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The Polynomial Conjecture for Restrictions of Some Nilpotent Lie Groups Representations

Ali Baklouti, Hidenori Fujiwara and Jean Ludwig

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Abstract. Let G be a connected and simply connected nilpotent Lie group, K an analytic subgroup of G and π an irreducible unitary representation of G whose coadjoint orbit of G is denoted by $\Omega(\pi)$. Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}$ designating the Lie algebra of G. We consider the algebra $(\mathcal{U}(\mathfrak{g})/\ker\pi)^K$ of the K-invariant elements of $\mathcal{U}(\mathfrak{g})/\ker\pi$. It turns out that this algebra is commutative if and only if the restriction $\pi|_K$ of π to K has finite multiplicities (cf. A. Baklouti and H. Fujiwara, *Commutativité des opérateurs différentiels sur l'espace des représentations restreintes d'un groupe de Lie nilpotent*, J. Math. Pures Appl. 83 (2004) 137–161). In this article we suppose this eventuality and we study the polynomial conjecture asserting that our algebra is isomorphic to the algebra $\mathbb{C}[\Omega(\pi)]^K$ of the K-invariant polynomial functions on $\Omega(\pi)$. We give a proof of the conjecture in the case where $\Omega(\pi)$ admits a normal polarization of G and in the case where K is abelian. This problem was partially tackled previously by A. Baklouti, H. Fujiwara, J. Ludwig, *Analysis of restrictions of unitary representations of a nilpotent Lie group*, Bull. Sci. Math. 129 (2005) 187–209.

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1. Introduction and preliminaries

Let $G = \exp\mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $K = \exp\mathfrak{k}$ an analytic subgroup of G. We denote by $\mathfrak{g}^*$ (resp. $\mathfrak{k}^*$) the dual vector space of $\mathfrak{g}$ (resp. $\mathfrak{k}$). Then, G (resp. K) acts on $\mathfrak{g}^*$ (resp. $\mathfrak{k}^*$) by the coadjoint action whose orbit space realizes by the orbit method [5], [8], [18] the unitary dual $\hat{G}$ (resp. $\hat{K}$) of G (resp. K). We denote by $\theta_G\colon \mathfrak{g}^* \to \hat{G}$ the Kirillov map and by $\Omega(\pi) = \Omega_G(\pi) = \theta_G^{-1}(\pi)$ the coadjoint orbit of G associated to $\pi \in \hat{G}$. Although we use the notation $\simeq$ for the unitary equivalence, we often identify an irreducible unitary representation with its equivalence class.

We know in the nilpotent case the canonical central disintegration of induced or restricted representations. Let $p\colon \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction mapping and $\sigma \in \hat{K}$. The canonical measure (cf. [5], Chap. II) on $\Omega_K(\sigma)$ and the Lebesgue measure on the annihilator $\mathfrak{k}^\perp = \{\ell \in \mathfrak{g}^*; \ell|_{\mathfrak{k}} = 0\}$ of $\mathfrak{k}$ in $\mathfrak{g}^*$ determine a measure μ_σ on the subvariety $p^{-1}(\Omega_K(\sigma))$ of $\mathfrak{g}^*$. We take a finite measure $\tilde{\mu_\sigma}$ on $\mathfrak{g}^*$ equivalent to μ_σ regarded as a measure on $\mathfrak{g}^*$, and consider its image $\nu_\sigma = (\theta_G)_*(\tilde{\mu_\sigma})$ on $\hat{G}$. We know

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[10], [13] the disintegration of the induced representation $\mathrm{ind}_K^G\sigma$:

$$\mathrm{ind}_K^G\sigma \simeq \int_{\hat{G}}^{\oplus} m_\sigma^\pi \pi d\nu_\sigma(\pi),$$

where the multiplicities m_σ^π is obtained as the number of the K-orbits contained in $\Gamma(\pi,\sigma)=\Omega_G(\pi)\cap p^{-1}(\Omega_K(\sigma))$.

On the other hand, we consider for $\pi\in\hat{G}$ a finite measure μ_π on $\mathfrak{g}^*$ equivalent to the canonical measure on the orbit $\Omega_G(\pi)$ which is regarded as a measure on $\mathfrak{g}^*$. Put this time $\nu_\pi=(\theta_K\circ p)_*(\mu_\pi)$. The restriction $\pi|_K$ of π to K is disintegrated [7], [14] as:

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m_\sigma^\pi \sigma d\nu_\pi(\sigma)$$

with the same multiplicities as in the case of the induction. The Frobenius reciprocity thus holds in these situations.

In other respects, it is well known ([7], [10]) that in these situations the multiplicities are either uniformly bounded almost everywhere or equal to infinity almost everywhere. According to these two eventualities, we say that $\mathrm{ind}_K^G\sigma$ (or $\pi|_K$) has either finite or infinite multiplicities. Concerning the induction mentioned above, K being monomial, $\sigma\in\hat{K}$ is induced by a unitary character χ of a subgroup H. Hence,

$$\mathrm{ind}_H^G\chi \simeq \mathrm{ind}_K^G(\mathrm{ind}_H^K\chi) \simeq \int_{\hat{G}}^{\oplus} m_\sigma^\pi \pi d\nu_\sigma(\pi).$$

In the frame of the monomial representation $\tau=\mathrm{ind}_H^G\chi$, we consider the algebra $D_\tau(G/H)$ of the G-invariant differential operators on the fiber space over G/H associated to the data (H,χ). In [9], Corwin and Greenleaf proved that $D_\tau(G/H)$ is commutative if τ has finite multiplicities, and presented the two following conjectures:

Conjecture 1.1. (Commutativity) The algebra $D_\tau(G/H)$ is commutative if and only if τ has finite multiplicities (see also [11]).

Conjecture 1.2. (Polynomial) When τ has finite multiplicities, the algebra $D_\tau(G/H)$ is isomorphic to the algebra $\mathbb{C}[f+\mathfrak{h}^\perp]^H$ of the H-invariant polynomial functions on the affine subspace $f+\mathfrak{h}^\perp$ of $\mathfrak{g}^*$. Here, $\mathfrak{h}$ denotes the Lie algebra of H and $f\in\mathfrak{g}^*$ an extension of $-id\chi\in\mathfrak{h}^*$.

Since, the commutativity conjecture was established in [17] and the polynomial conjecture is verified only in certain particular cases (cf. [4] and [15]).

For $\ell\in\mathfrak{g}^*$, let B_ℓ be the bilinear form on $\mathfrak{g}\times\mathfrak{g}$ defined by $B_\ell(X,Y)=\ell([X,Y])$. We denote by $\mathfrak{g}(\ell)$ the radical of B_ℓ, i.e. $\mathfrak{g}(\ell)=\{X\in\mathfrak{g}; B_\ell(X,Y)=0, \forall\, Y\in\mathfrak{g}\}$. Then $\mathfrak{g}(\ell)$ is nothing but the Lie algebra of the stabilizer $G(\ell)$ of ℓ in G. Let $S(\ell,\mathfrak{g})$ be the set of subalgebras $\mathfrak{h}$ isotropic for B_ℓ, i.e. verifying $B_\ell(\mathfrak{h},\mathfrak{h})=\{0\}$, and $M(\ell,\mathfrak{g})$ the set of (real) polarizations of $\mathfrak{g}$ at ℓ. Now $\mathfrak{h}\in S(\ell,\mathfrak{g})$ being given, we define a unitary character χ_ℓ of subgroup $H=\exp\mathfrak{h}$ by $\chi_\ell(\exp X)=e^{i\ell(X)}$ for all $X\in\mathfrak{h}$. The monomial representation $\tau=\mathrm{ind}_H^G\chi_f$ has finite multiplicities if and only if Q is a non-empty Zariski open set of $\Gamma=f+\mathfrak{h}^\perp$, where Q designates the set of the elements $\ell \in \Gamma$ such that $\mathfrak{h} + \mathfrak{g}(\ell)$ is a Lagrangian subspace of $\mathfrak{g}$ for B_ℓ (cf. [10]). In other respects, the polynomial conjecture holds for τ if there exists a common polarization $\mathfrak{b}$ for almost all $\ell \in Q$ with respect to the Lebesgue measure of Γ (cf. [15]).

In the present paper, we are interested in analogous subjects for the restriction $\pi|_K$. We denote by $\mathcal{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}_\mathbb{C}$ and let $\ker\pi$ be the primitive ideal of $\mathcal{U}(\mathfrak{g})$ associated to π. We introduce the algebra

$$\mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k} = \{A \in \mathcal{U}(\mathfrak{g}); [A, \mathfrak{k}] \subset \ker\pi\}$$

and its image
$$D_\pi(G)^K \cong \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}/\ker\pi \cong (\mathcal{U}(\mathfrak{g})/\ker\pi)^K,$$

where the last member designates the quotient algebra of K-invariant elements. The algebra $D_\pi(G)^K$ was the object of our three previous works [1], [2] and [3]. In particular, we proved [2] that our algebra $D_\pi(G)^K$ is commutative if and only if the restricted representation $\pi|_K$ has finite multiplicities. Here, we substantiate the following parallel polynomial conjecture:

Conjecture 1.3. (Polynomial) When $\pi|_K$ has finite multiplicities, the algebra $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega(\pi)]^K$ of K-invariant polynomial functions on $\Omega(\pi)$.

In our previous work [3], we studied this conjecture in many settings, especially when K is a normal subgroup of G and where the orbit $\Omega(\pi)$ is flat. In this paper, we further extend our study to the case where K is abelian and where $\Omega(\pi)$ admits a normal polarizing subgroup. Our main result is the following:

Theorem 1.4. *Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and π an irreducible unitary representation of G. We denote by Ω the coadjoint orbit of G associated to π. Let K be an analytic subgroup of G such that the restriction $\pi|_K$ of π to K has finite multiplicities. Then Conjecture* 1.3 *holds in the following two cases:*

(1) Ω *admits a normal polarization,*

(2) K *is abelian.*

Theorem 1.4 will be proved in Section 5 when assumption 1 holds (cf. Theorem 5.1) and in Section 6, when K is abelian (cf. Theorem 6.3). The rest outline of the paper is as follows: We introduce in the next section some backgrounds about the algebra $D_\pi(G)^K$ and some algebraic tools to describe its generators in term of the envelopping algebra of $\mathfrak{g}_\mathbb{C}$. This makes use of Pedersen's construction of the kernel $\ker\pi$, π being the Kirillov's model associated to $\Omega(\pi)$ (cf. [18]). Section 3 is devoted to generate an algorithm which allows to define a rational function P_W on $\Omega(\pi)$, for a given $W \in \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$. This result was also the subject of our previous paper [3]. We then state first elements of answers to the conjecture in Section 4. The last section is devoted to carry out some explicit computations through many examples. We trust that these explicitly made calculations turn out to be useful for readers and allow to understand how to construct the polynomial functions associated to the generators of the algebra $D_\pi(G)^K$.

2. Backgrounds

2.1. Orbits and saturation

Let N be a connected and simply connected nilpotent Lie group. We consider a unipotent representation of N on a real vector space V of finite dimension. Let $v \in V$ be an invariant vector by the action of N, i.e. $n{\cdot}v = v$ for all $n \in N$. Put for $x \in V$ arbitrarily fixed, $L_x = \{x + tv;\ t \in \mathbb{R}\}$, the straight line passing through x and having the direction of v. Then, there are two possibilities: either $L_x \cap N{\cdot}x = L_x$ or $L_x \cap N{\cdot}x = \{x\}$. According to these two possibilities, we shall say that the orbit $N{\cdot}x$ is either saturated or non-saturated in the direction $\mathbb{R}v$. We shall utilize in what follows this fact applied to the coadjoint representation of N (or a subgroup K of N), where the invariant vector v will be a linear form which vanishes on an ideal $\mathfrak{g}^0$ of codimension 1 of $\mathfrak{g}$. In this situation, we shall say that the orbit in question is either saturated or non-saturated with respect to $\mathfrak{g}^0$.

2.2. An algebraic description of $D_\pi(G)^K$

Let
$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g} \tag{1}$$
be a Jordan-Hölder sequence of $\mathfrak{g}$, i.e. an increasing sequence of ideals of $\mathfrak{g}$ such that $\dim(\mathfrak{g}_j) = j,\ \ j = 0, ..., n$. Let $\{X_1, ..., X_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$, associated to this Jordan-Hölder sequence, and $\{X_1^*, ..., X_n^*\}$ the basis of $\mathfrak{g}^*$ such that $X_i^*(X_j) = \delta_{i,j}, 1 \le i, j \le n$. Let $p_i\colon \mathfrak{g}^* \to \mathfrak{g}_i^*$ be the canonical projection which intertwines the actions of G on $\mathfrak{g}^*$ and $\mathfrak{g}_i^*$. For $\ell \in \mathfrak{g}^*$, we put $e_i(\ell) = \dim G{\cdot}p_i(\ell),\ e(\ell) = (e_1(\ell), ..., e_n(\ell))$ and $\mathcal{E} = \{e(\ell), \ell \in \mathfrak{g}^*\}$. For $e \in \mathcal{E}$, we define the G-invariant layer $U_e = \{\ell \in \mathfrak{g}^* :\ e(\ell) = e\}$. Putting $e_0 = 0$, we define

$$S(e) = \{i :\ e_i = 1 + e_{i-1}\},\ \mathfrak{g}_S^* = \mathbb{R}{-}\mathrm{vect}\{X_i^* :\ i \in S(e)\}$$

$$T(e) = \{i :\ e_i = e_{i-1}\},\ \mathfrak{g}_T^* = \mathbb{R}{-}\mathrm{vect}\{X_i^* :\ i \in T(e)\}.$$

Then $\mathfrak{g}^* = \mathfrak{g}_S^* \oplus \mathfrak{g}_T^*$. There exists an order among the elements of $\mathcal{E} = \{e^{(1)} > ... > e^{(k)}\}$ in such a manner that $U_{e^{(1)}}$ and $\cup_{j \le i} U_{e^{(j)}}$ are Zariski open sets of $\mathfrak{g}^*$ for every i. In this way all the layers U_e are semi-algebraic sets, i.e. difference of two Zariski open sets of $\mathfrak{g}^*$. Let U_e be an arbitrary layer, we write $S(e) = \{j_1 < \cdots < j_d\}$ where d designates the dimension of the G-orbits in U_e. Then there exist some functions $R_j^e\colon U_e \times \mathbb{R}^d \to \mathbb{R},\ j = 1, ..., n$ such that:

(a) For $f \in U_e$ fixed, $x = (x_1, ..., x_d) \mapsto R_j^e(f, x)\colon \mathbb{R}^d \to \mathbb{R}$ is a polynomial function in x and the coefficients are G-invariant function on U_e;

(b) $R_j^e(f, x) = x_k$ pour $j = j_k \in S(e),\ f \in U_e$;

(c) If $j_k \le j < j_{k+1}$, then $R_j^e(f, x)$ depends only on $x_1, ..., x_k$;

(d) For any $f \in U_e$, the coadjoint orbit $G{\cdot}f$ is given by (see [19]):

$$G{\cdot}f = \{\ \sum_{j=1}^n R_j^e(f, x) X_j^*; x \in \mathbb{R}^d\}.$$

Let $r_j^e(f)$ be the image in $\mathcal{U}(\mathfrak{g})$ by the symmetrization of the element

$$R_j^e(f, -iX_{j_1}, ..., -iX_{j_d})$$

in the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}_{\mathbb{C}}$, namely, we replace the variable x_k in $R_j^e(f, x)$ by $-iX_{j_k}$. Notice in particular that $r_{j_k}^e(f) = -iX_{j_k}$. Let V_e be the subspace of $S(\mathfrak{g})$ spanned by the elements of the form $X_{j_1}^{\alpha_1} \cdots X_{j_d}^{\alpha_d}$, $\alpha_1, ..., \alpha_d \in \mathbb{N}$, and let F_e be the image in $\mathcal{U}(\mathfrak{g})$ of V_e by the symmetrization. On the other hand, let E_e be the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by the elements of the form $X_{j_1}^{\alpha_1} \cdots X_{j_d}^{\alpha_d}, \alpha_1, ..., \alpha_d \in \mathbb{N}$. If $S(e) = \emptyset$, we put $V_e = F_e = E_e = \mathbb{C}\cdot 1$. Pedersen proved that the primitive ideal $\ker \pi$, where $\pi \in \hat{G}$ such that $f \in \Omega(\pi)$ is generated by the elements

$$u_j^e(f) = X_j - ir_j^e(f), \ j \in T(e)$$

and that

$$\mathcal{U}(\mathfrak{g}) = \ker \pi \oplus E_e = \ker \pi \oplus F_e$$

(see Theorem 2.1.1 and Theorem 2.2.1 in [19]). In the same way, the actions of π on E_e and F_e are faithful (see Lemma 2.2.12 and Lemma 2.2.13 in [19]). In this way, identifying E_e and F_e à $\mathcal{U}(\mathfrak{g})/\ker \pi$ and abusing notations, we have

$$D_\pi(G)^K \simeq E_e^K \simeq F_e^K \simeq \mathbb{C}[X_{j_1}, ..., X_{j_d}]^K.$$

These isomorphisms are simply isomorphisms of vector spaces.

