Journal of Lie Theory
Volume **29** (2019) 375–390

Trace Class Groups: the Case of Semi-Direct Products

Gerrit van Dijk

Communicated by K.-H. Neeb

Abstract. A Lie group G is called a trace class group if for every irreducible unitary representation π of G and every C^∞ function f with compact support the operator $\pi(f)$ is of trace class. In this paper we extend the study of trace class groups, begun in a previous paper, to special families of semi-direct products. For the case of a semisimple Lie group G acting on its Lie algebra $\mathfrak{g}$ by means of the adjoint representation we obtain a nice criterion in order that $\mathfrak{g} \rtimes G$ is a trace class group.

Mathematics Subject Classification: 22D10, 22E30, 43A80.

Key Words: Trace class group, Levi decomposition, semi-direct product, semisimple Lie group, orbit, invariant measure.

1. Introduction

In this paper we continue the study of trace class groups begun in [1]. An irreducible unitary representation π is said to be of trace class if for every C^∞ function f with compact support the operator $\pi(f)$ is of trace class. A group is said to be of trace class, or briefly, a trace class group, if every irreducible unitary representation is of trace class. Well-known examples of such groups are reductive Lie groups and unipotent Lie groups. It is known that the direct product of two trace class groups is again a trace class group, cf. [1], Proposition 1.10. This needs however not to be true for semi-direct products. Since in general each Lie group is a semi-direct product of a reductive and a unipotent Lie group, we will again take up the study of semi-direct products in this paper and start our investigations about a criterion for a semi-direct product in order to be a trace class group. We shall, for the time being, restrict to groups for which the unipotent part (the unipotent radical) is abelian. Several new examples of such groups are studied. One of the highlights is the theorem that the semi-direct product of a semisimple Lie group and its Lie algebra is a trace class group if and only if the group is compact. A nice generalization of this theorem would be provided by the case of a semisimple Lie group G acting on a real finite-dimensional vector space V by linear transformations and considering the semi-direct product of V and G. A proof is not yet available however at this time.

ISSN 0949–5932 / \$2.50

2. Levi decomposition

We begin with recalling the Chevalley or Levi decompostion of an algebraic linear group over $\mathbb{R}$ or $\mathbb{C}$.
Let G be such a group. The *unipotent radical* $\mathrm{Rad}_u(G)$ of G is the largest *normal* subgroup of G consisting of unipotent elements. The unipotent radical is a connected algebraic subgroup. If G is reductive then its unipotent radical consists of the identity element and conversily. The unipotent radical is itself a unipotent group.

Proposition 2.1. *There exists a reductive algebraic subgroup H of G such that $G = \mathrm{Rad}_u(G) \rtimes H$.*

This is the Chevalley or Levi decomposition of G. For a proof see [11], Ch. 1, 6.5. We conclude again that in order to find a criterion for a group to be of trace class, the study of semi-direct products is crucial.

3. Irreducible unitary representations of semi-direct products

We start with the more general case of semi-direct products of Lie groups.
Let G be a Lie group being the semi-direct product $G = N \rtimes H$ of a closed normal subgroup N and a closed subgroup H and assume that N is *abelian.* The group H acts continuously on N by $h \mapsto \alpha_h$ where $\alpha_h(n) = hnh^{-1}$ $(n \in N)$. Denote by $\widehat{N}$ the dual group of N and define an action of H on $\widehat{N}$ by $\alpha_h^*(\chi) = \chi \circ \alpha_{h^{-1}}$. The *orbit* O_χ of $\chi \in \widehat{N}$ is the set of all elements $\alpha_h^*(\chi)$, for h running through H. Define the *stabilizer* H_χ of χ as the set $H_\chi = \{h \in H \mid \alpha_h^*\chi = \chi\}$. Then $O_\chi \simeq H/H_\chi$.
We shall describe a wide class of irreducible unitary representations of G. One has the following proposition.

Proposition 3.1. *Let $G = N \rtimes H$ be a semi-direct product of N and H and assume that N is abelian, H unimodular and the action of H on N is unimodular. Let $\chi \in \widehat{N}$ and assume that the orbit O_χ carries a H-invariant measure. Then:*

(i) *If ρ is an irreducible unitary representation of H_χ, then $\chi \otimes \rho$ is an irreducible unitary representation of $N \rtimes H_\chi$.*

(ii) *The representation induced by $\chi \otimes \rho$ is an irreducible unitary representation of G.*

(iii) *If $O_{\chi_1} = O_{\chi_2}$, and ρ_1 is an irreducible unitary representation of H_{χ_1}, then there exists an irreducible unitary representation ρ_2 of H_{χ_2} such that the representations induced by $\chi_1 \otimes \rho_1$ and $\chi_2 \otimes \rho_2$ respectively are equivalent. If, in addition, any orbit O_χ carries an H-invariant measure, then one has:*

(iv) *If C is a set in $\widehat{N}$ meeting each orbit only once, then the representations induced by $\chi \otimes \rho$ $(\chi \in C)$ are mutually inequivalent.*

(v) *If one can choose C as a Borel subset, each irreducible unitary representation of G is equivalent with an induced representation as obtained above.*

For a proof we refer to [9] and [10].

If (v) holds, then one speaks of a *Mackey-regular semi-direct product.*

We remark that the requirement that any orbit O_χ admits a H-invariant measure, restricts the class of semi-direct products we consider. Indeed there exist semi-direct products where some orbits do not carry such a measure. We refer to [13], Sec. 4.

The next step is to determine whether the above induced representations have a trace and, if so, to establish a suitable formula for the trace.

4. A formula for the trace

Let again $G = N \rtimes H$ with N abelian, H unimodular and the action of H on N unimodular. Then G is unimodular too.

Fix $\chi_0 \in \widehat{N}$ and denote by H_0 the stabilizer of χ_0 in H. Assume that H/H_0 carries a H-invariant measure. Let ρ be an irreducible unitary representation of H_0 and define π as the irreducible unitary representation of G induced by $\chi_0 \otimes \rho$. One has:

Theorem 4.1. *For any function $\varphi \in C_c^\infty(G)$ one has*

$$\mathit{trace}\, \pi(\varphi) = \mathit{trace} \int_{H/H_0} \int_N \int_{H_0} \varphi(n, hh_0h^{-1})\, \chi_0(h^{-1}nh)\, \rho(h_0)\, dh_0\, dn\, d\dot{h}. \tag{1}$$

The left hand side is finite if and only if the right-hand side is.

The proof is for the greater part in [6], §13. Application of Mercer's theorem and the observation that any φ is a finite linear combination of functions of the form $\psi * \widetilde{\psi}$ completes the proof.

