Shintani Functions for the Holomorphic Discrete Series Representation of $GSp_4(\mathbb{R})$

Kohta Gejima

Communicated by T. Kobayashi

Abstract. Let π be the holomorphic discrete series representation of $GSp_4(\mathbb{R})$ and η the discrete series representation of $(GL_2 \times_{GL_1} GL_2)(\mathbb{R})$. We prove the uniqueness and an explicit formula of the Shintani functions for (π, η). As their application, we evaluate a local zeta integral of Murase-Sugano type, which turns out to be a quotient of the L-factors associated with π and η.

Mathematics Subject Classification: Primary 11F70, secondary 11F46, 22E50.

Key Words: Shintani functions, automorphic L-functions, zeta integrals.

1. Introduction

Murase-Sugano [11] introduced a global zeta integral for the reductive symmetric pair (O_n, O_{n-1}) of the orthogonal groups over an algebraic number field and, under certain assumptions, proved that the global zeta integral can be decomposed into an Euler product of local zeta integrals, which contain local Shintani functions as integrands. Moreover they proved that the unramified local zeta integral represents the standard L-factor of O_n by an unramified computation. On the other hand, we do not know what the local zeta integrals at the bad places represent.

In this paper, we consider Shintani functions for the real reductive symmetric pair $(GSp_4(\mathbb{R}), (GL_2 \times_{GL_1} GL_2)(\mathbb{R}))$ and prove the uniqueness and an explicit formula of the Shintani functions for the holomorphic discrete series representation of $GSp_4(\mathbb{R})$. Moreover we formulate an archimedean local zeta integral of Murase-Sugano type for $(GSp_4(\mathbb{R}), (GL_2 \times_{GL_1} GL_2)(\mathbb{R}))$ and, as an application of our explicit formula, prove that the local zeta integral represents the local L-factor associated with the holomorphic discrete series representation of $GSp_4(\mathbb{R})$.

Let $(G, G_0) = (GSp_4(\mathbb{R}), (GL_2 \times_{GL_1} GL_2)(\mathbb{R}))$. These groups are defined in Subsection 2.1. In order to explain our main result, we first recall the definition of Shintani functions for (G, G_0). Let $(\pi, \mathcal{H}_\pi)$ and (η, V_η) be admissible representations of G and G_0, respectively. We set

$$C_\eta^\infty(G_0 \backslash G) := \{f : G \xrightarrow{C^\infty} V_\eta^\vee \mid f(g_0 x) = \eta^\vee(g_0) f(x), \ \forall (g_0, x) \in G_0 \times G\},$$

where $(\eta^\vee, V_\eta^\vee)$ is the contragredient representation of η. Then G acts on $C_\eta^\infty(G_0 \backslash G)$

by right translation. We consider an intertwining space

$$\mathcal{I}_{\eta,\pi} := \mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C_\eta^\infty(G_0 \backslash G)).$$

Then the space $\mathcal{S}(\eta,\pi)$ of Shintani functions of type (η,π) is defined by

$$\mathcal{S}(\eta,\pi) := \mathrm{Image}(\mathcal{I}_{\eta,\pi} \otimes \mathcal{H}_\pi \to C_\eta^\infty(G_0 \backslash G)).$$

Let K be the maximal compact subgroup of G and τ a K-type of π. For an embedding $\imath \in \mathrm{Hom}_K(\tau, \pi)$, we define a $\mathbf{C}$-linear map $\imath^* : \mathcal{I}_{\eta,\pi} \to \mathrm{Hom}_K(\tau, C_\eta^\infty(G_0 \backslash G))$ by the pullback via $\imath$. Fix an embedding $\imath \in \mathrm{Hom}_K(\tau, \pi)$. Then we define the space $S(\eta,\pi,\tau)$ of Shintani functions of type (η,π,τ) by

$$\mathcal{S}(\eta,\pi,\tau) := \mathrm{Image}(\imath^* : \mathcal{I}_{\eta,\pi} \to \mathrm{Hom}_K(\tau, C_\eta^\infty(G_0 \backslash G))).$$

By virtue of the generalized Cartan decomposition $G = G_0 A K$, where A is a split torus of G, in order to give an explicit formula of Shintani functions of type (η,π,τ), it is sufficient to consider their values on A. Our main result is as follows. See Section 4 for more details. Note that Theorem 1.1 is related to a result of Sun-Zhu [13] for the symmetric pair $(SO_{3,2}(\mathbb{R}), SO_{2,2}(\mathbb{R}))$.

Theorem 1.1.   *Let π be the holomorphic discrete series representation of G with Blattner parameter (λ_1, λ_2) and τ the minimal K-type of π. Let $\eta = \eta_1 \boxtimes \eta_2$ be an irreducible admissible representation of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$.*

   (1) *We have $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} \leq 1$. If at least one of η_1 and η_2 is not a holomorphic discrete series representation of $GL_2(\mathbb{R})$, then $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} = 0$.*

   (2) *Assume that both of η_1 and η_2 are holomorphic discrete series representations of $GL_2(\mathbb{R})$. Then the restriction of the Shintani function of type (η,π,τ) to the split torus A of G can be expressed in terms of the hyperbolic functions.*

This paper is organized as follows. In Section 2, we recall the representation theory of $GSp_4(\mathbb{R})$ and $GL_2(\mathbb{R})$. In Section 3, we recall the definition of Shintani functions and derive a system of difference-differential equations which determines the Shintani functions for the holomorphic discrete series representation of $GSp_4(\mathbb{R})$. In Section 4, we prove the uniqueness and an explicit formula of the Shintani functions for the holomorphic discrete series representation by using the system of difference-differential equations. In Section 5, we formulate an archimedean local zeta integral of Murase-Sugano type following Tsuzuki [14], and evaluate it by using our explicit formula. As a result, it follows that the local zeta integral represents the local spin L-factor associated with the holomorphic discrete series representation of $GSp_4(\mathbb{R})$.

We mention several studies related to Shintani functions for the reductive symmetric pair $(GSp_4(\mathbb{R}), (GL_2 \times_{GL_1} GL_2)(\mathbb{R}))$. Moriyama [9] gave explicit formulas of Shintani functions for the semisimple symmetric pair $(Sp_4(\mathbb{R}), SL_2(\mathbb{R}) \times SL_2(\mathbb{R}))$ with respect to the large discrete series representation and the Jacobi induction of $Sp_4(\mathbb{R})$. Note that he also treated Shintani functions for another semisimple symmetric pair on $Sp_4(\mathbb{R})$ in [10]. Kato-Murase-Sugano [5] gave an explicit formula of the unramified (Whittaker-) Shintani functions for the symmetric pair (SO_n, SO_{n-1}) of the split special orthogonal groups over the nonarchimedean local field F of characteristic zero or an odd prime $p > 2$. In particular, their explicit formula for $n = 5$ implies

an explicit formula of the unramified Shintani functions for the symmetric pair $(GSp_4, GL_2 \times_{GL_1} GL_2)$ over F through the accidental isomorphism $SO_5 \simeq PGSp_4$. Later, in [2],[3], the author extended their explicit formula for $(GSp_4, GL_2 \times_{GL_1} GL_2)$ to the nonarchimedean local field of arbitrary characteristic and proved that an unramified local zeta integral of Murase-Sugano type represents the unramified spin L-factor of GSp_4.

2.   Preliminaries

For a real Lie algebra $\mathfrak{l}$, let $\mathfrak{l}_\mathbb{C}$ denote the complexification of $\mathfrak{l}$. For each positive integer n, let I_n denote the identity matrix of degree n. We denote by i the imaginary unit, as usual.

2.1. Lie groups and Lie algebras
Let G be the real symplectic similitude group $GSp_4(\mathbb{R})$, which is defined by

$$GSp_4(\mathbb{R}) = \left\{ g \in GL_4(\mathbb{R}) \,\middle|\, {}^t g \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} g = \nu(g) \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}, {}^\exists \nu(g) \in \mathbb{R}^\times \right\}.$$

Let $G' = Sp_4(\mathbb{R})$ be the subgroup consisting of $g \in G$ with $\nu(g) = 1$, and

$$\mathfrak{g}' = \mathfrak{sp}_4(\mathbb{R}) = \left\{ X \in M_4(\mathbb{R}) \mid JX + {}^t X J = 0 \right\}$$

the Lie algebra of G'. Then $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{R} C_4$ is the Lie algebra of G, where $C_n := I_n$. Let $\theta : G \to G, g \mapsto {}^t g^{-1}$ be a Cartan involution of G. Then $K := \{g \in G \mid \theta(g) = g\}$ is the maximal compact subgroup of G and $K' := G' \cap K$ is the maximal compact subgroup of G', which is isomorphic to $U(2)$ via $K' \to U(2) = \{g \in GL_2(\mathbb{C}) \mid {}^t\overline{g}g = I_2\}$,

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB.$$

The group K contains K' with index two. More precisely, we have $K = K' \rtimes \langle \epsilon \rangle$, where $\epsilon := \mathrm{diag}(I_2, -I_2)$. The differential of θ is denoted again by θ. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces of θ, respectively. Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $\mathfrak{k}$ is the Lie algebra of K.

Let σ be an involution of G given by

$$g \mapsto \mathrm{diag}(1, -1, 1, -1) \cdot g \cdot \mathrm{diag}(1, -1, 1, -1)^{-1}.$$

Then the fixed point subgroup of σ of G is given by

$$G_0 := \left\{ \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix} \,\middle|\, g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in GL_2(\mathbb{R}), \ \det(g_1) = \det(g_2) \right\}.$$

Setting $K_0 := G_0 \cap K$, we have

$$K_0 = \left\{ \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \end{array} \right) \;\middle|\; g_i = \left(\begin{array}{cc} a_i & b_i \\ c_i & d_i \end{array} \right) \in O(2), \; \det(g_1) = \det(g_2) \right\}.$$

Then $K_0 \simeq (SO(2) \times SO(2)) \rtimes \langle \epsilon \rangle$ is the maximal compact subgroup of G_0. We set

$$r(\theta_1, \theta_2) := \left(\begin{array}{cc|cc} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ \hline -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{array} \right) \in K_0, \quad \forall \theta_1, \theta_2 \in \mathbb{R}.$$

The differential of σ is denoted again by σ. Let $\mathfrak{g}_0$ and $\mathfrak{q}$ be the $+1$ and -1 eigenspaces of σ, respectively. Then we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{q}$. Setting $\mathfrak{k}_0 := \mathfrak{g}_0 \cap \mathfrak{k}$ and $\mathfrak{p}_0 := \mathfrak{g}_0 \cap \mathfrak{p}$, we have a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$. Note that $\mathfrak{k}_0$ is the Lie algebra of K_0.

Let $\mathfrak{h} \subset \mathfrak{k}' := \mathfrak{g}' \cap \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}'$ given by $\mathfrak{h} = \mathbb{R}T_1 \oplus \mathbb{R}T_2$, where

$$T_1 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 0 \\ \hline -1 & & & \\ & 0 & & \end{array} \right), \quad T_2 = \left(\begin{array}{cc|cc} & & 0 & \\ & & & 1 \\ \hline 0 & & & \\ & -1 & & \end{array} \right).$$

For a linear form $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$, we identify α with $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ if $\alpha(T_j) = i\alpha_j$ for each $j = 1, 2$. Then the root system $\Sigma = \Sigma(\mathfrak{g}_{\mathbb{C}}', \mathfrak{h}_{\mathbb{C}})$ of $(\mathfrak{g}_{\mathbb{C}}', \mathfrak{h}_{\mathbb{C}})$ is given by

$$\Sigma = \{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\}.$$

Let Σ_c denote the set of compact roots in Σ. Then $\Sigma_c = \{\pm(1,-1)\}$. For each root $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha'$ denote the root space for α. We fix a positive root system $\Sigma^+ = \{(2,0), (0,2), (1,1), (1,-1)\}$ and a compact positive root system $\Sigma_c^+ = \{(1,-1)\}$. We set

$$X_{(2,0)} := \left(\begin{array}{cc|cc} 1 & & i & \\ & 0 & & \\ i & & -1 & \\ \hline & & & 0 \end{array} \right), \quad X_{(1,1)} := \left(\begin{array}{cc|cc} 1 & & i & \\ 1 & & i & \\ \hline i & & -1 & \\ i & & -1 & \end{array} \right),$$

$$X_{(0,2)} := \left(\begin{array}{cc|cc} 0 & & & \\ \hline & 1 & & i \\ & 0 & & \\ & i & & -1 \end{array} \right), \quad X_{(1,-1)} := \left(\begin{array}{cc|cc} 1 & & -i & \\ -1 & & -i & \\ \hline i & & 1 & \\ i & & -1 & \end{array} \right),$$

and $X_{-\alpha} := {}^t\overline{X_\alpha}$ for $\alpha \in \Sigma^+$. Then we have $\mathfrak{g}_{\mathbb{C}}' = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha'$, where $\mathfrak{g}_\alpha' = \mathbb{C}X_\alpha$. We set $\mathfrak{p}_{\mathbb{C}}^+ := \mathbb{C}X_{(2,0)} \oplus \mathbb{C}X_{(1,1)} \oplus \mathbb{C}X_{(0,2)}$ and $\mathfrak{p}_{\mathbb{C}}^- := \mathbb{C}X_{(-2,0)} \oplus \mathbb{C}X_{(-1,-1)} \oplus \mathbb{C}X_{(0,-2)}$. Then $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^+ \oplus \mathfrak{p}_{\mathbb{C}}^- \oplus \mathbb{C}C_4$. A basis of $\mathfrak{k}_{\mathbb{C}} \simeq \mathfrak{gl}_2(\mathbb{C})$ is given by

$$Z_{\mathfrak{k}} := \left(\begin{array}{cc|cc} & & -i & \\ & & & -i \\ \hline i & & & \\ & i & & \end{array} \right), \quad H_{\mathfrak{k}} := \left(\begin{array}{cc|cc} & & -i & \\ & & & i \\ \hline i & & & \\ & & & -i \end{array} \right), \quad X_{\mathfrak{k}}^{\pm} := \frac{1}{2}X_{\pm(1,-1)}.$$

Then $\{X_{\mathfrak{k}}^{-}, H_{\mathfrak{k}}, X_{\mathfrak{k}}^{+}\}$ is an $\mathfrak{sl}_2$-triple, that is, $[H_{\mathfrak{k}}, X_{\mathfrak{k}}^{\pm}] = \pm 2 X_{\mathfrak{k}}^{\pm}$ and $[X_{\mathfrak{k}}^{+}, X_{\mathfrak{k}}^{-}] = H_{\mathfrak{k}}$. Note that $[Z_{\mathfrak{k}}, X] = 0$ for every $X \in \mathfrak{k}_{\mathbb{C}}$. A basis of $\mathfrak{g}_{0,\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} C_4$ is given by

$$
H_1 := \left(\begin{array}{c|c} \begin{matrix} & -i \\ & & 0 \\ \hline i & \\ & 0 \end{matrix} \end{array} \right), \quad X_1^+ := \frac{1}{2} X_{(2,0)}, \quad X_1^- := \frac{1}{2} X_{(-2,0)},
$$

$$
H_2 := \left(\begin{array}{c|c} \begin{matrix} & 0 \\ & & -i \\ \hline 0 & \\ & i \end{matrix} \end{array} \right), \quad X_2^+ := \frac{1}{2} X_{(0,2)}, \quad X_2^- := \frac{1}{2} X_{(0,-2)}
$$

and C_4. Then, for each $j = 1, 2$, $\{X_j^-, H_j, X_j^+\}$ are $\mathfrak{sl}_2$-triples. Table 1 may be convenient for later computations.

