^{m_i}) - d(a' \otimes u^{-\phi(r')}) \right) b' \\ &= \sum_{i \in I} \phi_i(s_i)m_i^{-1}u^{\phi(r)} \left(u_i^{-m_i}d(a \otimes u^{-\phi(r)}u_i^{m_i}) - d(a \otimes u^{-\phi(r)}) \right) b(a' \otimes b'). \end{aligned} \tag{14}$$

Now we add and subtract the term $d(aa' \otimes u^{-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')}$ to the right hand side of the equation (13). Then, using (14), we get that

$$D(aa' \otimes bb') = D(a \otimes b)(a' \otimes b') + (a \otimes b)D(a' \otimes b') + M - N$$

where
$$M = d(aa' \otimes u^{m(s,s')-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')-m(s,s')} - d(aa' \otimes u^{-\phi(s)-\phi(s')}bb')u^{\phi(s)+\phi(s')}$$

and
$$N = \sum_{i \in I_0} u^{\phi(r)+\phi(r')} \left(u_i^{-m_i}d(aa' \otimes u^{-\phi(r)-\phi(r')}u_i^{m_i}) - d(aa' \otimes u^{-\phi(r)-\phi(r')}) \right) bb'.$$

Thus, it is enough to show that $M - N = 0$. But, we have seen, in the proof of Lemma 3.6, that

$$M = (a \otimes b) \left(d(a' \otimes u^{m(s,s')-\phi(s')}b')u^{\phi(s')-m(s,s')} - d(a' \otimes u^{-\phi(s')}b')u^{\phi(s')} \right).$$

Moreover, we have

$$\begin{aligned} N &= \sum_{i \in I_0} d(a \otimes u^{-\phi(r)}u_i^{m_i})u^{\phi(r)}u_i^{-m_i}b(a' \otimes b') + (a \otimes b)d(a' \otimes u^{-\phi(r')}u^{\phi(r')}b') \\ &- \sum_{i \in I_0} d(a \otimes u^{-\phi(r)})u^{\phi(r)}b(a' \otimes b') + (a \otimes b)d(a' \otimes u^{-\phi(r')}u^{\phi(r')}b') \\ &= \sum_{i \in I_0} \left(d(a \otimes u^{-\phi(r)}u_i^{m_i})u^{\phi(r)}u_i^{-m_i} - d(a \otimes u^{-\phi(r)})u^{\phi(r)} \right) b(a' \otimes b'). \end{aligned}$$

Multiplying both sides of the formula (6) by b and putting $n = m$ in Lemma 3.6, we get that $M = N$ and we are done. ■

Remark 3.7. We note that the algebra \mathcal{A} considered in this section is not necessarily unital and the algebra \mathcal{B} is always associative and unital. In a recent paper, M. Brešar [10], considers derivations of tensor products of unital non-associative algebras.

4. Application to gradings induced by admissible automorphisms

Throughout this section I is an index set and $\{m_i\}_{i \in I}$ is a set of positive integers. We assume that for each i , \mathbb{k} contains a primitive m_i -th root of unity ω_i . For an algebra \mathcal{A} we denote by $\mathbf{cfa}_I(\mathcal{A})$ the collection of admissible families $\{\sigma_i\}_{i \in I}$ of commuting finite order automorphisms of \mathcal{A} ; *admissible* here means for every $a \in \mathcal{A}$, $\sigma_i(a) = a$ for all but finitely many i .

Let $\sigma := \{\sigma_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{A})$ and assume that $\sigma_i^{m_i} = 1$ for $i \in I$. Since $\{\sigma_i\}_{i \in I}$ is admissible, we have a $G := \bigoplus_{i \in I} \mathbb{Z}_{m_i}$ -grading

$$\mathcal{A} = \bigoplus_{\bar{r} \in G} \mathcal{A}_{\bar{r}}^{\sigma} \tag{15}$$

where for $\bar{r} = \{\bar{r}_i\} \in G$, $\mathcal{A}_{\bar{r}}^{\sigma} = \{a \in \mathcal{A} \mid \sigma_i(a) = \omega_i^{r_i} a \text{ for all } i \in I\}$. Moreover, σ induces a family $\sigma^* := \{\sigma_i^*\}_{i \in I}$ of automorphisms of \mathcal{A} defined by $\sigma_i^*(\varphi) = \sigma_i \varphi \sigma_i^{-1}$, for $\varphi \in \text{End}(\mathcal{A})$ and $i \in I$. Thus, we define

