

# Atiyah Classes of Lie Bialgebras

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Communicated by M. Schlichenmaier

**Abstract.** The Atiyah class was originally introduced by M. F. Atiyah. It has many developments in recent years. One important case is the Atiyah classes of Lie algebra pairs. In this paper, we study the Atiyah class of the Lie algebra pair associated with a Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$ . A simple description of the Atiyah class and the first scalar Atiyah class is given by the Lie algebra structures on  $\mathfrak{g}$  and  $\mathfrak{g}^*$ . As an application, the Atiyah classes for some special cases are investigated.

*Mathematics Subject Classification:* 17B56, 17B62.

*Key Words:* Atiyah class, Lie bialgebras, Lie algebra pair.

## 1. Introduction

The Atiyah class was originally introduced by M. F. Atiyah [1] in order to describe the obstruction to the existence of a holomorphic connection on a holomorphic vector bundle. In the late 1990's, Kontsevich [10] and Kapranov [9] revealed the relation between Atiyah class and Rozansky-Witten invariants. Subsequent works have appeared in many situations [2, 3, 4, 5, 6, 14] etc. One interesting case is the Atiyah class associated with a Lie algebra pair  $(L, A)$  and an  $A$ -module  $E$ . The geometric meaning of the Atiyah class of a Lie pair was studied in [16, 15]. There are recent developments of the study of the Atiyah classes of Lie pairs [3, 4, 6, 12]. In this paper, we investigate the Atiyah class of the Lie pair associated with a Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$ , or more precisely, the Lie algebra  $L = \mathfrak{g} \ltimes \mathfrak{g}^*$ , its subalgebra  $A = \mathfrak{g}$  and the  $\mathfrak{g}$ -module  $E = L/A \cong \mathfrak{g}^*$ . Let us denote by  $F$  the map

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{id} \otimes (-ad^*)} \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*).$$

Then  $F$  is a morphism between the  $\mathfrak{g}$ -modules  $\mathfrak{g} \otimes \mathfrak{g}$  and  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ . It induces a map

$$H^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{F_*} H^1(\mathfrak{g}, \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)) : F_*([\alpha]) = [F \circ \alpha],$$

for all cocycle  $\alpha \in \text{Hom}(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ . We have the following theorem for the Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \ltimes \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$ .

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\*Partially supported by NSFC grant 11401441.

**Theorem 1.1.** *Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a Lie bialgebra with the associated map  $\gamma: \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$ . Let  $\lambda \in \mathfrak{g}^* \otimes \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  be defined by  $\lambda(x, \xi) = ad_{ad_\xi^*(x)}^* \in \text{End}(\mathfrak{g}^*)$  for all  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ . Then*

- (1) *The map  $\lambda: \mathfrak{g} \mapsto \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  satisfies  $\lambda = -F \circ \gamma$ .*
- (2) *The cohomology class  $[\lambda] \in H^1(\mathfrak{g}, \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*))$  satisfies  $[\lambda] = -F_*[\gamma]$ .*
- (3) *the Atiyah class  $\alpha_E = [\lambda]$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$  vanishes if and only if  $[\gamma] \in \ker F_*$ .*

Given a Lie algebroid pair  $(L, A)$  and an  $A$ -module  $E$ , the scalar Atiyah classes  $c_k(E)$  are defined by Chen-Stiénon-Xu in [6]. Let  $\kappa \in \mathfrak{g}^*$  be the modular vector of the Lie algebra  $\mathfrak{g}$ , defined by

$$\kappa(x) = tr(ad_x), \quad \forall x \in \mathfrak{g}.$$

Let  $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be the cocycle associated with the Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$ . Let the map  $\iota_\kappa \gamma: \mathfrak{g} \rightarrow \mathfrak{g}$  be defined by

$$(\iota_\kappa \gamma)(x) = \iota_\kappa \gamma(x), \quad \forall x \in \mathfrak{g},$$

where  $\iota_\kappa \gamma(x)$  denotes by the contraction of  $\kappa \in \mathfrak{g}^*$  with the first part of  $\gamma(x) \in \mathfrak{g} \otimes \mathfrak{g}$ . Then we have the following theorem for the first scalar Atiyah class  $c_1(E)$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$ .

**Theorem 1.2.** *Let  $c_1(E)$  be the first scalar Atiyah class associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$ . Then we have*

(a) 
$$c_1(E) = -\frac{\sqrt{-1}}{2\pi} [\iota_\kappa \gamma]. \tag{1}$$

(b)  $c_1(E)$  vanishes if and only if there exists  $v \in \mathfrak{g}$  such that

$$ad_\kappa^* = ad_v \in \text{End}(\mathfrak{g}), \tag{2}$$

where  $ad_\kappa^* \in \text{End}(\mathfrak{g})$  is the dual map of  $ad_\kappa \in \text{End}(\mathfrak{g}^*)$ .

(c) Equation (2) is equivalent to 
$$ad_{\kappa+v}(\mathfrak{g}) = 0, \tag{3}$$

where  $ad_{\kappa+v}$  is considered as an element in  $\text{End}(L)$ , and  $\mathfrak{g}$  is considered as a subspace of  $L$ .

In [13], Lu-Weinstein give the Lie bialgebras  $(\mathfrak{g} = \mathfrak{u}, \mathfrak{g}^* = \mathfrak{b})$  and  $(\mathfrak{g} = \mathfrak{b}, \mathfrak{g}^* = \mathfrak{u})$  as shown in example 2.4. We investigate the Atiyah classes for both situations. In Example 3.9, we show that the Atiyah class associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g} = \mathfrak{u}, E = \mathfrak{g}^* = \mathfrak{b})$  does vanish. By contrast, in Proposition 3.13, we prove that the Atiyah class associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g} = \mathfrak{b}, E = \mathfrak{g}^* = \mathfrak{u})$  does not vanish.