Table 1: The table of the Lie bracket $[X, Y]$ for the basis of $\mathfrak{g}'_{\mathbb{C}}$

$X \backslash Y$	$X_{\mathfrak{k}}^{+}$	$H_{\mathfrak{k}}$	$X_{\mathfrak{k}}^{-}$	$Z_{\mathfrak{k}}$	$X_{(0,2)}$	$X_{(1,1)}$	$X_{(2,0)}$	$X_{(-2,0)}$	$X_{(-1,1)}$	$X_{(0,-2)}$
$X_{\mathfrak{k}}^{+}$	0	$-2X_{\mathfrak{k}}^{+}$	$H_{\mathfrak{k}}$	0	$X_{(1,1)}$	$2X_{(2,0)}$	0	$-X_{(-1,1)}$	$-2X_{(0,-2)}$	0
$H_{\mathfrak{k}}$	$2X_{\mathfrak{k}}^{+}$	0	$-2X_{\mathfrak{k}}^{-}$	0	$-2X_{(0,2)}$	0	$2X_{(2,0)}$	$-2X_{(-2,0)}$	0	$2X_{(0,-2)}$
$X_{\mathfrak{k}}^{-}$	$-H_{\mathfrak{k}}$	$2X_{\mathfrak{k}}^{-}$	0	0	0	$2X_{(0,2)}$	$X_{(1,1)}$	0	$-2X_{(-2,0)}$	$-X_{(-1,1)}$
$Z_{\mathfrak{k}}$	0	0	0	0	$2X_{(0,2)}$	$2X_{(1,1)}$	$2X_{(2,0)}$	$-2X_{(-2,0)}$	$-2X_{(-1,1)}$	$-2X_{(0,-2)}$
$X_{(0,2)}$	$-X_{(1,1)}$	$2X_{(0,2)}$	0	$-2X_{(0,2)}$	0	0	0	0	$4X_{\mathfrak{k}}^{-}$	$2(Z_{\mathfrak{k}} - H_{\mathfrak{k}})$
$X_{(1,1)}$	$-2X_{(2,0)}$	0	$-2X_{(0,2)}$	$-2X_{(1,1)}$	0	0	0	$4X_{\mathfrak{k}}^{-}$	$4Z_{\mathfrak{k}}$	$4X_{\mathfrak{k}}^{+}$
$X_{(2,0)}$	0	$-2X_{(2,0)}$	$-X_{(1,1)}$	$-2X_{(2,0)}$	0	0	0	$2(Z_{\mathfrak{k}} + H_{\mathfrak{k}})$	$4X_{\mathfrak{k}}^{+}$	0
$X_{(-2,0)}$	$X_{(-1,1)}$	$2X_{(-2,0)}$	0	$2X_{(-2,0)}$	0	$-4X_{\mathfrak{k}}^{-}$	$-2(Z_{\mathfrak{k}} + H_{\mathfrak{k}})$	0	0	0
$X_{(-1,1)}$	$2X_{(0,-2)}$	0	$2X_{(-2,0)}$	$2X_{(-1,1)}$	$-4X_{\mathfrak{k}}^{-}$	$-4Z_{\mathfrak{k}}$	$-4X_{\mathfrak{k}}^{+}$	0	0	0
$X_{(0,-2)}$	0	$-2X_{(0,-2)}$	$X_{(-1,1)}$	$2X_{(0,-2)}$	$-2(Z_{\mathfrak{k}} - H_{\mathfrak{k}})$	$-4X_{\mathfrak{k}}^{+}$	0	0	0	0

2.2. The unitary dual of $U(2)$

In this subsection, we recall the irreducible unitary representations of $K' \simeq U(2)$ Let $\mathcal{L}$ denote the set of Σ_c^+-dominant integral weights, that is,

$$
\mathcal{L} = \left\{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \mid \lambda_1 \geq \lambda_2 \right\}.
$$

By the highest weight theory, the set $\mathcal{L}$ parameterizes the equivalence classes of irreducible unitary representations of K'. Let $(\tau_\lambda, W_\lambda)$ denote the representation with highest weight $\lambda \in \mathcal{L}$. For each $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$, we set $d_\lambda := \lambda_1 - \lambda_2$. Then there exists a basis $\{w_j^\lambda\}_{j=0}^{d_\lambda}$ of W_λ such that the action τ_λ of $\mathfrak{k}'_{\mathbb{C}}$ is given by

$$
\tau_\lambda(Z_{\mathfrak{k}})w_j^\lambda = (\lambda_1 + \lambda_2)w_j^\lambda, \quad \tau_\lambda(H_{\mathfrak{k}})w_j^\lambda = (2j - d_\lambda)w_j^\lambda,
$$

$$
\tau_\lambda(X_{\mathfrak{k}}^{+})w_j^\lambda = (j+1)w_{j+1}^\lambda, \quad \tau_\lambda(X_{\mathfrak{k}}^{-})w_j^\lambda = (d_\lambda + 1 - j)w_{j-1}^\lambda
$$

for all $j = 0, \cdots, d_\lambda$. Here we put $w_{-1}^\lambda = w_{d_\lambda+1}^\lambda = 0$. Also the action of $SO(2) \times SO(2)$ on W_λ is given by

$$
\tau_\lambda(r(\theta_1, \theta_2))w_j^\lambda = \exp\left(i\big((\lambda_2 + j)\theta_1 + (\lambda_1 - j)\theta_2\big)\right)w_j^\lambda, \quad \forall \theta_1, \theta_2 \in \mathbb{R}
$$

for all $j = 0, \cdots, d_\lambda$. We call the basis $\{w_j^\lambda\}_{j=0}^{d_\lambda}$ the standard basis of $(\tau_\lambda, W_\lambda)$. For a K'-module (τ, W), we denote by $(\tau^\vee, W^\vee)$ its contragredient representation.

2.3. The unitary dual of K

In this subsection, we recall the irreducible unitary representations of $K \simeq K' \rtimes \langle \epsilon \rangle$. Let (τ, W) be a K-module and $w \in W$ a vector with weight (l_1, l_2), that is,

$$\tau(Z_{\mathfrak{k}} + H_{\mathfrak{k}})w = 2l_1 w, \quad \tau(Z_{\mathfrak{k}} - H_{\mathfrak{k}})w = 2l_2 w.$$

Then $\tau(\epsilon)w$ has weight $(-l_1, -l_2)$. Let $(\tau_\lambda, W_\lambda)$ be the irreducible representation of K' with highest weight $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$ and $\{w_j^\lambda\}_{j=0}^{d_\lambda}$ the standard basis of W_λ. Then each w_j^λ has weight $(j + \lambda_2, \lambda_1 - j)$. We consider the induced representation $\tau_{\lambda, K} := \mathrm{Ind}_{K'}^K(\tau_\lambda)$. Its representation space $W_{\lambda, K}$ is given by

$$\left\{ f \in L^2(K, W_\lambda) \mid f(k_0 k) = \tau_\lambda(k_0) f(k) \text{ for almost every pair } (k_0, k) \in K' \times K \right\}$$

and K acts on $W_{\lambda, K}$ by right translation. Then

$$W_{\lambda, K} = \{f \in W_{\lambda, K} \mid f(\epsilon) = 0\} \oplus \{f \in W_{\lambda, K} \mid f(I_4) = 0\}.$$

Each of the direct summands is stable under the action of K' via $\tau_{\lambda, K}|_{K'}$. More precisely, we have $\tau_{\lambda, K}|_{K'} \simeq \tau_\lambda \oplus \tau_\lambda^\vee$. We identify W_λ with the subspace

$$\{f \in W_{\lambda, K} \mid f(\epsilon) = 0\}.$$

Then a basis of $W_{\lambda, K}$ is given by $\{w_j^\lambda\}_{j=0}^{d_\lambda} \sqcup \{\tau_{\lambda, K}(\epsilon) w_j^\lambda\}_{j=0}^{d_\lambda}$. If $\lambda_1 \neq -\lambda_2$, then $\tau_{\lambda, K}$ is an irreducible K-module. If $\lambda_1 = -\lambda_2$, then $\tau_{\lambda, K}$ is not irreducible as a K-module. Indeed we have an isomorphism $\tau_{\lambda, K} \simeq \tau_\lambda^+ \oplus \tau_\lambda^-$ as a K-module, where

$$\tau_\lambda^+ = \bigoplus_{j=0}^{d_\lambda} \mathbb{C}\left(w_j^\lambda + (-1)^j \tau_{\lambda, K}(\epsilon) w_{d_\lambda - j}^\lambda\right), \quad \tau_\lambda^- = \bigoplus_{j=0}^{d_\lambda} \mathbb{C}\left(w_j^\lambda + (-1)^{j+1} \tau_{\lambda, K}(\epsilon) w_{d_\lambda - j}^\lambda\right).$$

The irreducible unitary representations of K are exhausted by

$$\left\{ \tau_{(\lambda, -\lambda)}^+, \tau_{(\lambda, -\lambda)}^- \mid \lambda \geq 0 \right\} \sqcup \left\{ \tau_{(\lambda_1, \lambda_2), K} \mid \lambda_1 \geq \lambda_2, \lambda_1 \geq 0, \lambda_1 \neq -\lambda_2 \right\}.$$

Lemma 2.1. *Let $\tau_{\lambda, K}^\vee$ be the contragredient representation of $\tau_{\lambda, K}$ with $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$. Then*

$$\tau_{\lambda, K}^\vee(Z_{\mathfrak{k}}) w_j^{\lambda\vee} = -(\lambda_1 + \lambda_2) w_j^{\lambda\vee}, \quad \tau_{\lambda, K}^\vee(Z_{\mathfrak{k}}) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee} = (\lambda_1 + \lambda_2) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee},$$

$$\tau_{\lambda, K}^\vee(H_{\mathfrak{k}}) w_j^{\lambda\vee} = (d_\lambda - 2j) w_j^{\lambda\vee}, \quad \tau_{\lambda, K}^\vee(H_{\mathfrak{k}}) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee} = -(d_\lambda - 2j) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee},$$

$$\tau_{\lambda, K}^\vee(X_{\mathfrak{k}}^+) w_j^{\lambda\vee} = -j w_{j-1}^{\lambda\vee}, \quad \tau_{\lambda, K}^\vee(X_{\mathfrak{k}}^+) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee} = (d_\lambda - j) \tau_{\lambda, K}^\vee(\epsilon) w_{j+1}^{\lambda\vee},$$

$$\tau_{\lambda, K}^\vee(X_{\mathfrak{k}}^-) w_j^{\lambda\vee} = -(d_\lambda - j) w_{j+1}^{\lambda\vee}, \quad \tau_{\lambda, K}^\vee(X_{\mathfrak{k}}^-) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee} = j \tau_{\lambda, K}^\vee(\epsilon) w_{j-1}^{\lambda\vee},$$

$$\tau_{\lambda, K}^\vee(r(\theta_1, \theta_2)) w_j^{\lambda\vee} = \exp\left(-i\left((\lambda_2 + j)\theta_1 + (\lambda_1 - j)\theta_2\right)\right) w_j^{\lambda\vee},$$

$$\tau_{\lambda, K}^\vee(r(\theta_1, \theta_2)) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee} = \exp\left(i\left((\lambda_2 + j)\theta_1 + (\lambda_1 - j)\theta_2\right)\right) \tau_{\lambda, K}^\vee(\epsilon) w_j^{\lambda\vee}.$$

Here $\{w_j^{\lambda\vee}\}_{j=0}^{d_\lambda}$ is the basis of $\tau_\lambda^\vee$ dual to the standard basis $\{w_j^\lambda\}_{j=0}^{d_\lambda}$ of τ_λ.

2.4. The discrete series representations of $Sp_4(\mathbb{R})$

First we recall the Harish-Chandra parametrization of the discrete series representations of G'. See [6, Chapter IX], for example. There exist four positive root systems of Σ which contain Σ_c^+. They are given by

$$\Sigma_{\mathrm{I}}^+ := \{(1,-1),(1,1),(2,0),(0,2)\}, \qquad \Sigma_{\mathrm{II}}^+ := \{(1,-1),(1,1),(2,0),(0,-2)\},$$
$$\Sigma_{\mathrm{III}}^+ := \{(1,-1),(-1,-1),(2,0),(0,-2)\}, \quad \Sigma_{\mathrm{IV}}^+ := \{(1,-1),(-1,-1),(-2,0),(0,-2)\}.$$

The set of Harish-Chandra parameters of the discrete series representations of G' is denoted by $\Xi = \bigcup_{J=\mathrm{I}}^{\mathrm{IV}} \Xi_J$, where

$$\Xi_{\mathrm{I}} = \{\Lambda = (\Lambda_1,\Lambda_2) \in \mathfrak{h}_{\mathbb{C}}^* \mid \Lambda_1,\Lambda_2 \in \mathbb{Z}, \Lambda_1 > \Lambda_2 > 0\},$$

$$\Xi_{\mathrm{II}} = \{\Lambda = (\Lambda_1,\Lambda_2) \in \mathfrak{h}_{\mathbb{C}}^* \mid \Lambda_1,\Lambda_2 \in \mathbb{Z}, \Lambda_1 + \Lambda_2 > 0 > \Lambda_2\},$$

$$\Xi_{\mathrm{III}} = \{\Lambda = (\Lambda_1,\Lambda_2) \in \mathfrak{h}_{\mathbb{C}}^* \mid \Lambda_1,\Lambda_2 \in \mathbb{Z}, \Lambda_1 > 0 > \Lambda_1 + \Lambda_2\},$$

$$\Xi_{\mathrm{IV}} = \{\Lambda = (\Lambda_1,\Lambda_2) \in \mathfrak{h}_{\mathbb{C}}^* \mid \Lambda_1,\Lambda_2 \in \mathbb{Z}, 0 > \Lambda_1 > \Lambda_2\},$$

and the Blattner parameter corresponding to $\Lambda = (\Lambda, J) \in \Xi_J \subset \Xi$ is given by

$$\lambda(\Lambda) = \Lambda - (1,-1) + \frac{1}{2}\sum_{\alpha \in \Sigma_J^+} \alpha = \begin{cases} \Lambda + (1,2) & (\text{if } J = \mathrm{I}), \\ \Lambda + (1,0) & (\text{if } J = \mathrm{II}), \\ \Lambda + (0,-1) & (\text{if } J = \mathrm{III}), \\ \Lambda + (-2,-1) & (\text{if } J = \mathrm{IV}). \end{cases}$$

Let Π_Λ be the discrete series representation with Harish-Chandra parameter $\Lambda \in \Xi$ and Blattner parameter $\lambda(\Lambda)$. Then Π_Λ is called holomorphic if $\Lambda \in \Xi_{\mathrm{I}}$, anti-holomorphic if $\Lambda \in \Xi_{\mathrm{IV}}$ and large if $\Lambda \in \Xi_{\mathrm{II}} \cup \Xi_{\mathrm{III}}$. If an irreducible representation τ_μ of K' is a K'-type of Π_Λ, then

$$\mu = \lambda(\Lambda) + \sum_{\alpha \in \Sigma_J^+ - \Sigma_c^+} m_\alpha \alpha \quad (m_\alpha \in \mathbb{Z}_{\geq 0}).$$

The irreducible representation $\tau_{\lambda(\Lambda)}$ is a K'-type of Π_Λ with multiplicity one. We call $\tau_{\lambda(\Lambda)}$ the minimal K'-type of Π_Λ.

Next we recall a construction of the (anti-) holomorphic discrete series representations of G'. Let $(\tau_\lambda, W_\lambda)$ be an irreducible representation of K' with highest weight $\lambda = (\lambda_1,\lambda_2) \in \mathcal{L}$. By letting $\mathfrak{p}_{\mathbb{C}}^-$ (resp. $\mathfrak{p}_{\mathbb{C}}^+$) act trivially on W_λ, we consider W_λ as a $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^-$ (resp. $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^+$) -module. We put

$$M(\lambda)^\pm = U(\mathfrak{g}'_{\mathbb{C}}) \otimes_{\mathfrak{k}_{\mathbb{C}} \otimes \mathfrak{p}_{\mathbb{C}}^\mp} W_\lambda$$
$$:= U(\mathfrak{g}'_{\mathbb{C}}) \otimes_{\mathbb{C}} W_\lambda / \left\langle YX \otimes w - Y \otimes Xw \mid Y \in \mathfrak{g}'_{\mathbb{C}}, X \in \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^\mp, w \in W_\lambda \right\rangle.$$

Then $M(\lambda)^+$ and $M(\lambda)^-$ are $\mathfrak{g}'_{\mathbb{C}}$-modules. Moreover we define an action of K' on $M(\lambda)^\pm$ by $k \cdot (X \otimes w) = \mathrm{Ad}(k)X \otimes \tau_\lambda(k)w$ for $k \in K', X \in \mathfrak{g}'_{\mathbb{C}}$ and $w \in W_\lambda$. Then $M(\lambda)^+$ and $M(\lambda)^-$ are $(\mathfrak{g}'_{\mathbb{C}}, K')$-modules. Let π_λ^+ (resp. π_λ^-) denote the $(\mathfrak{g}'_{\mathbb{C}}, K')$-action on $M(\lambda)^+$ (resp. $M(\lambda)^-$). The modules $M(\lambda)^+$ and $M(\lambda)^-$ have bases $\{\xi_{k,l,m,n}^{\lambda+}\}$ and $\{\xi_{k,l,m,n}^{\lambda-}\}$ given by

$$\xi_{k,l,m,n}^{\lambda+} := X_{(2,0)}^k X_{(1,1)}^l X_{(0,2)}^m \otimes (X_{\mathfrak{k}}^-)^n w_{d_\lambda}^\lambda \quad (k,l,m,n \geq 0, n \leq d_\lambda),$$
$$\xi_{k,l,m,n}^{\lambda-} := X_{(0,-2)}^k X_{(-1,-1)}^l X_{(-2,0)}^m \otimes (X_{\mathfrak{k}}^+)^n w_0^\lambda \quad (k,l,m,n \geq 0, n \leq d_\lambda),$$

respectively. Note that the vector $\xi_{k,l,m,n}^{\lambda+}$ (resp. $\xi_{k,l,m,n}^{\lambda+}$) has weight

$$(\lambda_1 + l + 2k + n, \lambda_2 + l + 2m - n) \quad (\text{resp. } (\lambda_2 - l - 2m + n, \lambda_1 - l - 2k - n)).$$

It is well-known that $M(\lambda)^+$ (resp. $M(\lambda)^-$) has a unique irreducible quotient $L(\lambda)^+ = M(\lambda)^+/S^+$ (resp. $L(\lambda)^- = M(\lambda)^-/S^-$), where S^+ (resp. S^-) is the sum of the $\mathfrak{g}'_{\mathbb{C}}$-submodules of $M(\lambda)^+$ (resp. $M(\lambda)^-$) which are distinct from $M(\lambda)^+$ (resp. $M(\lambda)^-$). The module $L(\lambda)^+$ (resp. $L(\lambda)^-$) is finite dimensional if and only if $0 \geq \lambda_1 \geq \lambda_2$ (resp. $\lambda_1 \geq \lambda_2 \geq 0$).

Proposition 2.2.   *Let $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$.*

  (i) *$L(\lambda)^+$ (resp. $L(\lambda)^-$) is unitarizable if and only if $\lambda_2 \geq 1$ (resp. $\lambda_1 \leq -1$) or $\lambda = (0,0)$;*

  (ii) *$L(\lambda)^+$ (resp. $L(\lambda)^-$) is tempered if and only if $\lambda_2 \geq 2$ (resp. $\lambda_1 \leq -2$);*

  (iii) *$L(\lambda)^+$ (resp. $L(\lambda)^-$) is square integrable if and only if $\lambda_2 \geq 3$ (resp. $\lambda_1 \leq -3$).*

If $\lambda_2 \geq 2$ (resp. $\lambda_1 \leq -2$), then $M(\lambda)^+$ (resp. $M(\lambda)^-$) is irreducible, that is, $M(\lambda)^+ = L(\lambda)^+$ (resp. $M(\lambda)^- = L(\lambda)^-$).