$$\text{grEnd}(\mathcal{A}) := \bigoplus_{\bar{r} \in G} \text{End}(\mathcal{A})_{\bar{r}}^{\sigma^*}$$

where $\text{End}(\mathcal{A})_{\bar{r}}^{\sigma^*} = \{\varphi \in \text{End}(\mathcal{A}) \mid \sigma_i^*(\varphi) = \omega_i^{r_i} \varphi \text{ for all } i \in I\}$, $\bar{r} \in G$. Note that

$$\text{End}(\mathcal{A})_{\bar{r}}^{\sigma^*} = \text{End}(\mathcal{A})_{\bar{r}} = \{\varphi \in \text{End}(\mathcal{A}) \mid \varphi(\mathcal{A}_{\bar{s}}) \subseteq \mathcal{A}_{\bar{s} + \bar{r}} \text{ for all } \bar{s} \in G\}.$$

In addition, for each i , the restriction of σ_i^* to $\mathcal{C}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ induces automorphisms on them, of period m_i , whence we set

$$\text{gr}\mathcal{C}(\mathcal{A}) := \bigoplus_{\bar{r} \in G} \mathcal{C}(\mathcal{A})_{\bar{r}}^{\sigma^*} \text{ and } \text{gr}\mathcal{D}(\mathcal{A}) := \bigoplus_{\bar{r} \in G} \mathcal{D}(\mathcal{A})_{\bar{r}}^{\sigma^*},$$

where, for $\bar{r} \in G$, $\mathcal{C}(\mathcal{A})_{\bar{r}}^{\sigma^*} = \text{End}(\mathcal{A})_{\bar{r}}^{\sigma^*} \cap \mathcal{C}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})_{\bar{r}}^{\sigma^*} = \text{End}(\mathcal{A})_{\bar{r}}^{\sigma^*} \cap \mathcal{D}(\mathcal{A})$. We also note that if $\eta := \{\eta_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{B})$, then $\sigma \otimes \eta := \{\sigma_i \otimes \eta_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{A} \otimes \mathcal{B})$. Hence if both σ_i and η_i have the same period m_i , $i \in I$, then we have a G -grading,

$$\mathcal{A} \otimes \mathcal{B} = \bigoplus_{\bar{r} \in G} (\mathcal{A} \otimes \mathcal{B})_{\bar{r}}^{\sigma \otimes \eta} \quad \text{where} \quad (\mathcal{A} \otimes \mathcal{B})_{\bar{r}}^{\sigma \otimes \eta} = \sum_{\bar{s} \in G} \mathcal{A}_{\bar{r} - \bar{s}}^{\sigma} \otimes \mathcal{B}_{\bar{s}}^{\eta}.$$

From now on, if the underlying automorphisms are clear from the context, we will write $\mathcal{A}_{\bar{r}}$, $\text{End}(\mathcal{A})_{\bar{r}}$, $\mathcal{A}_{\bar{r} - \bar{s}} \otimes \mathcal{B}_{\bar{s}}$, etc, instead of $\mathcal{A}_{\bar{r}}^{\sigma}$, $\text{End}(\mathcal{A})_{\bar{r}}^{\sigma^*}$, $\mathcal{A}_{\bar{r} - \bar{s}}^{\sigma} \otimes \mathcal{B}_{\bar{s}}^{\eta}$, etc, respectively.

