The Pro-Lie Group Aspect of Weakly Complete Algebras and Weakly Complete Group Hopf Algebras

Rafael Dahmen and Karl Heinrich Hofmann

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Dedicated to Joachim Hilgert on the occasion of his 60th birthday.

Abstract. A weakly complete vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is isomorphic to $\mathbb{K}^X$ for some set X algebraically and topologically. The significance of this type of topological vector spaces is illustrated by the fact that the underlying vector space of the Lie algebra of any pro-Lie group is weakly complete. In this study, weakly complete real or complex associative algebras are studied because they are necessarily projective limits of finite dimensional algebras. The group of units A^{-1} of a weakly complete algebra A is a pro-Lie group with the associated topological Lie algebra A_{Lie} of A as Lie algebra and the globally defined exponential function $\exp\colon A \to A^{-1}$ as the exponential function of A^{-1}. With each topological group, a weakly complete group algebra $\mathbb{K}[G]$ is associated functorially so that the functor $G \mapsto \mathbb{K}[G]$ is left adjoint to $A \mapsto A^{-1}$. The group algebra $\mathbb{K}[G]$ is a weakly complete Hopf algebra. If G is compact, the $\mathbb{R}[G]$ contains G as the set of grouplike elements. The category of all real Hopf algebras A with a compact group of grouplike elements whose linear span is dense in A is shown to be equivalent to the category of compact groups. The group algebra $A = \mathbb{R}[G]$ of a compact group G contains a copy of the Lie algebra $\mathfrak{L}(G)$ in A_{Lie}; it also contains a copy of the Radon measure algebra $M(G, \mathbb{R})$. The dual of the group algebra $\mathbb{R}[G]$ is the Hopf algebra $\mathcal{R}(G, \mathbb{R})$ of representative functions of G. The rather straightforward duality between vector spaces and weakly complete vector spaces thus becomes the basis of a duality $\mathcal{R}(G, \mathbb{R}) \leftrightarrow \mathbb{R}[G]$ and thus yields a new aspect of Tannaka duality.

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Introduction

A weakly complete vector space is a topological vector space isomorphic in the category of topological vector spaces to $\mathbb{R}^X$ for some set X with the Tychonoff topology. These vector spaces appear prominently in any Lie theory of pro-Lie groups, for instance, of all compact groups, or even of all locally compact groups which are compact modulo connectivity. This text will present a general study of weakly complete associative unital *algebras* over the fields of real and complex numbers, including eventually all weakly complete Hopf algebras. It will turn out that all weakly complete algebras are automatically projective limits of finite dimensional ones, and their groups of units (that is, multiplicatively invertible elements) are

always pro-Lie groups. This theory permits the introduction and study of weakly complete topological group algebras over the real and the complex field and their major properties. In particular, they are automatically topological Hopf algebras. Regarding compact groups, for example, we shall see that the weakly complete group algebra $\mathbb{R}[G]$ of a compact group G not only contains G but also its Lie algebra $\mathfrak{L}(G)$ and its (Radon) measure algebra $M(G, \mathbb{R})$. Indeeed let $\mathcal{G}$ denote the category of compact groups and $\mathcal{H}$ the category of weakly complete real Hopf algebras A whose set G of grouplike elements is compact such that its linear span $\mathrm{span}(G)$ is dense in A.

One of our central results is that $\mathcal{G}$ and $\mathcal{H}$ are equivalent categories. The dualtity between the category of abstract real vector spaces and the category of weakly complete vector spaces is of the Pontryagin type, is natural, and is easily accessible as has been detailed in the literature. Thus it is natural and relatively easy to pass from the category $\mathcal{H}$ to its dual category of real (abstract) Hopf algebras. In this process, the dual of $\mathbb{R}[G]$ emerges as the real (abstract) Hopf algebra $\mathcal{R}(G, \mathbb{R})$ of reprsentative functions. While this so-called *Tannaka duality* between a compact group G and the function vector space $\mathcal{R}(G, \mathbb{R})$ is standard in the literature but also although somewhat arduous to follow through, the natural equivalence $G \leftrightarrow \mathbb{R}[G]$ between the categories $\mathcal{G}$ and $\mathcal{H}$ however, now appears as a natural link that newly enlightens Tannaka duality.

1. Preliminaries: Vector spaces and their duality

In this section we collect the preliminary concepts which secure the language we use for dealing with the structure of a technically simple concept, namely, that of topological associative unital algebras whose underlying vector space is what we shall call weakly complete. A crucial fact therefore is the duality theory of weakly complete vector spaces.

1.1. Duality Reviewed

For the sake of completeness we recall the formal definition of a duality. We assume sufficient familiarity with category theory. A convenient reference is [19], pp. 745–818. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and consider two contravariant functors (see [19], Definition 3.25) $F \colon \mathcal{A} \to \mathcal{B}$ and $U \colon \mathcal{B} \to \mathcal{A}$.

Definition 1.1.   (a)   It is said that F and U are *adjoint on the right* if for each object A in $\mathcal{A}$ there is a natural $\mathcal{A}$-morphism $\eta_A \colon A \to UFA$ such that for each $\mathcal{A}$-morphism $f \colon A \to UB$ there is a unique $\mathcal{B}$-morphism $f' \colon B \to FA$ such that $f = (Uf') \circ \eta_A$.

$$
\begin{array}{ccccc}
\mathcal{A} & & & & \mathcal{B} \\
\hline
A & \xrightarrow{\;\eta_A\;} & UFA & & FA \\
{\scriptstyle \forall f}\downarrow & & \downarrow{\scriptstyle Uf'} & & \uparrow{\scriptstyle \exists! f'} \\
UB & \xrightarrow{\;\mathrm{id}\;} & UB & & B.
\end{array}
$$

The natural morphism η_A is called the *evaluation morphism.*

(b) *The categories $\mathcal{A}$ and $\mathcal{B}$ are said to be dual* if there are two contravariant functors F and U, adjoint on the right, such that η_A is an isomorphism for all objects A of $\mathcal{A}$.

In the general literature, authors frequently speak of "duality" as soon as they have two "naturally defined" contravariant functors that are adjoint on the right without the postulate that the natural morphism η_A is an isomorphism in all cases – for example if they consider the category of Banach spaces as dual to itself under the passage of a Banach space to the dual Banach space.

1.2. Vector spaces reviewed

It is a basic fact that a perfect duality theory exists between the category of real or complex vector spaces $\mathcal{V}$ and the linear maps between them on the one hand and the so-called weakly complete topological vector spaces $\mathcal{W}$ and the continuous linear maps between them on the other (see [6, 17, 19, 20]). Yet this fact is less commonly mentioned than one might expect. In the following let $\mathbb{K}$ denote the field of either the real or the complex numbers. Most of the duality works with arbitrary topological fields, but for applications in the theory of, say, compact groups, a restriction to these fields suffices.

Definition 1.2. A topological $\mathbb{K}$-vector space V is called *weakly complete* if there is a vector space E such that V is isomorphic as a topological vector space to the space $E^* \overset{\text{def}}{=} \operatorname{Hom}(E, \mathbb{K})$ of linear scalar valued functionals endowed with the topology of pointwise convergence from the containment $\operatorname{Hom}(E, \mathbb{K}) \subseteq \mathbb{K}^E$.

Together with all continuous linear maps between them, weakly complete topological vector spaces form a category denoted $\mathcal{W}$. In particular, since every $\mathbb{K}$-vector space E is isomorphic to the direct sum $\mathbb{K}^{(J)}$ for some set J due to the existence of a basis via the Axiom of Choice we have from these definitions the following insight:

Remark 1.3. Every weakly complete vector space is isomorphic as a topological vector space to $\mathbb{K}^J$ for some set J.

According to this remark the cardinality of J is the only characteristic invariant of a weakly complete vector space.

Conversely, if a weakly complete vector space V is given, a vector space E such as it is required by Definition 1.2, is easily obtained by considering the (algebraic) vector space of its topological dual $V' = \operatorname{Hom}(V, \mathbb{K})$ of all continuous linear functionals. Indeed, we have:

Theorem 1.4. *The categories $\mathcal{V}$ and $\mathcal{W}$ are dual to each other with the contravariant functors*

$$E \mapsto E^* : \mathcal{V} \to \mathcal{W}, \ \text{respectively,} \quad V \mapsto V' : \mathcal{W} \to \mathcal{V}.$$

For a vector space E, the evaluation morphism $\eta_E \colon E \to (E^*)'$, for $v \in E$ and $\omega \in (E^*)'$, is given by $\eta_E(v)(\omega) = \omega(v)$. In an analogous way one obtains the evaluation morphism $\eta_V \colon V \to (V')^*$.

For proofs see [19], Theorem 7.30, and [17], Appendix 2: weakly complete topological vector spaces.

This duality theorem is one of the "good" ones for the category $\mathcal{W}$ of weakly complete topological vector spaces since it can be used for the understanding of the category $\mathcal{W}$

by transporting statements in and about it to well known purely algebraic statements on vector spaces and linear maps.

1.3. Monoidal categories

The category $\mathcal{V}$ has a "multiplication," namely, the tensor product $(E, F) \mapsto E \otimes F$. Indeed this product is associative and commutative and it has any one dimensional vector space as identity object (represented conveniently by $\mathbb{K}$ itself). In order to meet the demands of general category theory, there is a formalism around these concepts which allows us to speak of *commutative monoidal categories* as is exemplified by a discussion in [19], Appendix 3, Paragraphs A3.61–A3.92, or in [16], Section 2. *Associativity* of the product $\otimes$ is implemented by a natural isomorphism $\alpha_{EFG} \colon E \otimes (F \otimes G) \to (E \otimes F) \otimes G$ and *commutativity* by a natural isomorphism $\kappa_{EF} \colon E \otimes F \to F \otimes E$. (Cf. [19], pp. 787ff.) The simplest example of a commutative monoidal category is the category of sets and functions with the cartesian binary product $(X, Y) \mapsto X \times Y$ with the singleton sets as identity elements.

If a functor between two monoidal categories respects the available monoidal structures, it will be called *multiplicative*. (See [19], Definition A3.66.)

For the present duality theory of vector spaces it is relevant that not only the category $\mathcal{V}$ of vector spaces has a tensor product, but that the category $\mathcal{W}$ of weakly complete vector spaces has a tensor product $(V, W) \mapsto V \otimes W$ as well (see [11] (1997) and [6] (2007)). The tensor product of weakly complete vector spaces has universal properties which are completely analogous to those well known for vector spaces. Indeed, we have the following proposition:

Proposition 1.5. *The respective tensor products endow each of $\mathcal{V}$ and $\mathcal{W}$ with the structure of a commutative monoidal category, and the contravariant functors*

$$E \mapsto E^* \colon \mathcal{V} \to \mathcal{W} \text{ and } V \mapsto V' \colon \mathcal{W} \to \mathcal{V}$$

are multiplicative.