5. Application

We give an alternative proof of [1], Proposition 1.13:

Proposition 5.1. *The group $\mathbb{R}^n \rtimes \mathrm{SL}(n, \mathbb{R})$ is not a trace class group.*

Proof. Set $N = \mathbb{R}^n$ and $H = \mathrm{SL}\,(n, \mathbb{R})$. Let $\chi_0(x) = \mathrm{e}^{-2\pi i (x \,|\, e_1)}$ $(x \in \mathbb{R}^n)$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and let $\rho = id$, the identity representation of H_0. We shall show that $\pi = \mathrm{Ind}_{N \rtimes H_0}^G \chi_0 \otimes \rho$ has no finite trace. Observe that $H/H_0 \simeq \mathbb{R}^n \backslash \{0\}$ carries a H-invariant measure, namely the ususal Lebesgue measure. We have to consider the right-hand side of equation (1). We get

$$\int_{H/H_0} \int_H \int_{H_0} \varphi(n, hh_0h^{-1})\, \chi_0(h^{-1}nh)\, dh_0\, dn\, d\dot{h}. \tag{2}$$

Clearly H_0 consists of the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ x & B \end{pmatrix}$$

where $x \in \mathbb{R}^{n-1}$ and $B \in \mathrm{SL}\,(n-1, \mathbb{R}^n)$. Setting

$$M_0 := \left\{ m = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \right\} \quad \text{and} \quad U := \left\{ u = u(x) = \begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix} \right\}$$

we may write $H_0 = M_0 U$. Both M_0 and U are unimodular. Select Haar measues dm on M_0 and dx on U. Then $dh_0 = dmdx$. Let A be the subgroup of $\mathrm{SL}\,(n, \mathbb{R})$ consisting of the matrices of the form

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu I \end{pmatrix}$$

with $\lambda \neq 0$, $\mu \neq 0$, $\mu^{n-1} = \lambda$. Let $K = \mathrm{SO}\,(n, \mathbb{R})$. One has $H = KAM_0U$ and $dh = |\lambda|^{n-1}\, dk\, d\lambda\, dm\, dx$, hence $d\dot{h} = |\lambda|^{n-1} dk d\lambda$. Moreover, it is easily checked that M_0 and A commute. For a function φ of the form $\varphi(n,h) = \varphi_1(n)\, \varphi_2(h)$, where $n \in N$, $h \in H$, φ_1 and φ_2 are assumed to be K-invariant, we obtain from (2):

$$\int \widehat{\varphi_1}(\lambda e_1) \cdot \left\{ \int \varphi_2(m\, u(\lambda\mu\, x_1, \lambda\mu\, x_2, \ldots, \lambda\mu\, x_{n-1}))\, dm\, dx_1\, dx_2 \ldots dx_{n-1} \right\} |\lambda|^{n-1} d\lambda$$

$$= \int_{-\infty}^{\infty} \widehat{\varphi_1}(\lambda e_1). \left\{ \int_{MU} \varphi_2(mu(x))\, dm\, dx \right\} \frac{d\lambda}{|\lambda|}.$$

This expression is clearly divergent for suitable φ_1 and φ_2. Therefore the trace of π does not exist. ■

If $n = 2$, so if $G = \mathbb{R}^2 \rtimes \mathrm{SL}\,(2, \mathbb{R})$, the representation π considered above is the only one without a trace.

The method of proof of Proposition 5.1 can be applied in several other situations as well. See Section 6.

6. More examples

For details about the structure of the groups considered below, see [3].

Let $\mathbb{F}$ stand for the field $\mathbb{R}$ or $\mathbb{C}$ and let H be one of the groups $\mathrm{O}(p,q)$ or $\mathrm{U}(p,q)$, abbreviated by $U(p,q;\mathbb{F})$. Set $n = p + q$. Denote by $E = E_n$ the space $\mathbb{F}^n$. The group H acts on E in the standard way. Let us consider the semi-direct product $G = E \rtimes H$. If the elements of G are written as pairs $g = (v,h)$ $(v \in E,\, h \in H)$ then the product in G is given by

$$(v,h)\,(v',h') = (v + h \cdot v',\, hh') \quad (v, v' \in E; h, h' \in H).$$

Notice that the group $G = E \rtimes H$ is unimodular, that H is unimodular and acts on E in a unimodular way.

Proposition 6.1. *The group $\mathbb{F}^n \rtimes \mathrm{U}(p,q)$ is not a trace class group unless $p = 0$ or $q = 0$.*

Proof. Define the groups N and M as follows:

$$M = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} : u \in \mathbb{F}, |u| = 1,\, v \in U(p-1, q-1, \mathbb{F}) \right\},$$

$$N = \left\{ n(z,w) = \begin{pmatrix} 1 + w - \frac{1}{2}[z,z] & z^* & -w + \frac{1}{2}[z,z] \\ z & I & -z \\ w - \frac{1}{2}[z,z] & z^* & 1 - w + \frac{1}{2}[z,z] \end{pmatrix} \right\},$$

where $w \in \mathbb{F}$, $w + \overline{w} = 0$, $z^* = -\,{}^t\overline{z} I_{p-1,q-1}$ and if

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n-2} \end{pmatrix}, \quad z' = \begin{pmatrix} z'_1 \\ \vdots \\ z'_{n-2} \end{pmatrix},$$

then $[z,z'] = \overline{z}'_1 z_1 + \cdots + \overline{z}'_{p-1} z_{p-1} - \overline{z}'_p z_p - \cdots - \overline{z}'_{n-2} z_{n-2} \quad (n = p+q)$. Here $I_{p-1,q-1}$ denotes the diagonal matrix with diagonal starting with $(p-1)$ times 1 and ending with $(q-1)$ times -1. One has

$$n(z,w)\cdot n(z',w') = n(z+z', w+w'+Im[z,z']).$$

The group M acts on N:

$$m\cdot n(z,w)\cdot m^{-1} = n(u^{-1}vz, w), \quad \text{where } m = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \in M.$$

Then MN is clearly a (unimodular) Lie group, being the semi-direct product of M and N (or N and M). Let $\xi^0 = (1,0,\ldots,0,1) \in E$. The stabilizer of ξ^0 in H is the group $H_0 = M_0N$ where M_0 is the subgroup of M with $u = 1$, so $M_0 \simeq \mathrm{U}(p-1,q-1)$. Let A be the group of matrices of the form