2.3. e-central elements

In [9], Corwin and Greenleaf showed that Pedersen's construction of the kernel $\ker \pi_\ell$, where π_ℓ designates the Kirillov's model [18] which represents the class $\theta_G(\ell)$, for $\ell \in U_e$ leads to construct e-central elements (cf. Theorem 3.1 in [9]). These are elements A of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ such that the operators $\pi_\ell(A)$ are scalars for $\ell \in U_e$. Then $\pi_{\ell'}(A) = \pi_\ell(A)$ for all $\ell' \in G\cdot\ell$. More precisely, let $U_e \subset \mathfrak{g}^*$ be one of the layers constructed above. Then there exists a Zariski open set $Z \subset \mathfrak{g}^*$ such that $Z \cap U_e$ is non-empty G-invariant and for all $j \in T(e)$ there exists an e-central element $A_j \in \mathcal{U}(\mathfrak{g}_j)$ on $Z \cap U_e$, i.e. the operators $\pi_\ell(A_j)$ are scalars for all $\ell \in Z \cap U_e$ with the following properties:

(1) $A_j = P_j X_j + Q_j$, where P_j, Q_j are in $\mathcal{U}(\mathfrak{g}_{j-1})$.

(2) P_j is e-central on $Z \cap U_e$ and does not belong to $\ker \pi_\ell$.

(3) $\pi_\ell(A_j) = \phi_j(\ell) Id$ for $\ell \in Z \cap U_e$, where $\phi_j(\ell) = \tilde{p}_j(\tilde{\ell})\ell(X_j) + \tilde{q}_j(\tilde{\ell})$, $\tilde{p}_j$ and $\tilde{q}_j$ being non-singular rational functions on $Z \cap U_e$ depending only on $(\ell(X_1), ..., \ell(X_{j-1}))$. While the rational function $\tilde{p}_j(\tilde{\ell})$ is G-invariant and never vanishes on $Z \cap U_e$. Moreover, we easily see that the system $\{A_j;\ j \in T(e)\}$ of these e-central elements separates the orbits in $Z \cap U_e$.

Having given the construction of A_j, Corwin-Greenleaf [9] remarked the following: Dropping out the Zariski open set $Z \cap U_e$ from U_e, we notice that, $U_e \backslash Z$ being G-invariant and semi-algebraic, the parametrization of the orbits in U_e is carried out and retains all its properties on this sub-layer in U_e. We are able to repeat the whole process starting from $U_e \backslash Z$. Since U_e is semi-algebraic, the ascendent chain condition for the ideals in $\mathbb{C}[\mathfrak{g}^*]$ assures that the process terminates after a finite number of steps. So, patching the pieces together, we may suppose that $Z \cap U_e = U_e$. Let ρ be a unitary representation of G. We denote by $\mathcal{H}_\rho$, $\mathcal{H}_\rho^\infty$ and $\mathcal{H}_\rho^{-\infty}$ respectively the space of ρ, that of its differentiable vectors and the anti-dual of $\mathcal{H}_\rho^\infty$ (cf. [6] and [20]). For $a \in \mathcal{H}_\rho^{\pm\infty}$ and $b \in \mathcal{H}_\rho^{\mp\infty}$, we denote by $\langle a, b\rangle$ the image of b by a, so that $\langle a, b\rangle = \overline{\langle b, a\rangle}$. Given a subgroup H of G and its unitary character χ, put

$$\left(\mathcal{H}_\rho^{-\infty}\right)^{H,\chi} = \left\{a \in \mathcal{H}_\rho^{-\infty}; \rho(h)a = \chi(h)a, \ \forall h \in H\right\}.$$

3. The function P_W on $\Omega(\pi)$

Recall once again our situation. Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, $K = \exp \mathfrak{k}$ an analytic subgroup of G and π an irreducible unitary representation of G whose coadjoint orbit is denoted by $\Omega(\pi)$. For $\ell \in \Omega(\pi)$, we designate by $\mathfrak{b}[\ell|_\mathfrak{k}]$ a polarization of $\mathfrak{k}$ at $\ell|_\mathfrak{k} \in \mathfrak{k}^*$. We know [2] that $\pi|_K$ has finite multiplicities if and only if $\mathfrak{b}[\ell|_\mathfrak{k}] + \mathfrak{g}(\ell)$ is a Lagrangian subspace for the form B_ℓ at μ_π-almost all ℓ in $\Omega(\pi)$.

At the flag of ideals (1) of $\mathfrak{g}$, let $\mathcal{I} = \{i_1 < \cdots < i_d\}$ where $d = \dim \mathfrak{k}$, be the set of indices $1 \le i \le n$ such that $\mathfrak{k} \cap \mathfrak{g}_i \ne \mathfrak{k} \cap \mathfrak{g}_{i-1}$ and put

$$\mathcal{J} = \{j_1 < \cdots < j_q\} = \{1, 2, \ldots, n\} \setminus \mathcal{I}$$

with $q = \dim(\mathfrak{g}/\mathfrak{k})$. Putting $\mathfrak{k}_d = \mathfrak{k}$ and $\mathfrak{k}_{d+r} = \mathfrak{k} + \mathfrak{g}_{j_r}$ for $1 \le r \le q$, we obtain a sequence of subalgebras of $\mathfrak{g}$:

$$\mathfrak{k} = \mathfrak{k}_d \subset \mathfrak{k}_{d+1} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g}, \ \dim(\mathfrak{k}_r/\mathfrak{k}_{r-1}) = 1. \tag{2}$$

Furthermore, considering $\mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{g}_{i_s}$ $(1 \le s \le d)$, we get a flag of ideals of $\mathfrak{k}$:

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k}_d = \mathfrak{k}, \ \dim \mathfrak{k}_s = s. \tag{3}$$

Extracting a vector $Y_s \in \mathfrak{k}_s \setminus \mathfrak{k}_{s-1}$ for $1 \le s \le d$, we form a Jordan-Hölder basis $\{Y_1, \ldots, Y_d\}$ of $\mathfrak{k}$. Similarly, extracting a vector $\tilde{X}_r \in \mathfrak{k}_{d+r} \setminus \mathfrak{k}_{d+r-1}$ for $1 \le r \le q$, we form a Malcev basis $\{\tilde{X}_1, \ldots, \tilde{X}_q\}$ of $\mathfrak{g}$ relative to $\mathfrak{k}$.

Let $\ell \in \Omega(\pi)$. Taking there a real polarization $\mathfrak{b}[\ell]$ of $\mathfrak{g}$, we realize π as $\pi = \mathrm{ind}_{B[\ell]}^G \chi_\ell$ with $B[\ell] = \exp(\mathfrak{b}[\ell])$. On the other hand, by means of the flag (3), we construct [5] the Vergne polarization $\mathfrak{b}[\ell|_\mathfrak{k}]$ of $\mathfrak{k}$ at $\ell|_\mathfrak{k} \in \mathfrak{k}^*$. Put $B[\ell|_\mathfrak{k}] = \exp(\mathfrak{b}[\ell|_\mathfrak{k}])$. It is easy to verify [3] that the formula

$$\langle a_\ell, \varphi\rangle = \int_{B[\ell|_\mathfrak{k}]/(B[\ell|_\mathfrak{k}] \cap B[\ell])} \overline{\varphi(b)\chi_\ell(b)} d\dot{b} \quad (\forall \varphi \in \mathcal{H}_\pi^\infty), \tag{4}$$

$d\dot{b}$ designating an invariant measure on the homogeneous space $B[\ell|_\mathfrak{k}]/(B[\ell|_\mathfrak{k}] \cap B[\ell])$, gives us a semi-invariant generalized vector a_ℓ in $(\mathcal{H}_\pi^{-\infty})^{B[\ell|_\mathfrak{k}], \chi_\ell}$.

We suppose hereafter that $\pi|_K$ has finite multiplicities. This would say just as in the case of the monomial representations, that $\mathfrak{b}[\ell|_\mathfrak{k}] + \mathfrak{g}(\ell)$ is a Lagrangian subspace of $\mathfrak{g}$ for B_ℓ at almost all $\ell \in \Omega(\pi)$ with respect to the invariant measure. Then, it results μ_π-almost everywhere in $\Omega(\pi)$ that a_ℓ is an eigen vector for all the elements of $D_\pi(G)^K$ acting on $\mathcal{H}_\pi^{-\infty}$ by continuity. This also means that for every $W \in \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$ we have

$$\pi(W)a_\ell = \lambda_\ell(W)a_\ell$$

with a certain scalar $\lambda_\ell(W)$ (cf. [3]). Remark that this scalar $\lambda_\ell(W)$ does not depend on the choice of the polarization $\mathfrak{b}[\ell]$ (cf. [12], Proposition 3).

In this way, for $W \in \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$ we define μ_π-almost everywhere on $\Omega(\pi)$ the function $P_W \colon \ell \mapsto \lambda_\ell(W)$. Next, we verify [3] that the identity $P_W(\ell) = 0$ leads to the belonging $W \in \ker\pi$ and $P_W(\ell)$ extends to a K-invariant rational function on $\Omega(\pi)$. Besides, the homomorphism $P \colon \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k} \ni W \mapsto P_W(\ell)$ supplies us by passing to the quotient an embedding of $D_\pi(G)^K$ in the field $\mathbb{C}(\Omega(\pi))^K$ of the K-invariant rational functions on $\Omega(\pi)$. We can say even more. Aligning the two sequences (2) and (3), we have a sequence of subalgebras of $\mathfrak{g}$:

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_d = \mathfrak{k} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g}. \tag{5}$$

Relative to this sequence, let us extract again a vector $X_k \in \mathfrak{k}_k \setminus \mathfrak{k}_{k-1}$ and put $\ell_k = \ell(X_k)$ for $1 \le k \le n$. Consider the action of K on the sequence (5) and define two sets S_K, T_K of jump and non-jump indices. Namely, we denote by $e_j^K(\ell)$ the dimension of the K-orbit of $\ell|_{\mathfrak{k}_j} \in \mathfrak{k}_j^*$ for every $1 \le j \le n$. Then we agree $e_0^K(\ell) = 0$. For each index j, the same possibility of the alternative $e_j^K(\ell) = e_{j-1}^K(\ell) + 1$ or $e_j^K(\ell) = e_{j-1}^K(\ell)$ happens μ_π-almost everywhere on $\Omega(\pi)$. We denote by S_K the set of the indices $1 \le j \le n$ which verify the first eventuality and by T_K that of the indices of the second eventuality. Put $\mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} = \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k} \cap \mathcal{U}(\mathfrak{k}_j)$. The following theorem is proved in [2].

Theorem 3.1. *We keep the same notations and hypotheses. Then:*

(1) *If $j \in S_K$, then $\mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} = \mathcal{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k} + \mathcal{U}(\mathfrak{k}_j)\,(\mathcal{U}(\mathfrak{k}_{j-1}) \cap \ker\pi)$.*

(2) *If $j \in T_K$, then $\mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} \neq \mathcal{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k} + \mathcal{U}(\mathfrak{k}_j)\,(\mathcal{U}(\mathfrak{k}_{j-1}) \cap \ker\pi)$ and there exists $W_j \in \mathcal{U}_\pi(\mathfrak{k}_j)^\mathfrak{k}$ having the form $W_j = aX_j + b$ $(a, b \in \mathcal{U}(\mathfrak{k}_{j-1}))$, $a \in \mathcal{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k}$ with $\pi(a) \neq 0$.*

(3) *For $j \in T_K$ and $\ell \in \Omega(\pi)$, $P_{W_j}(\ell) = \varphi_j(\ell)\ell_j + \psi_j(\ell)$, where $\varphi_j(\ell)$, $\psi_j(\ell)$ are two rational functions of $\ell_1, \ldots, \ell_{j-1}$.*

As a direct consequence of this result, we obtain as in [3] the:

Proposition 3.2. (1) *Let A be an element of $\mathcal{U}_\pi(\mathfrak{k}_m)^\mathfrak{k}$ for $1 \le m \le n$ satisfying $\pi(A) \neq 0$. Then there exist two non-zero polynomials β_A and γ_A of the elements $\{W_j; j \in T_K,\ j \le m\}$ such that $\beta_A A \equiv \gamma_A$ modulo $\ker\pi$.*

(2) *The functions $\left\{P_{W_j}(\ell); j \in T_K\right\}$ rationally generate the field $\mathbb{C}\left(\Omega(\pi)\right)^K$.*

Evoking Conjecture 1.2 mentioned above for the monomial representations, it is natural to pose the following conjecture for the restrictions of representations:

Conjecture 3.3. Let $G = \exp\mathfrak{g}$ be a connected and simply connected nilpotent Lie group and keep all the previous noations. When $\pi|_K$ has finite multiplicities, the mapping $\Theta \colon \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k} \ni W \mapsto P_W$ gives by passing to the quotient an isomorphism of algebras from $D_\pi(G)^K$ to the algebra $\mathbb{C}\left[\Omega(\pi)\right]^K$ of the K-invariant polynomial functions on the orbit $\Omega(\pi)$.

We have consequently the following:

Remark 3.4. Let S_Ω (resp. T_Ω) be the set of jump (resp. non-jump) indices of $\Omega = \Omega(\pi)$ concerning the flag (1). Assume that $\mathcal{I} \subset S_\Omega$. Then, the projection image of $\Omega(\pi)$ on $\mathfrak{k}^*$ coincides with whole $\mathfrak{k}^*$ and our algebra $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is isomorphic to the center of $\mathcal{U}(\mathfrak{k})$. In fact, let $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ and j_0 the minimal index such that $W \in \mathcal{U}(\mathfrak{k}_{j_0})$ modulo $\ker\pi$ in the sequence (5). Assume $j_0 \geq d+1$, $j_0 = d + r_0$ for instance. Namely, $\mathfrak{k}_{j_0} = \mathfrak{k} + \mathfrak{g}_{j_{r_0}}$ with $j_{r_0} \in \mathcal{J}$. Then, for generic $\ell \in \Omega$, there exists [2] in $\mathfrak{g}(\ell)$ an element $X(\ell) + Y(\ell)$, where $X(\ell) \in \mathfrak{g}_{i_0}$ with $i_0 \in \mathcal{I}$ and $Y(\ell) \in \mathfrak{g}_{j_{r_0}}$. Since $\mathcal{I} \subset S_\Omega$, we see $i_0 < j_{r_0}$ and $j_{r_0} \in T_\Omega$. But this contradicts the minimality of j_0 and hence $j_0 \leq d$. It follows that the algebra $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega(\pi)]^K$.

4. First elements of answers to Conjecture 3.3

4.1. Rational functions on the orbit $\Omega(\pi)$

As in the section 1, we start from the flag of ideals (1) of $\mathfrak{g}$ to parameterize the orbit $\Omega = \Omega(\pi)$ and denote there by S_Ω and T_Ω respectively the sets of jump and non-jump indices. Let $\{X_1, \dots, X_n\}$ be a Malcev basis adapted to the flag (1), $\ell_j = \ell(X_j)$ $(1 \leq j \leq n)$ for $\ell \in \Omega$, $S_\Omega = \{s_1 < \cdots < s_r\}$, $r = \dim\Omega$ and $x_k = \ell_{s_k}$ for $1 \leq k \leq r$. Describe as in the section 1 the orbit Ω by the polynomial relations

$$\ell_j = F_j(x_1, \dots, x_k),\ s_k < j < s_{k+1}, \tag{6}$$

where $x = (x_1, \dots, x_r)$ runs through $\mathbb{R}^r$. In these circumstances the rational functions on Ω are nothing but the rational functions of the variables $(x_1, \dots, x_r)$.

For $1 \leq k \leq r$, let $I^{(k)}$ be the set of the K-invariant polynomial functions on Ω, which depend only on the variables $\{x_i; i \leq k\}$. The arguments developed in the pages 60 - 61 of [21] make us see that every R in $\mathbb{C}(\Omega)^K$ verifying

$$\frac{\partial R}{\partial x_k} \neq 0 \text{ and } \frac{\partial R}{\partial x_i} = 0\ (i > k)$$

is written in the form P/Q, where P and Q belong to $I^{(k)}$. Therefore, the existence of such an element R means that $I^{(k-1)}$ is strictly contained in $I^{(k)}$. Next, let $Q = \sum_{i=0}^m Q_i x_k^i$ $(m > 0)$ be an element of $I^{(k)} \backslash I^{(k-1)}$, where Q_i $(0 \leq i \leq m)$ designate polynomials of $(x_1, \dots, x_{k-1})$ verifying $Q_m \neq 0$. We then confirm that Q_m and $mQ_m x_k + Q_{m-1}$ are K-invariant polynomials.

4.2. An inductive proof

In order to study the conjecture, we proceed by induction on $\dim G$. When $G = K$, the answer is immediate. Consider the flag of algebras (5) of $\mathfrak{g}$ and for sake of simplicity of notation, denote $\mathfrak{g}_{n-1}$ instead of $\mathfrak{k}_{n-1}$ which contains $\mathfrak{k}$. Put $\mathfrak{g}' = \mathfrak{g}_{n-1}$, $G' = \exp\mathfrak{g}'$ and suppose that Conjecture 3.3 holds for G'.

4.2.1. Case where the ideal $\mathfrak{g}'$ is of non-saturation

Suppose that the orbit Ω is non-saturated with respect to $\mathfrak{g}'$, namely that $n \in T_\Omega$. Then the projection $pr : \mathfrak{g}^* \to \mathfrak{g}'^*$ turns out to be a K-equivariant homeomorphism between Ω and $\omega = pr(\Omega)$ which is a G'-orbit. Hence, $\mathbb{C}[\Omega]^K \cong \mathbb{C}[\omega]^K$. On the other hand, $\pi' = \pi|_{G'}$ is irreducible and there exists in $\ker \pi$ an element W having the form $W = X_n + A$ with $A \in \mathcal{U}(\mathfrak{g}')$ which serves us to identify $D_\pi(G)^K$ with $D_{\pi'}(G')^K$. Since ω is the coadjoint orbit of G' associated to π', the induction hypothesis assures the desired result.