(See e.g. [6]). In particular, there are natural isomorphisms

$$(E_1 \otimes E_2)^* \cong (E_1)^* \otimes (E_2)^* \text{ and } (V_1 \otimes V_2)' \cong (V_1)' \otimes (V_2)'.$$

1.4. Monoids and comonoids – algebras and coalgebras

Let us consider a commutative monoidal category $(\mathcal{A}, \otimes)$. Good examples for the purpose of the discussion are the categories of sets with the cartesian product or, alternatively, the category of compact spaces and the (topological) cartesian product.

A *multiplication* on an object A of $\mathcal{A}$ is an $\mathcal{A}$-morphism $m \colon A \otimes A \to A$. Its *associativity* can be expressed in terms of $\mathcal{A}$-morphisms in a commutative diagram

$$
\begin{array}{ccc}
A \otimes (A \otimes A) & \xrightarrow{\alpha_{AAA}} & (A \otimes A) \otimes A \\
\downarrow{\scriptstyle \mathrm{id}_A \otimes m} & & \downarrow{\scriptstyle m \otimes \mathrm{id}_A} \\
A \otimes A & & A \otimes A \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
A & \xrightarrow[\mathrm{id}_a]{} & A.
\end{array}
$$

The multiplication m is called *commutative* if the following diagram commutes:

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\ \kappa_{AA}\ } & A \otimes A \\
\ \ \downarrow{\scriptstyle m} & & \ \ \downarrow{\scriptstyle m} \\
A & \xrightarrow[\text{id}_A]{} & A.
\end{array}
$$

(Cf. [19], Definition A3.62).

A set X with an associative multiplication is commonly called a *semigroup*, a topological space C with an associative multiplication is a *topological semigroup*.

A category $\mathcal{A}$ is said to have *identity objects* E (such as singletons in the case of the categories of sets or topological spaces), if for each object A there are natural isomorphisms $\iota_A \colon \mathsf{E} \otimes A \to A$ and $\iota'_A \colon A \otimes \mathsf{E} \to A$.

If an object A has a multiplication $m \colon A \otimes A \to A$, then an *identity* of (A, m) is a morphism $u \colon \mathsf{E} \to A$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathsf{E} \otimes A & \xrightarrow{\ u \otimes \text{id}_A\ } & A \otimes A & \xleftarrow{\ \text{id}_A \otimes u\ } & A \otimes \mathsf{E} \\
\ \ \downarrow{\scriptstyle \iota_A} & & \ \ \downarrow{\scriptstyle m} & & \ \ \downarrow{\scriptstyle \iota'_A} \\
A & \xrightarrow[\text{id}_A]{} & A & \xleftarrow[\text{id}_A]{} & A.
\end{array}
$$

Definition 1.6. In a commutative monoidal category $\mathcal{A}$, an object A with an associative multiplication $m \colon A \otimes A \to A$ and an identity $u \colon \mathsf{E} \to A$ is called a *monoid* in $\mathcal{A}$.

If we take for $(\mathcal{A}, \otimes)$ the category $(\mathcal{V}, \otimes)$ of vector spaces with the tensor product then *a monoid in $(\mathcal{V}, \otimes)$ is precisely a unital associative algebra over $\mathbb{K}$*, (in other words, an associative $\mathbb{K}$-algebra with identity). The multiplication $m \colon A \otimes A \to A$ is expressed as usual in the form

$$
(a, b) \mapsto ab \overset{\text{def}}{=} m(a \otimes b) : A \times A \to A,
$$

while the identity $u \colon \mathbb{R} \to A$ gives rise to the element $1_A = u(1)$ satisfying $a1_A = 1_A a = a$ for all $a \in A$.

In exactly the same spirit we consider the category $\mathcal{W}$ of weakly complete vector spaces. Here *a monoid in $\mathcal{W}$ is precisely a unital associative weakly complete topological algebra over $\mathbb{K}$*.

2. Weakly complete associative unital algebras

It is perhaps useful for our purpose to emphasize the utter simplicity of this concept in a separate definition:

Definition 2.1. A *weakly complete unital algebra* is an associative algebra A over $\mathbb{K}$ with identity, whose underlying vector space is weakly complete, and whose multiplication $(a, b) \mapsto ab : A \times A \to A$ is continuous.

Example 2.2.   **(1a)** The product algebra $\mathbb{K}^J$ for a set J (with componentwise operations) is a weakly complete unital algebra.

(1b) More generally, if $\{A_j : j \in J\}$ is any family of finite dimensional unital $\mathbb{K}$-algebras, then $A = \prod_{j \in J} A_j$ is a weakly complete unital algebra over $\mathbb{K}$.

(1c) Even more generally: Let J be a directed set and

$$\{p_{jk} \colon A_k \to A_j \quad | \quad j \le k, \text{ in } J\}$$

a projective system of morphisms of finite dimensional unital $\mathbb{K}$-algebras. Then the projective limit $A = \lim_{j \in J} A_j$ is a weakly complete unital $\mathbb{K}$-algebra. (See [19], Definition 1.25ff., pp.17ff., also [17], pp.77ff.)

(2) The algebra $\mathbb{K}[[X]]$ of all formal power series $a_0 + a_1 X + a_2 X^2 + \cdots$ with $(a_0, a_1, a_2, \dots) \in \mathbb{K}^{\mathbb{N}_0}$ and the topology making

$$(a_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} a_n X^n : \mathbb{K}^{\mathbb{N}_0} \to \mathbb{K}[[X]]$$

an isomorphism of weakly complete vector spaces is a weakly complete unital algebra.

(3) Let V be any weakly complete topological vector space. Endow $A = V \times \mathbb{K}$ with componentwise addition and the multiplication

$$(v_1, r_1)(v_2, r_2) = (r_1 \cdot v_2 + r_2 \cdot v_1, r_1 r_2).$$

Then A is a weakly complete unital algebra.   ∎

The weakly complete unital algebras form a category $\mathcal{WA}$ in a natural way with morphisms preserving multiplication and identities in the usual way.

If $\mathcal{A}$ and $\mathcal{B}$ are commutative monoidal categories, then a functor $F \colon \mathcal{A} \to \mathcal{B}$ is a *functor of monoidal categories* or, more shortly, a *multiplicative* functor, if it firstly induces a functor on the underlying categories and if it secondly respects multiplication and identities so that there are natural isomorphisms

$$F(A_1 \otimes_{\mathcal{A}} A_2) \cong F(A_1) \otimes_{\mathcal{B}} F(A_2), \text{ and } F(\mathsf{E}_{\mathcal{A}}) \cong \mathsf{E}_{\mathcal{B}}.$$

It is then a routine to prove the following fact:

Proposition 2.3.   (a) *A multiplicative functor $F \colon \mathcal{A} \to \mathcal{B}$ between commutative monoidal categories maps monoids in $\mathcal{A}$ to monoids in $\mathcal{B}$ and induces a functor between the categories of monoids in the respective categories.*

(b) *In particular, if F implements an equivalence of categories* (see [19], Definition A3.39), *then the respective categories of monoids are likewise equivalent.*

Let us consider the two categories $\mathcal{W}$ and $\mathcal{V}$ and denote by $\mathcal{V}^{\mathrm{op}}$ the opposite category in which all arrows are formally reversed (cf. [19], Definition A3.25 and paragraph preceding it). Then by Theorem 1.4, the functors

$$E \mapsto E^* : \mathcal{V}^{\mathrm{op}} \to \mathcal{W}, \text{ respectively, } V \mapsto V' : \mathcal{W} \to \mathcal{V}^{\mathrm{op}}$$

implement an equivalence of categories, and these functors are multiplicative by Proposition 1.5. Hence by Proposition 2.3 we have the following lemma:

Lemma 2.4. *The category of weakly complete unital topological algebras is equivalent to the category of monoids in $\mathcal{V}^{\mathrm{op}}$.*

But what is a monoid in $\mathcal{V}^{\mathrm{op}}$?

Definition 2.5. Let $\mathcal{A}$ be a commutative monoidal category. Then a *comonoid* in $\mathcal{A}$ is an object C together with morphisms $c\colon C \to C \otimes C$ and $k\colon C \to \mathsf{E}$ such that

$$
\begin{array}{ccc}
C \otimes (C \otimes C) & \xrightarrow{\ \alpha_{CCC}\ } & (C \otimes C) \otimes C \\
\uparrow{\scriptstyle \mathrm{id}_C \otimes c} & & \uparrow{\scriptstyle c \otimes \mathrm{id}_C} \\
C \otimes C & & C \otimes C \\
\uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c} \\
C & \xrightarrow[\ \mathrm{id}_c\ ]{} & C
\end{array}
\quad\text{and}\quad
\begin{array}{ccccc}
\mathsf{E} \otimes C & \xleftarrow{\ k \otimes \mathrm{id}_C\ } & C \otimes C & \xrightarrow{\ \mathrm{id}_C \otimes k\ } & C \otimes \mathsf{E} \\
\uparrow{\scriptstyle \iota_C} & & \uparrow{\scriptstyle c} & & \uparrow{\scriptstyle \iota'_C} \\
C & \xleftarrow[\ \mathrm{id}_C\ ]{} & C & \xrightarrow[\ \mathrm{id}_C\ ]{} & C
\end{array}
$$

are commutative (for natural isomorphisms $\iota_C\colon C \to \mathsf{E} \otimes C$ and $\iota'_C\colon C \to C \otimes \mathsf{E}$.) ∎

An almost trivial example of comonoids are obtained in any category $\mathcal{C}$ with finite products and a terminal object E (that is, an object such that $\mathrm{Hom}_{\mathcal{C}}(X, \mathsf{E})$ is singleton for all objects X: see [19], Definition A3.6). Indeed there is a unique "diagonal" morphism $c_X : X \to X \times X$, and a unique morphism $k_X\colon X \to \mathsf{E}$ and these two morphisms make *every* object X endowed with c_X and k_X a coassociative comonoid. This applies to the category of sets with the cartesian product and constant functions to the singleton set. (The verification is a straightforward exercise.)

From the definition it is clear that a comonoid in $\mathcal{A}$ is exactly a monoid in the opposite category $\mathcal{A}^{\mathrm{op}}$. If $\mathcal{A}$ is a category of vector spaces over some field, notably in the case of $\mathbb{R}$ or $\mathbb{C}$, another name is present in the literature (see [21]):

Definition 2.6.   A comonoid in the commutative monoidal category $\mathcal{V}$ of $\mathbb{K}$-vector spaces is called a *coalgebra* over $\mathbb{K}$.

Accordingly, Lemma 2.4 translates into the following statement:

Theorem 2.7.   *The category $\mathcal{WA}$ of weakly complete unital topological $\mathbb{K}$-algebras is dual to the category of $\mathbb{K}$-coalgebras $\mathcal{CA}$.*

These observations conclude our collection of preliminary concepts. Theorem 2.7 is all but profound. So let us pay attention where it may lead us.