Proof.   See [12, §2], for example.   ∎

Let $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$. Then $(\pi_\lambda^+, M(\lambda)^+)$ (resp. $(\pi_\lambda^-, M(\lambda)^-)$) is equivalent to the holomorphic (resp. anti-holomorphic) discrete series representation Π_Λ with Harish-Chandra parameter $\Lambda = (\lambda_1 - 1, \lambda_2 - 2)$ (resp. $\Lambda = (\lambda_1 + 2, \lambda_2 + 1)$) if $\lambda_2 \geq 3$ (resp. $\lambda_1 \leq -3$). The representation $(\pi_\lambda^+, M(\lambda)^+)$ (resp. $(\pi_\lambda^-, M(\lambda)^-)$) is called the limit of holomorphic (resp. limit of anti-holomorphic) discrete series representations of G' with Harish-Chandra parameter $\Lambda = (\lambda_1 - 1, 0)$ (resp. $\Lambda = (0, \lambda_2 + 1)$) if $\lambda_2 = 2$ (resp. $\lambda_1 = -2$). Then we denote by Π_Λ the limit of (anti-) holomorphic discrete series representation with Harish-Chandra parameter Λ. We set $\Lambda(\lambda) := (\lambda_1 - 1, \lambda_2 - 2)$ for each $\lambda = (\lambda_1, \lambda_2) \in \overline{\mathcal{L}_+^{Hol}}$, where

$$\overline{\mathcal{L}_+^{Hol}} := \{\lambda = (\lambda_1, \lambda_2) \in \mathcal{L} \mid \lambda_2 \geq 2\}.$$

For each $\lambda \in \overline{\mathcal{L}_+^{Hol}}$, the irreducible unitary representation τ_λ of K' is the minimal K'-type of $\Pi_{\Lambda(\lambda)}$.

2.5. The holomorphic discrete series representations of $GSp_4(\mathbb{R})$

In this subsection, we recall the discrete series representations of G.
Let $Sp_4(\mathbb{R})^\pm$ be a subgroup of G defined by

$$Sp_4(\mathbb{R})^\pm = \{g \in G \mid |\nu(g)| = 1\} = Sp_4(\mathbb{R}) \sqcup \epsilon \cdot Sp_4(\mathbb{R}).$$

Then we have a short exact sequence

$$1 \to Sp_4(\mathbb{R})^\pm \to G \to \mathbb{R}_{>0}^\times \to 1,$$

where the third arrow $G \to \mathbb{R}_{>0}^\times$ is given by $g \mapsto |\nu(g)|$. This exact sequence splits by $\mathbb{R}_{>0}^\times \to G$, $t \mapsto \sqrt{t}I_4$. Thus we have $G \simeq Sp_4(\mathbb{R})^\pm \times \mathbb{R}_{>0}^\times$.

For each $\lambda = (\lambda_1, \lambda_2) \in \overline{\mathcal{L}_+^{Hol}}$, we consider the induced representation

$$\Pi_{\Lambda(\lambda)}^{\pm} := \mathrm{Ind}_{Sp_4(\mathbb{R})}^{Sp_4(\mathbb{R})^{\pm}}(\Pi_{\Lambda(\lambda)}).$$

Then $\Pi_{\Lambda(\lambda)}^{\pm}$ is an irreducible representation of $Sp_4(\mathbb{R})^{\pm}$ and this is decomposed as follows:

$$\Pi_{\Lambda(\lambda)}^{\pm}|_{Sp_4(\mathbb{R})} \simeq \Pi_{\Lambda(\lambda)} \oplus \Pi_{\Lambda(\lambda)^\vee}.$$

Here we set $\Lambda^\vee := (-\Lambda_2, -\Lambda_1)$ for each $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}^2$. For each $\gamma \in \mathbb{C}$, the representation $\Pi_{\Lambda(\lambda)}^{\pm}$ is extended to a representation $\Pi_{\Lambda(\lambda)}[\gamma]$ of G by

$$\Pi_{\Lambda(\lambda)}[\gamma]|_{Sp_4(\mathbb{R})^{\pm}} \simeq \Pi_{\Lambda(\lambda)}^{\pm}, \quad \Pi_{\Lambda(\lambda)}[\gamma](tI_4)v := t^\gamma v \quad (\forall t \in \mathbb{R}_{>0}^{\times}, \ \forall v \in \Pi_{\Lambda(\lambda)}[\gamma]).$$

We call $\Pi_{\Lambda(\lambda)}[\gamma]$ the holomorphic discrete series representation of G. For each $\lambda \in \overline{\mathcal{L}_+^{Hol}}$, the representation $\tau_{\lambda,K}$ is a K-type of $\Pi_{\Lambda(\lambda)}[\gamma]$ with multiplicity one. Note that for each $\lambda \in \overline{\mathcal{L}_+^{Hol}}$, if $w \in W_\lambda$ (resp. $w \in W_\lambda^\vee$), we have $\Pi_{\Lambda(\lambda)}[\gamma](\mathfrak{p}_{\mathbb{C}}^-)w = \{0\}$ (resp. $\Pi_{\Lambda(\lambda)}[\gamma](\mathfrak{p}_{\mathbb{C}}^+)w = \{0\}$).

2.6. The unitary dual of $O(2)$

In this subsection, we recall the irreducible unitary representations of $O(2) \simeq SO(2) \rtimes \langle \mathrm{diag}(1,-1) \rangle$ and K_0. We set

$$r(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2), \quad \forall \theta \in \mathbb{R}.$$

For each $n \in \mathbb{Z}$, we define an irreducible unitary representation $(\sigma_n, \mathbb{C}v_n')$ of $SO(2)$ by $\sigma_n(r(\theta))v_n' := e^{in\theta}v_n'$, where $v_n' := 1$. Then the irreducible unitary representations of $SO(2)$ are exhausted by $\{\sigma_n \,|\, n \in \mathbb{Z}\}$. For a nonnegative integer n, we consider the induced representation $\sigma_{n,O(2)} = \mathrm{Ind}_{SO(2)}^{O(2)}(\sigma_n)$. Then $\sigma_{n,O(2)}|_{SO(2)} \simeq \sigma_n \oplus \sigma_{-n}$. If $n \geq 1$, the representation $\sigma_{n,O(2)}$ is an irreducible $O(2)$-module.

Let $X_n := \mathbb{C}v_n' \oplus \mathbb{C}v_{-n}'$ denote the representation space of $\sigma_{n,O(2)}$. Note that there exists an inner product $\langle \cdot, \cdot \rangle_n$ on X_n such that $\{v_n', v_{-n}'\}$ becomes an orthonormal basis. The action of $\mathrm{diag}(1,-1)$ on X_n is given by $\sigma_{n,O(2)}(\mathrm{diag}(1,-1))v_{\pm n}' = v_{\mp n}'$. On the other hand, the representation $\sigma_{0,O(2)}$ is not irreducible as an $O(2)$-module.

Indeed, $\sigma_{0,O(2)}$ has a basis $\{v_0^+, v_0^-\}$ which satisfies $\sigma_{0,O(2)}(\mathrm{diag}(1,-1))v_0^{\pm} = \pm v_0^{\pm}$. Thus, setting $X_0^{\pm} := \mathbb{C}v_0^{\pm}$ and $\sigma_0^{\pm} := (\sigma_{0,O(2)}|_{X_0^{\pm}}, X_0^{\pm})$, we have $\sigma_{0,O(2)} \simeq \sigma_0^+ \oplus \sigma_0^-$ as an $O(2)$-module. The irreducible unitary representations of $O(2)$ are exhausted by

$$\left\{\sigma_{n,O(2)} \mid n \geq 1\right\} \sqcup \left\{\sigma_0^+, \sigma_0^-\right\}.$$

Similarly, the irreducible unitary representations of K_0 are given by

$$\left\{\sigma_{(k,l),K_0} \mid (k,l) \neq (0,0)\right\} \sqcup \{(\mathrm{triv}, \mathbb{C}), (\mathrm{sgn}, \mathbb{C})\}.$$

Here $\sigma_{(k,l),K_0}|_{SO(2) \times SO(2)} \simeq (\sigma_k \boxtimes \sigma_l) \oplus (\sigma_{-k} \boxtimes \sigma_{-l})$ and $\sigma_{(k,l),K_0}(\epsilon)(v_k' \otimes v_l') = v_{-k}' \otimes v_{-l}'$, and the action sgn on $\mathbb{C}$ is given by $\mathrm{sgn}(r(\theta_1, \theta_2)) = 1$ and $\mathrm{sgn}(\epsilon) = -1$. Note that $\sigma_{(-k,-l),K_0}$ is equivalent to $\sigma_{(k,l),K_0}$. The following lemma can be easily checked:

Lemma 2.3.   *For an irreducible unitary representation τ of K, we have the following irreducible decomposition as a K_0-module:*

$$
\tau|_{K_0} \simeq
\begin{cases}
\displaystyle\bigoplus_{j=0}^{\lambda_1-\lambda_2} \sigma_{(\lambda_2+j,\lambda_1-j),K_0} & (\text{if } \tau = \tau_{(\lambda_1,\lambda_2),K} \text{ and } \lambda_1 \neq -\lambda_2), \\[2em]
\mathrm{triv} \oplus \displaystyle\bigoplus_{j=0}^{\lambda-1} \sigma_{(\lambda-j,j-\lambda),K_0} \\
(\text{if } \tau = \tau^+_{(\lambda,-\lambda)} \text{ and } \lambda \equiv 0 \bmod 2, \text{ or } \tau = \tau^-_{(\lambda,-\lambda)} \text{ and } \lambda \equiv 1 \bmod 2), \\[1em]
\mathrm{sgn} \oplus \displaystyle\bigoplus_{j=0}^{\lambda-1} \sigma_{(\lambda-j,j-\lambda),K_0} \\
(\text{if } \tau = \tau^+_{(\lambda,-\lambda)} \text{ and } \lambda \equiv 1 \bmod 2, \text{ or } \tau = \tau^-_{(\lambda,-\lambda)} \text{ and } \lambda \equiv 0 \bmod 2).
\end{cases}
$$

2.7. Representation of $GL_2(\mathbb{R})$

In this subsection, we recall the (nonunitary) principal series representation of $GL_2(\mathbb{R})$. For $s \in \mathbb{C}$ and $\varepsilon = 0,1$, we define a character $\chi_{s,\varepsilon} : \mathbb{R}^\times \to \mathbb{C}^\times$ by $\chi_{s,\varepsilon}(a) := |a|^s (a/|a|)^\varepsilon$. The characters of $\mathbb{R}^\times$ are exhausted by $\{\chi_{s,\varepsilon} \mid s \in \mathbb{C}, \varepsilon \in \{0,1\}\}$. Let χ_1, χ_2 be characters of $\mathbb{R}^\times$. Then $\chi_1 \boxtimes \chi_2$ is regarded as a character of the Borel subgroup B consisting of the upper triangular matrices of $GL_2(\mathbb{R})$ as follows:

$$
(\chi_1 \boxtimes \chi_2)\left(\begin{pmatrix} a & * \\ & d \end{pmatrix} \right) = \chi_1(a)\chi_2(d), \quad \forall a,d \in \mathbb{R}^\times.
$$

The principal series representation $I(\chi_1 \boxtimes \chi_2)$ is realized on the space of all smooth functions $f : GL_2(\mathbb{R}) \to \mathbb{C}$ such that

$$
f\left(\begin{pmatrix} a & * \\ & d \end{pmatrix} g \right) = \chi_1(a)\chi_2(d) \left| \frac{a}{d} \right|^{1/2} f(g), \quad \forall a,d \in \mathbb{R}^\times, \ \forall g \in GL_2(\mathbb{R}),
$$

and $GL_2(\mathbb{R})$ acts on this space by right translation. For a character $\chi = \chi_{s_1,\varepsilon_1} \boxtimes \chi_{s_2,\varepsilon_2}$ of $\mathbb{R}^\times \times \mathbb{R}^\times$, we set $s(\chi) := s_1 - s_2$ and $\varepsilon(\chi) := |\varepsilon_1 - \varepsilon_2| \in \{0,1\}$. The principal series representation $I(\chi_1 \boxtimes \chi_2)$ is isomorphic to the space of all smooth functions $f : O(2) \to \mathbb{C}$ such that

$$
f\left(\begin{pmatrix} a & * \\ & d \end{pmatrix} k \right) = \chi_1(a)\chi_2(d) f(k), \quad \forall a,d \in \{\pm1\}, \ \forall k \in O(2).
$$

Then the Frobenius reciprocity law implies that

$$
I(\chi)|_{O(2)} \simeq
\begin{cases}
\sigma_0^+ \oplus \displaystyle\bigoplus_{\substack{n \equiv 0 \bmod 2 \\ n \geq 2}} \sigma_{n,O(2)} & (\text{if } \varepsilon(\chi) = 0, \varepsilon_2 = 0), \\[1.5em]
\sigma_0^- \oplus \displaystyle\bigoplus_{\substack{n \equiv 0 \bmod 2 \\ n \geq 2}} \sigma_{n,O(2)} & (\text{if } \varepsilon(\chi) = 0, \varepsilon_2 = 1), \\[1.5em]
\displaystyle\bigoplus_{\substack{n \equiv 1 \bmod 2 \\ n \geq 1}} \sigma_{n,O(2)} & (\text{if } \varepsilon(\chi) = 1).
\end{cases}
$$

For each $O(2)$-type σ of $I(\chi)$, an embedding $\sigma \hookrightarrow I(\chi)$ is given by $v \mapsto f_v^\chi$, where for every $k \in O(2)$ we set

$$f_v^\chi(k) := \begin{cases} \sigma_0^\pm(k)v & \text{(if } \sigma = \sigma_0^\pm\text{),} \\ \langle \sigma_{n,O(2)}(k)v, v_n + v_{-n} \rangle & \text{(if } \sigma = \sigma_{n,O(2)}, \ n \geq 1\text{).} \end{cases}$$

For any $n \geq 1$, we can take a basis $\{v_n, v_{-n}\}$ of X_n so that $\sigma_{n,O(2)}(\operatorname{diag}(1,-1))v_n = \chi_2(-1)v_n$. Setting

$$f_0^\chi := \begin{cases} f_{v_0^+}^\chi & \text{(if } \varepsilon_2 = 0\text{),} \\ f_{v_0^-}^\chi & \text{(if } \varepsilon_2 = 1\text{)} \end{cases}$$

and $f_p^\chi := f_{v_p}^\chi$, a basis of $I(\chi)$ is given by $\{f_p = f_p^\chi \mid p \in \mathbb{Z}, p \equiv \varepsilon(\chi) \bmod 2\}$. We define a basis of $\mathfrak{gl}_2(\mathbb{C})$ by

$$H := \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad X_\pm := \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}, \quad C_2 := I_2.$$

Then $\{X_-, H, X_+\}$ is an $\mathfrak{sl}_2$-triple, and $\mathfrak{gl}_2(\mathbb{C})$ and $O(2)$ act on $I(\chi)$ as follows:

$$Hf_p = pf_p, \quad X_\pm f_p = \frac{s(\chi) + 1 \pm p}{2} f_{p\pm 2}, \quad C_2 f_p = (s_1 + s_2)f_p,$$

$$r(\theta)f_p = e^{ip\theta}f_p, \quad \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} f_p = \chi_2(-1)f_{-p}.$$

In particular, $I(\chi)$ is irreducible, except when $s(\chi) = l - 1 \in \mathbb{Z} - \{0\}$ and $l \equiv \varepsilon(\chi) \bmod 2$. If $s(\chi) = l - 1 \in \mathbb{Z} - \{0\}$ and $l \equiv \varepsilon(\chi) \bmod 2$, we have the following exact sequences of $(\mathfrak{gl}_2(\mathbb{R}), O(2))$-modules:

$$0 \to \mathcal{D}_l[s_1 + s_2] \to I(\chi) \to (\chi_2 \circ \det)|\det|^{1/2}\operatorname{Sym}^{l-2}(\operatorname{st}_2) \to 0 \quad \text{(if } l - 1 > 0\text{),}$$

$$0 \to (\chi_1 \circ \det)|\det|^{1/2}\operatorname{Sym}^{-l}(\operatorname{st}_2) \to I(\chi) \to \mathcal{D}_{-l+2}[s_1 + s_2] \to 0 \quad \text{(if } l - 1 < 0\text{).}$$

Here, for all $l \geq 1$, we denote by $\mathcal{D}_l[c]$ the discrete series representation of $GL_2(\mathbb{R})$ characterized by

$$\mathcal{D}_l[c]|_{SL_2(\mathbb{R})} = \mathcal{D}_l^+ \oplus \mathcal{D}_l^-, \quad \mathcal{D}_l[c](tI_2)v = t^c v \quad (\forall t > 0),$$

where $\mathcal{D}_l^+$ (resp. $\mathcal{D}_l^-$) is the (limit of) holomorphic (resp. anti-holomorphic) discrete series representation of $SL_2(\mathbb{R})$ with Blattner parameter l (resp. $-l$). Note that $I(\chi) \simeq \mathcal{D}_1[s_1 + s_2]$ if $s(\chi) = l - 1 = 0$ and $l \equiv \varepsilon(\chi) \bmod 2$. The discrete series representation $\mathcal{D}_l[c]$ has a basis $\{\zeta_q = \zeta_q^{(l,c)} \mid |q| \geq l, q \equiv l \bmod 2\}$ with

$$H\zeta_q = q\zeta_q, \quad X_\pm \zeta_q = \frac{l \pm q}{2}\zeta_{q\pm 2}, \quad C_2\zeta_q = c\zeta_q,$$

$$r(\theta)\zeta_q = e^{iq\theta}\zeta_q, \quad \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\zeta_q = \zeta_{-q}.$$

Let $\mathcal{D}_l[c]^\vee$ be the contragredient representation of $\mathcal{D}_l[c]$ and

$$\{\zeta_q^\vee = \zeta_q^{(l,c)\vee} \mid |q| \geq l, q \equiv l \bmod 2\}$$

the basis of $\mathcal{D}_l[c]^\vee$ dual to $\{\zeta_q \,|\, |q| \geq l, q \equiv l \bmod 2\}$. We set $\xi_q^\vee = \xi_q^{(l,c)\vee} := d(q)\zeta_{-q}^\vee$, where

$$d(q) := \begin{cases} (\frac{q-l}{2})!/(l)_{\frac{q-l}{2}} & \text{(if } q \geq l), \\ d(-q) & \text{(if } q \leq -l). \end{cases}$$

Here $(a)_n$ is the Pochhammer symbol. Then $\mathcal{D}_l[c]^\vee$ is equivalent to $\mathcal{D}_l[-c]$ via $\xi_q^\vee \mapsto \zeta_q$. For two characters χ and χ' of $\mathbb{R}^\times \times \mathbb{R}^\times$, we set $I(\chi, \chi') := I(\chi) \boxtimes I(\chi')$ and $f_{p,q} = f_{p,q}^{\chi,\chi'} := f_p^\chi \otimes f_q^{\chi'}$. If $\chi = \chi_{s_1,\varepsilon_1} \boxtimes \chi_{s_2,\varepsilon_2}$ and $\chi' = \chi_{s_1',\varepsilon_1'} \boxtimes \chi_{s_2',\varepsilon_2'}$, then $\{f_{p,q}\}_{(p,q)\in E_{\chi,\chi'}}$ is a basis of $I(\chi, \chi')$, where

$$E_{\chi,\chi'} := \left\{ (p,q) \in \mathbb{Z}^2 \mid p \equiv \varepsilon(\chi) \bmod 2, q \equiv \varepsilon(\chi') \bmod 2 \right\}.$$

The representation $I(\chi, \chi')$ is called the (nonunitary) principal series representation of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$. The restriction of $I(\chi, \chi')$ to G_0 is denoted again by $I(\chi, \chi')$.