Example 4.1. Let L be a complex Lie algebra and \mathcal{H} be a toral subalgebra of L . Assume that S is the root system of L with respect to \mathcal{H} and $\mathcal{Q}(S)$ denotes the

\mathbb{Z} -span of S . Now, consider any group homomorphism $\phi: \mathcal{Q}(S) \rightarrow \mathbb{C} \setminus \{0\}$. One checks that ϕ induces an automorphism σ on L such that

$$\sigma|_{L_\alpha} = \phi(\alpha) \text{Id}_{L_\alpha} \text{ for all } \alpha \in S.$$

Next, assume that $\{\alpha_i\}_{i \in I}$ is a \mathbb{Z} -basis of the root lattice, namely $\mathcal{Q}(S) = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$. For each i , define $\phi_i: \mathcal{Q}(S) \rightarrow \mathbb{C} \setminus \{0\}$ by $\phi_i(\alpha_i) = \omega_i$ and $\phi_i(\alpha_j) = 1$, for all $j \in I \setminus \{i\}$. Let σ_i be the automorphism on L induced by ϕ_i . Then σ_i has period m_i and $\{\sigma_i\}_{i \in I} \in \mathbf{cfa}_I(L)$

The following proposition follows immediately from Proposition 3.3.

Proposition 4.2. *Assume that \mathcal{A} is perfect, \mathcal{B} is unital, commutative and associative, and the map ψ in (2) is an isomorphism. Assume that $\{\sigma_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{A})$ and $\{\eta_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{B})$ with periods $\{m_i\}_{i \in I}$. Then, setting $G := \bigoplus_{i \in I} \mathbb{Z}_{m_i}$, we have*

$$\text{gr } \mathcal{D}(\mathcal{A} \otimes \mathcal{B}) = \sum_{g \in G} \mathcal{D}(\mathcal{A} \otimes \mathcal{B})_g = \sum_{g \in G} (\mathcal{D}(\mathcal{A}) \overset{\leftarrow}{\otimes} \mathcal{B})_g \oplus (\mathcal{C}(\mathcal{A}) \overset{\rightarrow}{\otimes} \mathcal{D}(\mathcal{B}))_g$$

such that

$$(\mathcal{D}(\mathcal{A}) \otimes \mathcal{B})_g = \sum_{g' \in G} \mathcal{D}(\mathcal{A})_{g'} \overset{\leftarrow}{\otimes} \mathcal{B}_{g-g'} \text{ and } (\mathcal{C}(\mathcal{A}) \otimes \mathcal{D}(\mathcal{B}))_g = \sum_{g' \in G} \mathcal{C}(\mathcal{A})_{g'} \overset{\rightarrow}{\otimes} \mathcal{D}(\mathcal{B})_{g-g'}.$$

Theorem 4.3. *Assume that \mathcal{A} is perfect, \mathcal{B} is unital, commutative and associative, and the map ψ in (2) is an isomorphism. Assume that $\{\sigma_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{A})$ and $\{\eta_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{B})$ with periods $\{m_i\}_{i \in I}$, and that m_i^{-1} makes sense in \mathbb{k} for each i . For $i \in I$, set $G_i := \mathbb{Z}_{m_i}$ and $G := \bigoplus_{i \in I} \mathbb{Z}_{m_i}$, and suppose for some unit $\bar{q}_i \in G_i$ there exists a unit $u_i \in \mathcal{B}_{\iota(\bar{q}_i)}$. If $(\mathcal{A} \otimes \mathcal{B})_0$ and $\mathcal{D}(\mathcal{A} \otimes \mathcal{B})_0$ denote the fixed points of $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{D}(\mathcal{A} \otimes \mathcal{B})$ with respect to $\{\sigma_i \overset{\sim}{\otimes} \eta_i\}_{i \in I}$ and $\{\sigma_i^* \overset{\sim}{\otimes} \eta_i^*\}_{i \in I}$ respectively, then the restriction map*

$$\pi: \mathcal{D}(\mathcal{A} \otimes \mathcal{B})_0 \longrightarrow \mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0); \quad D \mapsto D|_{(\mathcal{A} \otimes \mathcal{B})_0}$$

is an isomorphism. In particular,

$$\mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0) \cong \sum_{\bar{r} \in G} (\mathcal{D}(\mathcal{A})_{\bar{r}} \overset{\leftarrow}{\otimes} \mathcal{B}_{-\bar{r}}) \oplus (\mathcal{C}(\mathcal{A})_{\bar{r}} \overset{\rightarrow}{\otimes} \mathcal{D}(\mathcal{B})_{-\bar{r}}).$$