4.2.2. Case where the ideal $\mathfrak{g}'$ is of saturation

Suppose now that Ω is saturated with respect to $\mathfrak{g}'$, namely that $n \in S_\Omega$. We have the following lemma:

Lemma 4.1. *There exists one and only one index $2 \le j \le n-1$ belonging to S_Ω and $b \in \mathcal{U}(\mathfrak{g}_{j-1})$ such that $X_j + b \in \mathcal{U}_\pi(\mathfrak{g}_j)^{\mathfrak{g}'}$.*

Proof. By Theorem 3.1 there exists one and only one index $2 \le j \le n-1$ belonging to S_Ω possessing the property that we find in $\mathcal{U}_\pi(\mathfrak{g}_j)^{\mathfrak{g}'}$ an element $W = aX_j + b$ with $a,\ b \in \mathcal{U}(\mathfrak{g}_{j-1})$ verifying $\pi(a) \neq 0$. Hence $a \in \mathcal{U}_\pi(\mathfrak{g}_{j-1})^{\mathfrak{g}'}$, otherwise there would exist in $\ker \pi$ an element of the form $[X, W] = [X, a]X_j + b'$ with $b' \in \mathcal{U}(\mathfrak{g}_{j-1})$, $X \in \mathfrak{g}'$ verifying $[X, a] \notin \ker \pi$, which contradicts the result of Pedersen [19] because $j \in S_\Omega$. Taking into account the fact that

$$\mathcal{U}_\pi(\mathfrak{g}_{j-1})^{\mathfrak{g}'} = \mathcal{U}_\pi(\mathfrak{g}_{j-1})^{\mathfrak{g}} \simeq \mathbb{C}$$

modulo $\ker \pi$, we can suppose that a is a constant modulo $\ker \pi$ and so that $a = 1$. ■

Likewise, if $j = s_i$ $(1 \le i \le r-1)$, there exists a G'-invariant polynomial function

$$y = x_i + \varphi(x_1, \dots, x_{i-1}) \tag{7}$$

on Ω, which separates the G'-orbits contained in $pr(\Omega)$. This means that $pr(\Omega) = \coprod_{y \in \mathbb{R}} \omega_y$, the disjoint union of G'-orbits ω_y. Accordingly,

$$\pi|_{G'} \simeq \int_{\mathbb{R}}^{\oplus} \pi'_t dt \tag{8}$$

with $\pi'_t = \theta_{G'}(\omega_t)$ for all $t \in \mathbb{R}$.

Let now $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Because $\pi|_{G'}$ has finite multiplicities, W belongs modulo $\ker \pi$ to $\mathcal{U}(\mathfrak{g}')$ (cf. [2]) and $P_W(\ell)$ is on Ω a rational function of x_k for $1 \le k \le r-1$, hence of y and of x_k for $1 \le k \le r-1,\ k \neq i$.

Proposition 4.2. *For every $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, $P_W(\ell)$ is a rational function of the variables x_k for $1 \le k \le r-1$, whose denominator is a polynomial of y.*

Proof. From the induction hypothesis applied to G', $P_W(\ell)$ restricted to ω_y is a polynomial of x_k $(k \neq i)$ for almost all $y \in \mathbb{R}$. For a multi-index $I = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{r-1})$ of non-negative integers, we write

$$x^I = x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_{i+1}^{\alpha_{i+1}} \cdots x_{r-1}^{\alpha_{r-1}}.$$

This leads to
$$P_W(\ell) = \frac{\sum_I p_I(y)x^I}{\sum_J q_J(y)x^J} = \sum_\Gamma r_\Gamma(y)x^\Gamma \tag{9}$$
for generic $\ell \in \Omega$, where $p_I(y)$ and $q_J(y)$ are polynomials of y and $r_\Gamma(y)$ are certain functions of y. It suffices then to confirm that the $r_\Gamma(y)$ are rational functions of y. For this purpose we introduce in the set of the $(r-2)$-series of non-negative integers a lexicographic order: For $I = (\alpha_1, \dots, \alpha_{r-2})$ and $J = (\beta_1, \dots, \beta_{r-2})$ we define
$$I < J \Leftrightarrow 1 \leq \exists k \leq r-2;\ \alpha_{r-2} = \beta_{r-2}, \dots, \alpha_{k+1} = \beta_{k+1}, \alpha_k < \beta_k.$$
Let us employ the induction on the order of the right member of the relation (9). When this member does not depend on x, it is clear that $r_{(0,\dots,0)}(y)$ is rational in y. Let I_0, J_0 and Γ_0 be the dominant multi-indices of I, J and Γ respectively which appear in (9). From
$$\sum_I p_I(y)x^I = \left(\sum_J q_J(y)x^J\right)\left(\sum_\Gamma r_\Gamma(y)x^\Gamma\right),$$
we obtain $p_{I_0}(y) = q_{J_0}(y)r_{\Gamma_0}(y)$, which implies that $r_{\Gamma_0}(y)$ is rational in y. Moving the term $r_{\Gamma_0}(y)x^{\Gamma_0}$ from the right to the left of (9), we get a new relation whose right member contains only multi-indices of order inferior to Γ_0 and we are able to apply there the induction hypothesis to reach the desired result. ■

Corollary 4.3. *Suppose that $\mathfrak{k}$ is contained in an ideal of codimension 2. Then for every $W \in \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$, the function $\ell \mapsto P_W(\ell)$ is polynomial.*

Proof. We can suppose that the orbit $\Omega(\pi)$ is twice saturated with respect to two distinct ideals. Consequently, the denominator of the rational function $P_W(\ell)$ is a polynomial on one side of $y = x_i + \varphi(x_1, \dots, x_{i-1})$ and on the other side of $z = x_j + \psi(x_1, \dots, x_{j-1})$ with $i \neq j$. It results that the function $P_W(\ell)$ is polynomial. ■

4.2.3. Surjectivity properties of the homomorphism Θ

Proposition 4.4. *Keep the same notations and hypotheses and let us denote y' the variable corresponding to the polynomial function defined as in equation (7). Then for every polynomial $\zeta(x) \in \mathbb{C}[\Omega]^K$, there exists a polynomial $s(y')$ of y' such that the product $s(y')\zeta(x)$ is in the image of Θ.*

Proof. Let E be the vector subspace of $\mathcal{U}(\mathfrak{g})$ spanned by the elements of the form $X^A = X_{s_1}^{\alpha_1} \cdots X_{s_r}^{\alpha_r}$, $A = (\alpha_1, \dots, \alpha_r)$ running through all r-series of non-negative integers. As we remarked it in the section 1, the action of π on E is faithful. Remark that all the elements of $D_\pi(G)^K \cong \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}/\ker\pi$ admit, being suitably modified by elements of $\ker\pi$, a representative in E. Keep the notations introduced above, and let $\zeta(x_1, \dots, x_{i-1}, y', x_{i+1}, \dots, x_{r-1})$ be a K-invariant polynomial function on Ω. Here, we know [4] that $y' = x_i + \varphi(x_1, \dots, x_{i-1}) = P_{Y'}(\ell)$ for a certain
$$Y' = -\sqrt{-1}X_{s_i} + \sum c_{k_1,\dots,k_{i-1}} X_{s_1}^{k_1} \cdots X_{s_{i-1}}^{k_{i-1}} \tag{10}$$

belonging to $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{g}'}$ with $c_{k_1,\ldots,k_{i-1}} \in \mathbb{C}$. For a $(r-2)$-series

$$I = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{r-1})$$

of non-negative integers, put

$$\hat{X}^I = X_{s_1}^{\gamma_1} \cdots X_{s_{i-1}}^{\gamma_{i-1}} X_{s_{i+1}}^{\gamma_{i+1}} \cdots X_{s_{r-1}}^{\gamma_{r-1}}$$

and consider the vector subspace E' of $\mathcal{U}(\mathfrak{g}')$ generated by $\hat{X}^I$. From the induction hypothesis applied to G', for almost all $y' \in \mathbb{R}$ there exists an element

$$W(y') = \sum_I r_I(y')\hat{X}^I \in \mathcal{U}_{\pi_{y'}}(\mathfrak{g}')^{\mathfrak{k}}$$

such that $$P_{W(y')}\left(\ell|_{\omega_{y'}}\right) = \zeta(x_1, \ldots, x_{i-1}, y', x_{i+1}, \ldots, x_{r-1})$$

for generic $\ell \in \Omega$. Here, $r_I(y')$ are certain functions of y'. On the other side, Proposition 3.2 says that there exists

$$U = \sum_J p_J(Y')\hat{X}^J \text{ and } V = \sum_\Gamma q_\Gamma(Y')\hat{X}^\Gamma$$

in $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ satisfying

$$P_U(\ell)\zeta(x_1, \ldots, x_{i-1}, y', x_{i+1}, \ldots, x_{r-1}) = P_V(\ell)$$

for generic $\ell \in \Omega$. Here, $p_J(Y')$ and $q_\Gamma(Y')$ designate polynomials of Y'. For almost all $y' \in \mathbb{R}$, $\pi_{y'}$ being faithful on E', the E'-component of

$$\left(\sum_J p_J(y')\hat{X}^J\right)\left(\sum_I r_I(y')\hat{X}^I\right)$$

is equal to $\sum_\Gamma q_\Gamma(y')\hat{X}^\Gamma$. Thus, we just verify as before that $r_I(y')$ are rational in y'. If so, there exists a certain polynomial $s(y')$ of y' such that $s(y')W(y') = \sum_I t_I(y')\hat{X}^I$ with polynomials $t_I(y')$ of y'. It follows that $Q = \sum_I t_I(Y')\hat{X}^I$ belongs to $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ and

$$P_Q(\ell) = s(y')\zeta(x_1, \ldots, x_{i-1}, y', x_{i+1}, \ldots, x_{r-1}). \qquad \blacksquare$$

4.3. Towards Conjecture 3.3

We keep the previous notations and still assume that $\pi|_K$ has finite multiplicities. Recall the sequence (5) of subalgebras which is a Jordan-Hölder sequence for the action of K. We consider the image $p(\Omega)$ of $\Omega = \Omega(\pi)$ by the projection $p\colon \mathfrak{g}^* \to \mathfrak{k}^*$ and the layer of K-orbits which encounters $p(\Omega)$ in a non-empty Zariski open set of $p(\Omega)$. Using the notations introduced just before Proposition 3.2 in Section 3, let

$$T_K \cap \{1, \ldots, d\} = \{k_1 < \cdots < k_u\}.$$

Theorem 4.5. *Assume that $\pi|_K$ has finite multiplicities. Then, any $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is algebraic modulo* $\ker\pi$ *on* $\{W_{k_1}, \dots, W_{k_u}\}$.

Proof. In the sequence (5) of subalgebras, W is supposed to be contained modulo $\ker\pi$ in $\mathcal{U}(\mathfrak{k}_p), d < p$ but not in $\mathcal{U}(\mathfrak{k}_{p-1})$. We also assume that the theorem holds for the elements of $\mathcal{U}(\mathfrak{k}_{p-1})$. Since $\pi|_K$ has finite multiplicities, $p \in T_K$ and there exists [2] in $\ker\pi$ an element $\kappa = UX_p^m + V$, where $U \in \mathcal{U}(\mathfrak{k}_{p-1}), \pi(U) \neq 0, m \geq 1$ and V is an element of $\mathcal{U}(\mathfrak{k}_p)$ whose degree with respect to X_p is smaller than or equal to $m-1$. Choose κ in such a way that $m \geq 1$ is the smallest possible. If necessary, we apply several times the adjoint action of $\mathfrak{k}$ on κ so that we can suppose $U \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Let us utilize the Corwin-Greenleaf e-central element $W_p = aX_p + b, a \in \mathcal{U}_\pi(\mathfrak{k}_{p-1})^{\mathfrak{k}}, b \in \mathcal{U}(\mathfrak{k}_{p-1})$ introduced before. From Proposition 3.2, it suffices for us to show that the theorem is true for $W = W_p$. Clearly,

$$UW_p^m - a^m\kappa = \lambda X_p^\iota + V',$$

where $\iota \leq m-1, \lambda \in \mathcal{U}(\mathfrak{k}_{p-1})$ and V' denotes an element of $\mathcal{U}(\mathfrak{k}_p)$ whose degree with respect to X_p is smaller than or equal to $m-2$, belongs to $\mathcal{U}_\pi(\mathfrak{k}_p)^{\mathfrak{k}}$. The minimality of m leads that λ belongs to $\mathcal{U}_\pi(\mathfrak{k}_{p-1})^{\mathfrak{k}}$. Next,

$$a^\iota(UW_p^m - a^m\kappa) - \lambda W_p^\iota$$

belongs to $\mathcal{U}_\pi(\mathfrak{k}_p)^{\mathfrak{k}}$ and its degree with respect to X_p is smaller than or equal to $\iota - 2$. Repeating this process, we arrive to the desired result. ■

For every $j \in T_K$ concerning the sequence (5), there exists in $\mathbb{C}[\Omega]^K$ a function

$$f_j(\ell) = g_j(\ell_1, \dots, \ell_{j-1})\ell_j + h_j(\ell_1, \dots, \ell_{j-1}),$$

where $\ell_i = \ell(X_i)(1 \leq i \leq j)$ and g_j, h_j are two polynomials verifying $0 \neq g_j \in \mathbb{C}[\Omega]^K$. A similar reasoning to the above proof brings us to:

Theorem 4.6. *Assume that $\pi|_K$ has finite multiplicities. Then, every $F(\ell) \in \mathbb{C}[\Omega]^K$ is algebraic on $\{f_{k_1}(\ell), \dots, f_{k_u}(\ell)\}$.*

On the other hand, let $W \in \mathcal{U}_\pi(\mathfrak{k}_v)^{\mathfrak{k}}$. If $v \leq d$, W belongs to $\mathcal{U}_\pi(\mathfrak{k})^{\mathfrak{k}}$ and the operator $\sigma(W)$ is a scalar for almost all $\sigma \in \hat{K}$ with respect to the measure ν_π used in the irreducible decomposition of $\pi|_K$. Then, we can apply Theorem 2.1.1 in [16] to get:

Proposition 4.7. *For any $W \in \mathcal{U}_\pi(\mathfrak{k})^{\mathfrak{k}}$, the function $P_W(\ell)$ is polynomial.*

5. Case where Ω admits a normal polarization

We assume in this section that there exists a common polarization $\mathfrak{b}$ for almost all $\ell \in \Omega(\pi)$ with respect to the invariant measure. In particular, this is the case if $\mathfrak{b}$ is either abelian or normal in $\mathfrak{g}$. In fact, under this hypothesis $\mathfrak{b}$ becomes essentially abelian. Indeed, the intersection $\mathfrak{a}$ of $\ker\ell$ for all $\ell \in \Omega(\pi)$ is G-invariant and contains $[\mathfrak{b}, \mathfrak{b}]$. If we try to show the polynomial conjecture by induction on $\dim G + \dim G/K$, we can suppose that $\mathfrak{k}$ contains $\mathfrak{a}$ which is included in $\ker\pi$. In this way, we can pass to the quotient $\mathfrak{g}/\mathfrak{a}$ to which applies the induction hypothesis. Hence, we can suppose that $\mathfrak{a}$ and consequently $[\mathfrak{b},\mathfrak{b}]$ are trivial. We shall consider the case where $\dim \mathfrak{g}/\mathfrak{b} \geq 2$, otherwise $\dim \Omega(\pi) \leq 2$ and the polynomial conjecture holds [3]. Now, our objective is:

Theorem 5.1. *Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and π an irreducible unitary representation of G. We denote by Ω the coadjoint orbit of G associated to π. Let K be an analytic subgroup of G such that the restriction $\pi|_K$ of π to K has finite multiplicities. Suppose that Ω admits a normal polarization, then Conjecture* 3.3 *holds.*

The proof will be made in several steps. We first show the following lemma which plays a key role to prove Proposition 5.3 below.

Lemma 5.2. *Let $H = \exp \mathfrak{h}$, $N = \exp \mathfrak{n}$ be subgroups of G such that $\mathfrak{n}$ contains $\mathfrak{h}_{j_0} = \mathfrak{h} + \mathfrak{g}_{j_r}$ and ρ an irreducible representation of N whose coadjoint orbit Ω_ρ is contained in the projection image of Ω on $\mathfrak{n}^*$. Suppose that $\rho|_H$ has finite multiplicities, $\alpha \in \mathcal{U}(\mathfrak{n})$, and $\rho(\alpha)$ is not a constant. Suppose further that the coadjoint orbit Ω_ρ of N admits a polarization $\mathfrak{p}$ which is an ideal of $\mathfrak{n}$. Then, if*

$$\mathcal{U}_\rho(\mathfrak{h}_{j_0})^{\mathfrak{h}} \not\subset \mathcal{U}_\rho(\mathfrak{h}_{j_0-1})^{\mathfrak{h}} + \ker\rho,$$

there exists an $\mathfrak{h}_{j_0}$-invariant element $\nu = aX_{j_0} + b$, $a, b \in \mathcal{U}(\mathfrak{h}_{j_0-1})$, namely ν is an e-central element of Corwin-Greenleaf for $p_{j_0}(\Omega)$, such that a is not divisible by α modulo $\ker\rho$.