**Acknowledgements.** We would like to thank Zhanqiang Bai, Zhuo Chen, Camille Laurent-Gengoux, Zhangju Liu, Yu Qiao, Yannick Voglaire and Ping Xu for helpful discussions and comments. The author would like to give his special thanks to the referee for the valuable opinions.

## 2. Preliminaries

### 2.1. Atiyah classes for Lie algebroid pairs

In [6], Chen, Stiénon and Xu introduced the Atiyah class for Lie algebroid pairs. A Lie algebroid pair  $(L, A)$  is a Lie algebroid  $L$  together with a Lie subalgebroid  $A$  over the same base manifold. Assume that  $E$  is an  $A$ -module, and  $\nabla$  is an  $L$ -connection on  $E$  extending its  $A$ -action. The curvature of  $\nabla$  is the bundle map  $R_E^\nabla: \wedge^2 L \rightarrow \text{End}(E)$  defined by

$$R_E^\nabla(l_1, l_2) = \nabla_{l_1} \nabla_{l_2} - \nabla_{l_2} \nabla_{l_1} - \nabla_{[l_1, l_2]} \tag{4}$$

for all  $l_1, l_2 \in \Gamma(L)$ . Since  $E$  is an  $A$ -module, the restriction of  $R_E^\nabla$  to  $\wedge^2 A$  vanishes. Hence the curvature induces a section  $R_E^\nabla \in \Gamma(A^* \otimes A^\perp \otimes \text{End}(E))$ , or equivalently, a bundle map  $R_E^\nabla: A \otimes (L/A) \rightarrow \text{End}(E)$  given by

$$R_E^\nabla(a, \bar{l}) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a, l]} \tag{5}$$

for all  $a \in \Gamma(A)$  and  $l \in \Gamma(L)$ .

**Theorem 2.1.** [6]

- (1) *The section  $R_E^\nabla$  of  $A^* \otimes A^\perp \otimes \text{End}(E)$  is a 1-cocycle for Lie algebroid  $A$  with values in the  $A$ -module  $A^\perp \otimes \text{End}(E)$ . We call  $R_E^\nabla$  the Atiyah cocycle associated with the  $L$ -connection  $\nabla$  that extends the  $A$ -module structure of  $E$ .*
- (2) *The cohomology class  $\alpha_E \in H^1(A, A^\perp \otimes \text{End}(E))$  of the cocycle  $R_E^\nabla$  does not depend on the choice of the  $L$ -connection extending the  $A$ -action. And the cohomology class  $\alpha_E \in H^1(A, A^\perp \otimes \text{End}(E))$  is called the Atiyah class of the  $A$ -module  $E$ .*
- (3) *The Atiyah class  $\alpha_E$  of  $E$  vanishes if and only if there exists an  $A$ -compatible  $L$ -connection on  $E$ .*

Given a Lie algebroid pair  $(L, A)$  and an  $A$ -module  $E$ , the scalar Atiyah classes  $c_k(E)$  is defined [6] by

$$c_k(E) = \frac{1}{k!} \left( \frac{\sqrt{-1}}{2\pi} \right)^k \text{tr}(\alpha_E^k) \in H^k(A, \wedge^k A^\perp). \tag{6}$$

Here  $\alpha_E^k$  denotes the image of  $\alpha_E \otimes \dots \otimes \alpha_E$  under the map

$$H^1(A, A^\perp \otimes \text{End}(E)) \wedge \dots \wedge H^1(A, A^\perp \otimes \text{End}(E)),$$

which is induced by the composition in  $\text{End}(E)$  and the wedge product in  $\wedge^\bullet A^\perp$ . Let  $(L, A)$  be a Lie algebroid pair. Then  $E = L/A$  naturally becomes an  $A$ -module, with the  $A$ -modules structure on  $E = L/A$  defined by

$$a \cdot \bar{l} = \overline{[a, l]}$$

for all  $a \in \Gamma(A)$  and  $l \in \Gamma(L)$ . In the special case of  $(L, A)$  being a Lie algebra pair, we can define the Atiyah class associated with  $(L, A, L/A)$  by Theorem 2.1.

### 2.2. Lie algebra modules and Lie bialgebras

We first recall some necessary knowledge of Lie algebras. Let  $\mathfrak{g}$  be a Lie algebra over the field  $\mathbf{k} = \mathbb{R}$  or  $\mathbb{C}$ . A *representation* of  $\mathfrak{g}$  on a  $\mathbf{k}$ -vector space  $V$  is a morphism  $\rho: \mathfrak{g} \rightarrow \text{End}V$  satisfying

$$\rho([x, y]) = [\rho(x), \rho(y)]$$

for all  $x, y \in \mathfrak{g}$ . The action map of  $\mathfrak{g}$  on  $V$

$$\mathfrak{g} \times V \rightarrow V: (x, v) \rightarrow x \cdot v = \rho(x)v, \quad \forall x \in \mathfrak{g}, v \in V$$

gives a  $\mathfrak{g}$ -module structure on  $V$ . Suppose that  $V, W$  are  $\mathfrak{g}$ -modules with the associated representation  $\rho_V$  and  $\rho_W$ . Then  $V^*$ ,  $\text{End}(V)$  and  $V \otimes W$  are all  $\mathfrak{g}$ -modules, with the corresponding representation given by

$$\rho_{V^*} = -\rho_V^*, \quad \rho_{\text{End}(V)} = [\rho_V, \cdot], \quad \rho_{V \otimes W} = \rho_V \otimes \text{id} + \text{id} \otimes \rho_W.$$