Proof. For $i \in I$, let $\bar{p}_i \in G_i$ be such that $\bar{p}_i \bar{q}_i = \bar{1}$. Then, $u_i \in \mathcal{B}_{\iota(\bar{q}_i)}$ is equivalent to $u'_i := u_i^{p_i} \in \mathcal{B}_{\iota(\bar{1})}$, for all i . For $i \in I$, we define $\phi_i: G_i \rightarrow \mathbb{Z}$ by requiring that $\phi_i(\bar{r}_i)$ be the preimage of \bar{r}_i in $\{0, \dots, m_i - 1\}$ under the canonical projection. Therefore, for each i , ϕ_i is m_i -additive and note that u'_i is a unit in \mathcal{B} such that $(u'_i)^{m_i} \in \mathcal{B}_0$ and $(u'_i)^{\phi_i(\bar{r}_i)} \in \mathcal{B}_{\iota(\bar{r}_i)}$, $\bar{r}_i \in G_i$. Therefore, all the assumptions of Theorem 3.4 hold and so we get the result. ■

Recall that the *differential centroid* of \mathcal{A} , denoted $d\mathcal{C}(\mathcal{A})$, is by definition the centralizer of $\mathcal{D}(\mathcal{A})$ in $\mathcal{C}(\mathcal{A})$, that is $d\mathcal{C}(\mathcal{A}) = \{\gamma \in \mathcal{C}(\mathcal{A}) \mid [\gamma, \mathcal{D}(\mathcal{A})] = \{0\}\}$. We may consider \mathcal{A} also as a $d\mathcal{C}(\mathcal{A})$ -module.

Corollary 4.4. Under the assumptions of Theorem 4.3, if one of the following conditions holds:

- (i) \mathcal{A} or \mathcal{B} is finite dimensional,
- (ii) $\mathcal{C}(\mathcal{A})$ is finite dimensional or \mathcal{B} is finitely generated, and one of the following is satisfied:
 - (a) \mathcal{A} is finitely generated over \mathbb{k} ,
 - (b) \mathcal{A} is finitely generated as a module over $\mathcal{C}(\mathcal{A})$ and $[\mathcal{D}(\mathcal{A}), \mathcal{C}(\mathcal{A})] = \{0\}$,
 - (c) \mathcal{A} is finitely generated as a module over $\mathrm{d}\mathcal{C}(\mathcal{A})$,

then
$$\mathcal{D}((\mathcal{A} \otimes \mathcal{B})_0) \cong \mathcal{D}(\mathcal{A} \otimes \mathcal{B})_0 = \sum_{\bar{r} \in G} (\mathcal{D}(\mathcal{A})_{\bar{r}} \otimes \mathcal{B}_{-\bar{r}}) \oplus (\mathcal{C}(\mathcal{A})_{\bar{r}} \otimes \mathcal{D}(\mathcal{B})_{-\bar{r}}).$$

Proof. The proof follows from [7, Theorem 2.9] and Theorem 4.3. ■

Remark 4.5. Note that, if $\mathcal{A} = \mathbb{k}1$ in Theorem 4.3, we get that $\mathcal{D}(\mathcal{B})_0 \cong \mathcal{D}(\mathcal{B}_0)$.

5. Derivations of extended multi-loop algebras

In this section, we assume that I is an index set, $S^I := \mathbb{k}[z_i^{\pm 1}]_{i \in I}$ is the algebra of Laurent polynomials in variables $\{z_i\}_{i \in I}$ over \mathbb{k} and we let \mathbb{k} contain a primitive m_i -th root of unity, for all $i \in I$. Set $G := \bigoplus_{i \in I} \mathbb{Z}_{m_i}$ and consider

$$r = \sum_{i \in I} r_i, s = \sum_{i \in I} s_i \in \bigoplus_{i \in I} \mathbb{Z}.$$

We set $z^r := \prod_{i \in I} z_i^{r_i}$, and note that $z^r z^s := \prod_{i \in I} z_i^{r_i + s_i}$. Also, let $\{\sigma_i\}_{i \in I} \in \mathbf{cfa}_I(\mathcal{A})$ and m_i be the period of σ_i for $i \in I$. Then, we have a G -grading on \mathcal{A} as in (15).