Proof. Let us proceed by induction on $\dim N$. Let $\mathfrak{n}' = \mathfrak{n} \cap \mathfrak{g}'$. Because $\rho(\alpha)$ is not a constant, $\mathfrak{n}'$ is an ideal of codimension 1 in $\mathfrak{n}$. It suffices for us to treat the case where $\dim(\mathfrak{n}/\mathfrak{p}) \geq 2$. Indeed, if $\dim(\mathfrak{n}/\mathfrak{p}) = 1$, then modulo $\ker\rho$, we can suppose that $\mathfrak{p}$ is abelian. In this case, we are essentially in the threadlike case, namely supposing that the center of $\mathfrak{n}$ is one-dimensional. If further $\mathcal{U}_\rho(\mathfrak{n})^{\mathfrak{h}} \neq \mathbb{C}$ modulo $\ker\rho$, then $\mathfrak{h} \subset \mathfrak{p}$ and at the level j_0 in question there is a new comer in $\ker\rho$ except the case where j_0 corresponds to the first jump index for ρ. This says that either our situation does not occur or $\nu = X_{j_0}$ suits us.

Take an ideal $\widetilde{\mathfrak{n}}$ of codimension 1 in $\mathfrak{n}$ different from $\mathfrak{n}'$ and containing $\mathfrak{p}$. Let $\widetilde{N} = \exp \widetilde{\mathfrak{n}}$, $\mathfrak{n} = \mathbb{R}X + \widetilde{\mathfrak{n}}$ and $\widetilde{p}$: $\mathfrak{n}^* \to \widetilde{\mathfrak{n}}^*$ the restriction mapping.

5.1. Case 1. $\mathfrak{h} \subset \widetilde{\mathfrak{n}}$

To simplify the notation, we agree that $\widetilde{p}(\Omega_\rho)$ is the disjoint union of a family $\{\omega_\lambda\}$ with one parameter $\lambda \in \mathbb{R}$ of coadjoint orbits of $\widetilde{N}$. Let ρ_λ be the irreducible unitary representation of $\widetilde{N}$ associated to ω_λ. Cutting the Jordan-Hölder sequence $\{\mathfrak{g}_j\}_{j=1}^{n}$ of $\mathfrak{g}$ by $\mathfrak{n}$ and $\mathfrak{h}$, we consider the Jordan-Hölder sequences $\{\mathfrak{n}_j\}_{j=1}^{n'}$, where $n' = \dim \mathfrak{n}$, of $\mathfrak{n}$ and $\{\mathfrak{h}_j\}_{j=1}^{d'}$, where $d' = \dim \mathfrak{h}$, of $\mathfrak{h}$. Put as before

$$J = \{j_1 < \cdots < j_{n'-d'}\} = \{1 \leq j \leq n' : \dim(\mathfrak{n}_j \cap \mathfrak{h}) = \dim(\mathfrak{n}_{j-1} \cap \mathfrak{h})\}$$

and $\mathfrak{h}_k = \mathfrak{h} + \mathfrak{n}_{j_{k-d'}}$ for $d' + 1 \leq k \leq n'$. Write again $\mathfrak{h}_k = \mathbb{R}X_k + \mathfrak{h}_{k-1}$ and $H_k = \exp(\mathfrak{h}_k)$. Put $\widetilde{\mathfrak{h}} = \mathfrak{h} \cap \widetilde{\mathfrak{n}}$, $\widetilde{H} = \exp \widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{h}}_{j_0} = \widetilde{\mathfrak{h}} + \mathfrak{g}_{j_r}$ and $\widetilde{H}_{j_0} = \exp(\widetilde{\mathfrak{h}}_{j_0})$. Since

$\rho_\lambda|_{\widetilde{H}_{j_0}}$ has finite multiplicities, there exists from the induction hypothesis an element

$$\widetilde{\nu}_\lambda = a(\lambda)X_{j_0} + b(\lambda)$$

which is $\widetilde{\mathfrak{h}}_{j_0}$-invariant for the projection image of ω_λ on $(\widetilde{\mathfrak{h}}_{j_0})^*$, where $a(\lambda), b(\lambda)$ are two polynomial functions of λ with values in $\mathcal{U}(\mathfrak{h}_{j_0-1}\cap\widetilde{\mathfrak{n}})$ because $\omega_\lambda = \exp(\lambda X)\cdot\omega_0$, and where $a(\lambda)$ is not divisible by α. For the projection image $\widetilde{p}(\Omega_\rho)$ of Ω_ρ on $(\widetilde{\mathfrak{n}})^*$, there exists an e-central element

$$\widetilde{\kappa} = X_{k_0} + u \ (k_0 \le n-1)$$

with a certain $u \in \mathcal{U}(\mathfrak{h}_{k_0-1}\cap\widetilde{\mathfrak{n}})$ such that $\rho(\widetilde{\kappa})$ is not a constant. Choose such $\widetilde{\kappa}$ in a manner such that k_0 is as small as possible. As we may suppose, by modifying $\widetilde{\kappa}$ by a scalar, that $\rho_\lambda(\widetilde{\kappa}) = \lambda$, the element $\widetilde{\nu} = a(\widetilde{\kappa})X_{j_0} + b(\widetilde{\kappa})$ verifies $[\widetilde{\mathfrak{h}}_{j_0}, \widetilde{\nu}] \subset \ker\rho$ and $a(\widetilde{\kappa})$ is not divisible by α modulo $\ker\rho$. Here, we are led to the case where $k_0 < j_0$. Indeed, let $k_0 > j_0$. If $\mathfrak{h}$ is contained in $\widetilde{\mathfrak{n}}$, $[\mathfrak{h}, \widetilde{\kappa}] \subset \ker\rho$ and, when we write $\widetilde{\nu}$ as a polynomial of $\widetilde{\kappa}$, all the coefficients commute with $\mathfrak{h}$ modulo $\ker\rho$ and there is at least one which suits us. Further if $k_0 = j_0$, the element $\widetilde{\kappa}$ itself is qualified as ν. That is how $\widetilde{\nu}$ supplies us a required element.

5.2. Case 2. $\mathfrak{h} \not\subset \widetilde{\mathfrak{n}}$

Suppose now that $\mathfrak{h}$ is not contained in $\widetilde{\mathfrak{n}}$ and $\widetilde{\nu}$ is not $\mathfrak{h}$-invariant modulo $\ker\rho$. We develop the arguments utilized in [2]. Let $\mathfrak{h} = \mathbb{R}X + \widetilde{\mathfrak{h}}$. As $[X, \widetilde{\kappa}]$ is e-central for $\widetilde{p}(\Omega_\rho)$, it is constant modulo $\ker\rho$. It results that $[X,[X,\widetilde{\kappa}]]$ belongs to $\ker\rho$. For a sufficiently large integer $v \in \mathbb{N}$, we consider $\psi = \widetilde{\nu} + F(\widetilde{\kappa})$, where $F(t)$ is a polynomial with one variable t of degree $2v$. For $k \in \mathbb{N}$, put

$$\psi_0 = \psi,\ \psi_k = (\mathrm{ad}X)^k(\psi).$$

Since $(\mathrm{ad}X)^k(F(\widetilde{\kappa}))$ are e-central for $\widetilde{p}(\Omega_\rho)$, if they find themselves outside of $\ker\rho$, they are not divisible by α. Otherwise, X' taken in $\widetilde{\mathfrak{n}}$, $[X', (\mathrm{ad}X)^k(F(\widetilde{\kappa}))]$ does not belong to $\ker\rho$. Now, the integer v being sufficiently large, we see that

$$\psi_{2v} \notin \ker\rho,\ \psi_{2v+1} \in \ker\rho.$$

Everything as in [2], let us construct an element of $\mathcal{U}_\rho(\mathfrak{n})^{\mathfrak{h}_{j_0}}$ by the formula

$$\nu = (\psi_0\psi_{2v} + \psi_{2v}\psi_0) - (\psi_1\psi_{2v-1} + \psi_{2v-1}\psi_1) + \cdots$$
$$+(-1)^{v-2}(\psi_{v-2}\psi_{v+2} + \psi_{v+2}\psi_{v-2}) + (-1)^{v-1}(\psi_{v-1}\psi_{v+1} + \psi_{v+1}\psi_{v-1}) + (-1)^v\psi_v^2.$$

Here, we remark that $k_0 < j_0$ because $[\widetilde{\kappa}, \widetilde{\mathfrak{h}}_{j_0}] \subset \ker\rho$. If $k_0 \ge j_0$, there would exist a new element of $\mathcal{U}_\rho(\mathfrak{n})^{\mathfrak{h}}$ coming at the level of $\mathfrak{h}_{k_0}$. Since $\rho|_H$ has finite multiplicities, this means that there exists at almost all $\ell \in \Omega_\rho$ an element in $\mathfrak{h}_{k_0}$ outside of $\mathfrak{h}_{k_0-1}$ which is written as $a(\ell) + b(\ell)$ with $a(\ell) \in \mathfrak{h}(\ell|_\mathfrak{h})$ and $b(\ell) \in \mathfrak{g}(\ell)$. But this is impossible, because $\mathfrak{h} \not\subset \widetilde{\mathfrak{n}}$. Remark once again the fact that v is sufficiently large. This assures us that ν is of degree 1 with respect to X_{j_0}. Finally, choosing suitably the polynomial F, we confirm that the coefficient of X_{j_0} in ν is not divisible by α modulo $\ker\rho$. Indeed, let

$$F(t) = \lambda_0 + \lambda_1 t + \cdots + \lambda_{2v-1}t^{2v-1} + t^{2v},\ \lambda_j \in \mathbb{C}\ (0 \le j \le 2v-1).$$

Suppose that $(\mathrm{ad}X)^k(a(\widetilde{\kappa}))$ $(0 \le k \le m)$ are not divisible modulo $\ker\rho$ by α but so do $(\mathrm{ad}X)^{m+1}(a(\widetilde{\kappa}))$ and hence $(\mathrm{ad}X)^k(a(\widetilde{\kappa}))$, $k \ge m+1$. Remark that, concerning the coefficient of X_{j_0} in ν, the coefficient λ_{2v-m} of F comes only from the terms

$$\sum_{k\ge m}(\psi_k\psi_{2v-k} + \psi_{2v-k}\psi_k) \equiv 2\sum_{k\ge m}\psi_{2v-k}\psi_k \pmod{\ker\rho}$$

and that λ_{2v-m} appears in the term $\psi_{2v-m}\psi_m$ as

$$\lambda_{2v-m}(\mathrm{ad}X)^{2v-m}(\widetilde{\kappa}^{2v-m})\psi_m.$$

If so, in order that the coefficient of X_{j_0} in ν is always divisible modulo $\ker\rho$ by α for any value of λ_{2v-m}, it is necessary that the coefficient of X_{j_0} in the element $(\mathrm{ad}X)^{2v-m}(\widetilde{\kappa}^{2v-m})\psi_m$ and hence in ψ_m must be divisible by α modulo $\ker\rho$, what contradicts the choice of the index m. In this way, we find in this situation the required element ν and this finishes the proof of the lemma. ■

We now prove the following:

Proposition 5.3. *For any $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, the function $\Omega \ni \ell \mapsto P_W(\ell)$ is polynomial.*

Proof. As usual we employ induction on $\dim G + \dim G/K$. Especially, we assume that the polynomial conjecture holds for proper analytic subgroups $G'' = \exp(\mathfrak{g}'')$ of G which properly contain K and such that $\mathfrak{b}\cap\mathfrak{g}''$ is a polarization of $\ell'' = \ell_{|\mathfrak{g}''}$. Assuming $G \ne K$, we consider the sequence of subalgebras

$$\mathfrak{k}_d = \mathfrak{k} \subset \mathfrak{k}_{d+1} \subset \cdots \subset \mathfrak{k}_n = \mathfrak{g},\ \dim(\mathfrak{k}_j) = \dim(\mathfrak{k}_{j-1}) + 1\ (d+1 \le j \le n)$$

and the restriction mappings $p_j\colon \mathfrak{g}^* \to {\mathfrak{k}_j}^*$ for $d \le j \le n-1$, where $p_d = p$. Let as before $\mathfrak{k}_j = \mathbb{R}X_j + \mathfrak{k}_{j-1}$ for $1 \le j \le n$ and put $\mathfrak{g}' = \mathfrak{k}_{n-1}, p' = p_{n-1}$ and $G' = \exp(\mathfrak{g}')$.

Suppose first that Ω is not saturated with respect to $\mathfrak{g}'$. Let $\mathfrak{g} = \mathbb{R}X' + \mathfrak{g}'$. Then, there exists in $\ker\pi$ an element $\eta = X' + \zeta$ with $\zeta \in \mathcal{U}(\mathfrak{g}')$. One can utilize η in order to eliminate from W the part which is outside of $\mathcal{U}(\mathfrak{g}')$. This means that we can suppose W in $\mathcal{U}(\mathfrak{g}')$. Since the restriction π' of π to G' is irreducible and $\Omega' = p'(\Omega)$ is the coadjoint orbit of G' associated to π', our objective derives immediately from the induction hypothesis.

Suppose next that Ω is saturated with respect to $\mathfrak{g}'$. Let $\mathfrak{g} \ne \mathfrak{k}+\mathfrak{b}$. Then, we are able to suppose that $\mathfrak{g}'$ contains the subalgebra $\mathfrak{k}+\mathfrak{b}$. As $\pi|_K$ has finite multiplicities, W belongs modulo $\ker\pi$ to $\mathcal{U}(\mathfrak{g}')$. Then, the induction hypothesis says that the denominator of the rational function $P_W(\ell)$ is a polynomial of the e-central element $P_\alpha(\ell), \alpha = X_{i_0} + c$, with a certain index $2 \le i_0 \le n-1$ and an element $c \in \mathcal{U}(\mathfrak{g}_{i_0-1})$, of Corwin-Greenleaf for G' which separates the one parameter family of G'-orbits in $\Omega' = p'(\Omega)$. Considering W multiplied by a certain polynomial of α if necessary, it suffices for us to study the case where the denominator of $P_W(\ell)$ is a polynomial of $P_\alpha(\ell)$ of degree 1.

Let j_0 be the smallest index j such that W belongs modulo $\ker\pi$ to $\mathcal{U}(\mathfrak{k}_j)$. When $j_0 = d$, W belongs to $\mathcal{U}(\mathfrak{k})^{\mathfrak{k}}$, namely it is an e-central element for $p(\Omega) \subset \mathfrak{k}^*$ and consequently the function $P_W(\ell)$ is polynomial from Proposition 4.7.

Suppose $j_0 \ge d+1$ and write again

$$W = v_sX_{j_0}^s + v_{s-1}X_{j_0}^{s-1} + \cdots + v_1X_{j_0} + v_0,\ v_i \in \mathcal{U}(\mathfrak{k}_{j_0-1})\ (0 \le i \le s)$$

with $v_s \notin \ker\pi$. Remark that v_s is $\mathfrak{k}$-invariant modulo $\ker\pi$. Let $\pi|_{G'} \simeq \int_{\mathbb{R}}^{\oplus} \pi'_t dt$ be the usual irreducible decomposition. As $\pi|_K$ has finite multiplicities, it is also the case for $\pi'_t|_K$, and W belongs to $\mathcal{U}(\mathfrak{g}')$ hence to $\mathcal{U}_{\pi'_t}(\mathfrak{g}')^{\mathfrak{k}}$ for generic $t \in \mathbb{R}$. Then the induction hypothesis says that $[\mathfrak{k}, v_s] \subset \ker\pi'_t$ for almost all $t \in \mathbb{R}$, what leads $[\mathfrak{k}, v_s] \subset \ker\pi$.

We now propose to show by induction on s that $P_W(\ell)$ is a polynomial function on Ω. If there exists in $\mathcal{U}(\mathfrak{k}_{j_0})^{\mathfrak{k}}$ a $\mathfrak{k}$-invariant element $\nu = aX_{j_0} + b$ which corresponds to a polynomial function, where a, b belong to $\mathcal{U}(\mathfrak{k}_{j_0-1})$ such that $a \notin \ker\pi$, it follows that $a, a^sW - v_s\nu^s \in \mathcal{U}(\mathfrak{k}_{j_0})^{\mathfrak{k}}$. As the degree of $a^sW - v_s\nu^s$ with respect to X_{j_0} is smaller than or equal to $s-1$, this implies that W multiplied by a certain power of a corresponds to a polynomial function. Thus, if $P_{W_1}(\ell)$ is the denominator of $P_W(\ell)$, W_1 appears as a factor γ of a power of a. That is to say, α is an irreducible factor of γ. In order to show that $P_W(\ell)$ is a polynomial function, it suffices for us to see the existence of such an element ν with a not divisible by α and such that the function $P_\nu(\ell)$ is polynomial.

Assume that $\mathfrak{k}_{j_0} = \mathfrak{k} + \mathfrak{g}_{j_r}$. If $r = 1$, there exists in $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ an element $A = uX_{d+1} + v$ with $u, v \in \mathcal{U}(\mathfrak{k})$. This means that $[X_{d+1}, W] \in \ker\pi$ and hence $[\mathfrak{k}_{d+1}, W] \subset \ker\pi$, which signifies from the induction hypothesis that $P_W(\ell)$ is a polynomial function, i.e. α is trivial. Thus, $2 \le r \le q$.