The Lie algebra  $\mathfrak{g}$  acts on itself by the adjoint action:  $x \in \mathfrak{g} \mapsto ad_x \in \text{End}(\mathfrak{g})$ , where  $ad_x(y) = [x, y]$  for all  $y \in \mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  acts on  $\mathfrak{g}^*$  by the coadjoint action:

$$x \in \mathfrak{g} \mapsto -ad_x^* \in \text{End}(\mathfrak{g}^*).$$

For a given Lie algebra  $\mathfrak{g}$ , the vector spaces  $\mathfrak{g} \otimes \mathfrak{g}$  and  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  are both  $\mathfrak{g}$ -modules, with the  $\mathfrak{g}$ -module structures given by

$$x \cdot (y \otimes z) = ad_x(y) \otimes z + y \otimes ad_x(z), \tag{7}$$

$$x \cdot (y \otimes T) = ad_x(y) \otimes T + y \otimes [-ad_x^*, T] \tag{8}$$

for all  $x, y, z \in \mathfrak{g}$  and  $T \in \text{End}(\mathfrak{g}^*)$ . The action of the Lie algebra  $\mathfrak{g}$  on  $\mathfrak{g} \otimes \mathfrak{g}$  in equation (7) is also called the *adjoint representation*, denoted by  $ad^{(2)}: \mathfrak{g} \mapsto \text{End}(\mathfrak{g} \otimes \mathfrak{g})$ :

$$x \rightarrow ad_x \otimes \text{id} + \text{id} \otimes ad_x \quad \text{for all } x \in \mathfrak{g}. \tag{9}$$

**Convention.** In this paper, we take the  $\mathfrak{g}$ -module structures above on the corresponding spaces without special explanation.

Given a  $\mathfrak{g}$ -module  $V$ , the *Lie algebra cohomology*  $H^*(\mathfrak{g}, V)$  is defined by the Chevalley-Eilenberg complex. The *coboundary* of  $f \in \text{Hom}(\wedge^k \mathfrak{g}, V)$  is an element  $\delta f \in \text{Hom}(\wedge^{k+1} \mathfrak{g}, V)$ , given by

$$\begin{aligned} (\delta f)(x_0, x_1, \dots, x_k) &= \sum_{i=0}^k (-1)^i \rho(x_i) f(x_0, \dots, \hat{x}_i, \dots, x_k) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k), \quad \text{for } x_0, x_1, \dots, x_k \in \mathfrak{g}. \end{aligned}$$

Next we recall some classical theory of Lie bialgebras (see [11]).

**Definition 2.2.** A Lie bialgebra is a Lie algebra  $\mathfrak{g}$  with a linear map  $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  such that

- (1) the dual map  $\gamma^*: \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  defines a Lie bracket on  $\mathfrak{g}^*$ , i.e., is a skew-symmetric bi-linear map satisfying the Jacobi identity, and
- (2)  $\gamma$  is a cocycle on  $\mathfrak{g}$  with values in  $\mathfrak{g} \otimes \mathfrak{g}$ , where  $\mathfrak{g}$  acts on  $\mathfrak{g} \otimes \mathfrak{g}$  by the adjoint representation  $ad^{(2)}$ .

The cocycle  $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  defines a Lie bracket on  $\mathfrak{g}^*$ , determined by

$$\langle [\xi, \eta], x \rangle = \langle \gamma(x), \xi \otimes \eta \rangle, \tag{10}$$

for  $\xi, \eta \in \mathfrak{g}^*$  and  $x \in \mathfrak{g}$ . In fact,  $\gamma$  is a linear map from  $\mathfrak{g}$  to  $\mathfrak{g} \wedge \mathfrak{g}$ . The map  $\gamma: \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$  is a cocycle, thus  $\gamma$  defines a cohomology class  $[\gamma] \in H^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ . If the cocycle  $\gamma$  is a coboundary, i.e.,  $\gamma = \delta r$ , with  $r \in \mathfrak{g} \otimes \mathfrak{g}$ , the corresponding Lie bialgebra is called a *coboundary Lie bialgebra*, and  $r \in \mathfrak{g} \otimes \mathfrak{g}$  is called a *r-matrix*.

**Definition 2.3.** A Manin triple consists of a triple of Lie algebras  $(L, \mathfrak{g}, \mathfrak{g}')$  and a nondegenerate symmetric invariant product  $\langle, \rangle$  on  $L$  such that

- (1) both  $\mathfrak{g}$  and  $\mathfrak{g}'$  are Lie subalgebras of  $L$ ,
- (2)  $L = \mathfrak{g} \oplus \mathfrak{g}'$  as vector spaces, and  $dim \mathfrak{g} = dim \mathfrak{g}'$ ,
- (3)  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isotropic with respect to the the product  $\langle, \rangle$ .

There is a one to one correspondence between Lie algebras and Manin triples.

Let  $(\mathfrak{g}, \gamma)$  be a Lie bialgebra. Then  $(L = \mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$  becomes a Manin triple by the following way. The Lie algebra structure on  $\mathfrak{g}^*$  is defined by Equation (10). The Lie bracket on  $L = \mathfrak{g} \oplus \mathfrak{g}^*$  be defined by

$$[x + \xi, y + \eta] = [x, y] + [\xi, \eta] - ad_x^* \eta + ad_\eta^* x - ad_\xi^* y + ad_y^* \xi. \tag{11}$$

for all  $x, y \in \mathfrak{g}$  and  $\xi, \eta \in \mathfrak{g}^*$ . The non-degenerate symmetric invariant product on  $L = \mathfrak{g} \oplus \mathfrak{g}^*$  is defined by

$$\langle x + \xi, y + \eta \rangle = \langle x, \eta \rangle + \langle \xi, y \rangle, \tag{12}$$

for all  $x, y \in \mathfrak{g}$  and  $\xi, \eta \in \mathfrak{g}^*$ . Then  $(L = \mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$  is a Manin triple

Conversely, let  $(L, \mathfrak{g}, \mathfrak{g}')$  be a Manin triple. By the condition 2 and condition 3 of Definition 2.3, we may identify  $\mathfrak{g}'$  with  $\mathfrak{g}^*$  by the product  $\langle, \rangle$  between  $\mathfrak{g}$  and  $\mathfrak{g}'$ . With the identification of  $\mathfrak{g}^*$  with  $\mathfrak{g}'$ ,  $\mathfrak{g}^*$  becomes a Lie algebra. Let the map  $\gamma: \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$  be defined as the dual map of the Lie bracket on  $\mathfrak{g}^*: \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ . Then  $(\mathfrak{g}, \gamma)$  becomes a Lie bialgebra.