Definition 5.1. We call the subalgebra

$$M(\mathcal{A}, \{\sigma_i\}_{i \in I}, \{m_i\}_{i \in I}) := \bigoplus_{r \in \bigoplus_{i \in I} \mathbb{Z}} \mathcal{A}_{\bar{r}} \otimes z^r$$

of $\mathcal{A} \otimes S^I$ the *extended multi-loop algebra* of $\{\sigma_i\}_{i \in I}$ based on \mathcal{A} , or simply, the *extended multi-loop algebra*.

Remark 5.2. (i) Let $|I| = n$. Then $M(\mathcal{A}, \{\sigma_i\}_{i \in I}, \{m_i\}_{i \in I}) = M(\mathcal{A}, \sigma_1, \dots, \sigma_n)$ is in fact the n -step multi-loop algebra of $\sigma_1, \dots, \sigma_n$ based on \mathcal{A} introduced in [4].

(ii) Let $\mathcal{L} := M(\mathcal{A}, \{\sigma_i\}_{i \in I}, \{m_i\}_{i \in I})$ be an extended multi-loop algebra. Consider $\{\eta_i\}_{i \in I} \in \mathbf{cfa}_I(S^I)$ defined by $\eta_i(z_i^t) = \omega_i^{-t} z_i^t$ and $\eta_i(z_j) = z_j$, for all $j \neq i$, where ω_i is a primitive m_i -th root of unity. Then \mathcal{L} is the space of fixed points of $\mathcal{A} \otimes S^I$ with respect to $\sigma \otimes \eta := \{\sigma_i \otimes \eta_i\}_{i \in I}$, i.e.,

$$\mathcal{L} = (\mathcal{A} \otimes S^I)_0^{\sigma \otimes \eta}. \tag{16}$$

(iii) The author in [6, Theorem 3.7] determines, using an induction based reasoning, the algebra of derivations of an n -step multi-loop algebra. The results here not only

allow us to pass from finite to infinite automorphisms but also provide a short and direct proof for [6, Theorem 3.7]. Indeed, if $|I| = n$ and $S^n = S^I$, \mathcal{A} is perfect and finitely generated as a $\mathcal{C}(\mathcal{A})$ -module and $\mathcal{C}(\mathcal{A})$ is finitely generated, then by Corollary 4.4 and (ii),

$$\begin{aligned} \mathcal{D}(M(\mathcal{A}, \sigma_1, \dots, \sigma_n)) &= \mathcal{D}((\mathcal{A} \otimes S^n)_0) \cong \mathcal{D}(\mathcal{A} \otimes S^n)_0 \\ &= \sum_{\bar{r} \in G} (\mathcal{D}(\mathcal{A})_{\bar{r}} \otimes S^n_{-\bar{r}}) \oplus ((\mathcal{C}(\mathcal{A})_{\bar{r}} \otimes \mathcal{D}(S^n)_{-\bar{r}})) \\ &= M(\mathcal{D}(\mathcal{A}), \sigma_1^*, \dots, \sigma_n^*) \oplus \sum_{\bar{r} \in G} \mathcal{C}(\mathcal{A})_{\bar{r}} \otimes \mathcal{D}(S^n)_{-\bar{r}}. \end{aligned}$$

Theorem 5.3. *Assume that \mathcal{A} is perfect and central, and consider the extended multi-loop algebra $\mathcal{L} := M(\mathcal{A}, \{\sigma_i\}_{i \in I}, \{m_i\}_{i \in I})$. Then*

$$\mathcal{D}(\mathcal{L}) = \sum_{\bar{r} \in G} (\mathcal{D}(\mathcal{A})_{\bar{r}} \overleftarrow{\otimes} S^I_{-\bar{r}}) \oplus \mathcal{D}((S^I)_0), \quad \text{where } (S^I)_0 = \mathbb{k}[z_i^{\pm m_i}]_{i \in I}. \quad (17)$$

Proof. Since $\mathcal{C}(\mathcal{A}) = \mathbb{k} \text{id}_{\mathcal{A}}$, then $\mathcal{C}(\mathcal{A})_0 = \mathbb{k} \text{id}_{\mathcal{A}}$ and $\mathcal{C}(\mathcal{A})_{\bar{r}} = \{0\}$, for all $\bar{r} \neq 0$. Thus, using Lemma 2.1, Remark 4.5, Remark 5.2(ii) and Theorem 4.3, we get the result. ■