When $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$, we are led as usual to the case where $\mathfrak{g}_1 = \mathbb{R}Z$ is the center of $\mathfrak{g}$, $\mathfrak{g}_2 = \mathbb{R}Y + \mathfrak{g}_1$ and $\mathfrak{g} = \mathbb{R}X + \mathfrak{g}'$ with the centralizer $\mathfrak{g}'$ of Y in $\mathfrak{g}$. Here, $X \in \mathfrak{k}$ satisfies $[X, Y] = Z$. Moreover, it is possible to take $\mathfrak{b}$ in $\mathfrak{g}'$. Writing $\mathfrak{k} = \mathbb{R}X + \mathfrak{k}'$ with $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$. Let also $G' = \exp(\mathfrak{g}')$ and $K' = \exp(\mathfrak{k}')$. Recall the usual decomposition (8) into irreducibles. Then the restriction $\pi'_t|_{K'}$ has finite multiplicities for almost all $t \in \mathbb{R}$.

First assume that $W \in \mathcal{U}(\mathfrak{g}')$. From the induction hypothesis, $\pi'_t(W)a_{\ell'} = P'_W(\ell')a_{\ell'}$ for almost all $t \in \mathbb{R}$ for some functions $P'_W(\ell')$ such that $P'_W(\ell')_{|\Omega_t}$ is a polynomial function, where Ω_t denotes the orbit associated to the representation π'_t. From the K-invariance of W, it comes out that $P_W(\ell) = P'_W(\ell')$ for generic $\ell \in \Omega$. As the denominator of $P_W(\ell)$ is a polynomial function upon the variable $\ell(Y)$, it turns out that it is indeed a polynomial function from the property of K-invariance.

Let $W \notin \mathcal{U}(\mathfrak{g}')$, this means that the generic K-orbits in $p(\Omega)$ are non-saturated with respect to $\mathfrak{k}'$. Hence, there exists an e-central element of Corwin-Greenleaf $\gamma = aX + b \in \mathcal{U}_\pi(\mathfrak{k})^{\mathfrak{k}}, a, b \in \mathcal{U}(\mathfrak{k}')$, where $a \notin \ker\pi$ such that $[\mathfrak{k}, a] \subset \ker\pi$. Making use of γ to reduce the degree of W relative to X, we see by induction hypothesis that a^mW corresponds to a polynomial function for a certain positive integer m. Hence, α comes this time from an irreducible factor of a. If so, we can make use again of the same arguments of Case 2 of Lemma 5.2, with Y playing the role of $\widetilde{\kappa}$. At the level of the desintegration (8), we take the mark point π'_0 so that $\pi'_0|_{K'}$ has finite multiplicities. From the induction hypothesis applied to the triplet (G', K', π'_0), there exists an element ν' in $\mathcal{U}_{\pi'_0}(\mathfrak{k}_{j_0})^{\mathfrak{k}'}$ of degree 1 with respect to X_{j_0} whose coefficient is not divisible by α. Then, it suffices for us to extend ν' to a $\mathfrak{k}$-invariant element. Namely, substituting $-Y$ to t in the expression of $\exp(tX)\cdot\nu'$, we obtain our desired element ν.

Taking $\mathfrak{n} = \mathfrak{g}, \rho = \pi, \mathfrak{h} = \mathfrak{k}$, we can apply Lemma 5.2 to conclude that the function $P_W(\ell)$ is in fact polynomial. Because in this situation, after the construction of

ν, the function $P_\nu(\ell)$ is a polynomial function on the orbit Ω, from which results Proposition 5.3. ■

Proof of Theorem 5.1 We look at the surjectivity statement of the homomorphism

$$\Theta\colon \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \ni W \mapsto P_W(\ell) \in \mathbb{C}[\Omega]^K.$$

Take an arbitrary $q(\ell)$ in $\mathbb{C}[\Omega]^K$. Just as before, we are brought to the last eventuality:

$$\mathfrak{g} = \mathbb{R}X + \widetilde{\mathfrak{g}},\ X \in \mathfrak{k}$$

with an ideal $\widetilde{\mathfrak{g}}$ of codimension 1, for which the orbit Ω is saturated. Let $\widetilde{\kappa} \in \mathcal{U}(\widetilde{\mathfrak{g}})$ be the e-central element for $\widetilde{p}(\Omega)$ which we utilized above. Then, there exists a polynomial $F(y')$ of $y' = P_{\widetilde{\kappa}}(\ell)$, $\ell \in \Omega$, such that $F(y')q(\ell)$ is in the image of the homomorphism Θ by means of Proposition 4.4.

Suppose that the function $q(\ell)$ depends only on $x_1 = \ell(X_1), \dots, x_{j_0} = \ell(X_{j_0})$ and $\ell|_{\mathfrak{k}}$. Write $q(\ell) = r(\ell)x_{j_0}^v + s(\ell)$ with a certain integer $v \in \mathbb{N}$, where $r(\ell)$ does not depend on x_{j_0} and $s(\ell)$ is of degree smaller than or equal to $v-1$ with respect to x_{j_0}. Let α be a polynomial of $\widetilde{\kappa}$ such that $P_\alpha(\ell)$ is an irreducible factor of $F(y')$. In this situation the algebra $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ extends when we pass through $\mathfrak{k}_{j_0-1}$ to $\mathfrak{k}_{j_0}$ and as it is proved above, there exists in $\mathcal{U}_\pi(\mathfrak{k}_{j_0})^{\mathfrak{k}}$ an element

$$\sigma = aX_{j_0} + b \quad (a, b \in \mathcal{U}(\mathfrak{k}_{j_0-1})),$$

where a is not divisible by α modulo $\ker\pi$. Then,

$$(P_a(\ell))^v q(\ell) - r(\ell)P_{\sigma^v}(\ell)$$

belongs to the image of Θ by induction hypothesis on the degree v as well as $(P_a(\ell))^v q(\ell)$ because $r(\ell)$ does by induction hypothesis. Let

$$P_{W_1}(\ell) = F(y')q(\ell),\ P_{W_2}(\ell) = (P_a(\ell))^v q(\ell)$$

with certain $W_1, W_2 \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Hence,

$$F(y')(P_a(\ell))^v q(\ell) = (P_a(\ell))^v P_{W_1}(\ell) = F(y')P_{W_2}(\ell).$$

This implies $P_{a^vW_1}(\ell) = P_{F(\widetilde{\kappa})W_2}(\ell)$ and then $a^vW_1 \equiv F(\widetilde{\kappa})W_2$ modulo $\ker\pi$. Since a is not divisible modulo $\ker\pi$ by α, W_1 must be divided by α modulo $\ker\pi$ and we can eliminate α from $F(\widetilde{\kappa})$. In this way we conclude that $q(\ell)$ belongs to the image of homomorphism Θ and this achieves the proof of the theorem. ■

The case where $\mathfrak{g}$ is two-step nilpotent is treated in our article [3]. Remark more generally the:

Corollary 5.4. *Suppose that $\mathfrak{g}$ is three-step nilpotent. Then, the polynomial conjecture holds for $G = \exp\mathfrak{g}$.*

Proof. In this case, $[\mathfrak{g}, \mathfrak{g}]$ turns out to be an abelian ideal of $\mathfrak{g}$. This makes possible to choose a normal polarization at any linear form on $\mathfrak{g}$. ■

6. Case where K is abelian

We treat in this section the case where $\mathfrak{k}$ is an abelian subalgebra. First we have:

Lemma 6.1. *Assume that $K = \exp \mathfrak{k}$ is an abelian subgroup of G. Let $\pi \in \hat{G}$ realized as $\pi = ind_{B[\ell]}^G \chi_\ell$ by means of a Vergne polarization $\mathfrak{b}[\ell], B[\ell] = \exp(\mathfrak{b}[\ell])$, constructed at $\ell \in \Omega(\pi)$ from the flag of ideals (1). We define $a_\ell \in \mathcal{H}_\pi^{-\infty}$ by the formula*

$$\langle a_\ell, \varphi\rangle = \int_{K/(K\cap B[\ell])} \overline{\varphi(k)\chi_\ell(k)} d\dot{k},\ \varphi \in \mathcal{H}_\pi^\infty.$$

Let $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ such that $W \cdot a_\ell = P_W(\ell) a_\ell$ on a non-empty Zariski open set of $\Omega(\pi)$ with a certain function P_W. Then, $P_W(\ell)$ is a rational function.

Proof. The polarizations $\mathfrak{b}[\ell]$ and the subalgebras $\mathfrak{k} \cap \mathfrak{b}[\ell]$ vary rationally in $\ell \in \Omega(\pi)$. We can find a non-empty Zariski open set $\mathcal{U}$ of $\Omega(\pi)$ and a family of elements $\mathfrak{B} = \{X_1, \cdots, X_r\}$ of $\mathfrak{g}$ which constitutes a Malcev basis of $\mathfrak{g}$ modulo the polarization $\mathfrak{b}[\ell]$ at ℓ and such that $\mathfrak{B} \cap \mathfrak{k}$ is a Malcev basis of $\mathfrak{k}$ modulo $\mathfrak{k} \cap \mathfrak{b}[\ell]$ for $\ell \in \mathcal{U}$. Consider the maps $E_{\mathfrak{B}} \colon \mathbb{R}^r \to G$ defined by

$$E_{\mathfrak{B}}(t_1, \cdots, t_r) = \exp(t_1X_1) \cdots \exp(t_rX_r)$$

and $E_{\mathfrak{B}\cap K} \colon \mathbb{R}^m \to K$ defined by

$$E_{\mathfrak{B}\cap K}(t) = \exp(t_1X_{i_1}) \cdots \exp(t_mX_{i_m}),$$

where $\{X_{i_1}, \cdots, X_{i_m}\} := \mathfrak{k} \cap \mathfrak{B}$. For any $g \in G$ we can write

$$g = E_{\mathfrak{B}}(t(g, \ell)) \cdot b(g, \ell)$$

with $t(g, \ell) \in \mathbb{R}^r$ and $b(g, \ell) \in B[\ell]$. The maps $(g, \ell) \mapsto t(g, \ell)$ and $(g, \ell) \mapsto b(g, \ell)$ are then polynomial in g and rational in $\ell \in \Omega(\pi)$ and the mapping $G \to \mathbb{R}^r \times B[\ell] \colon g \mapsto (t(g, \ell), b(g, \ell))$ is bi-polynomial in g and bi-rational in ℓ.
Fix a Schwartz function $\psi \colon \mathbb{R}^r \to \mathbb{C}$. Then for any $\ell \in \mathcal{U}$ the function defined on G by

$$\psi_\ell(E_{\mathfrak{B}}(t) \cdot b_\ell) = \psi(t)\chi_\ell(b_\ell^{-1}),\ t \in \mathbb{R}^r, b_\ell \in B[\ell],$$

is a C^∞-vector of the representation $\pi_\ell = \mathrm{ind}_{B[\ell]}^G \chi_\ell$. If $V \in \mathcal{U}(\mathfrak{g})$, then there exist for $\alpha, \beta \in \mathbb{N}^r$, some rational functions $p_{\alpha,\beta}^V$ in $\ell \in \Omega(\pi)$ such that

$$d\pi_\ell(V)(\psi_\ell)(E_{\mathfrak{B}}(t)) = \sum_{\alpha,\beta} p_{\alpha,\beta}^V(\ell) t^\alpha \partial^\beta \psi(t),\ \ell \in \mathcal{U}, t \in \mathbb{R}^r, \psi \in \mathcal{S}(\mathbb{R}^r).$$

In this way, for $W \in \mathcal{U}(\mathfrak{g})$ and $\psi_\ell \in \mathcal{H}_{\pi_\ell}^\infty$ we see that

$$\langle W \cdot a_\ell, \psi_\ell\rangle = \sum_{\alpha,\beta} \overline{p_{\alpha,\beta}^{W^*}(\ell)} \int_{\mathbb{R}^m} t^\alpha \overline{(\partial^\beta \psi_\ell(E_{\mathfrak{B}\cap K}(t)))\chi_\ell(E_{\mathfrak{B}\cap K}(t))} dt.$$

Hence, for $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ of our lemma and for $\ell \in \Omega(\pi)$, we have that

$$\begin{aligned}\langle W \cdot a_\ell, \psi_\ell\rangle &= P_W(\ell) \int_{\mathbb{R}^m} \overline{\psi_\ell(E_{\mathfrak{B}\cap K}(t))\chi_\ell(E_{\mathfrak{B}\cap K}(t))} dt \\ &= \sum_{\alpha,\beta} \overline{p_{\alpha,\beta}^{W^*}(\ell)} \int_{\mathbb{R}^m} t^\alpha \overline{(\partial^\beta \psi_\ell(E_{\mathfrak{B}\cap K}(t)))\chi_\ell(E_{\mathfrak{B}\cap K}(t))} dt,\end{aligned}$$

where W^* denotes the image of W by the anti-automorphism of $\mathcal{U}(\mathfrak{g})$. This tells us that the function P_W is rational. ■

We next prove

Lemma 6.2. *Assume that $\mathfrak{k}$ is abelian and let $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Then $W \in \ker \pi$ if and only if $W\cdot a_\ell = \pi(W)a_\ell = 0$ for generic $\ell \in \Omega(\pi)$.*

Proof. When $W \in \ker \pi$, it is evident that $W\cdot a_\ell = 0$ because $a_\ell \in \mathcal{H}_\pi^{-\infty}$. Suppose now that $W\cdot a_\ell = 0$ for generic $\ell \in \Omega = \Omega(\pi)$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Dividing if necessary by $\mathfrak{z}\cap\ker\pi$, we are led as usual to the case where $\dim\mathfrak{z} = 1$ and $\mathfrak{z}$ is not contained in $\ker\pi$. Thus, in the Jordan-Hölder sequence (1), $\mathfrak{g}_1 = \mathfrak{z}$ and $\mathfrak{g}_2 = \mathbb{R}Y + \mathfrak{z}$. Put $\mathfrak{g} = \mathbb{R}X + \mathfrak{g}'$, where $\mathfrak{g}'$ denotes the centralizer of Y in $\mathfrak{g}$, $G' = \exp(\mathfrak{g}'), \pi' = \mathrm{ind}_{B[\ell]}^{G'}\chi_\ell$ and denote by ℓ' the restriction of $\ell \in \Omega$ to $\mathfrak{g}'$. We make use of vectors $\varphi \in \mathcal{H}_\pi^\infty$ having the form

$$\varphi(g) = \phi(t)\psi(g'),\ g = \exp(tX)g' \in G,\ t \in \mathbb{R},\ g' \in G'$$

with $\phi \in \mathcal{S}(\mathbb{R}), \psi \in \mathcal{H}_{\pi'}^\infty$. If $\mathfrak{k} \subset \mathfrak{g}'$, write

$$W = \sum_{j=0}^m X^j w_j,\ w_j \in \mathcal{U}(\mathfrak{g}')\ \ (0 \le j \le m).$$

We have
$$\langle \pi(W)a_\ell, \varphi\rangle = \sum_{j=0}^m (-1)^j \frac{d^j\overline{\phi}}{dt^j}(0)\langle \pi'(w_j)a_{\ell'}, \psi\rangle. \tag{11}$$

Hence, $\pi(W)a_\ell = 0$ entails $\pi'(w_j)a_{\ell'} = 0$ for $0 \le j \le m$. Remark that the $\mathfrak{k}$-invariance of W induces the same property for w_m. Then, the induction hypothesis means $w_m \in \ker\pi'$. As this happens at almost every point $\ell \in \Omega$, we conclude that $X^m w_m \in \ker\pi$. Repeating this argument, we see that W belongs to $\ker\pi$.

If $\mathfrak{k} \not\subset \mathfrak{g}'$, we choose X in $\mathfrak{k}$ so that $X\cdot a_\ell = i\ell(X)a_\ell$. Let $\mathfrak{k} = \mathbb{R}X + \mathfrak{k}', \mathfrak{k}' = \mathfrak{k}\cap\mathfrak{g}'$ and $K' = \exp(\mathfrak{k}')$. Let us write

$$W = \sum_{j=0}^m w_j X^j,\ w_j \in \mathcal{U}(\mathfrak{g}')\ (0 \le j \le m).$$

Since W is $\mathfrak{k}$-invariant, all $w_j, 0 \le j \le m$, are $\mathfrak{k}$-invariant too. Then, we have

$$\langle \pi(W)a_\ell, \varphi\rangle = \sum_{j=0}^m (i\ell(X))^j (\int_{\mathbb{R}} \overline{\phi(t)e^{it\ell(X)}}dt)\langle \pi'(w_j)a_{\ell'}, \psi\rangle,$$

where
$$\langle a_{\ell'}, \psi\rangle = \int_{K'/(K'\cap B[\ell])} \overline{\psi(k')\chi_\ell(k')}d\dot{k}'.$$

Hence, $\pi(W)a_\ell = 0$ entails $\pi'(w_j)a_{\ell'} = 0$ for $0 \le j \le m$. Then, the induction hypothesis says that $w_j, 0 \le j \le m$, belong to $\ker\pi'$ at almost every point $\ell \in \Omega$ and finally W belongs to $\ker\pi$. ■

We now prove our main result of this section.