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{p-1,q-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}$$

where $t \in \mathbb{R}$. Then $P = MAN$ is a parabolic subgroup of H, being the stabilizer of the line $\mathbb{F}\,\xi_0$. The group A normalizes N:

$$a_t\, n(z,w)\, a_{-t} = n(\mathrm{e}^t z, \mathrm{e}^{2t} w),$$

and commutes with M. Let $K = \mathrm{U}(p)\times\mathrm{U}(q)$. Then one has $H = KAMN = KAM_0N$. Select Haar measures dk on K, dm on M_0, dt on A and $dn = dzdw$ on N. Then $dh = \mathrm{e}^{2\rho t}\, dkdtdmdn$, where $2\rho = n-2$ ($\mathbb{F}=\mathbb{R}$), $2\rho = 2n-2$ ($\mathbb{F}=\mathbb{C}$). Define the character χ_0 of E as follows. Set

$$[v,v'] = \overline{v}'_1 v_1 + \cdots + \overline{v}'_p v_p - \overline{v}'_{p+1} v_{p+1} - \cdots - \overline{v}'_n v_n$$

for $v, v' \in E$. Then $\chi_0(v) = \mathrm{e}^{-2\pi i \mathrm{Re}\,[v,\xi^0]}$, $(v \in E)$. Consider again $\pi = \mathrm{Ind}_{E\rtimes H_0}^G \chi_0 \otimes id$. Similarly to the proof of Proposition 5.1 we again meet a divergent integral for trace π. ∎

Other examples are $\mathbb{C}^n \rtimes O(n,\mathbb{C})$ and $\mathbb{C}^n \rtimes SL(n,\mathbb{C})$.

7. Semi-direct product of abelian groups

The following result is in the thesis of Klamer [7].

Let $G = N \rtimes H$ be a semi-direct product of the locally compact abelian groups N and H, and assume that the Haar measure on N is H-invariant. Then G is a unimodular group.

Let $\chi_0 \in \widehat{N}$ and set H_0 for the stabilizer of χ_o in H. Select $\rho \in \widehat{H}_0$ and define the representation $\pi = \pi_{\chi_0,\rho}$ as

$$\pi = \mathrm{Ind}_{N\rtimes H_0}^G\, \chi_0 \otimes \rho.$$

We know that π is irreducible and unitary. Denote by O_{χ_0} the H-orbit of χ_0 in $\widehat{N}$. Since $O_{\chi_0} \simeq H/H_0$, the orbit obviously carries a H-invariant measure.

Theorem 7.1. *The representation $\pi_{\chi_0,\rho}$ has a trace if and only if the H-invariant measure on O_{χ_0} naturally extends to a tempered H-invariant measure on $\widehat{N}$.*

Proof. Denote by μ the H-invariant meassure on O_{χ_0}. If μ extends to a tempered H-invariant measure on $\widehat{N}$, then clearly $\operatorname{trace}\pi$ exists. Indeed, apply formula (1) and observe that

$$n \mapsto \int_{H_0} \varphi(n, h_0)\,\rho(h_0)\,dh_0$$

is in $C_c^\infty(N)$, and therefore its Fourier transform is in $\mathcal{S}(\widehat{N})$.

For the converse direction assume that $\operatorname{trace}\pi(\varphi)$ exists for all $\varphi \in C_c^\infty(G)$. Taking φ of the form $\varphi = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in C_c^\infty(N)$, $\varphi_2 \in C_c^\infty(H)$, we see from the formula for the trace that $\int_{O_{\chi_0}} \widehat{\varphi}_1(\chi)\,d\mu(\chi)$ exists for all $\varphi_1 \in C_c^\infty(N)$, and, in addition, that this expression defines a distribution T on N, since $\operatorname{trace}\pi$ is a distribution.

Observe that for all $\chi \in \widehat{N}$ there exists a function $\varphi_1 \in C_c^\infty(N)$ with $\widehat{\varphi}_1(\chi) \neq 0$. It follows that μ is a locally finite measure on $\widehat{N}$, hence a (Radon) measure on $\widehat{N}$. The distribution T is clearly positive-definite, hence tempered ([12], Théorème VII, p. 242). Thus μ is a tempered measure on $\widehat{N}$, since $\varphi_1 \mapsto \widehat{\varphi}_1$ is an isomorphism of $\mathcal{S}(\widehat{N})$ onto $\mathcal{S}(N)$. ■

This interesting theorem enables us to find examples of non-trace class groups in an easy way. Take for instance $G = \mathbb{R}^2 \rtimes O(1,1)$.

On the positive side we have the following examples of semi-direct products of abelian groups.

Example 7.2. (Heisenberg groups)
Consider the group of real $(n+2)\times(n+2)$-matrices of the form

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & I_n & {}^t b \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

with $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $c \in \mathbb{R}$ and I_n the $n \times n$-diagonal matrix with only 1's on the diagonal. Let us denote by N and H the subgroups defined by $a_1 = \ldots = a_n = 0$ and $c = b_1 = \ldots = b_n = 0$ respectively. Then both N and H are closed abelian subgroups of G. N is a normal subgroup isomophic to $\mathbb{R}^{n+1}$ and H is isomorphic to $\mathbb{R}^n$. Moreover $G = NH$ and $N \cap H = \{1\}$. G is a semi-direct product of N and H. It turns out that it is even a regular semi-direct product.

Example 7.3. (The group of 4×4 upper triangular unipotent matrices)
Let G denote the group of matrices

$$G = \left\{ \begin{pmatrix} 1 & x & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, y, a, b, c, d \in \mathbb{R} \right\}$$

and set
$$N = \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Then N is a closed abelian normal subgroup of G isosmorphic to $\mathbb{R}^4$. Let
$$H = \left\{ \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Then H is a closed abelian subgroup isomorphic to $\mathbb{R}^2$. One has $G = NH$, $N \cap H = \{1\}$. G is naturally a semi-direct product of N and H. It turns out that it is even a regular semi-direct product. In both examples G is clearly a trace class group, hence the H-invariant measures on the H-orbits in $\widehat{N}$ are tempered. This fact can of course also be checked directly.

8. Compact stabilizers

Let $G = N \rtimes H$, N abelian, H unimodular and the action of H on N also unimodular. Suppose that for $\chi_0 \in \widehat{N}$ the stabilizer of χ_0 in H is *compact*. Select $\rho \in \widehat{H}_0$ and define $\pi = \pi_{\chi_0,\rho}$ as above.

Theorem 8.1. *The representation π has a trace if the H-invariant measure on O_{χ_0} naturally extends to a tempered H-invariant measure on $\widehat{N}$.*

Proof. Apply formula (1) and observe that for all $\psi \in C_c^\infty(G)$
$$\text{trace} \int_{H_0} \psi(n, hh_0h^{-1})\,\rho(h_0)\,dh_0$$
is a bounded continuous function of n and $\dot{h}$, of compact support with respect to n. In case $N = \mathbb{R}^n$ take for instance $\psi = (1 + \Delta_N)^l\,\varphi$ where Δ is the usual Laplace operator on $\mathbb{R}^n$, $\varphi \in C_c^\infty(G)$ and l a sufficiently large integer. ∎

As application consider the group $\mathbb{R}^{n+1} \rtimes O(1, n)$ and set for instance $\chi_0(x) = \mathrm{e}^{-2\pi i x_0}$ for all $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$.