3. Shintani functions

3.1. Definition of Shintani functions

In this subsection, we introduce the space of Shintani functions for the reductive symmetric pair (G, G_0). Let $(\pi, \mathcal{H}_\pi)$ and (η, V_η) be admissible representations of G and G_0, respectively. We set

$$C_\eta^\infty(G_0\backslash G) := \{f : G \xrightarrow{C^\infty} V_\eta^\vee \mid f(g_0 x) = \eta^\vee(g_0)f(x), \ \forall (g_0, x) \in G_0 \times G\},$$

where $(\eta^\vee, V_\eta^\vee)$ is the contragredient representation of η. Then G acts on $C_\eta^\infty(G_0\backslash G)$ via right translation ρ. We consider the intertwining space

$$\mathcal{I}_{\eta,\pi} := \mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C_\eta^\infty(G_0\backslash G)).$$

Then the space $\mathcal{S}(\eta, \pi)$ of Shintani functions of type (η, π) is defined by

$$\mathcal{S}(\eta, \pi) := \mathrm{Image}(\mathcal{I}_{\eta,\pi} \otimes \mathcal{H}_\pi \to C_\eta^\infty(G_0\backslash G)).$$

Let τ be a K-type of π. For an embedding $\imath \in \mathrm{Hom}_K(\tau, \pi)$, we define a $\mathbf{C}$-linear map $\imath^* : \mathcal{I}_{\eta,\pi} \to \mathrm{Hom}_K(\tau, C_\eta^\infty(G_0\backslash G))$ by the pullback via $\imath$. Fix an embedding $\imath \in \mathrm{Hom}_K(\tau, \pi)$. We define the space $\mathcal{S}(\eta, \pi, \tau)$ of Shintani functions of type (η, π, τ) by

$$\mathcal{S}(\eta, \pi, \tau) := \mathrm{Image}(\imath^* : \mathcal{I}_{\eta,\pi} \to \mathrm{Hom}_K(\tau, C_\eta^\infty(G_0\backslash G))).$$

For an admissible representation (η, V_η) of G_0 and a finite dimensional representation (τ, W_τ) of K, let $C_{\eta,\tau}^\infty(G_0\backslash G/K)$ denote a vector space

$$\left\{ F : G \xrightarrow{C^\infty} V_\eta^\vee \otimes W_\tau^\vee \mid F(g_0 x k) = \eta^\vee(g_0) \otimes \tau^\vee(k)^{-1} F(x), \ \forall (g_0, x, k) \in G_0 \times G \times K \right\}.$$

Then $\mathrm{Hom}_K(\tau, C_\eta^\infty(G_0\backslash G))$ is isomorphic to $C_{\eta,\tau}^\infty(G_0\backslash G/K)$. Thus $\mathcal{S}(\eta, \pi, \tau)$ is regarded as a subspace of $C_{\eta,\tau}^\infty(G_0\backslash G/K)$.

Let $\pi = \Pi_\Lambda[\gamma]$ be the holomorphic discrete series representation of G with Blattner parameter $\lambda = \lambda(\Lambda) \in \overline{\mathcal{L}_+^{Hol}}$, and $\eta = I(\chi, \chi')^\vee$ the contragredient representation of the principal series representation $I(\chi, \chi')$ of G_0. We are especially interested in Shintani functions of type $(\eta, \pi, \tau_{\lambda,K})$ and investigate them in Section 4.

Proposition 3.1. *Let $\pi = \Pi_{\Lambda(\lambda)}[\gamma]$ be the holomorphic discrete series representation of G and η an admissible representation of G_0. Then $\mathcal{I}_{\eta,\pi} \simeq \mathcal{S}(\eta, \pi, \tau_{\lambda,K})$.*

Proof. Let $\iota \colon \tau_{\lambda,K} \to \pi$ be a K-embedding. It is sufficient to show that the surjective linear map $\iota^* \colon \mathcal{I}_{\eta,\pi} \to \mathcal{S}(\eta, \pi, \tau_{\lambda,K})$ is injective. Let $T \in \mathcal{I}_{\eta,\pi}$. Assume $\iota^* T = 0$. Namely we have $T(\iota(w)) = 0$ for every $w \in W_{\lambda,K}$. Note that π is generated by

$$\left\{ \pi(X_{(\pm 2,0)})^k \pi(X_{(\pm 1,\pm 1)})^l \pi(X_{(0,\pm 2)})^m \iota(w) \mid k, l, m \geq 0, w \in W_{\lambda,K} \right\}.$$

Since
$$T(\pi(X_{(\pm 2,0)})^k \pi(X_{(\pm 1,\pm 1)})^l \pi(X_{(0,\pm 2)})^m \iota(w))$$
$$= \rho(X_{(\pm 2,0)})^k \rho(X_{(\pm 1,\pm 1)})^l \rho(X_{(0,\pm 2)})^m T(\iota(w)) = 0$$

for all $k, l, m \geq 0$ and all $w \in W_{\lambda,K}$, we have $T = 0$. This completes the proof. $\blacksquare$

Let $\mathrm{pr}_+ \colon V_\eta^\vee \otimes W_{\lambda,K}^\vee \to V_\eta^\vee \otimes W_\lambda^\vee$ (resp. $\mathrm{pr}_- \colon V_\eta^\vee \otimes W_{\lambda,K}^\vee \to V_\eta^\vee \otimes \tau_{\lambda,K}^\vee(\epsilon)W_\lambda^\vee$) be the projection. We set $\mathrm{pr}_\pm(F)(x) := \mathrm{pr}_\pm(F(x))$. If $F \in C_{\eta,\tau_{\lambda,K}}^\infty(G_0 \backslash G / K)$ is expressed as

$$F(x) = \sum_{j=0}^{d_\lambda} \left(F^{(j,+)}(x) \otimes w_j^{\lambda\vee} + F^{(j,-)}(x) \otimes \tau_{\lambda,K}^\vee(\epsilon) w_j^{\lambda\vee} \right), \tag{1}$$

where $F^{(j,\pm)} \in C_\eta^\infty(G_0 \backslash G)$, then $\mathrm{pr}_+(F)$ and $\mathrm{pr}_-(F)$ are given by

$$\mathrm{pr}_+(F)(x) = \sum_{j=0}^{d_\lambda} F^{(j,+)}(x) \otimes w_j^{\lambda\vee}, \quad \mathrm{pr}_-(F)(x) = \sum_{j=0}^{d_\lambda} F^{(j,-)}(x) \otimes \tau_{\lambda,K}^\vee(\epsilon) w_j^{\lambda\vee}.$$

For a smooth function F on G, we set $[R(X \otimes c)F](g) := c \cdot \frac{d}{dt}\big|_{t=0} F(g \exp(tX))$ for $X \otimes c \in \mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. If $\pi = \Pi_\Lambda[\gamma]$, $\mathcal{S}(\eta, \pi, \tau_{\lambda,K})$ is regarded as a subspace of

$$\mathcal{S}_{\eta,\tau_{\lambda,K}}^{Hol}(G_0 \backslash G / K)^{[\gamma]}$$
$$= \left\{ F \in C_{\eta,\tau_{\lambda,K}}^\infty(G_0 \backslash G / K) \;\middle|\; \begin{matrix} F(rx) = r^\gamma F(x) \;\; (\forall (r,x) \in \mathbb{R}_{>0}^\times \times G), \\ R(\mathfrak{p}_-)\mathrm{pr}_+(F) = \{0\}, R(\mathfrak{p}_+)\mathrm{pr}_-(F) = \{0\} \end{matrix} \right\}.$$

In Section 4, we prove $\dim_\mathbb{C} \mathcal{S}_{\eta,\tau_{\lambda,K}}^{Hol}(G_0 \backslash G / K)^{[\gamma]} \leq 1$ and an explicit formula of elements of $\mathcal{S}_{\eta,\tau_{\lambda,K}}^{Hol}(G_0 \backslash G / K)^{[\gamma]}$. It is easy to see the following proposition:

Proposition 3.2. *Let η be an admissible representation of G_0 with central character ω_η. If $r^\gamma \omega_\eta(rI_4) \neq 1$ for some $r > 0$, then $\mathcal{S}_{\eta,\tau_{\lambda,K}}^{Hol}(G_0 \backslash G / K)^{[\gamma]} = \{0\}$.*

3.2. The generalized Cartan decomposition

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ given by

$$\mathfrak{a} := \mathbb{R}Y_\mathfrak{a}, \quad Y_\mathfrak{a} := \begin{pmatrix} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & -1 & \end{pmatrix}$$

and A the Lie subgroup of G corresponding to the Lie algebra $\mathfrak{a}$, that is,

$$
A := \left\{ a_t := \exp(tY_{\mathfrak{a}}) = \left(\begin{array}{cc|cc} \cosh t & \sinh t & & \\ \sinh t & \cosh t & & \\ \hline & & \cosh t & -\sinh t \\ & & -\sinh t & \cosh t \end{array} \right) \,\middle|\, t \in \mathbb{R} \right\}.
$$

We set

$$
\begin{aligned}
N_{K_0}(\mathfrak{a}) &:= \{ x \in K_0 \mid \mathrm{Ad}(x)\mathfrak{a} = \mathfrak{a} \} = \{ r(\theta, -\theta), r(\theta, -\theta + \pi) \mid \theta \in \mathbb{R} \} \cup \{\epsilon\}, \\
Z_{K_0}(\mathfrak{a}) &:= \{ x \in K_0 \mid \mathrm{Ad}(x)X = X, \ \forall X \in \mathfrak{a} \} = \{ r(\theta, -\theta) \mid \theta \in \mathbb{R} \} \cup \{\epsilon\}.
\end{aligned}
$$

Since $r(\theta, -\theta + \pi)r(-\theta, \theta) = r(0, \pi)$ for all $\theta \in \mathbb{R}$, the order of the Weyl group $W_{K_0}(\mathfrak{a}) = N_{K_0}(\mathfrak{a})/Z_{K_0}(\mathfrak{a})$ is two. We take $w_0 := r(0, \pi) = \mathrm{diag}(1, -1, 1, -1)$ as a nontrivial representative of $W_{K_0}(\mathfrak{a})$.

Proposition 3.3. *The multiplication map* $\Phi : G_0 \times A \times K \to G,\ (h, a, k) \mapsto hak$ *has the following three properties:*

  (i) *The map* Φ *is* C^∞*-surjective,*

  (ii) $G = G_0 A K = G_0 \overline{A^+} K$*, where* $\overline{A^+} := \{ a_t \mid t \geq 0 \}$*,*

  (iii) $\Phi^{-1}(hak) = \begin{cases} \{ (hx^{-1}, I_4, xk) \mid x \in K_0 \} & (\text{if } a = I_4), \\ \{ (hx^{-1}, xax^{-1}, xk) \mid x \in N_{K_0}(\mathfrak{a}) \} & (\text{if } a \neq I_4). \end{cases}$

Proof. See [4, p.108], for example. ∎

Let η be an admissible representation of G_0 and τ a finite dimensional representation of K. Since we have $G = G_0 \overline{A^+} K$, any $F \in C^\infty_{\eta, \tau}(G_0 \backslash G / K)$ is determined by its restriction $F|_{\overline{A^+}}$. We call $F|_{\overline{A^+}}$ the radial part of F. Let $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$. For each $p, q \in \mathbb{Z}$, we set $d_\lambda(p, q) := (d_\lambda + p - q)/2$.

Proposition 3.4. *Let* $\eta = I(\chi, \chi')^\vee$ *be the contragredient representation of* $I(\chi, \chi')$ *with* $\chi = \chi_{s_1, \varepsilon_1} \boxtimes \chi_{s_2, \varepsilon_2}$ *and* $\chi' = \chi_{s'_1, \varepsilon'_1} \boxtimes \chi_{s'_2, \varepsilon'_2}$*, and* $\tau = \tau_{\lambda, K}$ *the irreducible unitary representation of* K *with* $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$*. If* $F \in C^\infty_{\eta, \tau}(G_0 \backslash G / K)$ *is expressed as (1) and each* $F^{(j,\pm)}$ *is expressed as*

$$
F^{(j,\pm)}(x) = \sum_{(p,q) \in E_{\chi, \chi'}} F^{(j,\pm)}_{p,q}(x) f_{\pm p, \pm q}, \quad \forall x \in G, \tag{2}
$$

then $F^{(j,\pm)}_{p,q}$ *has the following four properties:*

  (i) *If* $d_\lambda \not\equiv \varepsilon(\chi) - \varepsilon(\chi') \bmod 2$*, then* $C^\infty_{\eta,\tau}(G_0 \backslash G / K) = \{0\}$*. If* $d_\lambda \equiv \varepsilon(\chi) - \varepsilon(\chi') \bmod 2$*, then* $F^{(j,\pm)}_{p,q} \equiv 0$ *on* G *for* $j \neq d_\lambda(p, q)$*;*

  (ii)

$$
F^{(j,\pm)}_{p,q}(a^{-1}) = \begin{cases} F^{(j,\pm)}_{p,q}(a) & (\text{if } j \equiv \lambda_1,\ \varepsilon(\chi') \equiv 0 \bmod 2, \\ & \quad \text{or } j \equiv \lambda_1 + 1, \varepsilon(\chi') \equiv 1 \bmod 2), \\ -F^{(j,\pm)}_{p,q}(a) & (\text{if } j \equiv \lambda_1,\ \varepsilon(\chi') \equiv 1 \bmod 2, \\ & \quad \text{or } j \equiv \lambda_1 + 1, \varepsilon(\chi') \equiv 0 \bmod 2) \end{cases}
$$

    for all $a \in A$*;*

(iii) if $F_{p,q}^{(d_\lambda(p,q),\pm)}(I_4) \neq 0$, then $p + q = \lambda_1 + \lambda_2$;

(iv) $F_{p,q}^{(d_\lambda(p,q),-)} = (-1)^{\varepsilon_2 + \varepsilon_2'} F_{p,q}^{(d_\lambda(p,q),+)}$.

Setting $\mathbf{F}_{p,q} := F_{p,q}^{(d_\lambda(p,q),+)}$ and

$$\Psi_\lambda^{\chi,\chi'}(p,q) := f_{p,q}^{\chi,\chi'} \otimes w_{d_\lambda(p,q)}^{\lambda\vee} + (-1)^{\varepsilon_2 + \varepsilon_2'} f_{-p,-q}^{\chi,\chi'} \otimes \tau_{\lambda,K}^\vee(\epsilon) w_{d_\lambda(p,q)}^{\lambda\vee},$$

we immediately have the following from Proposition 3.4:

Corollary 3.5.   If $F \in C_{\eta,\tau}^\infty(G_0 \backslash G / K)$ is expressed as (1) and each $F^{(j,\pm)}$ is expressed as (2), then

$$F(x) = \sum_{\substack{(p,q) \in E_{\chi,\chi'} \\ 0 \leq d_\lambda(p,q) \leq d_\lambda}} \mathbf{F}_{p,q}(x) \Psi_\lambda^{\chi,\chi'}(p,q). \tag{3}$$

Proposition 3.4 is a consequence of the following two lemmas:

Lemma 3.6.   Let $\eta = \eta_1 \boxtimes \eta_2$ be an admissible representation of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ and $\tau = \tau_{\lambda,K}$ the irreducible unitary representation of K with $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$. If $F \in C_{\eta,\tau}^\infty(G_0 \backslash G / K)$ is expressed as (1), each $F^{(j,\pm)} \in C_\eta^\infty(G_0 \backslash G)$ has the following three properties for all $j = 0, \cdots, d_\lambda$:

(i) $e^{\pm i(d_\lambda - 2j)\theta} \left(\eta_1(r(\theta)) \otimes \eta_2(r(-\theta)) \right) F^{(j,\pm)}(a) = F^{(j,\pm)}(a), \quad \forall (a,\theta) \in A \times \mathbb{R}$,

(ii) $F^{(j,\pm)}(a^{-1}) = \begin{cases} (\mathrm{id} \otimes \eta_2(-I_2)) F^{(j,\pm)}(a) & (\text{if } j \equiv \lambda_1 \bmod 2) \\ -(\mathrm{id} \otimes \eta_2(-I_2)) F^{(j,\pm)}(a) & (\text{if } j \equiv \lambda_1 + 1 \bmod 2) \end{cases} \quad (\forall a \in A),$

(iii) $e^{\mp i\left((\lambda_2+j)\theta_1 + (\lambda_1-j)\theta_2\right)} \left(\eta_1(r(\theta_1)) \otimes \eta_2(r(\theta_2)) \right) F^{(j,\pm)}(I_4) = F^{(j,\pm)}(I_4), \quad \forall \theta_1, \theta_2 \in \mathbb{R}$.