Theorem 6.3. *Let $G = \exp\mathfrak{g}$ be a nilpotent Lie group and $K = \exp\mathfrak{k}$ an abelian subgroup of G. Let $\pi \in \hat{G}$ realized as $\pi = ind_{B[\ell]}^{G}\chi_\ell$ by means of the Vergne polarization $\mathfrak{b}[\ell], B[\ell] = \exp(\mathfrak{b}[\ell])$, constructed from the flag (1) at the point $\ell \in \Omega = \Omega(\pi)$. Let us define $a_\ell \in (\mathcal{H}_\pi^{-\infty})^{K,\chi_\ell}$ by the formula*

$$\langle a_\ell, \varphi\rangle = \int_{K/(K\cap B[\ell])} \overline{\varphi(k)\chi_\ell(k)}d\dot{k},\ \varphi \in \mathcal{H}_\pi^\infty.$$

Let $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ satisfying $W\cdot a_\ell = P_W(\ell)a_\ell$ almost everywhere on Ω with a rational function $P_W(\ell)$. Then, $P_W(\ell)$ is a polynomial function. In particular, Conjecture 3.3 *holds when $\mathfrak{k}$ is abelian.*

Proof. Let us proceed as usual by induction on $n = \dim G$. When $n = 1$ or $G = K$, there is nothing to do. Suppose $n > 1$ and $G \neq K$. Besides, we are led to the case where the center $\mathfrak{z}$ of $\mathfrak{g}$ is one-dimensional and is not contained in $\ker\pi$. Thus, $\mathfrak{g}_1 = \mathfrak{z}$ in the flag (1). We put there $\mathfrak{g}_2 = \mathbb{R}Y + \mathfrak{z}$ and let $\mathfrak{g}'$ be the centralizer of Y in $\mathfrak{g}$. Let $G' = \exp(\mathfrak{g}')$ and $p'\colon \mathfrak{g}^* \to (\mathfrak{g}')^*$ be the restriction mapping. Suppose first that $\mathfrak{k} \subset \mathfrak{g}'$. Because $W\cdot a_\ell = P_W(\ell)a_\ell$ almost everywhere on Ω, W belongs modulo $\ker\pi$ to $\mathcal{U}(\mathfrak{g}')$. Then, following the proof of Proposition 4.2, the induction hypothesis says that the denominator of the rational function $P_W(\ell)$ is a polynomial of the e-central element $\ell(Y)$ of Corwin-Greenleaf for G' which separates the G'-orbits of one parameter family in $\Omega' = p'(\Omega)$.

Suppose next that $\mathfrak{k} \not\subset \mathfrak{g}'$. Let $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}', K' = \exp(\mathfrak{k}')$ and $\mathfrak{k} = \mathbb{R}X + \mathfrak{k}'$. Then, for $\varphi \in \mathcal{H}_\pi^\infty$ having the form

$$\varphi(\exp(tX)g') = \phi(t)\psi(g')\ (t \in \mathbb{R}, g' \in G')$$

with $\phi \in \mathcal{S}(\mathbb{R})$ and $\psi \in \mathcal{H}_{\pi'}^\infty$, where $\pi' = \mathrm{ind}_{B[\ell]}^{G'}\chi_\ell$, we have

$$\langle a_\ell, \varphi\rangle = \int_{\mathbb{R}} \overline{\phi(t)e^{it\ell(X)}}dt \int_{K'/(K'\cap B[\ell])} \overline{\psi(g')\chi_\ell(g')}d\dot{g'}.$$

Now, if we write $$W = \sum_{j=0}^{r'} W_j X^j,\ W_j \in \mathcal{U}(\mathfrak{g}')\ \ (0 \le j \le r'),$$

we see immediately that $[\mathfrak{k}, W_j] \subset \ker\pi (0 \le j \le r')$, because $\mathfrak{k}$ is abelian. Hence,

$$P_W(\ell) = \sum_{j=0}^{r'} (i\ell(X))^j P_{W_j}(\ell).$$

From the $\mathfrak{k}$-invariance, W_j acts on the functions $\psi \in \mathcal{H}_{\pi'}^\infty$ and the induction hypothesis says that the denominator of $P_W(\ell)$ is a polynomial function in the variable $\ell(Y)$. Now we choose another ideal $\widetilde{\mathfrak{g}}$ of codimension 1 in $\mathfrak{g}$ and put $\widetilde{G} = \exp(\widetilde{\mathfrak{g}})$.

Suppose first that Ω is non-saturated with respect to $\widetilde{\mathfrak{g}}$. Let $\mathfrak{g} = \mathbb{R}\widetilde{X} + \widetilde{\mathfrak{g}}$. Then, there exists in $\ker\pi$ an element $\eta = \widetilde{X} + \zeta$ with $\zeta \in \mathcal{U}(\widetilde{\mathfrak{g}})$. Using this η, we can eliminate from W the part being outside of $\mathcal{U}(\widetilde{\mathfrak{g}})$. That is to say, we can suppose that W belongs to $\mathcal{U}(\widetilde{\mathfrak{g}})$. The restriction $\widetilde{\pi}$ of π to $\widetilde{G} = \exp(\widetilde{\mathfrak{g}})$ being irreducible, our objective follows directly from the induction hypothesis if $\mathfrak{k} \subset \widetilde{\mathfrak{g}}$.

Suppose that $\mathfrak{k} \not\subset \widetilde{\mathfrak{g}}$. Let $\widetilde{\mathfrak{k}} = \mathfrak{k} \cap \widetilde{\mathfrak{g}}, \mathfrak{k} = \mathbb{R}\widetilde{X} + \widetilde{\mathfrak{k}}$ and $\widetilde{K} = \exp(\widetilde{\mathfrak{k}})$. Let $\widetilde{\ell} = \ell|_{\widetilde{\mathfrak{g}}}, \mathfrak{b}[\widetilde{\ell}] = \mathfrak{b}[\ell] \cap \widetilde{\mathfrak{g}}, B[\widetilde{\ell}] = \exp(\mathfrak{b}[\widetilde{\ell}])$ for $\ell \in \Omega$ and $\widetilde{\Omega} = \Omega|_{\widetilde{\mathfrak{g}}}$. We remark that $\mathfrak{b}[\widetilde{\ell}]$ is the Vergne polarization at $\widetilde{\ell}$ constructed by means of the flag of $\widetilde{\mathfrak{g}}$ obtained from the flag (1). We identify $\mathcal{H}_{\pi}^{-\infty}$ with $\mathcal{H}_{\widetilde{\pi}}^{-\infty}$. Taking a Malcev basis in $\mathfrak{g}$ relative to $\mathfrak{b}[\ell]$ which contains a Malcev basis in $\mathfrak{k}$ relative to $\mathfrak{k}\cap\mathfrak{b}[\ell]$, we identify the space $\mathcal{H}_\pi$ of π with $\mathbb{R}^m, m=\dim(\mathfrak{g}/\mathfrak{b}[\ell])$. Since

$$K/(K\cap B[\ell]) \simeq KB[\ell]/B[\ell] = \exp(\mathbb{R}\widetilde{X})\widetilde{K}B[\ell]/B[\ell],$$

there are two eventualities: either

$$K/(K\cap B[\ell]) \simeq \widetilde{K}/(\widetilde{K}\cap B[\widetilde{\ell}])$$

or for almost all $\ell\in\Omega$,

$$K/(K\cap B[\ell]) \simeq \exp(\mathbb{R}\widetilde{X})\widetilde{K}B[\ell]/B[\ell] \simeq \exp(\mathbb{R}\widetilde{X})\times\widetilde{K}/(\widetilde{K}\cap B[\ell])$$

In the first case, the generalized vector a_ℓ of π is identified to the generalized vector $a_{\widetilde{\ell}}$ of $\widetilde{\pi}$. In the second case, for $\varphi\in\mathcal{H}_\pi^\infty$ verifying

$$\varphi(\exp(t\widetilde{X})\widetilde{g}) = \phi(t)\psi(\widetilde{g}),\ t\in\mathbb{R}, \widetilde{g}\in\widetilde{G},$$

with $\phi\in C_c(\mathbb{R}), \psi\in\mathcal{H}_{\widetilde{\pi}}^\infty$, we have by the $\mathfrak{k}$-invariance of W that

$$\langle W\cdot a_\ell,\varphi\rangle = (\int_{\mathbb{R}}\overline{\phi(t)e^{it\ell(\widetilde{X})}}dt)\langle W\cdot a_{\widetilde{\ell}},\psi\rangle = P_W(\widetilde{\ell})(\int_{\mathbb{R}}\overline{\phi(t)e^{it\ell(\widetilde{X})}}dt)\langle a_{\widetilde{\ell}},\psi\rangle = P_W(\widetilde{\ell})\langle a_\ell,\varphi\rangle.$$

In any way, the induction hypothesis tells us that $P_W(\ell)=P_W(\widetilde{\ell})$ is a polynomial function.

Suppose finally that Ω is saturated with respect to $\widetilde{\mathfrak{g}}$. We can assume that $\mathfrak{b}[\ell]$ is in $\widetilde{\mathfrak{g}}$. Put $\widetilde{\pi}=\mathrm{ind}_{B[\ell]}^{\widetilde{G}}\chi_\ell$. We first consider the case where $\mathfrak{k}\subset\widetilde{\mathfrak{g}}$. Because $W\cdot a_\ell = P_W(\ell)a_\ell$ for generic $\ell\in\Omega$, it follows that $W\in\mathcal{U}(\widetilde{\mathfrak{g}})$ modulo $\ker\pi$. Indeed, write $W=\sum_{j=0}^m X^j w_j, w_j\in\mathcal{U}(\mathfrak{g}')$ $(0\le j\le m)$. Coming back to formula (11) and choosing ϕ such that

$$\frac{d^j\overline{\phi}}{dt^j}(0)=0 \text{ for } j=0,...,m-1 \text{ and } \frac{d^m\overline{\phi}}{dt^m}(0)=1,$$

we arrive to the assertion $w_m\in\ker\pi$ by Lemma 6.2. From the disintegration

$$\pi|_{\widetilde{G}} \simeq \int_{\mathbb{R}}^{\oplus}\widetilde{\pi}_t dt,\ \widetilde{\pi}_t\in\widehat{\widetilde{G}}, t\in\mathbb{R},$$

here in fact $\widetilde{\pi}_0=\widetilde{\pi}, \widetilde{\pi}_t=\exp(t\widetilde{X})\cdot\widetilde{\pi}$, we deduce by induction hypothesis that the denominator of $P_W(\ell)$ is a polynomial of $\ell(\widetilde{\kappa})$, $\widetilde{\kappa}$ being an e-central element on the projection of Ω in $(\widetilde{\mathfrak{g}})^*$. But this means that $P_W(\ell)$ is a polynomial function, since $\widetilde{\kappa}$ is not a polynomial of Y. We next consider the case where $\mathfrak{k}\not\subset\widetilde{\mathfrak{g}}$ and write $\mathfrak{k}=\mathbb{R}\widetilde{X}+\widetilde{\mathfrak{k}}$ with $\widetilde{\mathfrak{k}}=\mathfrak{k}\cap\widetilde{\mathfrak{g}}$. Put $\widetilde{K}=\exp(\widetilde{\mathfrak{k}})$ as before. For $\varphi\in\mathcal{H}_\pi^\infty$ having the form

$$\varphi(\exp(t\widetilde{X})\widetilde{g}) = \phi(t)\psi(\widetilde{g}),\ t\in\mathbb{R}, \widetilde{g}\in\widetilde{G},$$

with $\phi\in\mathcal{S}(\mathbb{R}), \psi\in\mathcal{H}_{\widetilde{\pi}}^\infty$, we have

$$\langle a_\ell,\varphi\rangle = \int_{\mathbb{R}}\overline{\phi(t)e^{it\ell(\widetilde{X})}}dt\int_{\widetilde{K}/(\widetilde{K}\cap B[\ell])}\overline{\psi(\widetilde{g})\chi_\ell(\widetilde{g})}d\dot{\widetilde{g}}.$$

Therefore, if $W \in \mathcal{U}(\widetilde{\mathfrak{g}})$, we conclude as before that $P_W(\ell)$ is a polynomial function. If $W \notin \mathcal{U}(\widetilde{\mathfrak{g}})$, we write

$$W = \sum_{j=0}^{v} W_j \widetilde{X}^j,\ W_j \in \mathcal{U}(\widetilde{\mathfrak{g}}), 0 \le j \le v.$$

Since $\mathfrak{k}$ is abelian, we see for every $Y' \in \mathfrak{k}$ that

$$[Y', W] = \sum_{j=0}^{v} [Y', W_j] \widetilde{X}^j,$$

from which follows immediately that $W_j, 0 \le j \le v$, are all $\mathfrak{k}$-invariant. Then the above arguments prove that the denominators of $P_{W_j}(\ell)$ are polynomials functions of the variable $\ell(\widetilde{\kappa})$ and consequently $P_{W_j}(\ell)$ are polynomial functions. Hence,

$$P_W(\ell) = \sum_{j=0}^{v} (i\ell(\widetilde{X}))^j P_{W_j}(\ell)$$

is a polynomial function.

Now, let us return to the situation of our theorem. When $\pi|_K$ has finite multiplicities, the mapping $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \ni W \mapsto P_W$ supplies by passing to the quotient an injective homomorphism of algebras from $D_\pi(G)^K$ into the algebra $\mathbb{C}\,[\Omega(\pi)]^K$ of the K-invariant polynomial functions on the orbit $\Omega(\pi)$ (cf. [3]). It remains for us to show the surjectivity of the homomorphism

$$\Theta \colon \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \ni W \mapsto P_W(\ell) \in \mathbb{C}[\Omega]^K.$$

We first prove the following result.

Lemma 6.4. *Assume that $K = \exp \mathfrak{k}$ is abelian. Let $q(x) \in \mathbb{C}[\Omega]^K$. Suppose that there exists $u(x) \in \mathbb{C}[\Omega]$ belonging to the image of Θ such that $u(x)q(x)$ belongs to the image of Θ. Then, $q(x)$ itself belongs to the image of Θ.*

Proof. We keep the situation and the notations of the previous lemma. When $\mathfrak{k} \subset \mathfrak{g}'$, our condition requires that $q(x)$ and $u(x)$ do not contain the variable $x_r = \ell(X)$. Let

$$\pi|_{G'} \simeq \int_{\mathbb{R}}^{\oplus} \pi'_t dt,\ \Omega|_{\mathfrak{g}'} = \sqcup_{t \in \mathbb{R}} \omega_t,\ t = \ell(Y)$$

the usual decompositions. For almost all $t \in \mathbb{R}$, there exists by induction hypothesis $W_t \in \mathcal{U}_{\pi'_t}(\mathfrak{g}')^{\mathfrak{k}}$ satisfying $P_{W_t}(\ell') = q(x)|_{x_1 = t}$, where $x = (x_1, ...)$. Actually, x_1 corresponds to the variable corresponding to the vector Y. As the Vergne polarization depends rationally on ℓ, W_t depends rationally on t. In fact, we take a reference point $\ell_0 \in \Omega$ such that $\ell_0(Y) = 0$ and consider the section

$$\Omega/G' \simeq \mathbb{R} \ni t \mapsto \ell_t = \exp(-tX) \cdot \ell_0 \in \Omega.$$

Let $\{Y_1(t), \ldots, Y_{m_1}(t)\}$ $(m_1 = \dim \mathfrak{b}[\ell_t])$, be a basis of $\mathfrak{b}[\ell_t]$ and $\{X_1, \ldots, X_{m_2}\}$, $m_2 = \dim \mathfrak{k}/(\mathfrak{k} \cap \mathfrak{b}[\ell_t])$, a Malcev basis of $\mathfrak{k}$ relative to $\mathfrak{k} \cap \mathfrak{b}[\ell_t]$ for generic $t \in \mathbb{R}$. For $I = (i_1, \ldots, i_{m_2})$ and $J = (j_1, \ldots, j_{m_1})$, set

$$X^I = X_1^{i_1} \cdots X_{m_2}^{i_{m_2}},\ Y^J = (Y_1(t))^{j_1} \cdots (Y_{m_1}(t))^{j_{m_1}}.$$

Then, W_t is written as $W_t = \sum_{I,J} c_{I,J}(t)X^I Y^J$ with certain rational functions $c_{I,J}(t)$ of $t \in \mathbb{R}$. Multiplying W_t by its denominator and substituting there t for $-iY$, we see that there exists a polynomial $s(x_1)$ of $x_1 = \ell(Y)$ so that the product $s(x_1)q(x)$ belongs to the image of Θ. When $\mathfrak{k} \not\subset \mathfrak{g}'$, we write

$$q(x) = \sum_{j=0}^{m} q_j(x_1, \ldots, x_{r-1})x_r^j, \; u(x) = \sum_{j=0}^{m'} u_j(x_1, \ldots, x_{r-1})x_r^j$$

with certain polynomials $q_j(0 \leq j \leq m), u_j(0 \leq j \leq m')$ of $(x_1, \ldots, x_{r-1})$ which are all $\mathfrak{k}$-invariant. Hence, it follows that all the $u_j(0 \leq j \leq m')$ belong to the image of Θ. As

$$u(x)q(x) = \sum_{j=0}^{m+m'} p_j(x_1, \ldots, x_{r-1})x_r^j, \; p_j = \sum_{0 \leq k \leq j} u_k q_{j-k},$$

belongs to the image of Θ, $p_{m+m'} = u_{m'}q_m$ too. For generic $t \in \mathbb{R}$, there exists by induction hypothesis $w_m(t) \in \mathcal{U}_{\pi'_t}(\mathfrak{g}')^{\mathfrak{k}'}$ verifying $P_{w_m(t)}(\ell') = q_m(\ell')$ for almost all $\ell' \in \omega_t$. Since $q(x)$ is K-invariant,

$$P_{W_{\exp(tX)\cdot y'}}(\exp(tX)\cdot\ell) = q(\exp(tX)\cdot x) = q(x)$$

for all $t \in \mathbb{R}$. While, the definition of the function $P_{W'}(\ell'), W' \in \mathcal{U}_{\pi'_t}(\mathfrak{g}')^{\mathfrak{k}'}$ gives

$$P_{\exp(sX)\cdot w_m(t)}(\exp(sX)\cdot\ell') = P_{w_m(t)}(\ell'), \; \forall\, s \in \mathbb{R}.$$