Another application is provided by the semi-direct products $\mathfrak{h} \rtimes H$ where H is a connected semisimple Lie group with finite center and $\mathfrak{h}$ its Lie algebra. H acts on $\mathfrak{h}$ by the adjoint representation. It is well-known that any regular H-orbit on $\mathfrak{h}$ admits an invariant measure, that can be extended to $\mathfrak{h}$ as a tempered Radon measure. Some groups H admit a compact Cartan subgroup A, namely if H admits a discrete series, for example $\mathrm{SL}(2, \mathbb{R})$. Let $\mathfrak{a}$ be the Lie algebra of A and select a regular element X in $\mathfrak{a}$. Then the stabilizer of X is precisely A, so compact.

Corollary 8.2. *Let $G = N \rtimes H$, N abelian and H compact. Then every $\pi = \pi_{\chi_0,\rho}$ has a trace.*

Proof. This is obvious by Theorem 8.1 and the observation that any measure with compact support is tempered. ∎

9. The semi-direct product $\mathbb{R}^3 \rtimes \mathbf{SO_0(1,2)}$

Let us write $N = \mathbb{R}^3$, $H = \mathrm{SO}_0(1,2)$ and $G = N \rtimes H$. The H-orbits on $\mathbb{R}^3$ are given as follows. Set

$$[x,\, y] = x_0y_0 - x_1y_1 - x_2y_2 \quad (x, y \in \mathbb{R}^3).$$

Then the non-trivial orbits are given by $[x,\, x] = \alpha^2,\ x_0 > 0,\ [x,\, x] = \alpha^2,\ x_0 < 0,$ $[x,\, x] = 0,\ x_0 > 0,\ [x.\, x] = 0,\ x_0 < 0$, and $[x,\, x] = -\alpha^2$, where throughout α is a strictly positive real number. Call the latter orbit $\mathcal{O}_\alpha$. The first two orbits give compact stabilizers and it is well-known that the H-invariant measures naturally extend to tempered invariant measures on $\widehat{N}$. The third and fourth orbit lead to representations of G without trace. We will consider the case of the orbit $\mathcal{O}_\alpha$. The stabilzer of $\alpha\, e_1$ is equal to $H_0 = A$ where A is the group of matrices of the form

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}$$

with $t \in \mathbb{R}$. The irreducible unitary representations of H_0 are given by

$$\rho_s :\, a_t \mapsto \mathrm{e}^{-2\pi i s t} \quad (t \in \mathbb{R}).$$

Define
$$\chi_\alpha(x) = \mathrm{e}^{-2\pi 1\, [x,\, e_2]\alpha} \quad (x \in \mathbb{R}^3,\, \alpha > 0).$$

The stabilizer of χ_α is H_0 again. Set $\pi_{s,\alpha} = \mathrm{Ind}_{\mathbb{R}^3 \rtimes H_0}^G\, \chi_\alpha \otimes \rho_s$. We shall show that $\operatorname{trace} \pi_{s,\alpha}$ exists.

We start with formula (1) from Section 4. For any $\varphi \in C_c^\infty(G))$ one has

$$\operatorname{trace} \pi_{s,\alpha)}(\varphi) = \int_{H/H_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \varphi(x; h a_t h^{-1})\, \chi_\alpha(h^{-1} \cdot x)\, \rho_s(a_t)\, dt\, dx\, d\dot{h}.$$

Write $H = KUA$ (Iwasawa decomposition) with $K \simeq \mathrm{SO}(2)$ and

$$U = \left\{ u_z = \begin{pmatrix} 1 + \frac{z^2}{2} & z & -\frac{z^2}{2} \\ z & 1 & -z \\ \frac{z^2}{2} & z & 1 - \frac{z^2}{2} \end{pmatrix} :\, z \in \mathbb{R} \right\}.$$

Then $dh = dk\, dz\, dt$. We may always assume that φ is $\mathrm{Ad}(K)$-invariant. We obtain:

$$\operatorname{trace} \pi_{s,\alpha}(\varphi) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \varphi(x;\, u_z a_t u_{-z})\, \chi_\alpha(u_{-z} \cdot x)\, \rho_s(a_t)\, dt\, dx\, dz.$$

Let $D_x = \dfrac{1}{2\pi^2\alpha^2} \left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} \right)$, an $\mathrm{Ad}(K)$-invariant differential operator. We get:

$$\operatorname{trace} \pi_{s,\alpha}(\varphi) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \frac{1}{1+z^2}\, D_x \varphi(x;\, u_z a_t u_{-z} a_{-t} \cdot a_t\, \chi_\alpha(u_{-z} \cdot x)\, \rho_s(a_t)\, dt\, dx\, dz.$$

Since $u_z a_t u_{-z} a_{-t} \in U$, we see that a_t, hence t, varies in a compact set. This is a crucial point. Hence

$$|\operatorname{trace} \pi_{s,\alpha}(\varphi)| \le C \cdot \max_{s,h} |D_x\, \varphi(x; h)| \cdot \int_{\mathbb{R}} \frac{dz}{1+z^2} < \infty.$$

This shows that $\operatorname{trace} \pi_{s,\alpha}$ exists.

10. A special family of semi-direct products

In this section we shall prove the following theorem:

Theorem 10.1. *Let G be a connected real semisimple Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. Let the group G act on $\mathfrak{g}$ by means of the adjoint representation. Then the semi-direct product $\mathfrak{g} \rtimes G$ is a trace class group if and only if G is compact.*

Proof. It is well-known that G is unimodular and acts on $\mathfrak{g}$ in a unimodular way. Furthermore, every G-orbit on the dual of $\mathfrak{g}$ carries a G-invariant measure. Let us assume that G is non-compact. Choose a Cartan involution θ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its decomposition into ± 1-eigenspaces. Denote by K the compact subgroup with Lie algebra $\mathfrak{k}$. Select a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let Σ denote the set of roots of $(\mathfrak{g}, \mathfrak{a})$. Then Σ is a root system (with multiplicities). Let Δ be a set of simple roots and Σ^+ the set of positive roots with respect to Δ. Set $\mathfrak{u}$ for the space spanned by the positive root vectors. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{u}$ and similarly $G = KAU$, the Iwasawa decomposition of G. Let β be a maximal positive root, i.e. $\alpha + \beta$ is not a root for any $\alpha \in \Sigma^+$. Denote by $X_0 \neq 0$ a root vector for β and by G_0 the stabilizer of X_0 in G. Let $\langle\ ,\ \rangle$ denote the Killing form of $\mathfrak{g}$. Define $H_0 \in \mathfrak{a}$ by