Proof.   It follows from Proposition 3.3 that the radial part of $F \in C_{\eta,\tau}^\infty(G_0 \backslash G / K)$ has the following three properties:

(i) $(\eta(x) \otimes \tau^\vee(x)) F(a) = F(a), \quad \forall (a,x) \in A \times Z_{K_0}(\mathfrak{a})$,

(ii) $(\eta(w_0) \otimes \tau^\vee(\omega_0)) F(a) = F(a^{-1}), \quad \forall a \in A$,

(iii) $\eta(x) \tau^\vee(x) F(I_4) = F(I_4), \quad \forall x \in K_0$.

The assertions are obtained from these properties.   ∎

Lemma 3.7.   Let $\eta = \eta_1 \boxtimes \eta_2$ be an admissible representation of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$. Let $\tau = \tau_{\lambda,K}$ be the irreducible unitary representation of K with $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$. If $F \in C_{\eta,\tau}^\infty(G_0 \backslash G / K)$ is expressed as (1), each $F^{(j,\pm)} \in C_\eta^\infty(G_0 \backslash G)$ satisfies

$$\eta(\epsilon) F^{(j,\pm)}(a) = F^{(j,\mp)}(a), \quad \forall a \in A.$$

Proof.   Since $\epsilon \in G_0 \cap K$ and $\epsilon a = a\epsilon$ for all $a \in A$, we have immediately the lemma.   ∎

The following two lemmas can be easily checked by direct computation. In the next subsection, these lemmas are used to derive a system of difference-differential equations which determines elements of $\mathcal{S}^{Hol}_{\eta,\tau_{\lambda},K}(G_0\backslash G/K)^{[\gamma]}$.

Lemma 3.8.   *Let* $t \in \mathbb{R} - \{0\}$. *Then*

$$X_{(2,0)} = \mathrm{Ad}(a_t)^{-1}\left(\frac{2\cosh^2(t)}{\cosh(2t)}X_1^+ + \frac{2\sinh^2(t)}{\cosh(2t)}X_2^-\right) - \tanh(2t)X_{\mathfrak{k}}^+,$$

$$X_{(1,1)} = \frac{2}{\sinh(2t)}\mathrm{Ad}(a_t)^{-1}H_2 + Y_{\mathfrak{a}} + \frac{1}{\sinh(2t)}H_{\mathfrak{k}} - \frac{1}{\tanh(2t)}Z_{\mathfrak{k}},$$

$$X_{(0,2)} = \mathrm{Ad}(a_t)^{-1}\left(\frac{2\sinh^2(t)}{\cosh(2t)}X_1^- + \frac{2\cosh^2(t)}{\cosh(2t)}X_2^+\right) - \tanh(2t)X_{\mathfrak{k}}^-,$$

$$X_{(0,-2)} = \mathrm{Ad}(a_t)^{-1}\left(\frac{2\sinh^2(t)}{\cosh(2t)}X_1^+ + \frac{2\cosh^2(t)}{\cosh(2t)}X_2^-\right) + \tanh(2t)X_{\mathfrak{k}}^+,$$

$$X_{(-1,-1)} = -\frac{2}{\sinh(2t)}\mathrm{Ad}(a_t)^{-1}H_2 + Y_{\mathfrak{a}} - \frac{1}{\sinh(2t)}H_{\mathfrak{k}} + \frac{1}{\tanh(2t)}Z_{\mathfrak{k}},$$

$$X_{(-2,0)} = \mathrm{Ad}(a_t)^{-1}\left(\frac{2\cosh^2(t)}{\cosh(2t)}X_1^- + \frac{2\sinh^2(t)}{\cosh(2t)}X_2^+\right) + \tanh(2t)X_{\mathfrak{k}}^-.$$

In particular, we have $\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}(a_t)^{-1}\mathfrak{g}_{0,\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$ *for all* $t \in \mathbb{R} - \{0\}$.

Lemma 3.9.   *For each* $F \in C^\infty_\eta(G_0\backslash G)$, *we have*

(i) $[\rho(\mathrm{Ad}(g^{-1})X)F](g) = \eta^\vee(X)F(g), \quad \forall g \in G, \ \forall X \in \mathfrak{g}_0,$

(ii) $[\rho(Y_{\mathfrak{a}})F](a_t) = \dfrac{d}{dt}F(a_t), \quad \forall t \in \mathbb{R} - \{0\}.$

3.3. A system of difference-differential equations for the Shintani functions

In this subsection, we derive a system of difference-differential equations which determines elements of $\mathcal{S}^{Hol}_{\eta,\tau_{\lambda},K}(G_0\backslash G/K)^{[\gamma]}$.

Theorem 3.10.   *Let* $t > 0$. *Let* $\lambda = (\lambda_1, \lambda_2) \in \overline{\mathcal{L}^{Hol}_+}$ *and* $\eta = I(\chi, \chi')^\vee$ *be the contragredient representation of* $I(\chi, \chi')$ *with* $\chi = \chi_{s_1,\varepsilon_1} \boxtimes \chi_{s_2,\varepsilon_2}$, $\chi' = \chi_{s'_1,\varepsilon'_1} \boxtimes \chi_{s'_2,\varepsilon'_2}$. *Assume that* $d_\lambda \equiv \varepsilon(\chi) - \varepsilon(\chi') \bmod 2$. *If* $F \in \mathcal{S}^{Hol}_{\eta,\tau_{\lambda},K}(G_0\backslash G/K)$ *is expressed as* (3), *then* $\{\mathbf{F}_{p,q}\}_{p,q}$ *satisfies the following system of difference-differential equations:*

$$2(s(\chi) + p - 1)\sinh^2(t)\mathbf{F}_{p-2,q}(a_t) + (d_\lambda + p - q)\sinh(2t)\mathbf{F}_{p,q}(a_t)$$
$$+ 2(s(\chi') - q - 1)\cosh^2(t)\mathbf{F}_{p,q+2}(a_t) = 0,$$

$$\left(\frac{d}{dt} + \frac{\lambda_1 + \lambda_2}{\tanh(2t)} - \frac{p+q}{\sinh(2t)}\right)\mathbf{F}_{p,q}(a_t) = 0,$$

$$2(s(\chi') + q - 1)\sinh^2(t)\mathbf{F}_{p,q-2}(a_t) + (d_\lambda - p + q)\sinh(2t)\mathbf{F}_{p,q}(a_t)$$
$$+ 2(s(\chi) - p - 1)\cosh^2(t)\mathbf{F}_{p+2,q}(a_t) = 0.$$

Let $t > 0$ and $F \in \mathcal{S}^{Hol}_{\eta, \tau_{\lambda}, K}(G_0 \backslash G / K)^{[\gamma]}$ can be expressed as (3). Then from the second equation in Theorem 3.10 we immediately have

$$\mathbf{F}_{p,q}(a_t) = c_{p,q}(F) \times \frac{\sinh(t)^{(p+q-\lambda_1-\lambda_2)/2}}{\cosh(t)^{(p+q+\lambda_1+\lambda_2)/2}}$$

for some $c_{p,q}(F) \in \mathbb{C}$. Thus Theorem 3.10 implies the following

Corollary 3.11.   *The sequence* $\{c_{p,q}(F)\}_{p,q}$ *satisfies*

$$(s(\chi)+p+1)c_{p,q}(F)+(d_\lambda+p-q+2)c_{p+2,q}(F)+(s(\chi')-q-1)c_{p+2,q+2}(F) = 0, \quad (4)$$

$$(s(\chi')+q+1)c_{p,q}(F)+(d_\lambda-p+q+2)c_{p,q+2}(F)+(s(\chi)-p-1)c_{p+2,q+2}(F) = 0. \quad (5)$$

Remark 3.12.   Since $\mathbf{F}_{p,q}(a_t)$ should be smoothly extended to $t = 0$, we have $c_{p,q}(F) = 0$ if $\lambda_1 + \lambda_2 > p + q$. Also we have $c_{p,q}(F) = 0$ if $d_\lambda(p,q) < 0$ or $d_\lambda < d_\lambda(p,q)$, that is, $d_\lambda < |p - q|$. Thus it is sufficient to give $c_{p,q}(F)$ for all $(p,q) \in E_{\chi,\chi'}$ with $\lambda_1 + \lambda_2 \leq p + q$ and $|p - q| \leq d_\lambda$.

In order to obtain Theorem 3.10, we prove the following proposition:

Proposition 3.13.   *Let $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}$ and η be an admissible representation of G_0. If $F \in \mathcal{S}^{Hol}_{\eta, \tau_\lambda, K}(G_0 \backslash G / K)$ is expressed as (1), $\{F^{(j,\pm)}\}_j$ satisfies the following system of difference-differential equations:*

$$\left(2\sinh^2(t)\eta(X_1^\pm) + 2\cosh^2(t)\eta(X_2^\mp)\right) F^{(j,\pm)}(a_t) + (j+1)\sinh(2t)F^{(j+1,\pm)}(a_t) = 0,$$

$$\left(\frac{d}{dt} + \frac{\lambda_1+\lambda_2}{\tanh(2t)} + \frac{d_\lambda - 2j}{\sinh(2t)} \mp \frac{2}{\sinh(2t)}\eta(H_2)\right) F^{(j,\pm)}(a_t) = 0,$$

$$\left(2\cosh^2(t)\eta(X_1^\mp) + 2\sinh^2(t)\eta(X_2^\pm)\right) F^{(j,\pm)}(a_t) + (d_\lambda+1-j)\sinh(2t)F^{(j-1,\pm)}(a_t) = 0.$$

Here we understand $F^{(j,\pm)} = 0$ for $j < 0$ or $j > d_\lambda$.

Proof.   From Lemma 3.8, we have

$$[R(X_{(0,-2)})\mathrm{pr}_+(F)](a_t)$$

$$= \left[R\left(\mathrm{Ad}(a_t)^{-1}\left(\frac{2\sinh^2(t)}{\cosh(2t)}X_1^+ + \frac{2\cosh^2(t)}{\cosh(2t)}X_2^-\right) + \tanh(2t)X_{\mathfrak{k}}^+\right)\mathrm{pr}_+(F)\right](a_t)$$

$$= \sum_{j=0}^{d_\lambda} \left(\frac{2\sinh^2(t)}{\cosh(2t)}\eta(X_1^+) + \frac{2\cosh^2(t)}{\cosh(2t)}\eta(X_2^-)\right) F^{(j,+)}(a_t) \otimes w_j^{\lambda^\vee}$$

$$+ \sum_{j=0}^{d_\lambda} j\tanh(2t)F^{(j,+)}(a_t) \otimes w_{j-1}^{\lambda^\vee}$$

$$= \sum_{j=0}^{d_\lambda} \left[\left(\frac{2\sinh^2(t)}{\cosh(2t)}\eta(X_1^+) + \frac{2\cosh^2(t)}{\cosh(2t)}\eta(X_2^-)\right) F^{(j,+)}(a_t)\right.$$

$$\left. + (j+1)\tanh(2t)F^{(j+1,+)}(a_t)\right] \otimes w_j^{\lambda^\vee}.$$

On the other hand, we have $R(X)\mathrm{pr}_+(F) = 0$ for all $X \in \mathfrak{p}_\mathbb{C}^-$. Thus we have the first equation. The others are similar.   ∎

Proof of Theorem 3.10. From Proposition 3.13, we have

$$
0 = \left(2\sinh^2(t)\eta(X_1^\pm) + 2\cosh^2(t)\eta(X_2^\mp)\right)F^{(j,\pm)}(t) + (j+1)\sinh(2t)F^{(j+1,\pm)}(t)
$$

$$
= \sum_{(p,q)\in E_{\chi,\chi'}} \left\{(s(\chi)-1+p)\sinh^2(t)F_{p-2,q}^{(j,\pm)}(t) + (s(\chi')-1-q)\cosh^2(t)F_{p,q+2}^{(j,\pm)}(t)\right.
$$
$$
\left. + (j+1)\sinh(2t)F_{p,q}^{(j+1,\pm)}(t)\right\}f_{\pm p,\pm q}.
$$

Thus for all $(p,q)\in E_{\chi,\chi'}$ we have

$$
(s(\chi)-1+p)\sinh^2(t)F_{p-2,q}^{(j,\pm)}(t) + (j+1)\sinh(2t)F_{p,q}^{(j+1,\pm)}(t)
$$
$$
+ (s(\chi')-1-q)\cosh^2(t)F_{p,q+2}^{(j,\pm)}(t) = 0.
$$

Here we understand $F_{p,q}^{(j,\pm)} = 0$ for $j < 0$ or $j > d_\lambda$. The others are similar. ∎

4. An explicit formula of the Shintani functions

Let $\eta = I(\chi,\chi')^\vee$ be the contragredient representation of $I(\chi,\chi')$ with

$$
\chi = \chi_{s_1,\varepsilon_1}\boxtimes\chi_{s_2,\varepsilon_2}, \quad \chi' = \chi_{s_1',\varepsilon_1'}\boxtimes\chi_{s_2',\varepsilon_2'},
$$

and $\tau = \tau_{\lambda,K}$ the irreducible unitary representation of K with $\lambda = (\lambda_1,\lambda_2)\in\overline{\mathcal{L}_+^{Hol}}$. In this section, we shall prove the uniqueness and an explicit formula of the Shintani functions for the holomorphic discrete series representation of G. Since $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$ if $\gamma \neq s_1 + s_2 + s_1' + s_2'$ or $d_\lambda \not\equiv \varepsilon(\chi) - \varepsilon(\chi') \bmod 2$ by Proposition 3.2 and Proposition 3.4 (i), we may assume that $\gamma = s_1 + s_2 + s_1' + s_2'$ and $d_\lambda \equiv \varepsilon(\chi) - \varepsilon(\chi') \bmod 2$ throughout this section.

4.1. Multiplicity free theorem

In this subsection, we prove $\dim_{\mathbb{C}}\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} \leq 1$. As its corollary, we have $\dim_{\mathbb{C}}\mathcal{I}_{\eta,\pi} \leq 1$, where $\pi = \Pi_\Lambda[\gamma]$ is the holomorphic discrete series representation of G. For $(l_1,l_2)\in\mathbb{Z}^2$, we set

$$
E_{l_1,l_2}(\lambda) := \left\{(p,q)\in\mathbb{Z}^2 \mid p\equiv l_1,\ q\equiv l_2 \bmod 2,\ |p-q|\leq d_\lambda,\right.
$$
$$
\left. p\geq l_1,\ q\geq l_2,\ p+q\geq\lambda_1+\lambda_2\right\}
$$

and $E_{\chi,\chi'}(\lambda) := E_{l_\chi,l_{\chi'}}(\lambda)\subset E_{\chi,\chi'}$.

Theorem 4.1. Let $\eta = I(\chi,\chi')^\vee$ and $\tau = \tau_{\lambda,K}$ with $\lambda\in\overline{\mathcal{L}_+^{Hol}}$. Assume that $d_\lambda\equiv\varepsilon(\chi)-\varepsilon(\chi') \bmod 2$ and $\gamma = s_1 + s_2 + s_1' + s_2'$.

  (i) *Assume that* $s(\chi) = l_\chi - 1, s(\chi') = l_{\chi'} - 1\in\mathbb{Z}$ *and* $l_\chi\equiv\varepsilon(\chi), l_{\chi'}\equiv\varepsilon(\chi') \bmod 2$. *If* $(l_\chi,l_{\chi'})\notin E_{\chi,\chi'}(\lambda)$, *then* $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$. *If* $(l_\chi,l_{\chi'})\in E_{\chi,\chi'}(\lambda)$, *then* $\dim_{\mathbb{C}}\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}\leq 1$. *In particular,* $\dim_{\mathbb{C}}\mathcal{I}_{\eta,\pi}\leq 1$.

  (ii) *Assume that at least one of* $I(\chi)$ *and* $I(\chi')$ *is the irreducible principal series representation of* $GL_2(\mathbb{R})$. *Then we have* $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$. *In particular,* $\dim_{\mathbb{C}}\mathcal{I}_{\eta,\pi} = 0$.

Remark 4.2. There exist several studies related to Theorem 4.1:

(i) Sun-Zhu [13] proved that for certain symmetric pairs (H, H_0) of classical Lie groups every irreducible Casselman-Wallach representation of H_0 occurs with multiplicity at most one in every irreducible Casselman-Wallach representation of H.

(ii) Kobayashi-Oshima [8] obtained upper and lower bounds of the multiplicities of irreducible admissible representations of a semisimple Lie group H occurring in the induced representations $\mathrm{Ind}_{H_0}^{H}(\eta)$ from irreducible admissible representations η of a closed subgroup H_0 of H.

(iii) For a reductive symmetric pair (H, H_0) of Lie groups, Kobayashi [7] studied Shintani functions for a pair of spherical, admissible smooth representations of H and H_0.