Hence, $w_m(\exp(sX) \cdot \ell') = \exp(sX) \cdot (w_m(t))$ and

$$\begin{aligned}
&\langle P_{w_m(\exp(sX)\cdot\ell')}(\exp(sX) \cdot \ell')a_{\exp(sX)\cdot\ell'}, \varphi\rangle = \langle(\exp(sX) \cdot (w_m(\ell'))a_{\exp(sX)\cdot\ell'}, \varphi\rangle \\
&= \int_{K'/(K'\cap B[\exp(sX)\cdot\ell'])} \overline{\exp(sX) \cdot \overline{{}^t(w_m(\ell'))}\varphi(k')\chi_{\exp(sX)\cdot\ell'}(k')}d\dot{k'} \\
&= \int_{K'/(K'\cap B[\ell'])} \overline{\exp(sX) \cdot \overline{{}^t(w_m(\ell'))}\varphi(k')\chi_{\ell'}(k')}d\dot{k'} \\
&= \langle(\exp(sX) \cdot (w_m(\ell')))a_{\ell'}, \varphi\rangle. \text{ Namely,}
\end{aligned}$$

$$\begin{aligned}
&\langle P_{w_m(\exp(sX)\cdot\ell')}(\exp(sX) \cdot \ell')a_{\ell'}, \varphi\rangle = \langle(\exp(sX) \cdot (w_m(\ell')))a_{\ell'}, \varphi\rangle \\
&= \langle P_{\exp(sX)\cdot(w_m(\ell'))}(\ell')a_{\ell'}, \varphi\rangle
\end{aligned}$$

for all $\varphi \in \mathcal{H}^\infty_{\pi'}$. In this way,

$$P_{\exp(sX)\cdot(w_m(\ell'))}(\ell') = P_{w_m(\exp(sX)\cdot\ell')}(\exp(sX) \cdot \ell') = q(x).$$

Deriving this equality with respect to s, we find by the preceding lemma $[X, w_m(t)] \in \ker \pi'_t$ for almost all $t = \ell(Y)$, that is to say $w_m(t) \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Moreover, we find as before that $w_m(t)$ depends rationally on $t = \ell(Y) \in \mathbb{R}$. Summing up, there exists a polynomial $s_m(x_1)$ of $x_1 = \ell(Y)$ so that the product $s_m(x_1)q_m(x_1, \ldots, x_{r-1})$ belongs to the image of Θ. Next, $p_{m+m'-1} = u_{m'-1}q_m + u_{m'}q_{m-1}$ belongs to the image of Θ and $s_m(x_1)u_{m'}q_{m-1}$ too. We can repeat the same argument to see that there exists a polynomial $s_{m-1}(x_1)$ of x_1 so that $s_{m-1}q_{m-1}$ is in the image of Θ. We are able to continue these process in order to conclude that there exists a polynomial $s(x_1)$ of $x_1 = \ell(Y)$ so that the product $s(x_1)q(x)$ belongs to the image of Θ.

Now, we take another ideal $\widetilde{\mathfrak{g}} \neq \mathfrak{g}'$ of codimension 1 in $\mathfrak{g}$. Let $\widetilde{G} = \exp(\widetilde{\mathfrak{g}})$. For $\ell \in \mathfrak{g}^*$, put $\widetilde{\ell} = \ell|_{\widetilde{\mathfrak{g}}} \in (\widetilde{\mathfrak{g}})^*$. Suppose first that Ω is non-saturated with respect to $\widetilde{\mathfrak{g}}$. Let $\widetilde{\Omega}$ be the projection image of Ω in $(\widetilde{\mathfrak{g}})^*$, which is the $\widetilde{G}$-orbit corresponding to the irreducible representation $\widetilde{\pi} = \pi|_{\widetilde{G}}$ of $\widetilde{G}$. For $\ell \in \Omega$, put $\widetilde{\ell} = \ell|_{\widetilde{\mathfrak{g}}} \in \widetilde{\Omega}$. If $\mathfrak{k} \subset \widetilde{\mathfrak{g}}$, it suffices for us to apply the induction hypothesis to the triple $(\widetilde{G}, \widetilde{\pi}, K)$.

If $\mathfrak{k} \not\subset \widetilde{\mathfrak{g}}$, let $\widetilde{\mathfrak{k}} = \mathfrak{k} \cap \widetilde{\mathfrak{g}}$. The induction hypothesis says that there exists $W \in \mathcal{U}_{\widetilde{\pi}}(\widetilde{\mathfrak{g}})^{\widetilde{\mathfrak{k}}}$ verifying $P_W(\widetilde{\ell}) = q(x)$ almost everywhere on $\widetilde{\Omega}$. Since $q(x)$ is K-invariant, then so is the case for W, namely $W \in \mathcal{U}_{\pi}(\mathfrak{g})^{\mathfrak{k}}$. Suppose next that Ω is saturated with respect to $\widetilde{\mathfrak{g}}$. We can suppose that the Vergne polarization $\mathfrak{b}[\ell]$ is contained in $\widetilde{\mathfrak{g}}$ and $q(x)$ is rewritten [4] as

$$q(x) = q(x_1, \ldots, x_{i-1}, y', x_{i+1}, \ldots, x_{r-1})$$

with
$$y' = x_i + \varphi(x_1, \ldots, x_{i-1}) = P_{Y'}(\ell)$$

for a certain
$$Y' = -\sqrt{-1}X_{s_i} + \sum c_{k_1,\ldots,k_{i-1}} X_{s_1}^{k_1} \cdots X_{s_{i-1}}^{k_{i-1}}$$

belonging to $\mathcal{U}_{\pi}(\mathfrak{g})^{\widetilde{\mathfrak{g}}}$ with $c_{k_1,\ldots,k_{i-1}} \in \mathbb{C}$. Then there exists a polynomial $r(y')$ of y' so that $r(y')q(x)$ belongs to the image of Θ. Summing up, there exist two elements W_1, W_2 in $\mathcal{U}_{\pi}(\mathfrak{g})^{\mathfrak{k}} \cap E$ such that

$$P_{W_1}(\ell) = s(x_1)q(x), \ P_{W_2}(\ell) = r(y')q(x)$$

for generic $\ell \in \Omega$. Taking into account the previous lemma,

$$r(Y')W_1 \equiv s(-iY)W_2 \text{ modulo } \ker \pi. \tag{12}$$

As $\pi|_E$ is faithful and $r(Y')$ is not divisible by $s(-iY)$, W_1 must be divisible by $s(-iY)$ because Y is central in $\mathfrak{g}'$. To see this, it suffices to employ an induction on the degree d_1 of W_1. If $d_1 = 1$, nothing to prove. If $d_1 > 1$, we develop the identity (12) according to a vector $X_j \not\equiv Y$ modulo $\ker \pi$ and use the induction hypothesis on the appearing factors which are of inferior degrees.

Write now $W_1 = s(-iY)W_0$ and $s(-iY) = \sum_{m \geq 0} a_m Y^m$. We easily show that $W_0 \in \mathcal{U}_{\pi}(\mathfrak{g})^{\mathfrak{k}}$. Indeed, the result is immediate if $\mathfrak{k} \subset \mathfrak{g}'$. Otherwise we can suppose $X \in \mathfrak{k}$, and hence

$$\ker \pi \ni \sum_{m \geq 1} m a_m Y^{m-1} Z W_0 + \sum_{m \geq 0} a_m Y^m [X, W_0].$$

Since $\pi(Y)$ acts as a multiplication operator, the result follows. These observations imply that $q(x)$ itself belongs to the image of Θ, which achieves the proof of Lemma 6.4. ∎

Assume now that $\pi|_K$ has finite multiplicities and show the surjectivity of Θ. Consider an arbitrary element $q(x)$ in $\mathbb{C}[\Omega]^K$. This time instead of $\widetilde{\mathfrak{g}}$ we utilise $\mathfrak{k}_{n-1}$ appearing in the sequence (5). It is an ideal of codimension 1 in $\mathfrak{g}$ containing $\mathfrak{k}$. Then, by Proposition 4.4 there exists a polynomial $s(y')$ of $y' = P_{Y'}(\ell)$ so that $s(y')q(x)$ belongs to the image of Θ. Therefore, it results from the above lemma that $q(x)$ itself belongs to the image of Θ, which also achieves the proof of Theorem 6.3. ∎

7. Examples

Here, we give some examples.

Example 7.1. Let $\mathfrak{g} = \langle X, P_1, P_2, Q, Y, Z\rangle_{\mathbb{R}}$ with the non-zero brackets

$$[X, P_1] = P_2, [X, Y] = Z, [P_1, Q] = Y, [P_2, Q] = Z.$$

Let $\pi \in \hat{G}$ such that $Z^* \in \Omega(\pi)$ and $\mathfrak{k} = \langle P_1, Y, Z\rangle_{\mathbb{R}}$. Then, a direct computation shows that an element $\ell \in \Omega(\pi)$ is written as

$$\ell = Z^* + x_1Y^* + x_2Q^* + x_3(P_2)^* + x_1x_3(P_1)^* + x_4X^*,\ x_j \in \mathbb{R}(1 \le j \le 4).$$

From this, $P_1 - x_1P_2 - x_3Y \in \mathfrak{g}(\ell)$. On the other hand, we have for $\varphi \in \mathcal{H}_\pi^\infty$

$$\langle a_\ell, \varphi\rangle = \int_{\mathbb{R}} \overline{\varphi(\exp(tP_1))} e^{itx_1x_3} dt.$$

So,

$$\begin{aligned}\langle P_2 a_\ell, \varphi\rangle &= \frac{d}{ds}\int_{\mathbb{R}} \overline{\varphi(\exp(sP_2)\exp(tP_1))} e^{itx_1x_3} dt\Big|_{s=0}\\ &= \frac{d}{ds}\int_{\mathbb{R}} \overline{\varphi\left(\exp\left(\left((t+\frac{s}{x_1}\right)P_1\right)\exp\left(sP_2 - \frac{s}{x_1}P_1\right)\right)} e^{itx_1x_3} dt\Big|_{s=0}\\ &= \frac{d}{ds}e^{-is(x_3+1)}\int_{\mathbb{R}} \overline{\varphi\left(\exp\left(\left(t+\frac{s}{x_1}\right)P_1\right)\right)} e^{itx_1x_3} dt\Big|_{s=0}\\ &= \frac{d}{ds}\int_{\mathbb{R}} \overline{\varphi\left(\exp\left(\left(t+\frac{s}{x_1}\right)P_1\right)\right)}\Big|_{s=0}\\ &= \frac{d}{ds}\int_{\mathbb{R}} \overline{\varphi(\exp(tP_1))} e^{i\left(t-\frac{s}{x_1}\right)x_1x_3} dt\Big|_{s=0}\\ &= \left(\frac{d}{ds}e^{-isx_3}\right)\Big|_{s=0}\int_{\mathbb{R}} \overline{\varphi(\exp(tP_1))} e^{itx_1} dt = -ix_3\langle a_\ell, \varphi\rangle.\end{aligned}$$

for $\ell \in \Omega(\pi)$ satisfying $x_1 = \ell(Y) \neq 0$. Put $\Phi(x,t) = \varphi(\exp(xX)\exp(tP_2))$ for $(x,t) \in \mathbb{R}^2$ and compute:

$$\begin{aligned}(P_1\Phi)(x,t) &= \frac{d}{ds}\varphi(\exp(-sP_1)\exp(xX)\exp(tP_2))|_{s=0}\\ &= \frac{d}{ds}\varphi(\exp(xX)\exp(-sP_1 + sxP_2)\exp(tP_2))|_{s=0}\\ &= \frac{d}{ds}\varphi(\exp(xX)\exp((t+sx)P_2)\exp(-sP_1 + sx_1P_2\\ &\quad + sx_3Y)\exp(-sx_1P_2 - sx_3Y))|_{s=0}\\ &= \frac{d}{ds}\varphi(\exp(xX)\exp((t + sx - sx_1)P_2))|_{s=0} = (x - x_1)\frac{\partial\Phi}{\partial t}(x,t),\\ (P_2\Phi)(x,t) &= \frac{d}{ds}\varphi(\exp(-sP_2)\exp(xX)\exp(tP_2))|_{s=0} = -\frac{\partial\Phi}{\partial t}(x,t),\\ (Y\Phi)(x,t) &= \frac{d}{ds}\varphi(\exp(-sY)\exp(xX)\exp(tP_2))|_{s=0}\\ &= \frac{d}{ds}\varphi(\exp(xX)\exp(-sY + xsZ)\exp(tP_2))|_{s=0} = i(x_1 - x)\Phi(x,t).\end{aligned}$$

Taking these observations into account, we check that $P_1 + iYP_2$ belongs to $\ker\pi$.

Example 7.2. Let $\mathfrak{g} = \langle X, P_1, P_2, P_3, Q_1, Q_2, Y, Z\rangle_{\mathbb{R}}$ with the non-zero brackets

$$[X, P_1] = P_2,\ [X, P_2] = P_3,\ [P_1, Q_1] = Q_2,\ [X, Q_2] = [P_2, Q_1] = Y,\ [X, Y] = [P_3, Q_1] = Z.$$

Let $\pi \in \hat{G}$ such that $Z^* \in \Omega(\pi)$ and $\mathfrak{k} = \langle P_1, Y, Z\rangle_{\mathbb{R}}$. Then, a direct computation shows that an element $\ell \in \Omega(\pi)$ can be written as

$$\ell = Z^* + x_1 Y^* + \frac{x_1^2}{2} Q_2^* + x_2 Q_1^* + x_3 P_3^* + x_1 x_3 P_2^* + \frac{x_1^2 x_3}{2} P_1^* + x_4 X^*,$$

where $x_j \in \mathbb{R}$, $1 \le j \le 4$. Hence $P_1 - \frac{x_1^2}{2} P_3 - x_1 x_3 Y$ and $P_2 - x_1 P_3 - x_3 Y$ belong to $\mathfrak{g}(\ell)$. On the other hand we have

$$\langle a_\ell, \varphi\rangle = \int_{\mathbb{R}} \overline{\varphi(\exp(tP_1))} e^{\frac{i x_1^2 x_3}{2} t} dt$$

for $\varphi \in \mathcal{H}_\pi^\infty$. So,

$$\begin{aligned}
\langle P_3 a_\ell, \varphi\rangle &= \frac{d}{ds} \int_{\mathbb{R}} \overline{\varphi(\exp(sP_3)\exp(tP_1))} e^{\frac{i x_1^2 x_3}{2} t} dt \Big|_{s=0} \\
&= \frac{d}{ds} \int_{\mathbb{R}} \overline{\varphi\left(\exp(tP_1)\exp\left(\frac{2s}{x_1^2} P_1\right)\exp\left(sP_3 - \frac{2s}{x_1^2} P_1\right)\right)} e^{\frac{i x_1^2 x_3}{2} t} dt \Big|_{s=0} \\
&= \frac{d}{ds} \int_{\mathbb{R}} \overline{\varphi\left(\exp\left(\left(t + \frac{2s}{x_1^2}\right) P_1\right)\right)} e^{\frac{i x_1^2 x_3}{2} t} e^{2isx_3} dt \Big|_{s=0} \\
&= \frac{d}{ds} \int_{\mathbb{R}} \overline{\varphi(\exp(tP_1))} e^{\frac{i x_1^2 x_3}{2}\left(t - \frac{2s}{x_1^2}\right)} e^{2isx_3} dt \Big|_{s=0} \\
&= \left(\frac{d}{ds} e^{isx_3}\right) \Big|_{s=0} \int_{\mathbb{R}} \overline{\varphi(\exp(tP_1))} e^{\frac{i x_1^2 x_3}{2} t} dt = i x_3 \langle a_\ell, \varphi\rangle,
\end{aligned}$$

for $\ell \in \Omega(\pi)$ satisfying $x_1 = \ell(Y) \neq 0$. Put $\Phi(x, t) = \varphi(\exp(xX)\exp(tP_3))$ for $(x, t) \in \mathbb{R}^2$ and compute:

$$\begin{aligned}
(P_1 \Phi)(x, t) &= \frac{d}{ds} \varphi(\exp(-sP_1)\exp(xX)\exp(tP_3)) \Big|_{s=0} \\
&= \frac{d}{ds} \varphi\left(\exp(xX)\exp\left(-sP_1 + sxP_2 - \frac{sx^2}{2} P_3\right)\exp(tP_3)\right) \Big|_{s=0} \\
&= \frac{d}{ds} \varphi\left(\exp(xX)\exp\left(\left(t - \frac{sx^2}{2}\right) P_3\right)\exp(sxP_2 - sP_1)\right) \Big|_{s=0} \\
&= \frac{d}{ds} \varphi\left(\exp(xX)\exp\left(\left(t - \frac{sx^2}{2} + s\left(xx_1 - \frac{x_1^2}{2}\right)\right) P_3\right) \times\right. \\
&\qquad \left. \times \exp\left(sxP_2 - sP_1 - s\left(xx_1 - \frac{x_1^2}{2}\right) P_3\right)\right) \Big|_{s=0} \\
&= \frac{d}{ds} \varphi\left(\exp(xX)\exp\left(\left(t - \frac{sx^2}{2} + s\left(xx_1 - \frac{x_1^2}{2}\right)\right) P_3\right)\right) \Big|_{s=0} \\
&= \left(xx_1 - \frac{x_1^2}{2} - \frac{x^2}{2}\right) \frac{\partial \Phi}{\partial t}, \\
(P_3 \Phi)(x, t) &= \frac{d}{ds} \varphi(\exp(-sP_3)\exp(xX)\exp(tP_3)) \Big|_{s=0} = -\frac{\partial \Phi}{\partial t},
\end{aligned}$$

$$
\begin{aligned}
(Y\Phi)(x,t) &= \frac{d}{ds}\varphi(\exp(-sY)\exp(xX)\exp(tP_3))\Big|_{s=0}\\
&= \frac{d}{ds}\varphi(\exp(xX)\exp(tP_3)\exp(-sY+sxZ))\Big|_{s=0}\\
&= \left(\frac{d}{ds}e^{-i(-sx_1+sx)}\right)\Big|_{s=0}\Phi(x,t) = i(x_1-x)\Phi(x,t).
\end{aligned}
$$

Taking these observations into account, we check that $P_1 - \frac{Y^2}{2}P_3$ belongs to $\ker\pi$. Finally,

$$
\begin{aligned}
(P_2\Phi)(x,t) &= \frac{d}{ds}\varphi(\exp(-sP_2)\exp(xX)\exp(tP_3))\Big|_{s=0}\\
&= \frac{d}{ds}\varphi(\exp(xX)\exp(-sP_2+sxP_3)\exp(tP_3))\Big|_{s=0}\\
&= \frac{d}{ds}\varphi(\exp(xX)\exp((t+sx+-sx_1)P_3)\exp(-sP_2+sx_1P_3))\Big|_{s=0}\\
&= \frac{d}{ds}\Phi(x,t+s(x-x_1))\Big|_{s=0} = (x-x_1)\frac{\partial\Phi}{\partial t}.
\end{aligned}
$$

Accordingly, $P_2 + iYP_3$ belongs to $\ker\pi$.