3.   Weakly complete unital algebras and their group of units

There is a dominant theorem in the theory of coalgebras, called the Fundamental Theorem of Coalgebras (see [21], Theorem 4.12, p. 742), attributed to Cartier. For us, the following version is relevant. It should be clear that a vector subspace S of a coalgebra C is a subcoalgebra if $c_C(S) \subseteq S \otimes S$ and $k_C(S) = \mathbb{R}$.

Theorem 3.1.   (Fundamental Theorem of Coalgebras)   *Every coalgebra C is the directed union of the set of its finite dimensional subcoalgebras.*

This is sometimes formulated as follows: *Every coalgebra is the injective limit of its finite dimensional subcoalgebras.*

Now if we take Theorems 2.7 and 3.1 together, we arrive at the following theorem [2]. Its consequences, as we shall see, are surprising. (For a discussion of limits in the sense of category theory see [19], Definition A3.41 ff., and in the concrete case of limits and notably projective limits of topological groups see [17], pp. 63ff., respectively, pp. 77ff.)

A projective limit of of topological groups is *strict* if all bonding morphisms and all limit morphisms are surjective (see [19], 1.32 or [17], Definition 1.24). We shall call a projective limit of topological groups *a strict projective limit of quotients* if all bonding maps and all limit morphisms are surjective *and open*; that is, are quotient morphisms. In the situation of vector spaces, Theorem A2.12 of [17] implies that for an injective morphism $f\colon E_1 \to E_2$ in the category $\mathcal{V}$ of vector spaces, the dual morphism

$$f^*\colon E_2^* \to E_1^*$$

in the category $\mathcal{W}$ of weakly complete vector spaces is automatically surjective and open.

Theorem 3.2. (The Fundamental Theorem of Weakly Complete Topological Algebras) *Every weakly complete unital topological $\mathbb{K}$-algebra is the strict projective limit of a projective system of quotient morphisms between its finite dimensional unital quotient-algebras.*

The literature on locally compact groups shows considerable attention to structural results derived from the information that a group, say, is a projective limit of Lie groups. It is therefore remarkable that a result concluding the presence of a projective limit of additive Lie groups emerges out of the vector space duality between $\mathcal{V}$ and $\mathcal{W}$ and the Fundamental Theorem on Coalgebras.

For a weakly complete topological unital algebra A let $\mathbb{I}(A)$ denote the filter basis of closed two-sided ideals I of A such that $\dim A/I < \infty$. We apply Theorem 1.30 of [17] and formulate:

Corollary 3.3. *In a weakly complete topological unital algebra A each neighborhood of 0 contains an ideal $J \in \mathbb{I}(A)$. That is, the filter basis $\mathbb{I}(A)$ converges to 0. In short, $\lim \mathbb{I}(A) = 0$ and $A \cong \lim_{J \in \mathbb{I}(A)} A/J$.*

If A is an *arbitrary* unital $\mathbb{K}$-algebra and $\mathbb{I}(A)$ is the lattice of all of its two-sided ideals J such that $\dim A/J < \infty$, then $\lim_{J \in \mathbb{I}(A)} A/J$ is a weakly complete unital algebra. It is instructive to apply this procedure to the polynomial algebra $A = \mathbb{K}[X]$ in one variable and obtain the weakly complete algebra

$$\mathbb{K}\langle X \rangle \overset{\text{def}}{=} \lim_{J \in \mathbb{I}(\mathbb{K}[X])} \mathbb{K}[X]/J,$$

which we claim to satisfy the following universal property:

Lemma 3.4. *For any element a in a weakly complete $\mathbb{K}$-algebra A there is a unique morphism of weakly complete algebras $\varphi\colon \mathbb{K}\langle X \rangle \to A$ such that $\varphi(X) = a$.*

Proof.   As a first step, let us observe that the algebra generated by X in $\mathbb{K}\langle X \rangle = \lim_{J \in \mathbb{I}(\mathbb{K}[X])} \mathbb{K}[X]/J$ is a dense subset. This implies that the map φ in the statement of the Lemma is unique if it exists. It remains to show its existence.

Recall that $A \cong \lim_{J \in \mathbb{I}(A)} A/J$. Note that for $J \in \mathbb{I}(A)$ there is a unique morphism $\varphi_J \colon \mathbb{K}[X] \to A/J$ sending a polynomial p to $p(a) + J$. Let us look at the following diagram:

$$
\begin{array}{ccccc}
\mathbb{K}[X] & \xrightarrow{\ \text{inc}\ } & \mathbb{K}\langle X \rangle & \xrightarrow{\ \lambda\ } & A \\
{\scriptstyle\text{id}}\downarrow & & \downarrow{\scriptstyle\rho_J} & & \downarrow{\scriptstyle\text{quot}} \\
\mathbb{K}[X] & \xrightarrow[\ \text{quot}\ ]{} & \dfrac{\mathbb{K}[X]}{\ker \varphi_J} & \xrightarrow[\ \varphi'_J\ ]{} & A/J,
\end{array}
$$

where λ is the map we want to define, where φ'_J is induced by φ_J, and where ρ_J is the limit map from $\mathbb{K}\langle X \rangle = \lim_{I \in \mathbb{I}(\mathbb{K}[X])} \mathbb{K}[X]/I$. Define $\lambda_J = \varphi'_J \circ \rho_J$. The "diagonal" morphisms $\lambda_J \colon \mathbb{K}\langle X \rangle \to A/J$, $J \in \mathbb{I}(A)$ are seen to be compatible with the morphisms $A/J_2 \to A/J_1$ for $J_1 \supseteq J_2$ in $\mathbb{I}(A)$. Accordingly, by the universal property of the limit (cf. [19], Definition A3.41), the unique fill-in morphism λ exists as asserted. ∎

Let $\mathbb{P}$ denote the set of all irreducible polynomials p with leading coefficient 1. Then we have the following:

Lemma 3.5.   *There is an isomorphism of weakly complete $\mathbb{K}$-algebras*

$$
\mathbb{K}\langle X \rangle \cong \prod_{p \in \mathbb{P}} \mathbb{K}_p\langle x \rangle, \ \ \textit{where } \mathbb{K}_p\langle X \rangle = \lim_{k \in \mathbb{N}} \frac{\mathbb{K}[X]}{(p^k)}.
$$

Proof.   We recall that every ideal $J \in \mathbb{I}(\mathbb{K}[X])$ is generated by a nonzero polynomial $f = f(X)$, that is $J = (f)$, since $\mathbb{K}[X]$ is a principal ideal domain. Furthermore, each polynomial f admits a unique decomposition into irreducible factors:

$$
\mathbb{I}(\mathbb{K}[X]) = \left\{ \left(\prod_{p \in \mathbb{P}} p^{k_p} \right) : (k_p)_{p \in \mathbb{P}} \in (\mathbb{N}_0)^{(\mathbb{P})} \right\}.
$$

Here, $(\mathbb{N}_0)^{(\mathbb{P})}$ denotes the set of all families of nonnegative integers where all but finitely many indices are zero. For each $f = \prod_{p \in \mathbb{P}} p^{k_p}$ we have

$$
\mathbb{K}[X]/(f) \cong \prod_{p \in \mathbb{P}} \mathbb{K}[X]/(p^{k_p})
$$

by the Chinese Remainder Theorem. This enables us to rewrite the projective limit in the definition of $\mathbb{K}\langle x \rangle$ as

$$
\lim_{J \in \mathbb{I}(\mathbb{K}[X])} \mathbb{K}[X]/J \to \prod_{p \in \mathbb{P}} \left(\lim_{k \in \mathbb{N}} \frac{\mathbb{K}[X]}{(p^k)} \right). \qquad \blacksquare
$$

We remark that if $p \in \mathbb{P}$ is of degree 1, the algebra $\mathbb{K}_p\langle X \rangle$ is isomorphic to $\mathbb{K}[[X]]$, the power series algebra in one variable. Since for $\mathbb{K} = \mathbb{C}$, all $p \in \mathbb{P}$ are of degree 1, it follows that the algebra $\mathbb{C}\langle X \rangle$ is isomorphic to $\mathbb{C}[[X]]^{\mathbb{C}}$.

In the case $\mathbb{K} = \mathbb{R}$, the polynomials $p \in \mathbb{P}$ are of degree one or two. For $p = X - r$ with a number $r \in \mathbb{R}$ the algebra $\mathbb{R}_p\langle X \rangle$ is isomorphic to $\mathbb{R}[[X]]$. But for polynomials $p \in \mathbb{P}$ of degree two, the situation becomes more complicated.

3.1. The group of units: density

An element a in an algebra A is called a *unit* if it has a multiplicative inverse, that is, there exists an element $a' \in A$ such that $aa' = a'a = 1$. The set A^{-1} of units of an algebra is a group with respect to multiplication.

Lemma 3.6. *The group of units A^{-1} of a weakly complete unital algebra A is a topological group.*

Proof. We must show that the function $a \mapsto a^{-1} : A^{-1} \to A^{-1}$ is continuous. In every finite dimensional real or complex unital algebra, the group of units is a topologial group. This applies to each factor algebra A/I, for $I \in \mathbb{I}(A)$. Then $a \mapsto a^{-1}I : A^{-1} \to (A/I)^{-1}$ is continuous for all $I \in \mathbb{I}(A)$. Since the isomorphism $A \cong \lim_{I \in \mathbb{I}(A)} A/I$ holds also in the category of topological spaces, the continuity of $a \mapsto a^{-1}$ follows by the universal property of the limit (see [19], Definition A3.41). ∎

We remark that there exist topological algebras in which inversion in the group of units is discontinuous. (See e.g. [3], Example 3.12.)