**Convention.** In this paper, we make a mixture of Lie bialgebras and Manin triples by the one-to-one correspondence. We write Lie bialgebras as  $(\mathfrak{g}, \mathfrak{g}^*)$  instead of  $(\mathfrak{g}, \gamma)$ .

**Example 2.4.** [13] Let  $L_{\mathbb{C}}$  be a semi-simple Lie algebra over  $\mathbb{C}$  and let  $L$  be the Lie algebra  $L_{\mathbb{C}}$  considered as a Lie algebra over  $\mathbb{R}$ . Let  $\mathfrak{u}$  be any compact real form of  $L$ , and let  $\mathfrak{t}$  be any maximal abelian subalgebra of  $\mathfrak{u}$ . Then  $\mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$

is a Cartan subalgebra of  $L_{\mathbb{C}}$ . Let  $\Delta$  be the set of roots of  $L_{\mathbb{C}}$  with respect to  $\mathfrak{h}$ , and let  $\Delta_+$  be the set of positive roots with respect to some ordering of  $\Delta$ . Let  $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+} L_{\alpha}$ , where  $L_{\alpha}$  is the root space of  $\alpha$ . Let  $\mathfrak{b} = \mathfrak{n}_+ \oplus \sqrt{-1}\mathfrak{t}$ . Then  $\mathfrak{b}$  and  $\mathfrak{u}$  are real Lie subalgebras of  $L$ . Moreover, we have  $L = \mathfrak{b} \oplus \mathfrak{u}$  as real vector spaces, and  $\dim \mathfrak{b} = \dim \mathfrak{u}$ . Let  $K$  be the Killing form on  $L_{\mathbb{C}}$ , then  $Im(K)$ , the imaginary part of  $K$ , defined by

$$\langle X, Y \rangle = Im(K(X, Y)) \tag{13}$$

for all  $X, Y \in L$ , is a real-valued non-degenerate symmetric invariant product on  $L$ . Under this product,  $\mathfrak{u}$  and  $\mathfrak{b}$  are maximal isotropic subspaces of  $L$ . In this way,  $(L, \mathfrak{u}, \mathfrak{b})$  and  $(L, \mathfrak{b}, \mathfrak{u})$  become Manin triples. If We identify  $\mathfrak{u}$  and  $\mathfrak{b}$  as the dual space of each other by the non-degenerate symmetric product  $\langle \cdot, \cdot \rangle$ , then  $(\mathfrak{g} = \mathfrak{u}, \mathfrak{g}^* = \mathfrak{b})$  and  $(\mathfrak{g} = \mathfrak{b}, \mathfrak{g}^* = \mathfrak{u})$  are both Lie bialgebras, and we have  $L = \mathfrak{g} \bowtie \mathfrak{g}^*$  in both situations. Moreover,  $(\mathfrak{g} = \mathfrak{u}, \mathfrak{g}^* = \mathfrak{b})$  is a coboundary Lie bialgebra.

### 3. Atiyah classes of Lie bialgebras

#### 3.1. Atiyah class associated with the triple $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$

Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a Lie bialgebra. Let  $L = \mathfrak{g} \bowtie \mathfrak{g}^*$ ,  $A = \mathfrak{g}$  and  $E = \mathfrak{g}^* \simeq L/A$ . Then  $(L, A)$  is a Lie pair,  $E$  is an  $A$ -module. The  $A$ -action on  $E = \mathfrak{g}^*$  is the coadjoint action, and the  $A$ -action on  $A^{\perp} \simeq (L/A)^* \simeq \mathfrak{g}$  is the adjoint action. Let  $\nabla: L \mapsto End(E)$  be an  $A$ -compatible  $L$ -connection on  $E$ . The map  $\nabla: L \mapsto End(E)$  splits into two parts:

$$\nabla|_{\mathfrak{g}}: \mathfrak{g} \mapsto End(\mathfrak{g}^*) \quad \text{and} \quad \nabla|_{\mathfrak{g}^*}: \mathfrak{g}^* \mapsto End(\mathfrak{g}^*),$$

where  $\nabla|_{\mathfrak{g}}: \mathfrak{g} \mapsto End(\mathfrak{g}^*)$  is exactly the coadjoint action of  $\mathfrak{g}$  on  $\mathfrak{g}^*$ . Let us denote by  $S$  the linear map  $\nabla|_{\mathfrak{g}^*}: \mathfrak{g}^* \mapsto End(\mathfrak{g}^*)$ . Then  $R_E^{\nabla}: A \otimes L/A \mapsto End(E)$  becomes  $R_E^{\nabla}: \mathfrak{g} \otimes \mathfrak{g}^* \mapsto End(\mathfrak{g}^*)$ . For all  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ , recall that  $[x, \xi] = -ad_x^* \xi + ad_{\xi}^* x$ . By Equation (5), the curvature  $R_E^{\nabla}: \mathfrak{g} \otimes \mathfrak{g}^* \mapsto End(\mathfrak{g}^*)$  can be written as

$$R_E^{\nabla}(x, \xi) = -ad_x^* S(\xi) + S(\xi) ad_x^* + S(ad_x^*(\xi)) + ad_{ad_{\xi}^*(x)}^*. \tag{14}$$

Applying Theorem 2.1 in this case, we obtain

- Theorem 3.1.** (1) *The element  $R_E^{\nabla} \in \mathfrak{g}^* \otimes \mathfrak{g} \otimes End(\mathfrak{g}^*)$  is a 1-cocycle for the Lie algebra  $\mathfrak{g}$  with values in the  $\mathfrak{g}$ -module  $\mathfrak{g} \otimes End(\mathfrak{g}^*)$ ;*
- (2) *the corresponding cohomology class  $\alpha_E \in H^1(\mathfrak{g}, \mathfrak{g} \otimes End(\mathfrak{g}^*))$ , called the Atiyah class associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$ , does not depend on the linear map  $S: \mathfrak{g}^* \mapsto End(\mathfrak{g}^*)$ ;*
- (3) *the Atiyah class  $\alpha_E$  vanishes if and only if there exists a linear map  $S: \mathfrak{g}^* \mapsto End(\mathfrak{g}^*)$  such that  $R_E^{\nabla} = 0$ , or in other words,*

$$-ad_x^* S(\xi) + S(\xi) ad_x^* + S(ad_x^*(\xi)) + ad_{ad_{\xi}^*(x)}^* = 0 \tag{15}$$

for all  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ .

In Theorem 3.1, the  $\mathfrak{g}$ -module structure on  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  is defined by Equation (8). The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.2.** *Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a Lie bialgebra. Then the following conditions are equivalent:*

- (a) *the annihilator of  $\text{Center}(\mathfrak{g})$  is not an ideal of  $\mathfrak{g}^*$ ;*
- (b) *the  $\text{Center}(\mathfrak{g})$  is not an invariant subspace of  $\mathfrak{g}$  under the coadjoint action of  $\mathfrak{g}^*$  on  $\mathfrak{g} \cong (\mathfrak{g}^*)^*$ .*

*If one of the above equivalent conditions is satisfied, then the Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$  does not vanish.*

**Proof.** (1) We first prove the equivalence of the conditions.

(a) $\Rightarrow$ (b): If the  $\text{Center}(\mathfrak{g})^\perp$  is not an ideal of  $\mathfrak{g}^*$ , then there exist  $\xi \in \mathfrak{g}^*$  and  $\eta \in (\text{Center}(\mathfrak{g}))^\perp \subset \mathfrak{g}^*$ , such that  $[\xi, \eta] \notin (\text{Center}(\mathfrak{g}))^\perp$ . Hence there exists  $x \in \text{Center}(\mathfrak{g})$ , such that  $\langle x, [\xi, \eta] \rangle \neq 0$ , which implies that  $\langle \text{ad}_\xi^*(x), \eta \rangle = \langle x, [\xi, \eta] \rangle \neq 0$ . Since  $\eta \in (\text{Center}(\mathfrak{g}))^\perp$ , we obtain that  $\text{ad}_\xi^*(x) \notin \text{Center}(\mathfrak{g})$ . Thus  $\text{Center}(\mathfrak{g})$  is not an invariant subspace of  $\mathfrak{g}$  under the coadjoint action of  $\mathfrak{g}^*$  on  $\mathfrak{g} \cong (\mathfrak{g}^*)^*$ .

(b) $\Rightarrow$ (a): The proof is similar as above. We skip it.

(2) If one of the equivalent conditions is satisfied, i.e., the condition (b) is satisfied, there exists  $x \in \text{Center}(\mathfrak{g})$  and  $\xi \in \mathfrak{g}^*$  satisfying  $\text{ad}_\xi^*(x) \notin \text{Center}(\mathfrak{g})$ . The condition  $x \in \text{Center}(\mathfrak{g})$  implies that  $\text{ad}_x^* = 0$ . And the condition  $\text{ad}_\xi^*(x) \notin \text{Center}(\mathfrak{g})$  implies that  $\text{ad}_{\text{ad}_\xi^*(x)}^* \neq 0$ . Therefore it does not exist the map  $S: \mathfrak{g}^* \mapsto \text{End}(\mathfrak{g}^*)$  satisfying Equation (15). By Theorem 3.1, the Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$  does not vanish. ■

**Example 3.3.** (An example with non-vanishing Atiyah class.) Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a 3-dimensional Lie bialgebra (see [7]), with Lie brackets on  $\mathfrak{g}$  and  $\mathfrak{g}^*$  being defined as

$$\begin{aligned} [x_1, x_2] &= x_3, & [x_2, x_3] &= 0, & [x_3, x_1] &= 0; \\ [\xi^1, \xi^2] &= \xi^2, & [\xi^2, \xi^3] &= 0, & [\xi^3, \xi^1] &= -\xi^3; \end{aligned}$$

where  $\{x_1, x_2, x_3\}$  is a basis of  $\mathfrak{g}$ , and  $\{\xi^1, \xi^2, \xi^3\}$  is the dual basis of  $\mathfrak{g}^*$ . The center of the Lie algebra  $\mathfrak{g}$  is spanned by  $x_3$ . As  $\text{ad}_{\xi^3}^*(x_3) = -x_1 \notin \text{Center}(\mathfrak{g})$ ,  $\text{Center}(\mathfrak{g})$  is not an invariant subspace of  $\mathfrak{g}$  under the coadjoint action of  $\mathfrak{g}^*$  on  $\mathfrak{g} \cong (\mathfrak{g}^*)^*$ , by Corollary 3.2, the Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$  does not vanish.

Choosing  $S = 0$  in Theorem 3.1, we get another version of Theorem 3.1.

**Theorem 3.4.** *Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a Lie bialgebra. Let  $\lambda \in \mathfrak{g}^* \otimes \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  be defined by  $\lambda(x, \xi) = \text{ad}_{\text{ad}_\xi^*(x)}^* \in \text{End}(\mathfrak{g}^*)$ , for all  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ . Then*

- (1)  *$\lambda \in \mathfrak{g}^* \otimes \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  is a 1-cocycle for the Lie algebra  $\mathfrak{g}$  with values in the  $\mathfrak{g}$ -module  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ ,*
- (2) *the cohomology class  $[\lambda] \in H^1(\mathfrak{g}, \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*))$  coincides with the Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$ .*

(3) the cohomology class  $[\lambda]$  vanishes if and only if there exists a linear map  $S: \mathfrak{g}^* \mapsto \text{End}(\mathfrak{g}^*)$  such that, for all  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ ,

$$\lambda(x, \xi) = ad_x^* \cdot S(\xi) - S(\xi) \cdot ad_x^* - S(ad_x^*(\xi)). \tag{16}$$

**Remark 3.5.** If we consider  $S: \mathfrak{g}^* \mapsto \text{End}(\mathfrak{g}^*)$  as an element in  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ , Equation (16) can then be written as the coboundary condition

$$\lambda(x) = -x \cdot S, \tag{17}$$

where the action of  $x \in \mathfrak{g}$  on  $S \in \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  is defined in Equation (8). ■

Let us denote by  $F$  the map  $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{id} \otimes (-ad^*)} \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ . (18)

It is easy to verify that  $F$  is a morphism between the two  $\mathfrak{g}$ -modules  $\mathfrak{g} \otimes \mathfrak{g}$  and  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ . Thus  $F$  induces a map

$$H^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{F_*} H^1(\mathfrak{g}, \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)) : F_*(\alpha)(x) = F(\alpha(x)), \tag{19}$$

for all  $\alpha \in H^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$  and  $x \in \mathfrak{g}$ .