Corollary 5.4. *Let $\mathcal{L} := M(\mathfrak{g}, \{\sigma_i\}_{i \in I}, \{m_i\}_{i \in I})$ be an extended multi-loop algebra based on a locally finite dimensional central simple Lie algebra \mathfrak{g} . Then*

$$\mathcal{D}(\mathcal{L}) = \sum_{\bar{r} \in G} (\mathcal{D}(\mathfrak{g})_{\bar{r}} \overleftarrow{\otimes} S^I_{-\bar{r}}) \oplus \mathcal{D}((S^I)_0). \quad (18)$$

Proof. Since \mathfrak{g} is perfect and central, the proof is a consequence of Theorem 5.3. ■

In the above corollary there are two ingredients which should be described for determining the derivation algebra of a particular extended multi-loop algebra, namely $\mathcal{D}(\mathfrak{g})$ and $\mathcal{D}((S^I)_0)$. In Lemma 5.5 below, we give a description of $\mathcal{D}(S^I)$ (hence $\mathcal{D}((S^I)_0)$), and for $\mathcal{D}(\mathfrak{g})$ we refer the reader to [15, Lemma I.1] in which elements of $\mathcal{D}(\mathfrak{g})$ are presented as inverse limits of some inner derivations. To be precise, let \mathfrak{g} be a locally finite simple Lie algebra which is a direct union of its finite dimensional simple Lie subalgebras $\{\mathfrak{g}_j\}_{j \in J}$, i.e., $\mathfrak{g} = \cup_{j \in J} \mathfrak{g}_j$ where J is an ordered set and $\mathfrak{g}_i \subseteq \mathfrak{g}_j$ if $i \leq j$. If $D \in \mathcal{D}(\mathfrak{g})$, by [15, Lemma I.1], for each $j \in J$, there exists a unique element $x_j \in [\mathfrak{g}_j, \mathfrak{g}]$ with $D|_{\mathfrak{g}_j} = \text{ad } x_j|_{\mathfrak{g}_j}$. Therefore, we get the following description of $\mathcal{D}(\mathfrak{g})$:

$$\mathcal{D}(\mathfrak{g}) \cong \varprojlim [\mathfrak{g}_j, \mathfrak{g}] = \{(x_j)_{j \in J} \in \prod_{j \in J} [\mathfrak{g}_j, \mathfrak{g}] \mid p_{jk}(x_k) = x_j, \text{ for } j \leq k\}$$

where $p_{jk} : [\mathfrak{g}_k, \mathfrak{g}] \rightarrow [\mathfrak{g}_j, \mathfrak{g}]$ is the linear projection with kernel $C_{[\mathfrak{g}_k, \mathfrak{g}]}(\mathfrak{g}_j)$; the centralizer of \mathfrak{g}_j in $[\mathfrak{g}_k, \mathfrak{g}]$, and $j \leq k$ means $\mathfrak{g}_j \subseteq \mathfrak{g}_k$. Indeed, each $D \in \mathcal{D}(\mathfrak{g})$ is associated with $(x_j)_{j \in J} \in \varprojlim [\mathfrak{g}_j, \mathfrak{g}]$, where $D|_{\mathfrak{g}_j} = \text{ad } x_j|_{\mathfrak{g}_j}$, for all $j \in J$.

In the following lemma, we give a description of $\mathcal{D}(S^I)$ as an S^I -module in terms of summable families (see Definition 3.1).

Lemma 5.5. *Consider $\mathcal{D}(S^I)$ as an S^I -module by left multiplication. Then the set $P := \{\frac{\partial}{\partial z_i} \mid i \in I\}$ forms a summable S^I -basis for $\mathcal{D}(S^I)$.*

Proof. First note that the set $P \subseteq \mathcal{D}(S^I)$ is summable and linearly independent over S^I . Now, let $d \in \mathcal{D}(S^I)$. Then using a standard argument one sees that for each i there exists $F_i \in S^I$ such that the restriction map $d : \mathbb{k}[z_i^{\pm 1}] \rightarrow S^I$ coincides with $d = F_i \frac{\partial}{\partial z_i}$. Thus, for each $j \in I$, we have

$$d(z_j) = F_j \frac{\partial}{\partial z_j}(z_j) = \left(\sum_{i \in I} F_i \frac{\partial}{\partial z_i} \right)(z_j).$$