The prescription which assigns to a weakly complete unital algebra A its group of units A^{-1} is a functor from the category of weakly complete unital algebras to the category of topological groups. This functor preserves products and intersections, hence arbitrary limits (see e.g. [19], Proposition A3.53). Thus $A \cong \lim_{J \in \mathbb{I}(A)} A/J$ implies $A^{-1} \cong \lim_{J \in \mathbb{I}(A)} (A/J)^{-1}$. Since the group of units of a finite dimensional unital algebra is a (finite dimensional) linear Lie group (see [19], Definition 5.32) we have

Lemma 3.7. *The group of units A^{-1} of a weakly complete unital real or complex algebra A is a projective limit of linear Lie groups.*

Due to a Theorem of Mostert and Shields [22], the group of units of a topological monoid on a locally euclidean space has an open group of units. This applies, in particular, to the multiplicative semigroup $(A, \cdot)$ of any finite dimensional real or complex algebra A. Howeever, in this case, one has more elementary linear algebra arguments to be aware of this fact. Indeed, let $(A, +)$ denote the vector space underlying A and $a \mapsto L_a : A \to \mathrm{Hom}((A, +), (A, +))$ the representation of A into the algebra of all vector space endomorphisms of $(A, +)$ given by $L_a(x) = ax$. If we set $\delta(a) = \det(L_a)$, then we obtain a morphism of multiplicative monoids $\delta : (A, \cdot) \to (\mathbb{R}, \cdot)$ in such a fashion that $A \setminus A^{-1} = \delta^{-1}(\{0\})$. This set is a closed nowhere dense algebraic hypersurface in A. Thus we have

Lemma 3.8. *If A is a finite dimensional real or complex unital algebra, then the group A^{-1} of units is a dense open subgroup of the monoid $(A, \cdot)$.*

It may be helpful to consider some examples:

Example 3.9. (a) In the weakly complete algebra $A = \mathbb{R}^{\mathbb{N}}$ of Example 2.2.1a, the identity $1 = (e_n)_{n \in \mathbb{N}}$, $e_n = 1 \in \mathbb{R}$ is the limit of the sequence of $(a_m)_{m \in \mathbb{N}}$, $a_m = (a_{mn})_{n \in \mathbb{N}}$ of nonunits, where

$$a_{mn} = \begin{cases} 0, & \text{if } n = m, \\ 1, & \text{otherwise.} \end{cases}$$

Hence A^{-1} fails to be open in A while it is dense in A.

(b) In the Examples 2.2.2 and 2.2.3 of weakly complete unital algebras A, we have a maximal ideal M (in case (2) containing all elements with $a_0 = 0$ and in case (3) equalling $V \times \{0\}$) such that $A^{-1} = A \setminus M$. Thus in these cases A^{-1} is open (regardless of dimension). ∎

In order to establish a first important result on the group of units A^{-1} of a weakly complete unital algebra A, namely, its density in A, we need to prove it in the special case

$$(\#) \qquad\qquad A = \mathbb{K}_p\langle X \rangle = \lim_{k \in \mathbb{N}} \mathbb{K}[X]/(p^k)$$

of Lemma 3.12.

Lemma 3.10. (Density Lemma) *For each irreducible polynomial p over $\mathbb{K}$ with leading coefficient 1, the weakly complete algebra $A \stackrel{\mathrm{def}}{=} \mathbb{K}_p\langle X \rangle$ is a local ring and its group A^{-1} of units is open and dense in A.*

Proof. Let $\pi \colon A \to \mathbb{K}[X]/(p)$ denote the limit morphism for $k = 1$ in $(\#)$ and let $J \stackrel{\mathrm{def}}{=} \ker \pi$. For every $f \in J$, the series $\sum_{m=0}^{\infty} f^m$ converges in A to $(1 - f)^{-1}$. So

$$1 - J \subseteq A^{-1}. \tag{1}$$

Now let $f \in A \setminus J$. Since $F \stackrel{\mathrm{def}}{=} \mathbb{K}[X]/(p)$ is a field, $\pi(f)$ has an inverse in F. Thus there is an element $g \in A$ with $h \stackrel{\mathrm{def}}{=} fg \in 1 - J$. By (1) h^{-1} exists and $fgh^{-1} = 1$. Hence f is invertible. This shows that

$$A \setminus J \subseteq A^{-1}. \tag{2}$$

Trivially $A^{-1} \cap J = \emptyset$ and so equality holds in (2).

This shows that the closed ideal J is maximal and thus A is a local ring. Moreover, $A^{-1} = A \setminus J = \pi^{-1}(F \setminus \{0\})$ is open and dense as the inverse of a dense set under an open surjective map. ∎

After the Density Lemma 3.10 we notice that for an irreducible polynomial p the algebra $\mathbb{R}_p\langle X \rangle$ is a local ring with maximal ideal $p\mathbb{R}_p\langle X \rangle$ such that

$$\frac{\mathbb{R}_p\langle X \rangle}{p\mathbb{R}_p\langle X \rangle} \cong \begin{cases} \mathbb{R} & \text{if } \deg p = 1, \\ \mathbb{C} & \text{if } \deg p = 2. \end{cases}$$

For $p = p(X) = X$ we have $\mathbb{R}_p\langle X \rangle = \mathbb{R}[[X]]$ as in Example 2.2.2. For $J = \mathbb{R}$, Example 2.2.1 can be obtained as a quotient of $\mathbb{R}\langle X \rangle$, firstly, by taking for each $r \in \mathbb{R}$

the irreducible polynomial $p = X - r$, secondly, by noticing that $\mathbb{R} \cong \frac{\mathbb{R}[X]}{X-r} = \frac{\mathbb{R}[X]}{p}$ is a quotient of $\frac{\mathbb{R}[X]}{p^k}$ for each $k \in \mathbb{N}$ and thus of $\lim_{k \in \mathbb{N}} \frac{\mathbb{R}[X]}{p^k}$, and, finally, by determining a quotient morphism

$$\mathbb{R}\langle X \rangle \cong \prod_{p \in \mathbb{P}} \lim_{k \in \mathbb{N}} \frac{\mathbb{R}[X]}{(p^k)} \to \prod_{r \in \mathbb{R}} \frac{\mathbb{R}[X]}{(X - r)} \cong \mathbb{R}^{\mathbb{R}}.$$

In passing, we note here again the considerable "size" of $\mathbb{K}\langle X \rangle$.

Theorem 3.11. (The First Fundamental Theorem on the Group of Units) *For any weakly complete unital $\mathbb{K}$-algebra A, the group A^{-1} of units is dense in A.*

Proof. Let $0 \neq a \in A$ and let V denote an open neighborhood of a of A. According to Lemma 3.4 there is a morphism $\varphi \colon \mathbb{K}\langle X \rangle \to A$ with $\varphi(X) = a$. Then $U \stackrel{\mathrm{def}}{=} \varphi^{-1}(V)$ is an open neighborhood of X in $\mathbb{K}\langle X \rangle$. If we find a unit $u \in \mathbb{K}\langle X \rangle^{-1}$ in U, then $\varphi(u) \in V \cap A^{-1}$ is a unit, and this will prove the density of A^{-1} in A. By Lemma 3.5 we have $\mathbb{K}\langle X \rangle \cong \prod_{p \in \mathbb{P}} \mathbb{K}_p\langle X \rangle$, and so the problem reduces to finding a unit near X in $\mathbb{K}_p\langle X \rangle$ for each $p \in \mathbb{P}$. The preceding Density Lemma 3.10 says that this is possible. ∎

3.2. The exponential function

Every finite dimensional unital $\mathbb{K}$-algebra is, in particular, a unital Banach algebra over $\mathbb{K}$ with respect to a suitable norm. By [19], Proposition 1.4, in any unital Banach algebra A over $\mathbb{K}$ the group A^{-1} of units is an open subgroup of the monoid $(A, \cdot)$, and it is a (real) linear Lie group with Lie algebra $\mathfrak{L}(A) = A_{\mathrm{Lie}}$, the real vector space underlying A with the Lie bracket given by $[x, y] = xy - yx$, with the exponential function $\exp \colon \mathfrak{L}(A^{-1}) \to A^{-1}$ given by the everywhere absolutely convergent power series $\exp x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$. (For $\mathbb{K} = \mathbb{R}$ this is discussed extensively in [19], Chapter 5, notably [19], Definition 5.32.)

Now let A be a weakly complete unital $\mathbb{K}$-algebra. Every closed (2-sided) ideal J of A is a closed Lie algebra ideal of A_{Lie}. We apply the Theorem 3.2 and note that the Lie algebra A_{Lie} is (up to natural isomorphism of topological Lie algebras) the strict projective limit of quotients

$$\lim_{J \in \mathbb{I}(A)} \left(\frac{A}{J} \right)_{\mathrm{Lie}} \subseteq \prod_{J \in \mathbb{I}(A)} \left(\frac{A}{J} \right)_{\mathrm{Lie}}$$

of its finite dimensional quotient algebras and therefore is a pro-Lie algebra. Each of these quotient Lie algebras is the domain of an exponential function

$$\exp_{A/J} \colon A_{\mathrm{Lie}}/J \to (A/J)^{-1} \subseteq A/J, \quad (\forall a_J \in A/J) \; \exp_{A/J} a_J = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot a_J^n.$$

This yields a componentwise exponential function on $\prod_{J \in \mathbb{I}(A)} A/J$ which respects the bonding morphisms of the subalgebra $\lim_{J \in \mathbb{I}(A)} A/J$. Thus we obtain the following basic result which one finds in [2, 20].

Theorem 3.12. (The Second Fundamental Theorem on the Group of Units) *If A is a weakly complete unital $\mathbb{K}$-algebra, then the exponential series $1 + a + \frac{1}{2!}a^2 + \cdots$ converges on all of A and defines the exponential function*

$$\exp_A \colon A_{\mathrm{Lie}} \to A^{-1}, \quad \exp_A a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$$

of the pro-Lie group A^{-1}. The Lie algebra $\mathfrak{L}(A^{-1})$ of the pro-Lie group A^{-1} may be identified with the topological Lie algebra A_{Lie}, whose underlying weakly complete vector space is the underlying weakly complete vector space of A.

It is instructive to observe that while we saw any weakly complete associative unital algebra to be a projective limit of finite dimensional quotient algebras, it would be incorrect to suspect that every weakly complete real Lie algebra was a projective limit of finite dimensional quotient algebras. Indeed consider the weakly complete vector space $L \overset{\mathrm{def}}{=} \mathbb{R}^{\mathbb{N}} \times \mathbb{R}$ with the Lie bracket $[((x_n)_{n\in\mathbb{N}}, s), ((y_n)_{n\in\mathbb{N}}, t)] = (s{\cdot}(y_{n+1})_{nin\mathbb{N}} - t{\cdot}(x_{n+1})_{n\in\mathbb{N}}, 0))$. Then L is a weakly complete Lie algebra which can be shown to have no arbitrarily small cofinite dimensional ideals and so cannot be the projective limit of finite dimensional quotients.

3.3. The group of units of weakly complete algebras

For the investigation of the group of units of a weakly complete unital algebra, this opens up the entire Lie theory of pro-Lie groups G, for which in the case of $\mathbb{K} = \mathbb{R}$ we refer to [17]. For instance, Lemma 3.29 of [17] tells us that A^{-1} is a pro-Lie group.

The units in finite dimensional real or complex algebras. For a finite dimensional unital algebra A let $G \overset{\mathrm{def}}{=} A^{-1}$ the subset of units, i.e. multiplicatively invertible elements. the set of pairs $(a, b) \in A \times A$ such that $ab = 1$ where 1 is the identity of A is the set of solutions of a finite sequence of equations over $\mathbb{K}$ and is the graph of the function

$$a \mapsto a^{-1} : G \to G$$

and therefore is a algebraic variety over $\mathbb{K}$. It is homeomorphic to G in the topology induced from the unique $\mathbb{K}$-vector space topology of A. For $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ this identifies G as a real Lie group. This raises the question about information on G/G_0 where G_0 is the identity component of G. Recall that a topological group G is called *almost connected* if the factor group G/G_0 is compact.