**Proof of Theorem 1.1:** For any  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ , let us denote by  $\iota_\xi \gamma(x)$  the contraction of  $\xi \in \mathfrak{g}^*$  with the first part of  $\gamma(x) \in \mathfrak{g} \otimes \mathfrak{g}$ . For any  $\eta \in \mathfrak{g}^*$ , we have

$$\langle \iota_\xi \gamma(x), \eta \rangle = \langle x, [\xi, \eta] \rangle = \langle ad_\xi^* x, \eta \rangle.$$

Thus we get  $\iota_\xi \gamma(x) = ad_\xi^* x$ . (20)

As  $F$  is defined by  $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{id} \otimes (-ad^*)} \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ ,  $F \circ \gamma(x)$  is an element in  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ . Let us denote by  $\iota_\xi(F \circ \gamma(x))$  the contraction of  $\xi \in \mathfrak{g}^*$  with the first part of  $F \circ \gamma(x) \in \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ . Then we have

$$\iota_\xi(F \circ \gamma(x)) = \iota_\xi((\text{id} \otimes (-ad^*))\gamma(x)) = (\iota_\xi \otimes (-ad^*))\gamma(x) = -ad_{\iota_\xi \gamma(x)}^*.$$

By Equation (20), it follows

$$\iota_\xi(F \circ \gamma(x)) = -ad_{\iota_\xi \gamma(x)}^* = -ad_{ad_\xi^* x}^* = -\lambda(x, \xi).$$

Thus we obtain  $\lambda = -F \circ \gamma$ . As  $F$  is a morphism of  $\mathfrak{g}$ -modules, we get  $[\lambda] = -F_*[\gamma]$ . By Theorem 3.1, the Atiyah class  $\alpha_E = [\lambda]$ . It vanishes if and only if  $[\gamma] \in \ker F_*$ . ■

**Remark 3.6.** Notice that in Theorem 1.1, the map  $\gamma$  is only related to the Lie algebra structures on  $\mathfrak{g}^*$ , and  $F$  is only related to the Lie algebra structures on  $\mathfrak{g}$ .

**Corollary 3.7.** Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a coboundary Lie bialgebra with the  $r$ -matrix, i.e.,  $\gamma = \delta r$ , where  $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  is the cocycle associated with  $(\mathfrak{g}, \mathfrak{g}^*)$  and  $r \in \mathfrak{g} \otimes \mathfrak{g}$ .

(1) The Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$  vanishes.

(2) Let the map  $S: \mathfrak{g}^* \rightarrow \text{End}(\mathfrak{g}^*)$  be defined by

$$S(\xi) = -ad_{r(\xi)}^* \tag{21}$$

for all  $\xi \in \mathfrak{g}^*$ , where  $r(\xi) \in \mathfrak{g}$  denotes by the contraction of  $\xi$  with the first part of  $r \in \mathfrak{g} \otimes \mathfrak{g}$ . Then  $S$  satisfies the equation (16).

**Remark 3.8.** The first part of the Corollary 3.7 is due to Khaoula Abdeljellil and Camille Laurent-Gengoux by private communication.

**Proof.** (1) For a coboundary Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$ , the corresponding cohomology class  $[\gamma] = 0$ . By theorem 1.1, the Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$  vanishes.

(2) By Theorem 1.1, we have  $\lambda = -F \circ \gamma = -F \circ \delta(r)$ .

As  $F: \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{id} \otimes (-ad^*)} \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$  is a morphism of  $\mathfrak{g}$ -modules, where the  $\mathfrak{g}$ -module structures are defined in Equation (7) and Equation (8), we get  $F \circ \delta(r) = \delta(F(r))$ . As a consequence, we have  $\lambda = -\delta(F(r))$ .

On the other hand, we have

$$S(\xi) = -ad_{r(\xi)}^* = \iota_\xi(\text{id} \otimes (-ad^*)(r)) = \iota_\xi F(r),$$

which implies that  $S = F(r)$ , where  $S: \mathfrak{g}^* \rightarrow \text{End}(\mathfrak{g}^*)$  is considered as an element in  $\mathfrak{g} \otimes \text{End}(\mathfrak{g}^*)$ . Thus we have  $\lambda = -\delta S$ , which is equivalent to equation (16). ■

**Example 3.9.** As shown in example 2.4,  $(\mathfrak{g} = \mathfrak{u}, \mathfrak{g}^* = \mathfrak{b})$  is a coboundary Lie bialgebra. Hence by Corollary 3.7, the Atiyah class associated with  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g} = \mathfrak{u}, E = \mathfrak{g}^* = \mathfrak{b})$  vanishes.