$$\beta(H) = \langle H, H_0 \rangle \quad (H \in \mathfrak{a})$$

and set $A_1 = \exp \mathbb{R} H_0$. Define the subgroup P as the stabilizer of the half-line $\mathbb{R}_+^* X_0$. Then P is equal to the semi-direct product $P = A_1 G_0$, $A_1 \cap G_0 = \{1\}$, $a_1 G_o a_1^{-1} = G_0$ for all $a_1 \in A_1$. Let da_1 and dg_0 denote left Haar measures on A_1 and G_0 respectively. Then $dp = da_1 dg_0$ is a left-invariant measure on P. Since every G-orbit on $\mathfrak{g}$ carries a G-invariant measure, dg_0 is also right invariant. So $d_r p = \Delta(a_1)\, da_1\, dg_0$ is a right-invariant measure on P, where $\Delta(a_1) = |det \mathrm{Ad}\,(a_1)|_{\mathfrak{g}_0}|$. Since $A \subset P$ and $U \subset P$, we have $G = KP$ and, by [4], p. 17, $dg = dk\, d_r p$ is a Haar measure on G. Define the character χ_0 of $\mathfrak{g}$ by

$$\chi_0(X) = \mathrm{e}^{-2\pi i \langle X, X_0 \rangle} \quad (X \in \mathfrak{g}).$$

Then G acts on χ_0 by $g.\chi_0(X) = \chi_0(\mathrm{Ad}\,(g^{-1})X)$ $(X \in \mathfrak{g})$ and $\mathrm{Stab}\, \chi_0$ is equal to G_0. We will consider the representation

$$\pi = \mathrm{Ind}_{\mathfrak{g} \rtimes G_0 \uparrow \mathfrak{g} \rtimes G}\, \chi_0 \otimes 1,$$

and determine its trace. Therefore we apply formula (1). Set $G_1 = \mathfrak{g} \rtimes G$. We get

$$\mathrm{tr}\, \pi\, (\varphi) = \int_{G/G_0} \int_{\mathfrak{g}} \int_{G_0} \varphi(\mathrm{Ad}\ (g)X,\, g g_0 g^{-1})\, \chi_0(X)\, dg_0 dX d\dot{g}$$

for $\varphi \in C_c^\infty(G_1)$. Let us determine the Haar measures more explicitly. We have

$$dg = \Delta(a_1)\, da_1\, dk\, dg_0, \quad \text{and} \quad d\dot{g} = \Delta(a_1)\, dk\, da_1.$$

We thus obtain:

$$\mathrm{tr}\, \pi\, (\varphi) = \int_{A_1} \int_{\mathfrak{g}} \int_{G_0} \varphi(\mathrm{Ad}\,(a_1)X,\, a_1 g_0 a_1^{-1})\, \chi_0(X)\, \Delta(a_1)\, da_1\, dX\, dg_0$$

where we have taken φ to be $\mathrm{Ad}\,(K)$-invariant. Taking the function φ even of the form $\varphi(X,g)=\varphi_1(X)\,\varphi_2(g)$ $(X\in\mathfrak{g},\,g\in G)$ with $\varphi_1\in C_c^\infty(\mathfrak{g})$, $\varphi_2\in C_c^\infty(G)$ we get:

$$\operatorname{tr}\pi\,(\varphi)=\int_0^\infty \widehat{\varphi}_1(\lambda^{\|H_0\|^2}X_0)\,\frac{d\lambda}{\lambda}\cdot\int_{G_0}\varphi_2(g_0)\,dg_0.$$

Setting $\mu=\lambda^{\|H_0\|^2}$ we get, up to the constant $\|H_0\|^{-2}$

$$\operatorname{tr}\pi\,(\varphi)=\int_0^\infty \widehat{\varphi}_1(\mu\,X_0)\,\frac{d\mu}{\mu}\cdot\int_{G_0}\varphi_2(g_0)\,dg_0.$$

This clearly gives a divergent first integral for suitable functions φ_1. Thus G must be compact. The converse follows from Corollary 8.2 and the observation that $\mathfrak{g}\rtimes G$ is a Mackey-regular semi-direct product for G compact. See e.g. [7], p. 105. ∎

Though the group $\mathfrak{g}\rtimes G$ is not trace class if G is noncompact, it is still, at least if G is algebraic, a type I group by a result of Dixmier [2]. Therefore sufficiently many trace class representations exist to compose the Plancherel formula. The next section is devoted to this topic. We start with complex groups. The general case requires more technical details.

11. Plancherel formula

Let G be a complex, semisimple, connected Lie group. Then G is algebraic and has finite center. In particular G is a type I group. Let $\mathfrak{g}$ be the Lie algebra of G. Select a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ and let U be the compact real analytic subgroup of G with Lie algebra $\mathfrak{u}$.

Select, in addition, a maximal abelian subspace $\mathfrak{a}$ of $i\mathfrak{u}$ and set $\mathfrak{t}=i\mathfrak{a}+\mathfrak{a}$. Then $\mathfrak{t}$ is the Lie algebra of a complex maximal torus T, $T=\exp\mathfrak{t}=\exp i\mathfrak{a}\cdot\exp\mathfrak{a}$. Observe that $\exp\mathfrak{ia}$ is compact. The elements of the dual group $\widehat{T}$ are given by the the unitary characters of the form $\rho_{n,s}$ with $n\in L$, $L\subset i\mathfrak{a}^*$ being the weight lattice of $(\mathfrak{u},\,i\mathfrak{a})$, and $s\in\mathfrak{a}^*$,

$$\rho_{n,s}(t)=\rho_{n,s}(\exp H)=\mathrm{e}^{-2\pi i n(H_1)}\cdot\mathrm{e}^{-2\pi i s(H_2)},$$

where $H=H_1+H_2, H_1\in\mathfrak{ia}$, $H_2\in\mathfrak{a}$. For $X\in\mathfrak{g}$ set

$$\det\,(z-\operatorname{ad}X)=\eta(X)\,z^l+\cdots(\text{terms of higher degree in } z),$$

where z is an indeterminate and l is the complex rank of $\mathfrak{g}$. The element X is called *regular* if $\eta(X)\neq 0$. The set of regular elements of $\mathfrak{g}$ is usually denoted by $\mathfrak{g}'$.