Lemma 3.13. *For a finite dimensional real unital algebra A the group of units A^{-1} has finitely many components and thus is almost connected.*

Proof. See [1], Theorem 2.1.5. ∎

If $\mathbb{K} = \mathbb{R}$ is the field of real numbers and $A = \mathbb{R}^n$, then $A^{-1}/(A^{-1})_0 \cong \mathbb{Z}(2)^n$ has 2^n components.

Lemma 3.14. (Mahir Can's Lemma) *For a finite dimensional complex unital algebra A the group of units A^{-1} is connected.*

Proof. We write $G = A^{-1}$ for the group of units of A and let N denote the nilradical of the algebra A. Then for some natural number $k \geq 2$ we have $N^k = \{0\}$ and so for each $x \in N$ we have $(1-x)^{-1} = 1 + x + \cdots + x^{k-1}$. Thus $1 + N$ is a subgroup of G which is immediately seen to be connected. Since N is an ideal, for any $g \in G$ we have $N = gg^{-1}N \subseteq gN \subseteq N$ and so $gN = N$ and $g(1+N) = g + N \in A/N$, that is, $G/(1+N) \subseteq A/N$ for the semisimple factor algebra A/N of A modulo its nilradical.

By a classical theorem of Wedderburn there is an m-tuple $(n_1, \ldots, n_m)$ of nonnegative integers such that

$$A/N \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_m}(\mathbb{C}),$$

where $M_n(\mathbb{C})$ denotes the $n \times n$ complex matrix algebra ($\cong \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n)$). (See e.g. Corollary 2.66 in [5]). The group $\operatorname{GL}(n, \mathbb{C}) = M_n(\mathbb{C})^{-1}$ of units in the full matrix algebra is connected, and so the group of units $H \overset{\text{def}}{=} (A/N)^{-1}$ of the algebra A/N is connected. (See e.g. [10], Theorem 2.2.5.)

Let $p \colon A \to A/N$ be the quotient map. We saw that p induces the morphism $p|G \colon G \to H$ of complex Lie groups. By Lemma 3.8, G is dense and open in A and H is open in A/N. The function p is open, and thus the function $p|G$ is an open morphism of topological groups so that $p(G)$ is open, hence closed in H. The density of G in A implies the density of $p(G)$ in $p(A) = A/N$ and so in H. Therefore $p(G) = H$. Thus we have an exact sequence

$$1 \to 1 + N \to G \xrightarrow{p|G} H \to 1.$$

So, since both $1 + N$ and H are connected, G is connected which we had to show. ∎

Quotient morphisms of algebras. Let $f \colon A \to B$ be a surjective morphism of weakly complete unital algebras. The restriction and corestriction of this function $f|A^{-1} \colon A^{-1} \to B^{-1}$ is a morphism of pro-Lie groups.

Then $\mathfrak{L}(f|A^{-1}) \colon \mathfrak{L}(A^{-1}) \to \mathfrak{L}(B^{-1})$ is f itself considered as a surjective morphism of pro-Lie algebras. Then by [17], Corollary 4.22.iii, we have

$$f(\langle \exp_A(A) \rangle) = (f|A^{-1})\langle \exp_A(A) \rangle = \langle \exp_B(B) \rangle. \tag{3}$$

Assume now that $\dim B < \infty$. Then B^{-1} is a Lie group and so

$$\langle \exp_B(B) \rangle = (B^{-1})_0. \tag{4}$$

Thus $(B^{-1})_0 \subseteq f(A^{-1})$ and so

$$f(A^{-1}) \text{ is open and closed in } B^{-1}. \tag{5}$$

By Theorem 3.11, the group of units A^{-1} is dense in A, whence $f(A^{-1})$ is dense in $f(A) = B$, and so, in particular, $f(A^{-1}) \subseteq B^{-1}$ is dense in B^{-1}. But $f(A^{-1})$ is also closed in B^{-1} by (3), and thus $f(A^{-1}) = B^{-1}$.

Let $I = \ker f$. Then $A^{-1} \cap (1+I) = \ker f|A^{-1}$. Thus we have a unique bijective morphism

$$f' \colon \frac{A^{-1}}{A^{-1} \cap (1+I)} \to B^{-1}$$

such that $\qquad (\forall x \in A^{-1})\, f'(x(A^{-1} \cap (1+I))) = f(x).$

Since $A/I \cong B$, as vector spaces, we know that $A^{-1}/(A^{-1} \cap (1+I)) \cong (A/I)^{-1}$ is a Lie group with finitely many components by Lemma 3.13, which is, therefore, σ-compact. Hence f' is an isomorphism by the Open Mapping Theorem for Locally Compact Groups (see e.g. [19], Exercise EA1.21 on p.704). If $q \colon A^{-1} \to A^{-1}/\ker(f|A^{-1})$ is the quotient morphism, then $f|A^{-1} = f' \circ q$. Hence $f|A^{-1}$ is an open morphism, and we have shown the following

Lemma 3.15.   *Let $f \colon A \to B$ be a surjective morphism of weakly complete unital algebras and assume that $\dim B < \infty$. Then f induces a quotient morphism $f|A^{-1} \colon A^{-1} \to B^{-1}$ from the pro-Lie group A^{-1} onto the Lie group B^{-1}.*

Keep in mind that a quotient morphism is an open map! We also note that Lemma 3.15 remains true for $\mathbb{K} = \mathbb{C}$. We apply this to any weakly complete unital algebra A and any finite dimensional quotient algebras $B = A/J$ for $J \in \mathbb{I}(A)$ and thus sharpen Lemma 3.7 in a significant way:

Theorem 3.16.   (The Third Fundamental Theorem on Units) *For a weakly complete unital $\mathbb{K}$-algebra A let $\mathbb{I}(A)$ be the set of two-sided closed ideals J such that $\dim A/J < \infty$. Then $\mathbb{I}(A)$ is a filter basis converging to 0 and the group of units A^{-1} is a strict projective limit of quotients of linear almost connected Lie groups $A^{-1}/(A^{-1} \cap (1+J))$ isomorphic to $(A/J)^{-1}$ via the map $g(A^{-1} \cap (1+J)) \mapsto g + J$ as J ranges through $\mathbb{I}(A)$, such that each limit morphism agrees with the natural quotient morphism.*

With this theorem, at the latest, it becomes clear that we had to resort to the structure theory of pro-Lie groups with almost connected Lie groups quotients. In [18] the extensive structure and Lie theory of *connected* pro-Lie groups of [17] was lifted to *almost connected* pro-Lie group. Yet it was still unknown whether a projective limit of almost connected finite dimensional Lie groups was complete and is, therefore, an almost connected pro-Lie group. However, this is now clear with the result recorded in [7].

4.   Applying pro-Lie group theory to the group of units of a weakly complete algebra

At this point we continue the theory of weakly complete algebras and prove the following result:

Theorem 4.1.   (The Fourth Fundamental Theorem on the Group of Units) *In any weakly complete unital algebra A, the multiplicative group A^{-1} of invertible elements is almost connected if the ground field is real and is connected if the ground field is complex.*

Proof.   (a) Let us abbreviate $G \overset{\text{def}}{=} A^{-1}$. Assume that the ground field $\mathbb{K}$ is $\mathbb{R}$. By Theorem 3.16, $A = \lim_{J \in \mathbb{I}(A)} A/J$ where $\dim A/J < \infty$ and $G = \lim_{J \in \mathbb{I}(A)} (A/J)^{-1}$ with finite dimensional Lie groups $(A/J)^{-1}$. By Lemma 3.13, the group $(A/J)^{-1}$ is almost connected. Let $L = G/N$ be any Lie group quotient of G. Let U be an identity neighborhood in G such that $UN = U$ and that U/N has no nonsingleton

subgroup. Since $\lim \mathbb{I}(A) = 0$ we have $\lim_{J \in \mathbb{I}(A)} 1 + J = 1$. So there is a $J \in \mathbb{I}(A)$ such that $1 + J \subseteq U$. Since $G \cap (1+J)$ is a multiplicative subgroup, $G \cap (1+J) \subseteq N$. Hence L is a homomorphic image of $G/(G \cap (1 + J)) \cong (A/J)^{-1}$. This group has finitely many components, and so L is almost connected. Now Theorem 1.1 of [7] shows that G is almost connected in the real case. This means that G/G_0 is a compact totally disconnected group and thus has arbitrarily small open-closed subgroups.

(b) Assume now $\mathbb{K} = \mathbb{C}$. In particular, A is a real weakly complete unital algebra and thus G/G_0 has arbitrarily small open-closed subgroups. Suppose that $G \neq G_0$. Then there is a proper open subgroup $H \subseteq G$.

According to Theorem 3.16 G is the strict projective limit of the complex Lie group quotients $G/(G \cap (1 + J)) = (A/J)^{-1}$, $J \in \mathbb{I}(A)$. Thus we find a $J \in \mathbb{I}(A)$ so small that $G \cap (1+J) \subseteq H$. Then $H/(G \cap (1+J))$ is a proper open subgroup of $(A/J)^{-1}$. However this complex Lie group is connected by Lemma 3.14. This contradiction shows that the assumption $G \neq G_0$ is false, and thus that G is connected. This completes the proof of the theorem. ∎

After this information is secured we invoke basic results of pro-Lie groups combined from [17], 12.81 on p.551, and [18]. We let A denote any weakly complete unital algebra (see Definition 2.1) and denote by G its group of units A^{-1}.

Theorem 4.2. *The group of units G of a weakly complete unital algebra A contains a maximal compact subgroup C and A contains up to four closed vector subspaces $V_1, \ldots, V_m$ such that*

$$(c, X_1, \ldots, X_m) \mapsto c \exp X_1 \cdots \exp X_m \colon C \times V_1 \times \cdots \times V_m \to G$$

is a homeomorphism. Every compact subgroup of G has a conjugate in C. The group G is homeomorphic to a product space $\mathbb{R}^J \times C$ for a suitable set J.

One has more detailed structural information on G when needed: If $N(G)$ denotes the nilradical of G_0 (see [17], Definition 10.40 on p. 447), then for each $1 \leq k \leq m$ the product $N(G) \exp V_k$ is a prosolvable closed subgroup (see Definition 10.12, p. 424 of [17]).

Since G is homeomorphic to a space of the form $\mathbb{R}^J \times C$ for a weakly complete vector space $\mathbb{R}^J$ and a compact group C we know that all of the algebraic-topological properties of G are those of a compact group, since $\mathbb{R}^J \times C$ and C are homotopy equivalent. (Cf. [20].)

From Lemma 1.1(iii) in [7], further Theorem 4.2, and [19], Theorem 9.41 on p. 485, and [19], Corollary 10.38 on p. 572, we also derive the following facts.