### 3.2. The first scalar Atiyah class associated with the triple $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$

Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be a Lie bialgebra and let  $\alpha_E \in H^1(\mathfrak{g}, \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*))$  be the Atiyah class associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g}, E = \mathfrak{g}^*)$ . The map

$$tr: \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*) \xrightarrow{\text{id} \otimes tr} \mathfrak{g} \otimes k = \mathfrak{g}$$

is a morphism of  $\mathfrak{g}$ -modules. The first scalar Atiyah class  $c_1(E)$  [6] is defined by

$$c_1(E) = \frac{\sqrt{-1}}{2\pi} tr(\alpha_E) \in H^1(\mathfrak{g}, \mathfrak{g}). \tag{22}$$

Let  $\kappa \in \mathfrak{g}^*$  be the modular vector of Lie algebra  $\mathfrak{g}$ , defined by

$$\kappa(x) = tr(ad_x), \quad \forall x \in \mathfrak{g}. \tag{23}$$

Let  $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be the cocycle associated with the Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$ . We define the map  $\iota_\kappa \gamma: \mathfrak{g} \rightarrow \mathfrak{g}$  by

$$(\iota_\kappa \gamma)(x) = \iota_\kappa \gamma(x), \quad \forall x \in \mathfrak{g}, \tag{24}$$

where  $\iota_\kappa \gamma(x)$  denotes by the contraction of  $\kappa \in \mathfrak{g}^*$  with the first part of  $\gamma(x) \in \mathfrak{g} \otimes \mathfrak{g}$ .

**Lemma 3.10.** *The map  $\iota_\kappa\gamma: \mathfrak{g} \rightarrow \mathfrak{g}$  is a cocycle for the Lie algebra  $\mathfrak{g}$  with values in  $\mathfrak{g}$ .*

**Proof.** The map  $\iota_\kappa\gamma: \mathfrak{g} \rightarrow \mathfrak{g}$  is the composition of the map  $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  and the map  $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\kappa \otimes \text{id}} k \otimes \mathfrak{g} = \mathfrak{g}$ . The map  $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\kappa \otimes \text{id}} k \otimes \mathfrak{g} = \mathfrak{g}$  can be written as

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{tr}(ad^*) \otimes \text{id}} k \otimes \mathfrak{g} = \mathfrak{g}. \tag{25}$$

As the maps  $ad^*: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*)$  and  $\text{tr}: \text{End}(\mathfrak{g}^*) \rightarrow \mathbb{R}$  are both  $\mathfrak{g}$ -module morphisms, by simple computation we can verify that the map (25) is a  $\mathfrak{g}$ -module morphism. Since  $\gamma: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  is a cocycle, we get that the map  $\iota_\kappa\gamma: \mathfrak{g} \rightarrow \mathfrak{g}$  is also a cocycle. ■

By Lemma 3.10, the map  $\iota_\kappa\gamma$  defines a cohomology class  $[\iota_\kappa\gamma] \in H^1(\mathfrak{g}, \mathfrak{g})$ .

**Proof of Theorem 1.2.** (1) By Theorem 1.1, we obtain  $\alpha_E = [\lambda]$  and  $\lambda = -F \circ \gamma$ , where  $F$  is the map

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{id} \otimes (-ad^*)} \mathfrak{g} \otimes \text{End}(\mathfrak{g}^*).$$

Thus we have 
$$\text{tr}(\lambda) = -(\text{id} \otimes \text{tr}(-ad^*)) \circ \gamma = (\text{id} \otimes \text{tr}(ad^*)) \circ \gamma. \tag{26}$$

As  $\text{tr}(ad_x^*) = \text{tr}(ad_x) = \kappa(x)$  for all  $x \in \mathfrak{g}$ , by Equation (26) we get that

$$\text{tr}(\lambda) = (\text{id} \otimes \kappa) \circ \gamma. \tag{27}$$

On the other hand,  $\gamma$  is a map from  $\mathfrak{g}$  to  $\mathfrak{g} \wedge \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ , which implies

$$\iota_\kappa\gamma = -(\text{id} \otimes \kappa) \circ \gamma. \tag{28}$$

By Equations (27) and (28), we obtain

$$c_1(E) = \frac{\sqrt{-1}}{2\pi} [\text{tr}(\gamma)] = -\frac{\sqrt{-1}}{2\pi} [\iota_\kappa\gamma].$$

(2) By the arguments above,  $c_1(E)$  vanishes if and only if  $[\iota_\kappa\gamma] = 0$  in  $H^1(\mathfrak{g}, \mathfrak{g})$ , or equivalently, there exist  $v \in \mathfrak{g}$  such that

$$\iota_\kappa\gamma(x) = ad_v(x) \tag{29}$$

for all  $x \in \mathfrak{g}$ . By equation (20), we have  $\iota_\kappa\gamma(x) = ad_\kappa^*x$ . Therefore  $c_1(E)$  vanishes if and only if there exists  $v \in \mathfrak{g}$  such that  $ad_\kappa^* = ad_v$ .

(3) For all  $y \in \mathfrak{g}$ , we have

$$\langle ad_x^*\kappa, y \rangle = \langle \kappa, [x, y] \rangle = \text{tr}(ad_{[x,y]}) = \text{tr}([ad_x, ad_y]) = 0.$$

Thus we obtain that 
$$ad_x^*\kappa = 0 \tag{30}$$

for all  $x \in \mathfrak{g}$ . For any  $x \in \mathfrak{g}$  we have

$$[\kappa + v, x] = -ad_\kappa^*x + ad_x^*\kappa + [v, x] = (ad_v - ad_\kappa^*)x + ad_x^*\kappa.$$

By Equation (30), we have

$$[\kappa + v, x] = (ad_v - ad_\kappa^*)x. \tag{31}$$

As a consequence, we get that  $ad_{\kappa+v}(\mathfrak{g}) = 0$  if and only if  $ad_\kappa^* = ad_v$ . ■

**3.3. The Atiyah class of  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g} = \mathfrak{b}, E = \mathfrak{g}^* = \mathfrak{u})$**

As shown in Example 2.4,  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g} = \mathfrak{b}, \mathfrak{g}^* = \mathfrak{u})$  is a Manin triple.