Denote by $\langle\ ,\ \rangle$ the Killing form of $\mathfrak{g}$. Select an additve unitary character of the real vector space $\mathfrak{g}$ of the form

$$\chi_0(X)=\mathrm{e}^{-2\pi i\mathrm{Re}\,\langle X,H_0\rangle}\quad (X\in\mathfrak{g}),$$

with H_0 a regular element in $\mathfrak{t}$. Notice that $Z_G(H_0)=T$. Set

$$\pi=\pi_{H_0,n,s}=\operatorname{Ind}_{\mathfrak{g}\rtimes T\uparrow\mathfrak{g}\rtimes G}\chi_0\otimes\rho_{n,s}.$$

We shall prove that $\operatorname{tr}\pi$ exists and then show that the representations $\pi_{H_0,n,s}$ can be used to compose the Plancherel formula. We start again with formula (1):

$$\operatorname{tr}\,\pi(\varphi) = \int_{G/T}\int_{\mathfrak{g}}\int_T \varphi(\operatorname{Ad}(g)X, gtg^{-1})\,\chi_o(X)\,\rho_{n,s}(t)\,dt\,dX\,d\dot{g} \tag{3}$$

for $\varphi \in C_c^\infty(\mathfrak{g} \rtimes G)$. This can also be written as:

$$\operatorname{tr}\,\pi(\varphi) = \int_{G/T}\int_T \widehat{\varphi}(\operatorname{Ad}(g)H_0, gtg^{-1})\,\rho_{n,s}(t)\,dt\,d\dot{g}, \tag{4}$$

where $\widehat{\varphi}$ is the Fourier transform of φ with respect to the first argument. Set $A = \exp\mathfrak{a}$ and let $G = UNA$ be an Iwasawa decomposition of G. Then G/T can be identified with $U/\exp i\mathfrak{a}\cdot N$ and $d\dot{g} = d\dot{u}\,dn$. From (4) we thus obtain

$$\operatorname{tr}\,\pi(\varphi) = \int_N\int_T \widehat{\varphi}(\operatorname{Ad}(n)H_0, ntn^{-1})\,\rho_{n,s}(t)\,dt\,dn, \tag{5}$$

where φ is assumed to be $\operatorname{Ad}(U)$-invariant. Writing

$$ntn^{-1} = t\cdot(t^{-1}ntn^{-1}) = tn'$$

with $n' \in N$, we see that t varies in a compact set. Furthermore, similarly to [5], Ch. IV, Lemma 4.6, and [16], 7.5.15, we have

$$\int_N \widehat{\psi}(\operatorname{Ad}(n)\,H_0)\,dn = \int_{\mathfrak{n}} \widehat{\psi}(H_0 + X)\,|\eta\,(H_0)|^{-2}\,dX$$

for $\psi \in C_c^\infty(\mathfrak{g})$, where $\mathfrak{n}$ is the Lie algebra of N and dX the Lebesgue measure on $\mathfrak{n}$ corresponding to dn by the exponential mapping. These two facts together show that $\operatorname{tr}\pi(\varphi)$ exists. Let us now proceed to determine the Plancherel formula. A main ingredient will be the following integration formula ([16], 7.8.3):

$$\int_{\mathfrak{g}} \psi(X)\,dX = \frac{1}{|W_T|}\int_{\mathfrak{t}}\int_{G/T} |\eta(H)|^2\,\psi(\operatorname{Ad}(g)H)\,d\dot{g}dH, \tag{6}$$

for $\psi \in L^1(\mathfrak{g})$. In the constant $|W_T|$, W_T stands for the group $N_G(T)/Z_G(T)$. From (4) we obtain:

$$\int_{G/T} \widehat{\varphi}(\operatorname{Ad}(g)\,H_0, 1)\,d\dot{g} = \sum_n \int_{\mathfrak{a}^*} \operatorname{tr}\pi_{H_0,n.s}(\varphi)\,ds.$$

Multiplying both sides with $\frac{1}{|W_T|}\,|\eta(H_0)|^2$ and integrating over $\mathfrak{t}$ we obtain:

Theorem 11.1. (Plancherel formula) *For any $\varphi \in C_c^\infty(\mathfrak{g} \rtimes G)$ one has*

$$\varphi(0,1) = \frac{1}{|W_T|}\int_{\mathfrak{t}}\int_{\mathfrak{a}^*}\sum_{n\in L} \operatorname{tr}\pi_{H_0,n,s}(\varphi)\,|\eta(H_0)|^2\,ds\,dH_0.$$

12. Plancherel formula: the general case

Let $\mathbf{G}$ be a connected, complex, semisimple algebraic group defined over $\mathbb{R}$ and let $G = \mathbf{G}(\mathbb{R})$ denote the group of real points of $\mathbf{G}$. Call $\mathbf{g}$ the Lie algebra of $\mathbf{G}$ and $\mathfrak{g}$ the Lie algebra of G. It is known that $\mathfrak{g}$ is semisimple, that G has finite center and has finitely many connected components in the ordinary topology.

Let θ be a Cartan involution of $\mathfrak{g}$, $\widetilde{\theta}$ its extension to $\mathbf{g}$:

$$\widetilde{\theta}(X + iY) = \theta(X) - i\,\theta(Y) \quad (X, Y \in \mathfrak{g}).$$

Denote by $\mathfrak{u} \subset \mathbf{g}$ the real Lie algebra of fixed points of $\widetilde{\theta}$ and by U the real analytic subgroup of $\mathbf{G}$ corresponding to $\mathfrak{u}$. Then U is compact. Set $K = U \cap G$.

Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$, defined over $\mathbb{R}$, that is a connected maximal abelian subgroup of $\mathbf{G}$, defined over $\mathbb{R}$. Set $T = \mathbf{T}(\mathbb{R})$ for the group of real points of $\mathbf{T}$. We shall say that T is a maximal torus in G. Furthermore, denote by $\mathbf{t}$ the Lie algebra of $\mathbf{T}$, by $\mathfrak{t}$ the Lie algebra of T. It is known that any T is conjugate with respect to G to a θ-invariant one. Let us assume from now one that all $\mathfrak{t}$ are θ-invariant. It is also known that there are only finitely many non-conjugate θ-invariant $\mathfrak{t}$. Fix a maximal torus T with a θ-invariant Lie algebra $\mathfrak{t}$ and let A denote its maximal split torus. Then A is the split component of a parabolic subgroup $P = MAN$ of G ([4], Lemma 18). One also has $G = KP$.

Lemma 12.1. *The Haar measure dg on G can be normalized so that if dk is the normalized invariant measure on K, da an invariant measure on A, dm one on M and dn an invariant measure on N and if $f \in C_c(G)$:*

$$\int_G f(g)\,dg = \int_{K\times M\times A\times N} f(kman)a^{2\rho}dk\,dm\,da\,dn,$$

where $a^{2\rho} = |\det Ad(a)|_{\mathfrak{n}}|$ $(a \in A)$, $\mathfrak{n}$ being the Lie algebra of N.

We refer to [16], 7.6.4, for a similar result. For $X \in \mathfrak{g}$ set

$$\det(t - \mathrm{ad}X) = \eta(X)t^l + \dots \text{ (terms of higher degree in } t),$$

where t is an indeterminate and $l = \operatorname{rank}\mathfrak{g}$. The element X is called *regular* if $\eta(X) \neq 0$. The set of regular elements of $\mathfrak{g}$ is denoted by $\mathfrak{g}'$.