Corollary 4.3. *Let G be the group of units of a weakly complete unital algebra. Then*

(i) *G contains a profinite subgroup D such that $G = G_0 D$ while $G_0 \cap D$ is normal in G and central in G_0, and*

(i) *G contains a compact subspace Δ such that $(c, d) \mapsto cd : G_0 \times \Delta \to G$ is a homeomorphism.*

4.1. Limitations of these results

While we know from [19], Corollary 2.29 on p. 43 that every compact group is contained (up to isomorphy) in the group of units of a weakly compact unital algebra, there are even Lie groups of dimension as small as 3 which cannot have any isomorphic copies contained in the group of units of a weakly complete algebra.

Example 4.4.   Among the connected noncompact Lie groups G which are not linear, the following 3–dimensional examples are prominent in the literature:

(a) (Garret Birkhoff) Let N be the "Heisenberg group" of matrices

$$[x, y, z] \stackrel{\text{def}}{=} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \ x, y, z \in \mathbb{R}$$

and $Z = \{[0, 0, n] : n \in \mathbb{Z}\} \cong \mathbb{Z}$ a central cyclic subgroup of N. Then $G = N/Z$ is a 3-dimensional class 2 nilpotent group which is not linear as first observed by G. Birkhoff. (See e.g. [19], Example 5.67ff.) The group G is homeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$.

(b) Let G be the universal covering group of the special linear group $\mathrm{Sl}(2, \mathbb{R})$, the group of 2 by 2 real matrices of determinant 1. Since $\mathrm{Sl}(2, \mathbb{R})$ is homeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$, the three dimensional Lie group G is homeomorphic to $\mathbb{R}^3$. The 3-dimensional Lie group G is not linear (See e.g. [14], Example 9.5.18.) For a practical parametrization of G see e.g. [13], Theorem V.4.37 on p. 425 and the surrounding discussion.   ∎

We observe that the universal covering group G of $\mathrm{Sl}(2, \mathbb{R})$ cannot be a subgroup of the group A^{-1} of units of a weakly complete real unital algebra A. Indeed Hilgert and Neeb show in [14], Example 9.5.18., that every linear representation $\pi \colon G \to \mathrm{Gl}(n, \mathbb{R})$ factors through $p \colon G \to \mathrm{Sl}(2, \mathbb{R})$ with a representation $f \colon \mathrm{Sl}(2, \mathbb{R}) \to \mathrm{Gl}(n, \mathbb{R})$ as $\pi = f \circ p$. So if z is one of the two generators of the center of G which is isomorphic to $\mathbb{Z}$, then $p(z^2) = 1$ and thus $\langle z^2 \rangle$ is contained in the kernel of any finite dimensional linear representation of G. From a subgroup G of A^{-1} we would obtain an injective morphism $\gamma \colon G \to A^{-1}$. Every $J \in \mathbb{I}(A)$ yields a linear representation $q_J \colon A^{-1} \to (A/J)^{-1}$ by Theorem 3.6. Thus $q_J \circ \gamma \colon G \to (A/J)^{-1}$ is a linear representation which will annihilate $\langle z^2 \rangle$ for all $J \in \mathbb{I}(A)$. Thus the injective morphism γ would annihilate z^2 which is a contradiction.

We leave it as an exercise to show that the group G of Birkhoff's Example 4.15(a) cannot be a subgroup of any A^{-1} of a weakly complete unital algebra 1. (Cf. [19], Example 5.67.)

Still, in the next section we shall show that for each topological group G there is a "best possible" weakly complete unital algebra $\mathbb{R}[G]$ with a natural "best possible" morphism $G \to \mathbb{R}[G]^{-1}$, one that would be an embedding for every compact group but would fail to be injective for the universal covering group of $\mathrm{Sl}(2, \mathbb{R})$.

5.   The weakly complete group algebra of a topological group

Let us complement our discussion at this stage by describing a relevant pair of adjoint functors.

So let $\mathcal{WA}$ be the category of weakly complete unital $\mathbb{K}$-algebras and $\mathcal{TG}$ the category of topological groups. Then

$$\Omega = (A \mapsto A^{-1}) : \mathcal{WA} \to \mathcal{TG}$$

is a well defined functor after Lemma 3.6. It is rather directly seen to preserve arbitrary products and intersections. Hence by [19], Proposition A3.51 it is a continuous functor (see loc.cit. Definition A3.50), that is, it preserves arbitrary limits.

5.1. The solution set condition

In order to conclude from this information that Ω has in fact a left adjoint, we need to verify the so called *Solution Set Condition* (see [19], Definition A3.58 on p. 786).

For this purpose we claim that for any topological group G in there is a *set* $S(G)$ of pairs (φ, A) with a continuous group morphisms $\varphi \colon G \to A^{-1}$ for some object A of $\mathcal{WA}$ such that for every pair (f, B), $f \colon G \to B^{-1}$ with a weakly complete unital algebra B there is a pair (φ, A) in $S(G)$ and a $\mathcal{WA}$-embedding $e \colon A \to B$ such that

$$f = G \xrightarrow{\;\varphi\;} A^{-1} \xrightarrow{\;e|A^{-1}\;} B^{-1},$$

where $e|A^{-1}$ denotes the bijective restriction and corestriction of e.

Indeed if $f \colon G \to B^{-1}$ determines a unique smallest algebraic unital abstract subalgebra C of B generated by $f(G)$, then there is only a set of these "up to equivalence". Then on each of these there is only a set of algebra topologies and, a fortiori, only a set of them for which the corestriction is continuous; for each of these, there is at most a set of algebra completions up to isomorphism. So, up to equivalence there is only a set of pairs (φ, A), $\varphi \colon G \to A^{-1}$ such that the unital algebra generated by $\varphi(G)$ is dense in A. Any such set $S(G)$ will satisfy the claim.

Now we are in a position to apply the Left Adjoint Functor Existence Theorem (see [19], Theorem A3.60) to conclude that $\Omega \colon \mathcal{WA} \to \mathcal{G}$ has a left adjoint $\Lambda \colon \mathcal{G} \to \mathcal{WA}$. We write the weakly complete unital algebra $\Lambda(G)$ as $\mathbb{K}[G]$ and call it the *weakly complete group algebra of G*. We summarize this result:

Theorem 5.1. (The Weakly Complete Group Algebra Theorem) *To each topological group G there is attached functorially a weakly complete group algebra $\mathbb{K}[G]$ with a natural morphism $\eta_G \colon G \to \mathbb{K}[G]^{-1}$ such that the following universal property holds:*

For each weakly complete unital algebra A and each morphism of topological groups $f \colon G \to A^{-1}$ there exists a unique morphism of weakly complete unital algebras $f' \colon \mathbb{K}[G] \to A$ restricting to a morphism $f'' \colon \mathbb{K}[G]^{-1} \to A^{-1}$ of topological groups such that $f = f' \circ \eta_G$.

$$
\begin{array}{ccccc}
\text{top groups} & & & \text{wc algebras} \\
\hline
G & \xrightarrow{\;\eta_G\;} & \mathbb{K}[G]^{-1} & & \mathbb{K}[G] \\
\downarrow{\scriptstyle \forall f} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle \exists! f'} \\
A^{-1} & \xrightarrow[\;\text{id}\;]{} & A^{-1} & & A
\end{array}
$$

Fact. *If G is one of the two groups of* Example 4.4*, then the natural morphism $\eta_G \colon G \to \mathbb{K}[G]$ is not injective.*

However, the adjunction of the functors $A \mapsto A^{-1}$ (on the right) and $G \mapsto \mathbb{K}[G]$ (on the left) also has a back adjunction

$$\varepsilon_A \colon \mathbb{K}[A^{-1}] \to A$$

such that for each topological group G and each continuous algebra morphism $f \colon \mathbb{K}[G] \to A$ there is a unique morphism of topological groups $f' \colon G \to A^{-1}$ such that $f = \varepsilon_A \circ \mathbb{K}[f']$. (Cf. [19], Proposition A3.36, p. 777). The general theory of adjunctions (as e.g. in [19], Proposition A3.38, p. 777) now tells us that we may formulate

Corollary 5.2. *For any weakly complete unital algebras A and any topological groups G we have*

$$(\forall A) \left(A^{-1} \xrightarrow{\eta_{A^{-1}}} \mathbb{K}[A^{-1}]^{-1} \xrightarrow{(\varepsilon_A)^{-1}} A^{-1} \right) = \mathrm{id}_{A^{-1}}, \quad and$$

$$(\forall G) \left(\mathbb{K}[G] \xrightarrow{\mathbb{K}[\eta_G]} \mathbb{K}\big[\mathbb{K}[G]^{-1}\big] \xrightarrow{\varepsilon_{\mathbb{K}[G]}} \mathbb{K}[G] \right) = \mathrm{id}_{\mathbb{K}[G]}.$$

In other words: Many topological groups are semidirect factors of unit groups of weakly complete algebras, for instance if they are unit groups to begin with, and many weakly complete unital algebras are homomorphic retracts (semidirect summands) of weakly complete group algebras, for instance, if they are group algebras to begin with.

However, more importantly, the universal property implies conclusions which are relevant for the concrete structure theory of $\mathbb{K}[G]$:

Proposition 5.3. *Let G be a topological group. Then the subalgebra linearly spanned by $\eta_G(G)$ in $\mathbb{K}[G]$ is dense in $\mathbb{K}[G]$.*

Proof. Let $S = \overline{\mathrm{span}}\big(\eta_G(G)\big) \subseteq \mathbb{K}[G]$ be the closed subalgebra linearly spanned by $\eta_G(G)$. Let $f_S \colon G \to S^{-1}$ be a morphism of topological groups and $f \colon G \to \mathbb{K}[G]$ the coextension of f_S. Then by the universal property of $\mathbb{K}[G]$ there is a unique morphism $f' \colon \mathbb{K}[G] \to S$ of weakly complete unital algebras such that $f' \circ \eta_G = f$, implying that $(f'|S) \circ \eta_G^o = f_S$ with the corestriction $\eta_G^o \colon G \to S$ of η_G to S. Thus S has the universal property of $\mathbb{K}[G]$; then the uniqueness of $\mathbb{K}[G]$ implies $S = \mathbb{K}[G]$. ∎

We recall from [19], Corollary 2.29(ii) that it can be proved in any theory of compact groups at a very early stage that

> *every compact group has an isomorphic copy in the group of units of a weakly complete unital algebra.*

As a consequence we have

Theorem 5.4. (The Group Algebra of a Compact Group) *If G is a compact group, then $\eta_G \colon G \to \mathbb{R}[G]^{-1}$ induces an isomorphism of topological groups onto its image.*

In other words,

> *any compact group may be considered as a subgroup of the group of units of its weakly complete real group algebra.*

5.2. The group algebra functor $\mathbb{K}[-]$ is multiplicative

If A and B are weakly complete algebras, we have $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ which implies

$$A^{-1} \otimes B^{-1} \subseteq (A \otimes B)^{-1},$$

where we have used the natural inclusion function $j \colon A \times B \to A \otimes B$ and write $A^{-1} \otimes B^{-1}$ in place of $j(A^{-1} \times B^{-1})$.