**Lemma 3.11.** *Let  $(\mathfrak{g} = \mathfrak{b}, \mathfrak{g}^* = \mathfrak{u})$  be the Lie bialgebra as in Example 2.4. Let  $\kappa \in \mathfrak{g}^*$  be defined by Equation (23). Then we have (1)  $\kappa \neq 0$ , (2)  $\kappa \in \mathfrak{t}$ .*

**Proof.** (1) For any  $t \in \sqrt{-1}\mathfrak{t}$ , since  $\mathfrak{g} = \sqrt{-1}\mathfrak{t} \oplus \mathfrak{n}_+$ , we have

$$\langle \kappa, t \rangle = tr(ad_t) = \sum_{\alpha \in \Delta_+} \langle \alpha, t \rangle = \langle \sum_{\alpha \in \Delta_+} \alpha, t \rangle, \tag{32}$$

where  $ad_t \in End(\mathfrak{g})$ . Since  $\sum_{\alpha \in \Delta_+} \alpha$  is a nonzero vector in  $\mathfrak{h}^*$ , we get  $\kappa \neq 0$ .

(2) For any  $y \in \mathfrak{n}_+$  and  $\xi \in \mathfrak{t}$ , since  $y$  and  $\xi$  are orthogonal under the Killing form, we have  $\langle y, \xi \rangle = Im(K(y, \xi)) = 0$ . This implies  $\mathfrak{t} \subset \mathfrak{n}_+^\perp \subset \mathfrak{g}^* = \mathfrak{u}$ , where  $\mathfrak{n}_+^\perp$  denotes by the annihilator of  $\mathfrak{n}_+$  in  $\mathfrak{g}^*$ . As  $\mathfrak{g} = \sqrt{-1}\mathfrak{t} \oplus \mathfrak{n}_+$ , we have  $\dim \mathfrak{t} = \dim \mathfrak{g} - \dim \mathfrak{n}_+$ .

Therefore we have  $\mathfrak{t} = \mathfrak{n}_+^\perp$ . (33)

On the other hand,  $ad_y \in End(\mathfrak{g})$  is nilpotent for all  $y \in \mathfrak{n}_+$ . It implies

$$\kappa(y) = tr(ad_y) = 0$$

for all  $y \in \mathfrak{n}_+$ . Hence we have  $\kappa \in \mathfrak{n}_+^\perp = \mathfrak{t}$ . ■

**Lemma 3.12.** [8] *Let  $V$  be a finite dimensional vector space over  $\mathbf{k}$ ,  $x \in End(V)$ . There exist unique  $x_s, x_n \in End(V)$  satisfying the conditions:  $x = x_s + x_n$ ,  $x_s$  is semi-simple,  $x_n$  is nilpotent,  $x_s$  and  $x_n$  commute. The decomposition  $x = x_s + x_n$  is called the Jordan-Chevalley decomposition of  $x$ .*

**Proposition 3.13.** *Let  $(\mathfrak{g} = \mathfrak{b}, \mathfrak{g}^* = \mathfrak{u})$  be the Lie bialgebra as in Example 2.4. Then the first scalar Atiyah class  $c_1(E)$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g} = \mathfrak{b}, E = \mathfrak{g}^* = \mathfrak{u})$  does not vanish. As a consequence, the Atiyah class  $\alpha_E$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g} = \mathfrak{b}, E = \mathfrak{g}^* = \mathfrak{u})$  does not vanish.*

**Proof.** Assume that the first scalar Atiyah class  $c_1(E)$  associated with the triple  $(L = \mathfrak{g} \bowtie \mathfrak{g}^*, A = \mathfrak{g} = \mathfrak{b}, E = \mathfrak{g}^* = \mathfrak{u})$  vanishes. Then by Theorem 1.2, there exists  $v \in \mathfrak{g}$  such that

$$ad_{\kappa+v}(\mathfrak{g}) = 0. \tag{34}$$

As  $\mathfrak{g} = \sqrt{-1}\mathfrak{t} \oplus \mathfrak{n}_+$ ,  $v \in \mathfrak{g}$  can be written as  $v = v_1 + v_2$ , where  $v_1 \in \sqrt{-1}\mathfrak{t}$  and  $v_2 \in \mathfrak{n}_+$ . By Equation (34), we have  $[\kappa + v, v_2] = 0$ , which implies

$$[\kappa + v_1, v_2] = [\kappa + v_1 + v_2, v_2] = [\kappa + v, v_2] = 0. \tag{35}$$

By Lemma 3.11,  $\kappa \in \mathfrak{t}$ , thus we have  $\kappa + v_1 \in \mathfrak{h} = \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t}$  and  $v_2 \in \mathfrak{n}_+$ . By Equation (35) and Lemma 3.12, we have the Jordan-Chevalley decomposition

$$ad_{\kappa+v} = ad_{\kappa+v_1} + ad_{v_2}, \quad (36)$$

where  $ad_{\kappa+v_1} \in \text{End}(\mathfrak{g})$  is the semi-simple part, and  $ad_{v_2} \in \text{End}(\mathfrak{g})$  is the nilpotent part. Since  $ad_{\kappa+v}(\mathfrak{g}) = 0$ , the Jordan-Chevalley decomposition (36) implies

$$ad_{\kappa+v_1}(\mathfrak{g}) = 0.$$

Since  $\mathfrak{n}_+ \subset \mathfrak{g}$ , we have  $ad_{\kappa+v_1}(\mathfrak{n}_+) = 0$ . (37)

As  $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+} L_\alpha$ , by Equation (37), we get

$$\langle \kappa + v_1, \alpha \rangle = 0 \quad (38)$$

for all  $\alpha \in \Delta_+ \subset \mathfrak{h}^*$ . As  $\{\alpha \mid \alpha \in \Delta_+\}$  span the vector space  $\mathfrak{h}^*$ , by Equation (38), we get  $\kappa + v_1 = 0$ . Since  $\kappa \in \mathfrak{t}$  and  $v_1 \in \sqrt{-1}\mathfrak{t}$ , we have  $\kappa = 0$ , which contradicts Lemma 3.11.

Thus  $c_1(E)$  does not vanish. And consequently, the Atiyah class  $\alpha_E$  does not vanish. ■

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Received March 1, 2018  
and in final form November 28, 2018