Let $\mathfrak{t}$ be as above. Set $\mathfrak{t}'$ for the set of regular elements of $\mathfrak{t}$ and define ${\mathfrak{g}'}_T = \mathrm{Ad}\,(G)\,\mathfrak{t}'$.

Then:
$$\mathfrak{g}' = \bigcup_T {\mathfrak{g}'}_T.$$

Observe that this is actually a finite union and we can take representatives T with Lie algebra $\mathfrak{t}$ being θ-invariant.

Denote by $\langle\,,\,\rangle$ the Killing form of $\mathfrak{g}$. Observe that $-\langle X, \theta(Y)\rangle$ $(X, Y \in \mathfrak{g})$ is a scalar product on $\mathfrak{g}$. Choose Euclidean measures dX on $\mathfrak{g}$ and dH on $\mathfrak{t}$ and normalize the Haar measures dg on G and dt on T so that at the identity they correspond to dX and dH via the exponential mapping. Let $d\dot{g}$ denote the invariant measure on G/T such that $dg = d\dot{g}dt$. Then one has the following integration formula:

Lemma 12.2.

$$\int_{\mathfrak{g}'_T} f(X)dX = \frac{1}{|W_T|}\int_{\mathfrak{t}} |\eta(H)| \int_{G/T} f(\mathrm{Ad}(g)H)\, d\dot{g}dH \quad (f \in L^1(\mathfrak{g})).$$

where $|W_T|$ is the number of elements of $W_T = \widetilde{T}/T$, $\widetilde{T}$ being the normalizer of T in G.

We refer to [16], 7.8.3, and its proof. Observe that we might be dealing with another normalization of dg in Lemma 12.1.

For any T let ρ^T denote a continuous, unitary, character of T and let $d\rho^T$ denote the Haar measure on the dual group $\widehat{T}$, dual to the Haar measure dt on T.

Select, in addition, an additive character of the real vector space $\mathfrak{g}$ of the form

$$\chi_0^T(X) = \mathrm{e}^{-2\pi i\langle X,\, H_0\rangle} \quad (X \in \mathfrak{g}),$$

with H_0 a regular element of $\mathfrak{t}$, the Lie algebra of T. Notice that the centralizer of H_0 in G equals T. Set

$$\pi^T = \pi^T_{H_0,\rho^T} = \mathrm{Ind}_{\mathfrak{g}\rtimes T\uparrow\mathfrak{g}\rtimes G}\, \chi_0^T \otimes \rho^T.$$

We shall first show that $\mathrm{tr}\,\pi^T$ exists and then show that these representations are sufficient to compose the Plancherel formula. According to formula (1) we have:

$$\mathrm{tr}\,\pi^T(\varphi) = \int_{G/T}\int_{\mathfrak{g}}\int_T \varphi(\mathrm{Ad}\,(g)X, gtg^{-1})\,\chi_0^T(X)\,\rho^T(t)\,dt\,dX\,d\dot{g}, \tag{7}$$

for all $\varphi \in C_c^\infty(\mathfrak{g}\rtimes G)$. This can also be written as:

$$\mathrm{tr}\,\pi^T(\varphi) = \int_{G/T}\int_T \widehat{\varphi}(\mathrm{Ad}\,(g)H_0, gtg^{-1})\,\rho^T(t)\,dt\,d\dot{g}, \tag{8}$$

where $\widehat{\varphi}$ is the Fourier transform of φ with respect to the first argument.

To prove that $\mathrm{tr}\,\pi^T(\varphi)$ exists we apply the decomposition $G = KP = KNMA$ and $dg = dk\,dn\,dm\,da$. As observed before, we might be using here a different normalization of dg, but it is clear that the exsistence of $\mathrm{tr}\,\pi^T(\varphi)$ does not depend on the normalization used.

Let us write $T = AB$ with $B = T\cap K$ compact. Then $B \subset M\cap K$ and G/T can be identified with $KN\dot{M}$ where $\dot{M} \simeq M/B$. Furthermore $d\dot{g}$ can be identified with $dk\,dn\,d\dot{m}$. Invoking this in formula (8) we obtain:

$$\mathrm{tr}\,\pi^T(\varphi) = \int_N\int_{\dot{M}}\int_T \widehat{\varphi}(\mathrm{Ad}\,(nm)H_0, nmtm^{-1}n^{-1})\,\rho^T(t)\,dt\,d\dot{m}\,dn, \tag{9}$$

where φ is assumed to be $\mathrm{Ad}(K)$-invariant. Writing $t = ab$ $(a \in A, b \in B)$ and

$$nmtm^{-1}n^{-1} = nmabm^{-1}n^{-1} = a\cdot a^{-1}nam'n^{-1}m'^{-1}\,m' = a\cdot n'\cdot m'$$

with $m' = mbm^{-1}$, we see that a, and thus t, varies in a compact set. We also know, by a result of Harish-Chandra, that the the invariant measure $d\dot{g}$ on $\mathrm{Ad}(G)H_0$

extends to a tempered measure on $\mathfrak{g}$ (cf. [15], page 40, Corollary 10). These two facts together imply that $\operatorname{tr}\pi^T(\varphi)$ exists. From equation (8) we now obtain:

$$\int_{G/T} \widehat{\varphi}(\mathrm{Ad}\,(g)H_0, 1)\, d\dot{g} = \int_{\widehat{T}} \mathrm{tr}_{H_0,\rho^T}(\varphi)\, d\rho^T. \tag{10}$$

Multiplying both sides with $\frac{1}{|W_T|}\,|\eta(H_0)|$ and integrating over $\mathfrak{t}$ we obtain:

$$\int_{\mathfrak{g}'_T} \widehat{\varphi}(X)\, dX = \frac{1}{|W_T|}\int_{\widehat{T}}\int_{\mathfrak{t}} \mathrm{tr}\,\pi_{H_0,\rho^T}(\varphi)\,|\eta(H_0)|\, d\rho^T dH_0, \tag{11}$$

for any $\varphi \in C_c^\infty(\mathfrak{g} \rtimes G)$. Finally we obtain

Theorem 12.3. (Plancherel formula) *For any $\varphi \in C_c^\infty(\mathfrak{g} \rtimes G)$ one has*

$$\varphi(0,1) = \sum_T \frac{1}{|W_T|}\int_{\widehat{T}}\int_{\mathfrak{t}} \mathrm{tr}\,\pi_{H_0,\rho^T}(\varphi)\,|\eta(H_0)|\, d\rho^T dH_0,$$

where the summation is over a complete (finite) set of non-conjugate maximal tori T with θ-invariant Lie algebra.