Now let G and H be topological groups. Then

$$\eta_G(G) \otimes \eta_H(H) \subseteq \mathbb{K}[G]^{-1} \otimes \mathbb{K}[H]^{-1} \subseteq \big(\mathbb{K}[G] \otimes \mathbb{K}[H]\big)^{-1},$$

and so we have the morphism

$$G \times H \to \big(\mathbb{K}[G] \otimes \mathbb{K}[H]\big)^{-1},$$

$(g,h) \mapsto \eta_G(g) \otimes \eta_H(h)$ which, by the univeral property of $\mathbb{K}[-]$ gives rise to a unique morphism $\alpha \colon \mathbb{K}[G \times H] \to \mathbb{K}[G] \otimes \mathbb{K}[H]$ such that

$$(\forall (g,h) \in G \times H) \, \alpha(\eta_{G \times H}(g,h)) = \eta_G(g) \otimes \eta_H(h). \tag{6}$$

On the other hand, the morphisms $j_G \colon G \to G \times H$, $j_G(g) = (g, 1_H)$ and $p_G \colon G \times H \to G$, $p_G(g,h) = g$ yield $p_G j_G = \mathrm{id}_G$. Therefore $\mathbb{K}[p_G] \colon \mathbb{K}[G \times H] \to \mathbb{K}[G]$ is an algebra retraction, and via $\mathbb{K}[j_G]$ we may identify $\mathbb{K}[G]$ with a subalgebra of $\mathbb{K}[G \times H]$; likewise $\mathbb{K}[H]$ is an algebra retract of the algebra $\mathbb{K}[G \times H]$. Since $(g,1)(1,h) = (g,h)$ in $G \times H$, with the identifications of $\mathbb{K}[G], \mathbb{K}[H] \subseteq \mathbb{K}[G \times H]$ we have

$$(\forall (g,h) \in G \times H) \, \eta_G(g)\eta_H(h) = \eta_{G \times H}(g,h) \in \mathbb{K}[G \times H]. \tag{7}$$

The function $\qquad \mathbb{K}[G] \times \mathbb{K}[H] \to \mathbb{K}[G \times H], \quad (a,b) \mapsto ab$

is a continuous bilinear map of weakly complete vector spaces; therefore the universal property of the tensor product in $\mathcal{W}$ yields a unique $\mathcal{W}$-morphism

$$\beta \colon \mathbb{K}[G] \otimes \mathbb{K}[H] \to \mathbb{K}[G \times H]$$

such that $\qquad (\forall a \in \mathbb{K}[G], \, b \in \mathbb{K}[H]) \, \beta(a \otimes b) = ab \in \mathbb{K}[G \times H]. \tag{8}$

Now if for an arbitrary element $(g,h) \in G \times H$ we set $a = \eta_G(g)$ and $b = \eta_H(h)$, then we have

$$\beta\big(\eta_G(g) \otimes \eta_H(h)\big) = a \otimes b = ab = \eta_G(g)\eta_H(h) = \eta_{G \times H}(g,h). \tag{9}$$

By Proposition 5.3, $\eta_G(G)$ generates $\mathbb{K}[G]$ as weakly complete unital algebra and likewise $\eta_H(H)$ generates $\mathbb{K}[H]$ in this fashion, and the algebraic tensor product of

$\mathbb{K}[G]$ and $\mathbb{K}[H]$ is dense in $\mathbb{K}[G] \otimes \mathbb{K}[H]$. Therefore, (9) implies $\beta \circ \alpha = \mathrm{id}_{\mathbb{K}[G \times H]}$. In other words, the diagram

$$
\begin{array}{ccc}
\mathbb{K}[G \times H] & \xrightarrow{\ \mathrm{id}_{\mathbb{K}[G \times H]}\ } & \mathbb{K}[G \times H] \\[2pt]
{\scriptstyle\alpha}\big\downarrow & & \big\uparrow{\scriptstyle\beta} \\[2pt]
\mathbb{K}[G] \otimes \mathbb{K}[H] & \xrightarrow[\ \mathrm{id}_{\mathbb{K}[G] \otimes \mathbb{K}[H]}\ ]{} & \mathbb{K}[G] \otimes \mathbb{K}[H]
\end{array}
$$

commutes. Similarly, let us look at $\alpha \circ \beta : \mathbb{K}[G] \otimes \mathbb{K}[H] \to \mathbb{K}[G] \otimes \mathbb{K}[H]$: We recall (6) and (9) and verify

$$
\alpha\big(\beta(\eta_G(g) \otimes \eta_H(h)) = \alpha(\eta_{G \times H}(g, h)) = \eta_G(g) \otimes \eta_H(h)
$$

By the same argument as above we conclude $\alpha \circ \beta = \mathrm{id}_{\mathbb{K}[G] \otimes \mathbb{K}[H]}$.

Taking everything together, we have proved the following important result:

Theorem 5.5.   (Multiplicativity of the Group Algebra Functor $\mathbb{K}[-]$) *For two arbitrary topological groups G and H the natural morphisms of weakly complete unital algebras $\alpha \colon \mathbb{K}[G \times H] \to \mathbb{K}[G] \otimes \mathbb{K}[H]$ and $\beta \colon \mathbb{K}[G] \otimes \mathbb{K}[H] \to \mathbb{K}[G \times H]$ are isomorphisms which are inverses of each other.*

5.3. Multiplication and comultiplication on the group algebra $\mathbb{K}[G]$

Let G be a topological group and $\delta_G \colon G \to G \times G$ the diagonal morphism $\delta_G(g) = (g, g)$. Together with the constant morphism $k_G \colon G \to \mathsf{E} = \{1\}$ we have a comonoid (δ_G, k_G) according to the example following Definition 2.5. Since the group-algebra functor $\mathbb{K}[-]$ is multiplicative we have *morphisms of weakly complete unital algebras* $\mathbb{K}[\delta_G] \colon \mathbb{K}[G] \to \mathbb{K}[G \times G]$ and $\mathbb{K}[k_G] \colon \mathbb{K}[G] \to \mathbb{K}[\{1\}] = \mathbb{K}$. By Theorem 5.5 above we have an isomorphism $\alpha_G \colon \mathbb{K}[G \times G] \to \mathbb{K}[G] \otimes \mathbb{K}[G]$ which gives us the following observation:

Lemma 5.6.   *For any topological group G, the weakly complete group algebra $\mathbb{K}[G]$ supports a cocommutative and coassociative comultiplation*

$$
\gamma_G \colon \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G], \quad \gamma_G = \alpha_G \circ \mathbb{K}[\delta_G]
$$

which is a morphism of weakly complete unital algebras, and there is a co-identity $\kappa_G \colon \mathbb{K}[G] \to \mathbb{K}$ which is an algebra morphism.

The following fairly immediate remark will be relevant:

Remark 5.7.   Let G be a topological group and $x \in \mathbb{K}[G]$. If $x \in \eta_G(G)$, then the following statement holds:

$$
(\dagger) \hspace{4cm} \gamma_G(x) = x \otimes x \text{ and } \kappa(x) = 1.
$$

The set of elements satisfying $(\dagger)$ is linearly independent.

Proof.   We recall the definition of

$$
(*) \hspace{3cm} \gamma_G = (\mathbb{K}[G] \xrightarrow{\mathbb{K}[\delta_G]} \mathbb{K}[G \times G] \xrightarrow{\alpha_G} \mathbb{K}[G] \otimes \mathbb{K}[G]) .
$$

If $a = \eta_G(g)$ for some $g \in G$, then $c(a) = \alpha_G(a, a) = a \otimes a$ by $(*)$ and by (1) above. The linear independence is a an exercise in linear algebra which one finds in [16], pp. 66, 67. $\blacksquare$

For each topological group G the *opposite group* G^{op} is the underlying topological space of G together with the multiplication $(g, h) \mapsto g * h$ defined by $g * h = hg$. The groups G and G^{op} are isomorphic under the function $\mathrm{inv}_G \colon G \to G^{\mathrm{op}}$, $\mathrm{inv}_G(g) = g^{-1}$. Analogously, every topological algebra A gives rise to an opposite algebra A^{op} on the same underlying topological vectorspace but with the multiplication defined by $a * b = ba$, giving us

$$(A^{-1})^{\mathrm{op}} = (A^{\mathrm{op}})^{-1}$$

by definition, but not necessariy being isomorphic to A. Consequently,

$$((\mathbb{K}[G])^{-1})^{\mathrm{op}} = (\mathbb{K}[G]^{\mathrm{op}})^{-1}$$

and there are morphisms of topological groups $\eta_G \colon G \to \mathbb{K}[G]^{-1}$ and $\eta_{G^{\mathrm{op}}} \colon G^{\mathrm{op}} \to \mathbb{K}[G^{\mathrm{op}}]^{-1}$. Accordingly, we have an isomorphism $\mathbb{K}[\mathrm{inv}_G] \colon \mathbb{K}[G] \to \mathbb{K}[G^{\mathrm{op}}]$ of weakly complete topological algebras and, accordingly, an involutive isomorphism of topological groups $\mathbb{K}[\mathrm{inv}_G]^{-1} \colon \mathbb{K}[G]^{-1} \to \mathbb{K}[G^{\mathrm{op}}]^{-1}$. This gives us a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\eta_G} & \mathbb{K}[G]^{-1} \\
\mathrm{inv}_G \downarrow & & \downarrow \mathbb{K}[\mathrm{inv}_G]^{-1} \\
G^{\mathrm{op}} & \xrightarrow[\eta_{G^{\mathrm{op}}}]{} & \mathbb{K}[G^{\mathrm{op}}]^{-1}.
\end{array}
$$

producing an isomorphism of weakly complete algebras $\mathbb{K}[G] \to \mathbb{K}[G^{\mathrm{op}}]$. But we also have a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\eta_G} & \mathbb{K}[G]^{-1} \\
\mathrm{inv}_G \downarrow & & \downarrow \mathrm{inv}_{\mathbb{K}[G]^{-1}} \\
G^{\mathrm{op}} & \xrightarrow[\eta_{G^{\mathrm{op}}}]{} & (\mathbb{K}[G]^{-1})^{\mathrm{op}} = (\mathbb{K}[G]^{\mathrm{op}})^{-1}.
\end{array}
$$

Let us abbreviate $f \stackrel{\mathrm{def}}{=} (\eta_{G^{\mathrm{op}}}) \circ \mathrm{inv}_G \colon G \to (\mathbb{K}[G]^{\mathrm{op}})^{-1}$. So by the adjunction formalism, there is a unique involutive isomorphism $f' \colon \mathbb{K}[G] \to \mathbb{K}[G]^{\mathrm{op}}$ of weakly complete algebras such that $f = f'|\mathbb{K}[G]^{-1} \circ \eta_G$.