Assume that $s(\chi) = l_\chi - 1, s(\chi') = l_{\chi'} - 1 \in \mathbb{Z}$ and $l_\chi \equiv \varepsilon(\chi), l_{\chi'} \equiv \varepsilon(\chi') \bmod 2$. From (4) and (5) of Corollary 3.11, we have

$$(l_\chi + p - 2)c_{p-2,q-2}(F) + (d_\lambda + p - q + 2)c_{p,q-2}(F) + (l_{\chi'} - q)c_{p,q}(F) = 0, \qquad (6)$$

$$(l_{\chi'} + q - 2)c_{p-2,q-2}(F) + (d_\lambda - p + q + 2)c_{p-2,q}(F) + (l_\chi - p)c_{p,q}(F) = 0. \qquad (7)$$

For $M, k, l \in \mathbb{Z}$, we set $L_M(k, l) := \left\{ (p, q) \in \mathbb{Z}^2 \mid p + q = k + l + 2M \right\}$.

First we show the following lemma:

Lemma 4.3. *We have $c_{p,q}(F) = 0$ if $p < l_\chi$ or $q < l_{\chi'}$.*

Proof. We put $L_M^{(i)}(\lambda_1, \lambda_2)[l] = \{ (p_1, p_2) \in L_M(\lambda_1, \lambda_2) \mid p_i < l \}$. We shall see that, for any $q \in \mathbb{Z}$, $c_{p,q}(F) = 0$ if $p < l_\chi$. In order to prove it, it is sufficient to show that $c_{p,q}(F) = 0$ for all $M \in \mathbb{Z}$ and all $(p, q) \in L_M^{(1)}(\lambda_1, \lambda_2)[l_\chi]$. We shall show it by induction on $M \in \mathbb{Z}$. From Remark 3.12, if $(p, q) \in L_M(\lambda_1, \lambda_2)$ for $M < 0$, then $c_{p,q}(F) = 0$. Let $M \geq 0$. Assume that $c_{p,q}(F) = 0$ for all $N \in \mathbb{Z}$ with $N < M$ and all $(p, q) \in L_N^{(1)}(\lambda_1, \lambda_2)[l_\chi]$. Then, from (6) and the induction hypothesis, we have

$$c_{p,q}(F) = \frac{1}{p - l_\chi} \left\{ (l_{\chi'} + q - 2)c_{p-2,q-2}(F) + (d_\lambda - p + q + 2)c_{p-2,q}(F) \right\} = 0$$

for each $(p, q) \in L_M^{(1)}(\lambda_1, \lambda_2)[l_\chi]$. The other case can be proved in a similar way to the above. ∎

By Remark 3.12 and Lemma 4.3, we have $c_{p,q}(F) = 0$ for all $(p, q) \in E_{\chi,\chi'} - E_{\chi,\chi'}(\lambda)$. Thus each $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0 \backslash G / K)^{[\gamma]}$ can be expressed as

$$F(x) = \sum_{(p,q)\in E_{\chi,\chi'}(\lambda)} \mathbf{F}_{p,q}(x) \Psi_\lambda^{\chi,\chi'}(p, q) \qquad (8)$$

with $$\mathbf{F}_{p,q}(a_t) = c_{p,q}(F) \times \frac{\sinh(t)^{(p+q-\lambda_1-\lambda_2)/2}}{\cosh(t)^{(p+q+\lambda_1+\lambda_2)/2}},$$

where $c_{p,q}(F) \in \mathbb{C}$. Next we shall show the following proposition:

Proposition 4.4. *Let $\eta = I(\chi, \chi')^\vee$ and $\tau = \tau_{\lambda,K}$ with $\lambda \in \overline{\mathcal{L}_+^{Hol}}$. Assume that $s(\chi) = l_\chi - 1, s(\chi') = l_{\chi'} - 1 \in \mathbb{Z}$ and $l_\chi \equiv \varepsilon(\chi), l_{\chi'} \equiv \varepsilon(\chi') \bmod 2$. If $(l_\chi, l'_\chi) \notin E_{\chi,\chi'}(\lambda)$, $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$. If $(l_\chi, l'_\chi) \in E_{\chi,\chi'}(\lambda)$, then we have $l_\chi, l_{\chi'} \geq 2$. In particular, if at least one of $I(\chi)$ and $I(\chi')$ is the limit of discrete series representation of $GL_2(\mathbb{R})$, then $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$.*

Lemma 4.5. *Let $\eta = I(\chi, \chi')^\vee$ and $\tau = \tau_{\lambda,K}$ with $\lambda \in \overline{\mathcal{L}_+^{Hol}}$. Assume that $s(\chi) = l_\chi - 1, s(\chi') = l_{\chi'} - 1 \in \mathbb{Z}$ and $l_\chi \equiv \varepsilon(\chi), l_{\chi'} \equiv \varepsilon(\chi') \bmod 2$. Let $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}$ be expressed as (8). Then $F = 0$ if and only if $c_{l_\chi, l_{\chi'}}(F) = 0$.*

Proof. It is sufficient to show that $F = 0$ if $c_{l_\chi, l_{\chi'}}(F) = 0$. Assume that $c_{l_\chi, l_{\chi'}}(F) = 0$. We shall show $c_{p,q}(F) = 0$ for all $(p, q) \in E_{\chi,\chi'}(\lambda)$. In order to prove it, it is sufficient to show that $c_{p,q}(F) = 0$ for all $M \geq 0$ and all $(p, q) \in E_{\chi,\chi'}(\lambda) \cap L_M(l_\chi, l_{\chi'})$. We consider induction on $M \geq 0$.

Let $M = 0$. If $(p, q) \in E_{\chi,\chi'}(\lambda)$ satisfies $p + q = l_\chi + l_{\chi'}$, then $(p, q) = (l_\chi, l_{\chi'})$. Thus the statement is true for $M = 0$ by the assumption.

Let $M > 0$. Assume that the statement is true for all $n \in \mathbb{Z}$ with $0 \leq n < M$. Then $p > l_\chi$ or $q > l_{\chi'}$. Indeed, if $p = l_\chi$, we have $q = l_{\chi'} + 2M > l_{\chi'}$. By the induction hypothesis, it follows from (6) and (7) that the statement is true for M. Therefore we have $c_{p,q}(F) = 0$ for all $(p, q) \in E_{\chi,\chi'}$. ∎

Proof of Proposition 4.4. The first part of the proposition follows from Lemma 4.5 directly, and the second part of the proposition follows from $|l_\chi - l_{\chi'}| \leq d_\lambda$ and $\lambda_1 + \lambda_2 \leq l_\chi + l_{\chi'}$. ∎

Remark 4.6. Let $\chi = \chi_{s_1, \varepsilon_1} \boxtimes \chi_{s_2, \varepsilon_2}$ and $\chi' = \chi_{s'_1, \varepsilon'_1} \boxtimes \chi_{s'_2, \varepsilon'_2}$ be characters of the Borel subgroup B of $GL_2(\mathbb{R})$. Assume that $s(\chi) = l_\chi - 1, s(\chi') = l_{\chi'} - 1 \in \mathbb{Z}, l_\chi \equiv \varepsilon(\chi) \bmod 2$ and $l_{\chi'} \equiv \varepsilon(\chi') \bmod 2$. We also assume that $l_\chi, l_{\chi'} \geq 2$. Then $I(\chi, \chi')$ has an irreducible submodule $\mathcal{D}_{l_\chi}[s_1 + s_2] \boxtimes \mathcal{D}_{l_{\chi'}}[s'_1 + s'_2]$ (see Subsection 1.6).

Finally we consider the case where at least one of $I(\chi)$ and $I(\chi')$ is an irreducible principal series representation of $GL_2(\mathbb{R})$.

Proposition 4.7. *Let $\eta = I(\chi, \chi')^\vee$ and $\tau = \tau_{\lambda,K}$ with $\lambda \in \overline{\mathcal{L}_+^{Hol}}$. Assume that at least one of $I(\chi)$ and $I(\chi')$ is an irreducible principal series representation of $GL_2(\mathbb{R})$. Then we have $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$.*

Proof. Without loss of generality, we may assume that $I(\chi)$ is an irreducible principal series representation. Then we have $s(\chi) - p + 1 \neq 0$ for all $p \in \mathbb{Z}$ with $p \equiv \varepsilon(\chi) \bmod 2$. Let $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}$ can be expressed as (8). Then $c_{p,q}(F) = 0$ for all $(p, q) \in E_{\chi,\chi'}$. Indeed we can prove that $c_{p,q}(F) = 0$ for all $M \in \mathbb{Z}$ and all $(p, q) \in L_M(\lambda_1, \lambda_2)$ in a similar way to Lemma 4.3. Thus we have $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$. ∎

Proof of Theorem 4.1. By Proposition 4.4 and Proposition 4.7, we may assume that $(l_\chi, l'_\chi) \in E_{l_\chi, l_{\chi'}}(\lambda)$, $s(\chi) = l_\chi - 1, s(\chi') = l_{\chi'} - 1 \in \mathbb{Z}$, $l_\chi \equiv \varepsilon(\chi)$ and

$l_{\chi'} \equiv \varepsilon(\chi') \bmod 2$. Let $F^{(1)}$ and $F^{(2)}$ be nonzero elements of $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}$. Since $c_{l_\chi,l_{\chi'}}(F^{(1)}), c_{l_\chi,l_{\chi'}}(F^{(2)}) \neq 0$ from Lemma 4.5, there exists $\xi \in \mathbb{C}^\times$ such that $c_{l_\chi,l_{\chi'}}(F^{(1)}) = \xi c_{l_\chi,l_{\chi'}}(F^{(2)})$. Then $c_{l_\chi,l_{\chi'}}(F^{(1)} - \xi F^{(2)}) = 0$, that is, $F^{(1)} = \xi F^{(2)}$. $\blacksquare$

As a corollary of Theorem 4.1, we have the following:

Corollary 4.8. *Let $\eta = \mathcal{D}_{l_1}[c_1] \boxtimes \mathcal{D}_{l_2}[c_2]$ and $\tau = \tau_{\lambda,K}$ with $\lambda \in \overline{\mathcal{L}_+^{Hol}}$. If $(l_1,l_2) \notin E_{l_1,l_2}(\lambda)$, $d_\lambda \not\equiv l_1 - l_2 \bmod 2$ or $\gamma + c_1 + c_2 \neq 0$, then $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} = \{0\}$. Assume that $(l_1,l_2) \in E_{l_1,l_2}(\lambda)$, $d_\lambda \equiv l_1 - l_2 \bmod 2$ and $\gamma + c_1 + c_2 = 0$. Then $\dim_{\mathbb{C}} \mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} \leq 1$. In particular, $\dim_{\mathbb{C}} \mathcal{I}_{\eta,\pi} \leq 1$.*

4.2. An explicit formula for the holomorphic discrete series representations of $GSp_4(\mathbb{R})$

In this subsection, we give an explicit formula of Shintani functions of type $(\eta, \pi, \tau) = (I(\chi,\chi')^\vee, \Pi_{\Lambda(\lambda)}[\gamma], \tau_{\lambda,K})$ with $\lambda \in \overline{\mathcal{L}_+^{Hol}}$. Our main result is as follows:

Theorem 4.9. *Assume that $s(\chi) = l_\chi - 1$, $s(\chi') = l_{\chi'} - 1 \in \mathbb{Z}$, $l_\chi \equiv \varepsilon(\chi)$, $l_{\chi'} \equiv \varepsilon(\chi') \bmod 2$, $(l_\chi, l_{\chi'}) \in E_{l_\chi,l_{\chi'}}(\lambda)$, $d_\lambda \equiv l_\chi - l_{\chi'} \bmod 2$ and $\gamma = s_1 + s_2 + s_1' + s_2'$. If $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}$ is expressed as (8), then the coefficients $c_{p,q}(F)$ are given by*

$$c_{l_\chi+2M,l_{\chi'}+2N}(F) =$$

$$c_{l_\chi,l_{\chi'}}(F) \sum_{j=0}^{M} \binom{l_{\chi'} + N - 1}{j} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j}{M - j} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j}$$

for all $M, N \geq 0$ with $(l_\chi + 2M, l_{\chi'} + 2N) \in E_{\chi,\chi'}(\lambda)$. Here we set

$$\binom{a}{b} := \begin{cases} \dfrac{a!}{b!\,(a-b)!} & \text{if } a \geq b \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for every } (a,b) \in \mathbb{Z}^2.$$

Remark 4.10. We put

$$G_{M,N}(A,B,C) = \binom{C}{N}\binom{B+N}{M} {}_3F_2\left(\begin{matrix} -N,\, -M,\, 1 - A - N \\ -B - N,\, 1 + C - N \end{matrix} \,\middle|\, 1\right),$$

where ${}_3F_2$ is the generalized hypergeometric function:

$$ {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{z^n}{n!}.$$

Then the formula given in Theorem 4.9 can be written as

$$c_{l_\chi+2M,l_{\chi'}+2N}(F) = c_{l_\chi,l_{\chi'}}(F) G_{M,N}(l_{\chi'}, d_\lambda(l_{\chi'}, l_\chi), d_\lambda(l_\chi, l_{\chi'})).$$

In order to prove Theorem 4.9, we first shall show the following lemma:

Lemma 4.11. *Assume that $s(\chi) = l_\chi - 1, s(\chi') = l_{\chi'} - 1 \in \mathbb{Z}, l_\chi \equiv \varepsilon(\chi)$, $l_{\chi'} \equiv \varepsilon(\chi') \bmod 2$, $(l_\chi, l_\chi') \in E_{\chi,\chi'}(\lambda)$ and $d_\lambda \equiv l_\chi - l_{\chi'} \bmod 2$. If $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}$ is expressed as (8), then we have for all $N, M \geq 0$*

$$c_{l_\chi,l_{\chi'}+2N}(F) = c_{l_\chi,l_{\chi'}}(F) \binom{d_\lambda(l_\chi, l_{\chi'})}{N}, \quad c_{l_\chi+2M,l_{\chi'}}(F) = c_{l_\chi,l_{\chi'}}(F) \binom{d_\lambda(l_{\chi'}, l_\chi)}{M}.$$

Proof.   Let $F \in \mathcal{S}^{Hol}_{\eta,\tau}(G_0\backslash G/K)^{[\gamma]}$. From (6) and (7), we have

$$2(l_\chi - 1)c_{l_\chi-2,q}(F) + (d_\lambda + l_\chi - q)c_{l_\chi,q}(F) + (l_{\chi'} - q - 2)c_{l_\chi,q+2}(F) = 0,$$

$$2(l_{\chi'} - 1)c_{p,l_{\chi'}-2}(F) + (d_\lambda - p + l_{\chi'})c_{p,l_{\chi'}}(F) + (l_\chi - p - 2)c_{p+2,l_{\chi'}}(F) = 0.$$

These equations yield

$$(d_\lambda + l_\chi - q)c_{l_\chi,q}(F) + (l_{\chi'} - q - 2)c_{l_\chi,q+2}(F) = 0, \tag{9}$$

$$(d_\lambda - p + l_{\chi'})c_{p,l_{\chi'}}(F) + (l_\chi - p - 2)c_{p+2,l_{\chi'}}(F) = 0. \tag{10}$$

Since

$$c_{l_\chi,q+2}(F) = \frac{d_\lambda + l_\chi - q}{q - l_{\chi'} + 2}c_{l_\chi,q}(F), \quad \forall q \neq l_{\chi'} - 2, \tag{11}$$

$$c_{p+2,l_{\chi'}}(F) = \frac{d_\lambda + l_{\chi'} - p}{p - l_\chi + 2}c_{p,l_{\chi'}}(F), \quad \forall p \neq l_\chi - 2, \tag{12}$$

we have for all $N, M \geq 0$

$$c_{l_\chi,l_{\chi'}+2N}(F) = c_{l_\chi,l_{\chi'}}(F) \prod_{j=0}^{N-1} \frac{d_\lambda + l_\chi - l_{\chi'} - 2(N-j-1)}{2(N-j)}$$

$$= c_{l_\chi,l_{\chi'}}(F) \binom{(d_\lambda + l_\chi - l_{\chi'})/2}{N},$$

$$c_{l_\chi+2M,l_{\chi'}}(F) = c_{l_\chi,l_{\chi'}}(F) \prod_{j=0}^{M-1} \frac{d_\lambda - l_\chi + l_{\chi'} - 2(M-j-1)}{2(M-j)}$$

$$= c_{l_\chi,l_{\chi'}}(F) \binom{(d_\lambda - l_\chi + l_{\chi'})/2}{M}. \qquad \blacksquare$$

Proof of Theorem 4.9. Assume that F can be expressed as (8). We shall show that for all $M, N \geq 0$ the coefficients $c_{M,N} = c_{l_\chi+2M,l'_\chi+2N}(F)$ can be expressed as

$$c_{M,N} = \sum_{j=0}^{M} \binom{l_{\chi'} + N - 1}{j} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j}{M - j} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j} \tag{13}$$

by induction on $M \geq 0$. The statement is true for $M = 0$ by Lemma 4.11. Let $M > 0$. Assume that $c_{m,N}$ can be written as (13) for all $N \geq 0$ and all m with $0 \leq m < M$. Now $c_{M,N}$ satisfies (7), that is,

$$Mc_{M,N} = (l_{\chi'} + N - 1)c_{M-1,N-1} + (d_\lambda(l_{\chi'}, l_\chi) - M + N + 1)\, c_{M-1,N}. \tag{14}$$

Then, by the induction hypothesis, we have

$$Mc_{M,N} = (l_{\chi'} + N - 1)c_{M-1,N-1} + (d_\lambda(l_{\chi'}, l_\chi) - M + N + 1)\, c_{M-1,N}$$

$$= (l_{\chi'} + N - 1)$$

$$\times \sum_{j=0}^{M-1} \binom{l_{\chi'} + N - 2}{j} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j - 1}{M - j - 1} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j - 1}$$

$$+ (d_\lambda(l_{\chi'}, l_\chi) - M + N + 1)$$

$$\times \sum_{j=0}^{M-1} \binom{l_{\chi'} + N - 1}{j} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j}{M - j - 1} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j}$$

$$= (l_{\chi'} + N - 1)$$

$$\times \sum_{j=0}^{M} \binom{l_{\chi'} + N - 2}{j - 1} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j}{M - j} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j}$$

$$+ (d_\lambda(l_{\chi'}, l_\chi) - M + N + 1)$$

$$\times \sum_{j=0}^{M} \binom{l_{\chi'} + N - 1}{j} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j}{M - j - 1} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j}$$

$$= M \sum_{j=0}^{M} \binom{l_{\chi'} + N - 1}{j} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j}{M - j} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j}.$$