13. The special case $G = \mathrm{SL}(2,\mathbb{R})$

In this section we write down the explicit form of the Plancherel formula for $\mathfrak{g} \rtimes G$ with $G = \mathrm{SL}(2,\mathbb{R})$. Consider the following two tori in the group $G = \mathrm{SL}(2,\mathbb{R})$:

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0\right\} \quad \text{and} \quad B = \left\{\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} : 0 \leq \varphi < 2\pi\right\}.$$

Let us define characters of A by

$$\rho^A_{\varepsilon,s}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = |a|^{-2\pi i s}\left(\frac{a}{|a|}\right)^{\varepsilon}$$

for $s \in \mathbb{R}$, $\varepsilon = 0, 1$, and characters of B by

$$\rho^B_n\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \mathrm{e}^{-2\pi i n\varphi}$$

for $n \in \mathbb{Z}$. Define the following elements of the Lie algebra $\mathfrak{a}$ of A and the Lie algebra $\mathfrak{b}$ of B respectively:

$$H_u = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \in \mathfrak{a}, \quad H_v = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix} \in \mathfrak{b},$$

and take $u \neq 0, v \neq 0$. They give rise to the following additive characters of $\mathfrak{g}$:

$$\chi_u(X) = \mathrm{e}^{-2\pi i \mathrm{tr}\,(XH_u)}, \quad \chi_v(X) = \mathrm{e}^{-2\pi i \mathrm{tr}\,(XH_v)} \quad (X \in \mathfrak{g}).$$

Let us now consider the unitary representations:

$$\pi^A_{\varepsilon,s,u} = \mathrm{Ind}_{\mathfrak{g}\rtimes A\uparrow\mathfrak{g}\rtimes G}\, \chi_u \otimes \rho^A_{\varepsilon,s} \quad \text{and} \quad \pi^B_{n,v} = \mathrm{Ind}_{\mathfrak{g}\rtimes B\uparrow\mathfrak{g}\rtimes G}\, \chi_v \otimes \rho^B_n.$$

It can be shown that both sets of representations are irreducible and of trace class, and that the following theorem holds:

Theorem 13.1. (Plancherel theorem) *For any* $\varphi \in C_c^\infty(\mathfrak{g} \rtimes G)$ *one has*

$$\varphi(0,e) = 2\sum_{\varepsilon=0,1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{tr} \pi^A_{\varepsilon,s,u}(\varphi)\, |u|\, ds du + 4 \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \operatorname{tr} \pi^B_{n,v}(\varphi)\, |v|\, dv.$$

14. A generalization

In this section we formulate a conjecture which generalizes Theorem 10.1 in case of algebraic groups. Let us first fix some notation.

Let $\mathbf{G}$ be a connected, complex, semisimple, linear algebraic group defined over $\mathbb{R}$ and let $G = \mathbf{G}(\mathbb{R})$ its group of real points. Then G is a semisimple Lie group with finite center and finitely many connected components. Let V be a real finite-dimensional vector space and $\mathbf{V}$ its complexification. Assume that $\mathbf{G}$ acts on $\mathbf{V}$ by a representation ρ which is, in addition, an algebraic morphism defined over $\mathbb{R}$. Set $G_1 = V \rtimes G$. Let $\mathbf{H}$ be the kernel of ρ and set $H = \mathbf{H}(\mathbb{R})$.

Conjecture 14.1. G_1 is a trace class group if and only if G/H is compact.

As to the proof of this conjecture, we can only confirm the 'if'- part at the moment. We formulate it in the following lemma.

Lemma 14.2. *If* G/H *is compact then the sem-direct product* $G_1 = V \rtimes G$ *is a trace class group.*

Proof. First of all observe that that the orbit space $V^*\backslash G$ is equal to the space $V^*\backslash(G/H)$. Since G/H is compact we may conclude that $V \rtimes G$ is a Mackey-regular semi-direct product. See e.g. [7], p. 105.

Observe that H is a normal subgroup of G, hence semisimple with finitely many connected components, hence unimodular and a trace class group by [1], Proposition 1.10 (ii). Now apply [1], Theorem 4.1 with $Q = G$ and $Q_0 = H$. With the notation of this theorem, observe that for any unitary character χ of V its stability group Q_χ in Q is unimodular and trace class, because of [8], p. 470. Hence G_1 is a trace class group. ■

References

[1] A. Deitmar, G. van Dijk: *Trace class groups*, J. Lie Theory 26(1) (2016) 269–291.

[2] J. Dixmier: *Sur les représentations unitaires des groupes de Lie algébriques*, Ann. Inst. Fourier, Grenoble 7 (1957) 315–328.

[3] J. Faraut: *Distributions sphériques sur les espaces hyperboliques*, J. Math. Pures Appl. 58 (1979) 369–444.

[4] Harish-Chandra (Notes by G. van Dijk), *Harmonic Analysis on Reductive p-adic Groups*, Lecture Notes in Mathematics 162, Springer, Berlin et al. (1970).

[5] S. Helgason: *Differential Geometry and Symmetric Spaces*, Academic Press, New York (1967).

[6] A. A. Kirillov: *Elements of the Theory of Representations*, Springer, Berlin et al. (1976).

[7] F. J. M. Klamer: *Group Representations on Hilbert Subspaces of a Locally Convex Space*, PhD thesis, University of Groningen (1979).

[8] A. Kleppner, R. L. Lipsman: *The Plancherel formula for group extensions*, Ann. Scient. Éc. Norm. Sup. 5 (1972) 459–516.

[9] G. W. Mackey: *Induced representations of locally compact groups I*, Annals of Math. 55 (1952) 101–139.

[10] G. W. Mackey: *Induced Representations of Groups and Quantum Mechanics*, W. A. Benjamin, New York (1968).

[11] A. L. Onishchik, E. B. Vinberg (eds.), *Lie Groups and Lie Algebras III*, Springer, Berlin et al. (1990).

[12] L. Schwartz: *Théorie des Distributions*, Hermann, Paris (1978).

[13] G. van Dijk: *Orbits on real affine symmetric spaces I*, Indagationes Mathematicae 86(1) (1983) 51–66.

[14] G. van Dijk: *Introduction to Harmonic Analysis and Generalized Gelfand Pairs*, De Gruyter, Berlin et al. (2010).

[15] V. S. Varadarajan: *Harmonic Analysis on Real Reductive Groups*, Lecture Notes in Mathematics 576, Springer, Berlin et al. (1977).

[16] N. R. Wallach: *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York (1973).

Gerrit van Dijk
Mathematisch Instituut
Niels Bohrweg 1
2333 CA Leiden, The Netherlands
dijk@math.leidenuniv.nl

Received February 20, 2018
and in final form January 24, 2019