This implies that $d = \sum_{i \in I} F_i \frac{\partial}{\partial z_i}$ and we are done. ■

Remark 5.6. It is shown in [12, Chapter 8] that any affine Kac-Moody Lie algebra (up to adding some central elements and derivations) can be realized as a “loop algebra”, based on a finite dimensional simple Lie algebra and a finite order automorphism. Also, E. Neher in [17] showed how to build any extended affine Lie algebra out of a Lie torus. Since almost all centerless Lie tori can be realized using multi-loop algebras (see [2] and [3]), therefore, extended affine Lie algebras can be in general built out of multi-loop algebras, based on finite dimensional simple Lie algebras and a finite number of automorphisms. In particular, in the above realization problem, the space of derivations of a multi-loop algebra plays an important role. Regarding these, our interest in introducing “extended multi-loop algebras” and studying their derivation algebras arises from the study of centerless cores of locally extended affine Lie algebras, the so-called *centerless locally Lie tori*. Indeed, we expect that centerless locally Lie tori can be in general realized by extended multi-loop algebras.

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References

- [1] B. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola: *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. 603 (1997) 1–122.
- [2] B. Allison, S. Berman, J. Faulkner, A. Pianzola: *Multiloop realization of extended affine Lie algebras and Lie tori*, Trans. Amer. Math. Soc. 361 (2009) 4807–4842.
- [3] B. Allison, S. Berman, J. Faulkner, A. Pianzola: *Realization of graded-simple algebras as loop algebras*, Forum Mathematicum 20 (2008) 395–432.
- [4] B. Allison, S. Berman, A. Pianzola: *Iterated loop algebras*, Pac. J. Math. 227 (2006) 1–41.
- [5] S. Azam: *Construction of extended affine Lie algebras by the twisting process*, Comm. Algebra 28(6) (2000) 2753–2781.

- [6] S. Azam: *Derivations of multi-loop algebras*, Forum Math. 19 (2007) 1029–1045.
- [7] S. Azam: *Derivations of tensor product algebras*, Comm. Algebra 36 (2008) 905–927.
- [8] S. Azam, V. Khalili, M. Yousofzadeh: *Extended affine root systems of type BC*, J. Lie Theory 15(1) (2005) 145–181.
- [9] G. Benkart, E. Neher: *The centroid of extended affine and root graded Lie algebras*, J. Pure Appl. Algebra 205(1) (2006) 117–145.
- [10] M. Brešar: *Derivations of tensor products of nonassociative algebras*, Linear Algebra Appl. 530 (2017) 244–252.
- [11] R. Farnsteiner: *Derivations and central extensions of finitely generated graded Lie algebras*, J. Algebra 118 (1988) 33–45.
- [12] V. Kac: *Infinite Dimensional Lie Algebras*, 3rd ed., Cambridge University Press, Cambridge (1990).
- [13] N. Jacobson: *Lie Algebras*, Interscience Publishers, New York et al. (1962).
- [14] J. Morita, Y. Yoshii: *Locally extended affine Lie algebras*, J. Algebra 301(1) (2006) 59–81.
- [15] K.-H. Neeb: *Derivations of locally simple Lie algebras*, J. Lie Theory 15 (2005) 589–594.
- [16] E. Neher: *Lie tori*, C. R. Math. Acad. Sci. Soc. R. Can. 26(3) (2004) 84–89.
- [17] E. Neher: *Extended affine Lie algebras*, C. R. Math. Acad. Sci. Soc. R. Can. 26(3) (2004) 90–96.
- [18] E. Neher: *Extended affine Lie algebras and other generalizations of affine Lie algebras – a survey*, in: *Developments and Trends in Infinite-Dimensional Lie Theory*, K.-H. Neeb, A. Pianzola (eds.), Progress in Mathematics 288, Birkhäuser, Boston (2011) 53–126.
- [19] E. Neher, A. Pianzola: *Étale descent of derivations*, Transform. Groups 18(4) (2013) 1189–1205.
- [20] A. Pianzola: *Derivations of certain algebras defined by étale descent*, Math. Zeitschrift 264 (2010) 485–495.

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