Thus the statement is true for $M > 0$. ∎

Remark 4.12. In the same way as the proof of Theorem 4.9, it follows from induction on N that

$$\sum_{j=0}^{N} \binom{l_\chi + M - 1}{j} \binom{d_\lambda(l_\chi, l_{\chi'}) + M - j}{N - j} \binom{d_\lambda(l_{\chi'}, l_\chi)}{M - j}$$

is also a solution of the system of difference equations given by (6) and (7). Thus the uniqueness implies that the following identity on multiple binomial sums holds:

$$\sum_{j=0}^{M} \binom{l_{\chi'} + N - 1}{j} \binom{d_\lambda(l_{\chi'}, l_\chi) + N - j}{M - j} \binom{d_\lambda(l_\chi, l_{\chi'})}{N - j}$$

$$= \sum_{j=0}^{N} \binom{l_\chi + M - 1}{j} \binom{d_\lambda(l_\chi, l_{\chi'}) + M - j}{N - j} \binom{d_\lambda(l_{\chi'}, l_\chi)}{M - j}.$$

This is a special case of a relation

$$G_{M,N}(A, B, C) = G_{N,M}(A - B + C, C, B),$$

which can be obtained from Sheppard's transformation (see [1, Corollary 3.3.4], for example)

$${}_3F_2 \left(\begin{array}{c} -N, a, b \\ c, d \end{array} \middle| 1 \right) = \frac{(c - a)_N (d - a)_N}{(c)_N (d)_N} {}_3F_2 \left(\begin{array}{c} -N, a, a + b - c - d - N + 1 \\ a - c - N + 1, a - d - N + 1 \end{array} \middle| 1 \right).$$

As a corollary of Theorem 4.9, we can derive an explicit formula of Shintani functions of type $(\mathcal{D}_{l_1}[c_1] \boxtimes \mathcal{D}_{l_2}[c_2], \Pi_{\Lambda(\lambda)}[\gamma], \tau_{\lambda,K})$. We set

$$\Psi_\lambda^{\mathbf{l},\mathbf{c}}(p, q)^\vee := \xi_{p,q}^\vee \otimes w_{d_\lambda(p,q)}^{\lambda\vee} + \xi_{-p,-q}^\vee \otimes \tau_{\lambda,K}^\vee(\epsilon) w_{d_\lambda(p,q)}^{\lambda\vee},$$

where $\mathbf{l} = (l_1, l_2)$, $\mathbf{c} = (c_1, c_2)$ and $\xi_{p,q}^\vee := \xi_p^{(l_1,c_1)\vee} \otimes \xi_q^{(l_2,c_2)\vee}$.

Corollary 4.13. *Let $\eta = \mathcal{D}_{l_1}[c_1] \boxtimes \mathcal{D}_{l_2}[c_2]$ be the discrete series representation of $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ and $\tau = \tau_{\lambda,K}$ the irreducible unitary representation of K with $\lambda \in \overline{\mathcal{L}_+^{Hol}}$. Assume that $(l_1, l_2) \in E_{l_1,l_2}(\lambda)$, $d_\lambda \equiv l_1 - l_2 \bmod 2$ and $\gamma + c_1 + c_2 = 0$. Then each $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0 \backslash G / K)^{[\gamma]}$ can be expressed up to scalar multiplication as*

$$F(a_t) = \frac{\sinh(t)^{(l_1+l_2-\lambda_1-\lambda_2)/2}}{\cosh(t)^{(l_1+l_2+\lambda_1+\lambda_2)/2}} \sum \left(\frac{\sinh(t)}{\cosh(t)} \right)^{M+N}$$

$$\times \sum_{j=0}^{M} \binom{l_2+N-1}{j} \binom{d_\lambda(l_2,l_1)+N-j}{M-j} \binom{d_\lambda(l_1,l_2)}{N-j} \Psi_\lambda^{\mathbf{l},\mathbf{c}}(l_1+2M,\ l_2+2N)^\vee,$$

where the first sum runs over the integers M, N with $(l_1 + 2M, l_2 + 2N) \in E_{l_1,l_2}(\lambda)$.

5. An archimedean local zeta integral of Murase-Sugano type

In this section, we shall formulate a local zeta integral of Murase-Sugano type for the reductive symmetric pair (G, G_0) and evaluate it by using our explicit formula of Shintani functions.

5.1. Definition of a local zeta integral of Murase-Sugano type

In this subsection, we formulate a local zeta integral of Murase-Sugano type for (G, G_0). Let G_1 be a subgroup of $GL_4(\mathbb{R})$ given by

$$G_1 = \{g \in GL_4(\mathbb{R}) \mid \det(g) > 0\}.$$

Then G and G_0 are subgroups of G_1. Let P_1 be a maximal parabolic subgroup of G_1 given by

$$P_1 = \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ & * & & * \\ \hline * & * & * & * \\ & * & & * \end{array} \right) \in G_1 \right\},$$

and $P_1 = M_1 N_1$ a Levi decomposition of P_1, where

$$M_1 = \{\mathbf{m}_1(a,b) \mid a, b \in GL_2(\mathbb{R}), \det(ab) > 0\}, \quad N_1 = \left\{ \left(\begin{array}{cc|cc} 1 & * & & * \\ & 1 & & \\ \hline & * & 1 & * \\ & & & 1 \end{array} \right) \in G_1 \right\}.$$

Here, for $a, b \in GL_2(\mathbb{R})$, we set

$$\mathbf{m}_1(a,b) := \left(\begin{array}{cccc} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{array} \right) \left(\begin{array}{cc} a & \\ & b \end{array} \right) \left(\begin{array}{cccc} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{array} \right)^{-1}.$$

Each $\mathbf{m}_1(a,b) \in M_1$ has a factorization

$$[\mathbf{m}_1(a,b) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} \cdot a & \\ & b \end{pmatrix} \begin{pmatrix} \alpha \cdot I_2 & \\ & I_2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{pmatrix}^{-1},$$

where $\alpha \in \mathbb{R}^\times$ with $\qquad \alpha^2 = \dfrac{\det(ab)}{\det(b)^2} = \dfrac{\det(a)}{\det(b)}.$

Thus every $m_1 \in M_1$ has a factorization

$$m_1 = \beta(m_1)\mathrm{diag}(\alpha(m_1), 1, \alpha(m_1), 1)$$

for some $(\beta(m_1), \alpha(m_1)) \in G_0 \times \mathbb{R}^\times$. Note that such a factorization of m_1 is not unique. We set $K_1 := G_1 \cap O(4, \mathbb{R}) = SO(4, \mathbb{R})$. Then K_1 is the maximal compact subgroup of G_1. Each $g \in G_1$ has an Iwasawa decomposition

$$g = m_1(g)n_1(g)k_1(g)$$
$$= \beta(m_1(g))\mathrm{diag}(\alpha(m_1(g)), 1, \alpha(m_1(g)), 1)n_1(g)k_1(g)$$

for some $(m_1(g), n_1(g), k_1(g)) \in M_1 \times N_1 \times K_1$. For each $g \in G_1$, we fix such a factorization and set $\beta(g) := \beta(m_1(g))$ and $\alpha(g) := \alpha(m_1(g))$. The following lemma can be easily checked by direct computation:

Lemma 5.1. $\qquad P_1 \cap K_1 = K_0 = \{\mathbf{m}_1(a,b) \mid a, b \in O(2, \mathbb{R}), \ \det(ab) = 1\}.$

Let (τ, W), (τ_0, W_0) and (τ_1, W_1) be irreducible unitary representations of K, K_0 and K_1, respectively. Assume that $\mathrm{Hom}_{K_0}(\tau_0, \tau_1) \neq \{0\}$ and $\mathrm{Hom}_K(\tau^\vee, \tau_1^\vee) \neq \{0\}$. For $\gamma \in \mathbb{C}$, let $C_{\tau_0^\vee, \tau}^\infty(K_0 \backslash G / K Z_+)^{[\gamma]}$ denote the space of $(W_0 \otimes W^\vee)$-valued smooth functions $S : G \to W_0 \otimes W_\tau^\vee$ with

$$S(rk_0xk) = r^\gamma \tau_0(k_0) \otimes \tau^\vee(k)^{-1} S(x), \quad \forall(k_0, x, k, r) \in K_0 \times G \times K \times Z_+,$$

where Z_+ is the identity component of the center of G, that is, $Z_+ := \{rI_4 \mid r > 0\}$. We fix embeddings $\imath_{\tau_0} \in \mathrm{Hom}_{K_0}(\tau_0, \tau_1)$ and $\imath_\tau^\vee \in \mathrm{Hom}_K(\tau^\vee, \tau_1^\vee)$.

Proposition 5.2. *An invariant measure $d\dot{g}$ on $G_0 \backslash G$ is given by*

$$\int_{G_0 \backslash G} f(g) d\dot{g} = \int_0^\infty dt \int_K dk f(a_t k) \sinh(2t) \cosh^2(2t),$$

where dt is the Lebesgue measure on $\mathbb{R}$ and dk is the Haar measure on K normalized so that $\mathrm{vol}(K; dk) = 1$.

Proof. See [4, p.110], for example. ∎

For each $S \in C_{\tau_0^\vee, \tau}^\infty(K_0 \backslash G / K Z_+)^{[\gamma]}$, a local zeta integral of Murase-Sugano type is defined by

$$Z_{MS}(s; S) := \int_{G_0 \backslash G} \epsilon_{W_1} \circ (\mathrm{Id}_{W_1} \otimes \tau_1^\vee(k_1(g))) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\beta(g)^{-1}g) |\alpha(g)|^{s+3/2} d\dot{g}, \quad (15)$$

where $\epsilon_{W_1} : W_1 \otimes W_1^\vee \to \mathbb{C}$ is defined by $v \otimes w^\vee \mapsto \langle v, w^\vee \rangle_{W_1}$. In this formulation $\langle \cdot, \cdot \rangle_{W_1} : W_1 \times W_1^\vee \to \mathbb{C}$ is the natural pairing.

Lemma 5.3. *The value of the integrand of the right hand side of* (15) *is independent of the choice of the Iwasawa decomposition* $g = \beta(g)\operatorname{diag}(\alpha(g), 1, \alpha(g), 1)n_1(g)k_1(g)$.

Proof. Let $g \in G$. If $g = \beta'(g)\operatorname{diag}(\alpha'(g), 1, \alpha'(g), 1)n_1'(g)k_1'(g)$ is another Iwasawa decomposition of g, then, by Lemma 5.1, we have $\beta'(g) = \beta(g)k_0$, $k_1'(g) = k_0^{-1}k_1(g)$ for some $k_0 \in K_0$ and $\alpha'(g) = \alpha(g)$. Thus we have

$$
\begin{aligned}
&\epsilon_{W_1} \circ (\operatorname{Id}_{W_1} \otimes \tau_1^\vee(k_1'(g))) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\beta'(g)^{-1}g)|\alpha'(g)|^s \\
&= \epsilon_{W_1} \circ (\operatorname{Id}_{W_1} \otimes \tau_1^\vee(k_0^{-1}k_1(g))) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(k_0^{-1}\beta(g)^{-1}g)|\alpha(g)|^s \\
&= \epsilon_{W_1} \circ (\operatorname{Id}_{W_1} \otimes \tau_1^\vee(k_0^{-1}k_1(g))) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ (\tau_0(k_0)^{-1} \otimes \operatorname{Id}_{W^\vee}) \circ S(\beta(g)^{-1}g)|\alpha(g)|^s \\
&= \epsilon_{W_1} \circ (\tau_1(k_0)^{-1} \otimes \tau_1^\vee(k_0^{-1}k_1(g))) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\beta(g)^{-1}g)|\alpha(g)|^s \\
&= \epsilon_{W_1} \circ (\operatorname{Id}_{W_1} \otimes \tau_1^\vee(k_1(g))) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\beta(g)^{-1}g)|\alpha(g)|^s. \qquad \blacksquare
\end{aligned}
$$

We shall prove the following proposition. This proposition is used in Subsection 5.3 to evaluate the local zeta integral $Z_{MS}(s; S)$.

Proposition 5.4. *For each* $S \in C^\infty_{\tau_0^\vee, \tau}(K_0 \backslash G / K Z_+)^{[\gamma]}$, *we have*

$$
\begin{aligned}
&Z_{MS}(s; S) \\
&= \int_0^\infty dt \; \epsilon_{W_1} \circ (\tau_1(\kappa_{-t}) \otimes \operatorname{Id}_{W_1^\vee}) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(a_t) \sinh(2t) \cosh(2t)^{1/2 - \gamma/2 - s}.
\end{aligned}
$$

For $t > 0$, we set

$$
\kappa_t := \frac{1}{\sqrt{\cosh(2t)}} \left(\begin{array}{cc|cc} \cosh t & -\sinh t & & \\ \sinh t & \cosh t & & \\ \hline & & \cosh t & \sinh t \\ & & -\sinh t & \cosh t \end{array} \right) \in K_1.
$$

We can easily check the following lemma by direct computation:

Lemma 5.5. *An Iwasawa decomposition of* a_t *is given by*

$$
a_t = \sqrt{\cosh(2t)} \left(\begin{array}{cc|cc} 1/\cosh(2t) & & & \\ & 1 & & \\ \hline & & 1/\cosh(2t) & \\ & & & 1 \end{array} \right)
$$
$$
\times \left(\begin{array}{cc|cc} 1 & \sinh(2t) & & \\ & 1 & & \\ \hline & & 1 & -\sinh(2t) \\ & & & 1 \end{array} \right) \kappa_t. \quad (16)
$$

In particular, $\beta(a_t) = \sqrt{\cosh(2t)} \cdot I_4$ *and* $\alpha(a_t) = \cosh(2t)^{-1}$.

Proof of Proposition 5.4. From Proposition 5.2, we have

$$
\begin{aligned}
Z_{MS}(s; S) = \int_0^\infty dt \int_K dk \; \epsilon_{W_1} \circ (\operatorname{Id}_{W_1} \otimes \tau_1^\vee(k_1(a_t k))) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\beta(a_t k)^{-1} a_t k) \\
\times |\alpha(a_t k)|^{s+3/2} \sinh(2t) \cosh^2(2t).
\end{aligned}
$$

Thus, from Lemma 5.5, we have

$$
\begin{aligned}
Z_{MS}(s; S) &= \int_0^\infty dt \int_K dk \; \epsilon_{W_1} \circ (\mathrm{Id}_{W_1} \otimes \tau_1^\vee(\kappa_t k)) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\beta(a_t)^{-1} a_t k) \\
&\qquad \times |\alpha(a_t)|^{s+3/2} \sinh(2t) \cosh^2(2t) \\
&= \int_0^\infty dt \int_K dk \; \epsilon_{W_1} \circ (\mathrm{Id}_{W_1} \otimes \tau_1^\vee(\kappa_t)) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\beta(a_t)^{-1} a_t) \\
&\qquad \times |\alpha(a_t)|^{s+3/2} \sinh(2t) \cosh^2(2t) \\
&= \int_0^\infty dt \; \epsilon_{W_1} \circ (\mathrm{Id}_{W_1} \otimes \tau_1^\vee(\kappa_t)) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(\sqrt{\cosh(2t)}^{-1} a_t) \\
&\qquad \times |\alpha(a_t)|^{s+3/2} \sinh(2t) \cosh^2(2t) \\
&= \int_0^\infty dt \; \epsilon_{W_1} \circ (\mathrm{Id}_{W_1} \otimes \tau_1^\vee(\kappa_t)) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(a_t) \sinh(2t) \cosh(2t)^{-s-\gamma/2+1/2} \\
&= \int_0^\infty dt \; \epsilon_{W_1} \circ (\tau_1(\kappa_{-t}) \otimes \mathrm{Id}_{W_1^\vee}) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S(a_t) \sinh(2t) \cosh(2t)^{-s-\gamma/2+1/2}. \quad \blacksquare
\end{aligned}
$$

5.2. The unitary dual of $SO(4)$

In this subsection, we recall the irreducible unitary representations of $K_1 = SO(4)$. First we recall an accidental isomorphism $(SU(2) \times SU(2))/\{\pm(I_2, I_2)\} \simeq K_1$. The accidental isomorphism is given by $\xi : SU(2) \times SU(2) \to K_1$,

$$
\left(\begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 \\ -b_1 + ib_2 & a_1 - ia_2 \end{pmatrix}, \begin{pmatrix} c_1 + ic_2 & d_1 + ic_2 \\ -d_1 + id_2 & c_1 - ic_2 \end{pmatrix} \right)
$$

$$
\to \left(\begin{array}{cc|cc} a_1 & -b_1 & -a_2 & -b_2 \\ b_1 & a_1 & b_2 & -a_2 \\ \hline a_2 & -b_2 & a_1 & b_1 \\ b_2 & a_2 & -b_1 & a_1 \end{array} \right) \left(\begin{array}{cc|cc} c_1 & d_1 & c_2 & d_2 \\ -d_1 & c_1 & d_2 & -c_2 \\ \hline -c_2 & -d_2 & c_1 & d_1 \\ -d_2 & c_2 & -d_1 & c_1 \end{array} \right).
$$

For each $\alpha, \beta \in SU(2)$, we set $\mathbf{k}_1[\alpha, \beta] := \xi\big((\alpha, \beta)\{\pm(I_2, I_2)\}\big)$. For example,

$$
\epsilon = \mathbf{k}_1\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right],
$$

$$
\kappa_t = \mathbf{k}_1\left[\sqrt{\cosh(2t)}^{-1} \begin{pmatrix} \cosh(t) & \sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix}, I_2\right].
$$

Next we recall the unitary dual of $SU(2)$. Let V_n be the space of homogeneous polynomials with two variables z_1, z_2 of degree n over $\mathbb{C}$. Then $GL_2(\mathbb{C})$ acts on V_n as follows:

$$
[S_n(g)f](z_1, z_2) := f(g_{11}z_1 + g_{21}z_2, g_{12}z_1 + g_{22}z_2), \quad \forall g = (g_{ij}) \in GL_2(\mathbb{C}).
$$

It is well-known that the irreducible unitary representations of $SU(2)$ are exhausted by $S_n = S_n|_{SU(2)}$ $(n = 0, 1, \cdots)$. Thus those of K_1 are exhausted by

$$
T_{(n_1, n_2)} := S_{n_1 + n_2} \boxtimes S_{n_1 - n_2} \quad (n_1 \geq |n_2|).
$$

We set $h_{p,q}^{(n_1,n_2)} := f_p^{(n_1+n_2)} \otimes f_q^{(n_1-n_2)}$, where $\{f_p^{(n)} := \binom{n}{p} z_1^p z_2^{n-p}\}_{p=0,\cdots,n}$ is a basis of V_n. Then $\{h_{p,q}^{(n_1,n_2)}\}_{p,q}$ is a basis of $T_{(n_1,n_2)}$.