Now we give some illustrative examples on the polynomial conjecture.

Example 7.3. Let $\mathfrak{g} = \langle X,T,U,V,Y,Z\rangle_{\mathbb{R}}$ with the non-zero brackets $[X,Y]=Z$, $[X,T]=U$, $[T,V]=Y$, $[U,V]=Z$. Let $G=\exp\mathfrak{g}$, $f=Z^*\in\mathfrak{g}^*$, $\Omega = G{\cdot}f$, $\mathfrak{k} = \langle T,Y,Z\rangle_{\mathbb{R}}$ and $K=\exp\mathfrak{k}$. Then, $\mathfrak{k}$ is abelian and $\mathfrak{g}(f) = \mathbb{R}T\oplus\mathbb{R}Z$. It follows that $\dim\Omega = 4$ and $\mathfrak{b} = \mathbb{R}V+\mathfrak{k}$ is a polarization of $\mathfrak{g}$ at f. We construct the irreducible unitary representation π of G associated to Ω as $\pi = \mathrm{ind}_B^G\chi_f$ where $B=\exp\mathfrak{b}$. Simple calculations show

$$\Omega = \{\ell = \alpha X^* + \beta\lambda T^* + \beta U^* + \gamma V^* + \lambda Y^* + Z^*;\ \alpha,\beta,\gamma,\lambda\in\mathbb{R}\}.$$

For $\ell\in\Omega$, we have $\mathfrak{g}(\ell) = \mathbb{R}(T-\ell(Y)U-\ell(U)Y)+\mathbb{R}Z$.

While $\mathfrak{b}[\ell] = \langle T-\ell(Y)U, V, Y, Z\rangle_{\mathbb{R}}$ is a polarization of $\mathfrak{g}$ at ℓ and $\mathfrak{b}[\ell|_{\mathfrak{k}}] = \mathfrak{k}$ is that of $\mathfrak{k}$ at the restriction $\ell|_{\mathfrak{k}}\in\mathfrak{k}^*$. Let $p\colon\mathfrak{g}^*\to\mathfrak{k}^*$ be the restriction mapping. We immediately see that

$$p(\Omega) = \mathbb{R}T^* + \mathbb{R}Y^* + Z^*$$

except the subset $(\mathbb{R}\setminus\{0\})T^*+Z^*$. Furthermore, for $\delta = aT^*+bY^*+Z^*\in p(\Omega)$ ($a\in\mathbb{R}$, $b\in\mathbb{R}\setminus\{0\}$), $p^{-1}(\delta)$ is a K-orbit. In this way, the restriction $\pi|_K$ is without multiplicities. In more details, for $\delta = aT^*+bY^*+Z^*\in\mathfrak{k}^*$ ($a,b\in\mathbb{R}$), let χ_δ be the unitary character of K defined by

$$\chi_\delta(\exp(tT+yY+zZ)) = e^{i(at+by+z)}.$$

Then,
$$\pi|_K \simeq \int_{\mathbb{R}^2}^{\oplus}\chi_\delta\, da db.$$

Besides, we define the Penney distribution a_ℓ at $\ell\in\Omega$ verifying $\ell(Y)\neq 0$ by the formula

$$a_\ell(\varphi) = \int_{\mathbb{R}}\overline{\varphi(\exp(tT))e^{i\ell(T)t}}dt$$

for $\varphi \in \mathcal{H}_\pi^\infty$. Now we consider a Jordan-Hölder sequence

$$\mathfrak{g}_1 = \mathbb{R}Z \subset \mathfrak{g}_2 = \mathbb{R}Z + \mathbb{R}Y \subset \mathfrak{g}_3 = \langle V, Y, Z\rangle_{\mathbb{R}}$$
$$\subset \ \mathfrak{g}_4 = \langle U, V, Y, Z\rangle_{\mathbb{R}} \subset \mathfrak{g}_5 = \langle Z, Y, V, U, T\rangle_{\mathbb{R}} \subset \mathfrak{g}_6 = \mathfrak{g}.$$

of $\mathfrak{g}$ and the associated sequence of subalgebras

$$\mathfrak{k}_1 = \mathfrak{g}_1 \subset \mathfrak{k}_2 = \mathfrak{g}_2 \subset \mathfrak{k}_3 = \mathfrak{k} \subset \mathfrak{k}_4 = \mathfrak{k} + \mathbb{R}V \subset \mathfrak{k}_5 = \mathfrak{k}_4 + \mathbb{R}[X, T] \subset \mathfrak{k}_6 = \mathfrak{g}.$$

When we pass from $\mathfrak{k}_4$ to $\mathfrak{k}_5$, the element $\sigma = YU - ZT$ appears in the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ and satisfies the relation $\pi(\sigma) = 0$. Moreover, U is $\mathfrak{k}$-invariant and

$$\pi(Z)\cdot a_\ell = ia_\ell, \pi(Y)\cdot a_\ell = i\ell(Y)a_\ell, \pi(T)\cdot a_\ell = i\ell(T)a_\ell$$

for all $\ell \in \Omega$ verifying $\ell(Y) \neq 0$. It results from this that $\pi(U)\cdot a_\ell = i\ell(U)a_\ell$ and the algebra $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega]^K$.

Example 7.4. Let $\mathfrak{g} = \langle X, T, U, V, Y, Z\rangle_{\mathbb{R}}$ with non-zero brackets

$$[X, Y] = Z,\ [X, T] = U,\ [U, T] = Z,\ [T, V] = Y,\ [U, V] = Z.$$

Let $G = \exp\mathfrak{g}, f = Z^* \in \mathfrak{g}^*, \Omega = G\cdot f, \mathfrak{k} = \langle T, Y, Z\rangle_{\mathbb{R}}$ and $K = \exp\mathfrak{k}$. Then, $\mathfrak{k}$ is abelian and $\mathfrak{g}(f) = \mathbb{R}(T - V) \oplus \mathbb{R}Z$. It follows that $\dim\Omega = 4$ and $\mathfrak{b} = \mathbb{R}V + \mathfrak{k}$ is a polarization of $\mathfrak{g}$ at f. We construct the irreducible unitary representation π of G associated to Ω as $\pi = \mathrm{ind}_B^G\chi_f$ where $B = \exp\mathfrak{b}$. Simple calculations show

$$\Omega = \{\ell = \alpha X^* + (\beta\lambda + \gamma)T^* + \beta U^* + \gamma V^* + \lambda Y^* + Z^*; \alpha, \beta, \gamma, \lambda \in \mathbb{R}\}.$$

Let $p\colon \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction mapping. We immediately see that

$$p(\Omega) = \mathbb{R}T^* + \mathbb{R}Y^* + Z^*.$$

Furthermore, for $\delta = aT^* + bY^* + Z^* \in p(\Omega)$ $(a, b \in \mathbb{R})$, $p^{-1}(\delta)$ ia a K-orbit. In this way, the restriction $\pi|_K$ is without multiplicities.

Now we consider a Jordan-Hölder sequence

$$\mathfrak{g}_1 = \mathbb{R}Z \subset \mathfrak{g}_2 = \mathbb{R}Z + \mathbb{R}Y \subset \mathfrak{g}_3 = \langle V, Y, Z\rangle_{\mathbb{R}}$$
$$\subset \ \mathfrak{g}_4 = \langle U, V, Y, Z\rangle_{\mathbb{R}} \subset \mathfrak{g}_5 = \langle Z, Y, V, U, T\rangle_{\mathbb{R}} \subset \mathfrak{g}_6 = \mathfrak{g}$$

of $\mathfrak{g}$ and the associated sequence of subalgebras

$$\mathfrak{k}_1 = \mathfrak{g}_1 \subset \mathfrak{k}_2 = \mathfrak{g}_2 \subset \mathfrak{k}_3 = \mathfrak{k} \subset \mathfrak{k}_4 = \mathfrak{k} + \mathbb{R}V \subset \mathfrak{k}_5 = \mathfrak{k}_4 + \mathbb{R}U \subset \mathfrak{k}_6 = \mathfrak{g}.$$

From $\mathfrak{k}_4$ to $\mathfrak{k}_5$, the element $\sigma = YU - ZT + VZ$ appears in the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. The Hilbert space $\mathcal{H}_\pi$ of π is identified to $L^2(\mathbb{R}^2)$ by the mapping $\mathcal{H}_\pi \ni \varphi \mapsto \Phi \in L^2(\mathbb{R}^2)$ given by

$$\Phi(x, s) = \varphi(\exp(xX)\exp(sU))\ (x, s \in \mathbb{R}).$$

In this situation, simple calculations show

$$\pi(Z) = i, (\pi(Y)\Phi)(x,s) = -ix\Phi(x,s),$$
$$(\pi(V)\Phi)(x,s) = -is\Phi(x,s), (\pi(T)\Phi)(x,s) = \big(-is + x\frac{\partial}{\partial s}\big)\Phi(x,s)$$
$$(\pi(U)\Phi)(x,s) = -\frac{\partial\Phi}{\partial s}(x,s).$$

Hence, we confirm $\pi(\sigma) = 0$ and the $\mathfrak{k}$-invariant element $w = YU + VZ$ is nothing but the element ZT modulo $\ker\pi$.

Example 7.5. Let finally $\mathfrak{g} = \langle X, S, T, R, U, V, Y, Z\rangle_{\mathbb{R}}$ with non-zero brackets

$$[X,Y] = Z,\ [X,U] = V,\ [X,V] = Y,\ [T,U] = V$$
$$[X,T] = S,\ [X,S] = R,\ [S,U] = Y,\ [R,U] = Z.$$

Then, $[\mathfrak{g},\mathfrak{g}] = \langle S, R, V, Y, Z\rangle_{\mathbb{R}}$ is abelian. Let

$$G = \exp\mathfrak{g},\ f = Z^* \in \mathfrak{g}^*,\ \Omega = G{\cdot}f,\ \mathfrak{k} = \langle T, U, V, Y, Z\rangle_{\mathbb{R}}$$

and $K = \exp\mathfrak{k}$. We have $\mathfrak{g}(f) = \langle T, S, V, Z\rangle_{\mathbb{R}}$, $\dim\Omega = 4$ and $\mathfrak{b} = \mathbb{R}T + [\mathfrak{g},\mathfrak{g}]$ is a polarization of $\mathfrak{g}$ at f. We realize the irreducible unitary representation π of G associated to Ω as $\pi = \mathrm{ind}_B^G\chi_f$ where $B = \exp\mathfrak{b}$. Next, take a Jordan-Hölder sequence

$$\mathfrak{g}_1 = \mathbb{R}Z \subset \mathfrak{g}_2 = \mathbb{R}Z + \mathbb{R}Y \subset \mathfrak{g}_3 = \langle Z, Y, V\rangle_{\mathbb{R}}$$
$$\subset \mathfrak{g}_4 = \langle Z, Y, V, R\rangle_{\mathbb{R}} \subset \mathfrak{g}_5 = [\mathfrak{g},\mathfrak{g}]$$
$$\subset \mathfrak{g}_6 = [\mathfrak{g},\mathfrak{g}] + \mathbb{R}U \subset \mathfrak{g}_7 = [\mathfrak{g},\mathfrak{g}] + \mathbb{R}U + \mathbb{R}T \subset \mathfrak{g}_8 = \mathfrak{g}$$

of $\mathfrak{g}$ and the corresponding sequence of subalgebras

$$\mathfrak{k}_1 = \mathfrak{g}_1 \subset \mathfrak{k}_2 = \mathfrak{g}_2 \subset \mathfrak{k}_3 = \mathfrak{g}_3 \subset \mathfrak{k}_4 = \mathfrak{k}_3 + \mathbb{R}U \subset \mathfrak{k}_5 = \mathfrak{k}$$
$$\subset\ \mathfrak{k}_6 = \mathfrak{k} + \mathbb{R}R \subset \mathfrak{k}_7 = \mathfrak{k} + \mathbb{R}R + \mathbb{R}S \subset \mathfrak{k}_8 = \mathfrak{g}.$$

For
$$\ell = aT^* + bU^* + cV^* + dY^* + Z^* \in \mathfrak{k}^*\ \ (a,b,c,d \in \mathbb{R})$$

satisfying $c \neq 0$, $\mathfrak{b}[\ell] = \langle U, V, Y, Z\rangle_{\mathbb{R}}$ is the Vergne polarization relative to the Jordan-Hölder sequence $\{\mathfrak{k}_j\}_{j=1}^5$ of $\mathfrak{k}$. Now the Hilbert space $\mathcal{H}_\pi$ of π is identified to $L^2(\mathbb{R}^2)$ by the mapping $\mathcal{H}_\pi \ni \varphi \mapsto \Phi \in L^2(\mathbb{R}^2)$ given by

$$\Phi(a,b) = \varphi(\exp(aX)\exp(bU))\ (a,b \in \mathbb{R}).$$

So, simple calculations show

$$\pi(Z) = i, (\pi(Y)\Phi)(a,b) = -ia\Phi(a,b),$$
$$(\pi(V)\Phi)(a,b) = \frac{ia^2}{2}\Phi(a,b), (\pi(R)\Phi)(a,b) = ib\Phi(a,b)$$
$$(\pi(S)\Phi)(a,b) = -iab\Phi(a,b), (\pi(U)\Phi)(a,b) = -\big(\frac{\partial}{\partial b} + \frac{ia^3}{6}\big)\Phi(a,b),$$
$$(\pi(T)\Phi)(a,b) = \frac{ia^2b}{2}\Phi(a,b), (\pi(x)\Phi)(a,b) = -\frac{\partial\Phi}{\partial a}(a,b).$$

From this, $\ker\pi$ is spanned by

$$Z-i, 2ZV-Y^2, S+iYR, T+iVR.$$

On the other hand, our algebra $\mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$ is generated modulo $\ker\pi$ by $\{YT-VS,\ V,\ Y,\ Z\}$. But,

$$\begin{aligned} YT-VS &\equiv YT-\frac{Y^2}{2i}S \equiv \frac{Y}{2i}(2iT-YS) \equiv \frac{Y}{2i}(2iT+iY^2R) \\ &\equiv \frac{Y}{2}(2T+Y^2R) \equiv Y(T+iVR) \equiv 0 \end{aligned}$$

modulo $\ker\pi$. Other computations show for $a,b,c,d\in\mathbb{R}$

$$\Omega = \{\ell = aX^* + \frac{cd^2}{2}T^* + bU^* + cdS^* + cR^* + \frac{d^2}{2}V^* + dY^* + Z^*.$$

Here also the restriction $\pi|_K$ turns out to be without multiplicities. For $\ell\in\Omega$ verifying $\ell(V)\neq 0$, the Penney distribution a_ℓ is defined by the formula

$$a_\ell(\varphi) = \int_{\mathbb{R}} \overline{\varphi(\exp(\lambda U))\chi_\ell(\exp(\lambda U))}d\lambda\ (\varphi\in\mathcal{H}_\pi^\infty).$$

Lastly, we immediately see

$$\pi(Z)a_\ell = ia_\ell, \pi(Y)a_\ell = i\ell(Y)a_\ell, \pi(V)a_\ell = i\ell(V)a_\ell.$$

In this way, we confirm once again that our algebra $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega]^K$.

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Ali Baklouti
Faculté des Sciences de Sfax
Dép. de Mathématiques
Route de Soukra
BP 1171, Sfax 3038
Tunisia
ali.baklouti@fss.usf.tn

Hidenori Fujiwara
Fac. de Science et Technologie
Université de Kinki
Iizuka 820-8555
Japan
fujiwara6913@yahoo.co.jp

Jean Ludwig
Institut Elie Cartan
Université de Lorraine
3 rue Augustin Fresnel
57000 Metz
France
jean.ludwig@univ-lorraine.fr

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