Finally we observe the restriction of $T_{(n_1,n_2)}$ $(n_1 \geq |n_2|)$ to K. Each $k \in U(2)$ is identified with

$$
\mathbf{k}_1\left[\begin{pmatrix} e^{-i\theta/2} & \\ & e^{i\theta/2} \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\right]
$$

$$
= \left(\begin{array}{cc|cc}
\cos(\theta/2) & & \sin(\theta/2) & \\
& \cos(\theta/2) & & \sin(\theta/2) \\
\hline
-\sin(\theta/2) & & \cos(\theta/2) & \\
& -\sin(\theta/2) & & \cos(\theta/2)
\end{array}\right)
\left(\begin{array}{cc|cc}
\alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\
-\beta_1 & \alpha_1 & \beta_2 & -\alpha_2 \\
\hline
-\alpha_2 & -\beta_2 & \alpha_1 & \beta_1 \\
-\beta_2 & \alpha_2 & -\beta_1 & \alpha_1
\end{array}\right),
$$

where $\theta \in \mathbb{R}$, $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ are given by

$$
\det(k) = e^{i\theta}, \quad \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -\beta_1 + i\beta_2 & \alpha_1 - i\alpha_2 \end{pmatrix} = ke^{-i\theta/2} \in SU(2).
$$

We can easily check the following:

Lemma 5.6. $\quad T_{(n_1,n_2)}(Z_{\mathfrak{k}})h_{p,q}^{(n_1,n_2)} = (n_1 + n_2 - 2p)h_{p,q}^{(n_1,n_2)},$

$$
T_{(n_1,n_2)}(H_{\mathfrak{k}})h_{p,q}^{(n_1,n_2)} = (2q - n_1 + n_2)h_{p,q}^{(n_1,n_2)},
$$

$$
T_{(n_1,n_2)}(X_{\mathfrak{k}}^+)h_{p,q}^{(n_1,n_2)} = (q+1)h_{p,q+1}^{(n_1,n_2)},
$$

$$
T_{(n_1,n_2)}(X_{\mathfrak{k}}^-)h_{p,q}^{(n_1,n_2)} = (n_1 - n_2 + 1 - q)h_{p,q-1}^{(n_1,n_2)},
$$

$$
T_{(n_1,n_2)}(r(\theta_1,\theta_2))h_{p,q}^{(n_1,n_2)} = \exp\left(i\big((n_2 - p + q)\,\theta_1 + (n_1 - p - q)\,\theta_2\big)\right)h_{p,q}^{(n_1,n_2)},
$$

$$
T_{(n_1,n_2)}(\epsilon)h_{p,q}^{(n_1,n_2)} = (-1)^{p+q}h_{n_1+n_2-p,n_1-n_2-q}^{(n_1,n_2)}.
$$

Lemma 5.6 implies that $T_{(n_1,n_2)}$ is decomposed as a K-module as follows:

Lemma 5.7. $\quad T_{(n_1,n_2)}|_K$

$$
\simeq \begin{cases}
\bigoplus_{j=0}^{(n_1+n_2-1)/2} \tau_{(n_1-j,n_2-j),K} & (\textit{if } n_1 + n_2 \textit{ is odd}), \\
\bigoplus_{j=0}^{(n_1+n_2)/2-1} \tau_{(n_1-j,n_2-j),K} \oplus \tau_{((n_1-n_2)/2,(n_2-n_1)/2)}^+ & (\textit{if } n_1 + n_2 \equiv 0 \bmod 4), \\
\bigoplus_{j=0}^{(n_1+n_2)/2-1} \tau_{(n_1-j,n_2-j),K} \oplus \tau_{((n_1-n_2)/2,(n_2-n_1)/2)}^- & (\textit{if } n_1 + n_2 \equiv 2 \bmod 4).
\end{cases}
$$

5.3. Evaluation of the archimedean local zeta integral

Let $\lambda = (\lambda_1, \lambda_2) \in \overline{\mathcal{L}_+^{Hol}}$ and $\eta = \mathcal{D}_{l_1}[c_1] \boxtimes \mathcal{D}_{l_2}[c_2]$ be the discrete series representation of G_0 with $l_1 - l_2 \equiv d_\lambda \bmod 2$, $(l_1, l_2) \in E_{l_1,l_2}(\lambda)$. We put $\tau = \tau_{\lambda,K}, \tau_0 = \sigma_{(l_1,l_2),K_0}$ and $\tau_1 = T_\lambda$, and assume that $(l_1, l_2) = (\lambda_2 + \mu, \lambda_1 - \mu)$ for $\mu = 0$ or d_λ. Then we have $\mathrm{Hom}_K(\tau^\vee, \tau_1^\vee) \neq \{0\}$ and $\mathrm{Hom}_{K_0}(\tau_0^\vee, \tau_1) \neq \{0\}$. We can take the embeddings

$\imath_{\tau^\vee} \in \mathrm{Hom}_K(\tau^\vee, \tau_1^\vee)$ and $\imath_{\tau_0} \in \mathrm{Hom}_{K_0}(\tau_0^\vee, \tau_1)$ given by

$$\imath_{\tau^\vee}(w_\mu^{\lambda\vee}) = h_{0,\mu}^{\lambda\vee}, \quad \imath_{\tau^\vee}(\tau^\vee(\epsilon)w_\mu^{\lambda\vee}) = (-1)^\mu h_{\lambda_1+\lambda_2, d_\lambda - \mu}^{\lambda\vee},$$

$$\imath_{\tau_0}(\xi_{\lambda_2+\mu, \lambda_1-\mu}^\vee) = h_{0,\mu}^\lambda, \quad \imath_{\tau_0}(\xi_{-\lambda_2-\mu, -\lambda_1+\mu}^\vee) = (-1)^\mu h_{\lambda_1+\lambda_2, d_\lambda - \mu}^\lambda.$$

Let $\mathrm{pr}_{(l_1,l_2)}^\vee \in \mathrm{Hom}_{K_0}(\eta^\vee, \tau_0^\vee)$ be the projection. For each $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0 \backslash G/K)^{[\gamma]}$, we set $S_F(x) := (\mathrm{pr}_{(l_1,l_2)}^\vee \otimes \mathrm{Id})(F(x))$. Then $S_F \in C_{\tau_0,\tau}^\infty(K_0 \backslash G/KZ_+)^{[\gamma]}$. We may assume $\gamma + c_1 + c_2 = 0$ by Proposition 3.2. Under the above assumptions, we have the following:

Theorem 5.8. *Let $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}$ be an element with $F(I_4) = 1$. Then $Z_{MS}(s; S_F)$ converges absolutely for $\mathrm{Re}(s) > s + \frac{\mathrm{Re}(\gamma)}{2} + \frac{\lambda_1+\lambda_2}{2} - \frac{3}{2}$. Moreover we have*

$$Z_{MS}(s; S_F) = \frac{1}{s + \frac{\gamma}{2} + \frac{\lambda_1+\lambda_2}{2} - \frac{3}{2}}. \tag{17}$$

Remark 5.9. We set

$$L_\infty(s; \Pi_{\Lambda(\lambda)}) := \Gamma_{\mathbb{C}}\left(s + \frac{\gamma}{2} + \frac{\lambda_1+\lambda_2}{2} - \frac{3}{2}\right) \Gamma_{\mathbb{C}}\left(s + \frac{\gamma}{2} + \frac{\lambda_1-\lambda_2}{2} + \frac{1}{2}\right),$$

$$L_\infty(s; \mathcal{D}_{l_1}[c_1] \boxtimes \mathcal{D}_{l_2}[c_2]) :=$$

$$\Gamma_{\mathbb{C}}\left(s + \frac{c_1+c_2}{2} + \frac{l_1+l_2}{2} - 1\right) \Gamma_{\mathbb{C}}\left(s + \frac{c_1+c_2}{2} + \frac{|l_1-l_2|}{2}\right),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then $L_\infty(s; \Pi_{\Lambda(\lambda)})$ and $L_\infty(s; \mathcal{D}_{l_1}[c_1] \boxtimes \mathcal{D}_{l_2}[c_2])$ are the local L-factors associated with $\Pi_{\Lambda(\lambda)}[\gamma]$ and $\mathcal{D}_{l_1}[c_1] \boxtimes \mathcal{D}_{l_2}[c_2]$, respectively. With these notations, Theorem 5.8 can be written as

$$Z_{MS}(s; S_F) = \frac{L_\infty(s; \Pi_{\Lambda(\lambda)})}{L_\infty(s + \frac{1}{2}; \mathcal{D}_{l_1}[-c_1] \boxtimes \mathcal{D}_{l_2}[-c_2])}.$$

Now we shall prove Theorem 5.8. Assume that $\mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]} \neq \{0\}$. By Corollary 4.13, we can take $F \in \mathcal{S}_{\eta,\tau}^{Hol}(G_0\backslash G/K)^{[\gamma]}$ with the radial part expressed as

$$F(a_t) = \frac{\sinh(t)^{(l_1+l_2-\lambda_1-\lambda_2)/2}}{\cosh(t)^{(l_1+l_2+\lambda_1+\lambda_2)/2}} \sum_{M,N} \left(\frac{\sinh(t)}{\cosh(t)}\right)^{M+N}$$

$$\times \sum_{i=0}^M \binom{l_2+N-1}{i} \binom{d_\lambda(l_2,l_1)+N-i}{M-i} \binom{d_\lambda(l_1,l_2)}{N-i} \Psi_\lambda^{\mathbf{l},\mathbf{c}}(l_1+2M, l_2+2N)^\vee,$$

where the first sum runs over the integers M, N with $(l_1+2M, l_2+2N) \in E_{l_1,l_2}(\lambda)$. Then we have

$$S_F(a_t) = \frac{\sinh(t)^{(l_1+l_2-\lambda_1-\lambda_2)/2}}{\cosh(t)^{(l_1+l_2+\lambda_1+\lambda_2)/2}} \Psi_\lambda^{\mathbf{l},\mathbf{c}}(l_1, l_2)^\vee.$$

Assume that $(l_1, l_2) = (\lambda_2+\mu, \lambda_1-\mu)$ for $\mu = 0$ or d_λ. Then

$$d_\lambda(l_1, l_2) = d_\lambda(\lambda_2+\mu, \lambda_1-\mu) = \mu.$$

From Lemma 5.4, we have

$$
\begin{aligned}
Z_{MS}&(s; S_F) \\
&= \int_0^\infty dt\; \epsilon_{W_1} \circ (\tau_1(\kappa_{-t}) \otimes \mathrm{Id}_{W_1^\vee}) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \circ S_F(a_t) \sinh(2t) \cosh(2t)^{-s-\gamma/2+1/2} \\
&= \int_0^\infty dt\; \cosh(t)^{-\lambda_1-\lambda_2} \sinh(2t) \cosh(2t)^{-s-\gamma/2+1/2} \\
&\qquad \times \epsilon_{W_1} \circ (\tau_1(\kappa_{-t}) \otimes \mathrm{Id}_{W_1^\vee}) \circ (\imath_{\tau_0} \otimes \imath_\tau^\vee) \left(\Psi_\lambda^{\mathbf{l,c}}(\lambda_2 + \mu, \lambda_1 - \mu)^\vee \right) \\
&= \int_0^\infty dt\; \cosh(t)^{-\lambda_1-\lambda_2} \sinh(2t) \cosh(2t)^{-s-\gamma/2+1/2} \\
&\qquad \times \epsilon_{W_1} \circ (\tau_1(\kappa_{-t}) \otimes \mathrm{Id}_{W_1^\vee}) \left(h_{0,\mu}^\lambda \otimes h_{0,\mu}^{\lambda\vee} + h_{\lambda_1+\lambda_2,d_\lambda-\mu}^\lambda \otimes h_{\lambda_1+\lambda_2,d_\lambda-\mu}^{\lambda\vee} \right) \\
&= \int_0^\infty dt\; \cosh(t)^{-\lambda_1-\lambda_2} \sinh(2t) \cosh(2t)^{-s-\gamma/2+1/2} \\
&\qquad \times \left(\langle \tau_1(\kappa_{-t}) h_{0,\mu}^\lambda, h_{0,\mu}^{\lambda\vee} \rangle_{W_1} + \langle \tau_1(\kappa_{-t}) h_{\lambda_1+\lambda_2,d_\lambda-\mu}^\lambda \otimes h_{\lambda_1+\lambda_2,d_\lambda-\mu}^{\lambda\vee} \rangle_{W_1} \right).
\end{aligned}
$$

The following lemma can be easily checked:

Lemma 5.10.

$$
\tau_1(\kappa_{-t}) h_{0,\mu}^\lambda = \cosh(2t)^{-(\lambda_1+\lambda_2)/2} \sum_{i=0}^{\lambda_1+\lambda_2} (-1)^i (\sinh t)^i (\cosh t)^{\lambda_1+\lambda_2-i} h_{i,\mu}^\lambda,
$$

$$
\tau_1(\kappa_{-t}) h_{\lambda_1+\lambda_2,d_\lambda-\mu}^\lambda = \cosh(2t)^{-(\lambda_1+\lambda_2)/2} \sum_{i=0}^{\lambda_1+\lambda_2} (\sinh t)^{\lambda_1+\lambda_2-i} (\cosh t)^i h_{i,d_\lambda-\mu}^\lambda.
$$

Thus we have

$$
Z_{MS}(s; S_F) = 2 \int_0^\infty \sinh(2t) \cosh(2t)^{-s-\gamma/2-(\lambda_1+\lambda_2)/2+1/2} dt\,.
$$

Putting $x = \cosh(2t)^{-1}$, we have

$$
Z_{MS}(s; S_F) = \int_0^1 x^{s+\gamma/2+(\lambda_1+\lambda_2)/2-5/2} dx = \frac{1}{s + \frac{\gamma}{2} + \frac{\lambda_1+\lambda_2}{2} - \frac{3}{2}}
$$

for $\mathrm{Re}(s + \gamma/2 + (\lambda_1 + \lambda_2)/2 - 3/2) > 0$. This completes the proof of the theorem.

References

[1] G. E. Andrews, R. Askey, R. Roy: *Special functions*, Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, Cambridge (1999).

[2] K. Gejima: *An explicit formula of the unramified Shintani functions for* $(\mathbf{GSp}_4, \mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)$ *and its application*, Proc. Japan Acad. Ser. A 93 (2017) 105–110.

[3] K. Gejima: *An explicit formula of the unramified Shintani functions for* $(\mathbf{GSp}_4, \mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)$, J. Number Theory 183 (2018) 84–132.

[4] G. Heckman, H. Schlichtkrull: *Harmonic Analysis and Special Functions on Symmetric Spaces*, Perspectives in Mathematics 16, Academic Press, New York (1994).

[5] S. Kato, A. Murase, T. Sugano: *Whittaker-Shintani functions for orthogonal groups*, Tohoku Math. J. 55 (2003) 1–64.

[6] A. W. Knapp: *Representation Theory of Semisimple Groups*, Princeton Univ. Press, Princeton (1986).

[7] T. Kobayashi: *Shintani functions, real spherical manifolds, and symmetric breaking operators*, Developments Math. 37 (2014) 127–159.

[8] T. Kobayashi, T. Oshima: *Finite multiplicity theorems for induction and restriction*, Adv. Math. 248 (2013) 921–944.

[9] T. Moriyama: *Spherical functions with respect to the semisimple symmetric pair* $(Sp(2, \mathbf{R}), SL(2, \mathbf{R}) \times SL(2, \mathbf{R}))$, J. Math. Sci. Univ. Tokyo 6 (1999) 127–179.

[10] T. Moriyama: *Spherical functions for the semisimple symmetric pair* $(Sp(2, \mathbf{R}), SL(2, \mathbf{C}))$, Canad. J. Math. 54 (2002) 828–865.

[11] A. Murase, T. Sugano: *Shintani function and its application to automorphic L-functions for classical groups. I: The orthogonal group case*, Math. Ann. 299 (1994) 17–56.

[12] A. Pitale, A. Saha, R. Schmidt: *Lowest weight modules of $Sp_4(\mathbb{R})$ and nearly holomorphic Siegel modular forms*, (2016), preprint, https://arxiv.org/abs/1501.00524.

[13] B. Sun, C.-B. Zhu: *Multiplicity one theorems: the Archimedean case*, Ann. Math. 175 (2012) 23–44.

[14] M. Tsuzuki: *Real Shintani functions on $U(n, 1)$. II: Computation of zeta integrals*, J. Math. Sci. Univ. Tokyo 8 (2001) 689–719.

Kohta Gejima
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka 560-0043
Japan
kohta.gejima@gmail.com