We have a grounding functor $A \to |A|$ from the cagegory $\mathcal{WA}$ of weakly complete algebras to the category $\mathcal{W}$ of weakly complete vector spaces, where $|A|$ is simply the weakly complete vector space underlying the weakly complete algebra A. With this convention we formulate the following definition:

Definition 5.8. For each topological group G there is a morphism of weakly complete vector spaces

$$\sigma_G \stackrel{\mathrm{def}}{=} |f'| \colon |\mathbb{K}[G]| \to |\mathbb{K}[G]^{\mathrm{op}}| = |\mathbb{K}[G]|,$$

often called *symmetry* or *antipode* such that for any topological group G we have

$$(\forall g \in G) \, \sigma_G(\eta_G(g)) = \eta_G(g^{-1}) = \eta_G(g)^{-1}. \qquad \blacksquare$$

Equivalently, $(\forall x \in \eta_G(G)) \, \sigma_G(x) \cdot x = x \cdot \sigma_G(x) = 1$, and, using the bilinearity of multiplication $(x, y) \mapsto xy$ in $\mathbb{K}[G]$ and defining $\mu_G \colon \mathbb{K}[G] \otimes \mathbb{K}[G] \to \mathbb{K}[G]$ by $\mu_G(x \otimes y) = xy$ and remembering from Remark 5.7 that $\gamma_G(x) = x \otimes x$ for all $x = \eta_G(g)$ with some $g \in G$, once more, equivalently,

$$(+) \qquad\qquad (\forall x \in \eta_G(G)) \, (\mu_G \circ (\sigma_G \otimes \mathrm{id}_{\mathbb{K}[G]}) \circ \gamma_G)(x) = 1.$$

By Proposition 5.3, the weakly complete algebra $\mathbb{K}[G]$ is the closed linear span of $\eta_G(G)$, equation $(+)$ holds in fact for all elements or $\mathbb{K}[G]$. Thus we have shown the following

Proposition 5.9. *For any topological group G, the following diagram involving natural morphisms of weakly complete vector spaces commutes where $W_G = |\mathbb{K}[G]|$:*

$$
\begin{array}{ccc}
W_G \otimes W_G & \xrightarrow{\;\sigma_G \otimes \mathrm{id}_{W_G}\;} & W_G \otimes W_G \\
\gamma_G \uparrow & & \downarrow \mu_G \\
W_G & \xrightarrow[\iota_G \circ |\kappa_G|]{} & W_G,
\end{array}
$$

where $\kappa_G \colon \mathbb{K}[G] \to \mathbb{K} = \mathbb{K}[\mathsf{E}]$, $\mathsf{E} = \{1\}$ is induced by the constant morphism and $\iota_G \colon \mathbb{K} = \mathbb{K}[\mathsf{E}] \to \mathbb{K}[G]$ by the unique inclusion $\mathsf{E} \to G$.

In this diagram, the multiplication μ_G defines the algebra structure of the group algebra $\mathbb{K}[G]$ on W_G and the comultiplication γ_G is an algebra morphism.

5.4. Weakly complete bialgebras and Hopf-algebras

We need to view the basic result of Proposition 5.9 of the last section in its more general abstract frame work. In an arbitrary monoidal category $(\mathcal{A}, \otimes)$ we have monoids $A \otimes A \xrightarrow{m} A \xleftarrow{u} \mathsf{E}$ and comonoids $\mathsf{E} \xleftarrow{k} C \xrightarrow{c} C \otimes C$. Every pair of monoids A and B gives rise to a monoid $A \otimes B$. In particular, for each monoid A also $A \otimes A$ is a monoid. An $\mathcal{A}$-morphism $f \colon A \to B$ between monoids is a *monoid morphism* if they respect the monoid structure in an evident fashion. These matters are discussed in a comprehensive fashion in [19] in Appendix 3 in the section entitled "Commutative Monoidal Categories and their Monoid Objects"; see pages 787 ff., Example A3.61ff.

Definition 5.10. (a) A *bimonoid* in a commutative monoidal category is an object together with both a monoid structure (m, u) and comonoid structure (c, k),

$$A \xrightarrow{c} A \otimes A \xrightarrow{m} A \quad \text{and} \quad \mathsf{E} \xrightarrow{u} A \xrightarrow{k} \mathsf{E},$$

such that c is a monoid morphism.

(b) A *group* (or often *group object* in a commutative monoidal category) is a bimonoid with commutative comultiplication and with an $\mathcal{A}$-morphism $\sigma \colon A \to A$, called *inversion* or *symmetry* (as the case may be) which makes the following diagram commutative

$$
(\sigma) \qquad
\begin{array}{ccc}
A \otimes A & \xrightarrow{\;\sigma \otimes \mathrm{id}\;} & A \otimes A \\
c \uparrow & & \downarrow m \\
A & \xrightarrow[uk]{} & A,
\end{array}
$$

plus a diagram showing its compatibility with the comultiplication (see [19], Definition A3.64.ii).

(c) In our commutative monoidal categories $(\mathcal{V}, \otimes)$ and $(\mathcal{W}, \otimes)$ of $\mathbb{K}$-vector spaces, respectively, weakly complete $\mathbb{K}$-vectorspaces, a group object (A, m, c, u, k, σ) is called a *Hopf algebra*, respectively, *a weakly complete Hopf algebra.* ∎

In reality, the definition of a bimonoid is symmetric and the equivalent conditions that c be a monoid morphism, respectively, that m be a comonoid morphism can be expressed in one commutative diagram (see [19], Diagram following Definition A3.64, p. 793). Also it can be shown that in a group the diagram arising from the diagram (σ) by replacing $\sigma \otimes \mathrm{id}$ by $\mathrm{id} \otimes \sigma$ commutes as well.

Since the group objects and group morphisms in a commutative monoidal category form a category (see e.g. [19], paragraph following Exercise EA3.34, p. 793), the Hopf-algebras in $(\mathcal{V}, \otimes)$ and $(\mathcal{W}, \otimes)$ do constitute categories in their respective contexts.

Example 5.11. (a) In the category of sets (or indeed in any category with finite products and terminal object) every monoid is automatically a bimonoid if one takes as comultiplication the diagonal morphism $X \to X \times X$.

(b) A bimonoid in the category of vector spaces with the tensor product as multiplication is called a *bialgebra*. Sometimes it is also called a *Hopf-algebra*, and sometimes this name is reserved for a group object in this category. The identity $u: \mathbb{K} \to A$ of an algebra identifies $\mathbb{K}$ with the subalgebra $\mathbb{K} \cdot 1$ of A. The coidentity $k: A \to \mathbb{K}$ of a bialgebra may be considered as the endomorphism $uk: A \to A$ mapping A onto the subalgebra $\mathbb{K} \cdot 1$ and then is often called the *augmentation*. ∎

In the present context we do restrict our attention to the commutative monoidal categories $(\mathcal{V}, \otimes)$ and $(\mathcal{W}, \otimes)$ of $\mathbb{K}$-vector spaces, respectively, weakly complete $\mathbb{K}$-vector spaces with their respective tensor products. In terms of the general terminology which is now available to us, the result in Proposition 5.9 may be rephrased as follows:

Theorem 5.12. (The Hopf-Algebra Theorem for Weakly Complete Group Algebras) *For every topological group G, the group algebra $\mathbb{K}[G]$ is a weakly complete Hopf-algebra with a comultiplication γ_G and symmetry σ_G.*

6. Grouplike and primitive elements

In any theory of Hopf algebras it is common to single out two types of special elements, and we review them in the case of weakly complete Hopf algebras.

Definition 6.1. Let A be a weakly complete coassociative coalgebra with comultiplication c and coidentity k. Then an element $a \in A$ is called *grouplike* if $k(a) = 1$ and $c(a) = a \otimes a$. If A is a bialgebra, $a \in A$ is called *primitive*, if $c(a) = a \otimes 1 + 1 \otimes a$.

For any $a \in A$ with $c(a) = a \otimes a$, the conditions $a \neq 0$ and $k(a) = 1$ are equivalent.

These definitions apply, in particular, to any weakly complete Hopf algebra and thus especially to each weakly complete group algebra $\mathbb{K}[G]$. By Remark 5.7 the subset $\eta_G(G)$ is a linearly independent set of grouplike elements.

Remark 6.2. In at least one source on bialgebras in earlier contexts, the terminology conflicts with the one introduced here which is now commonly accepted. In [16], p. 66, Definition 10.17, the author calls a grouplike element in a coalgebra *primitive.* Thus some caution is in order concerning terminology. Primitive elements in the sense of Definition 6.1 do not occur in [16].

The following observations are proved straightforwardly:

Lemma 6.3. *The set G of grouplike elements of a weakly complete bialgebra A is a closed submonoid of $(A, \cdot)$ and the set L of primitive elements of A is a closed Lie subalgebra of A_{Lie}. If A is a Hopf algebra, then G is a closed subgroup of A^{-1}.*

For a morphism $f \colon W_1 \to W_2$ of weakly complete vector spaces let the dual morphism of vector spaces be denoted by $f' = \mathrm{Hom}(-, \mathbb{K}) \colon W_2' \to W_1'$.

For a weakly complete coalgebra A let $A' = \mathrm{Hom}(A, \mathbb{K})$ be the dual of A. Then A' is an algebra: If $c \colon A \to A \otimes A$ is its comultiplication, then $c' \colon A' \otimes A' \to A'$ is the multiplication of A'. For a unital algebra R and a weakly complete coalgebra A in duality let $(a, g) \mapsto \langle a, g \rangle \colon R \times A \to \mathbb{K}$ denote the pairing between R and A, where for $f \in \mathbb{R} = \mathrm{Hom}(A, \mathbb{K})$ and $a \in A$ we write $\langle f, a \rangle = f(a)$.

Definition 6.4. Let R be a unital algebra over $\mathbb{K}$. Then a *character* of R is a morphism of unital algebras $R \to \mathbb{K}$. The subset of $\mathbb{K}^R$ consisting of all algebra morphisms inherits the topology of pointwise convergence from $\mathbb{K}^R$ and as a topological space is called the *spectrum* of R and is denoted $\mathrm{Spec}(R)$.

If $k \colon R \to \mathbb{K}$ is a morphism of algebras, then an element $d \in \mathcal{V}(R, \mathbb{K})$ is called a *derivative* (sometimes also called a *derivation* or *infinitesimal character*) of R (with respect to k) if it satisfies

$$(\forall x, y \in R)\, d(xy) = d(x)k(y) + k(x)d(y).$$

The set of all derivatives of R is denoted $\mathrm{Der}(R)$. ∎

Now let R be a unital algebra and $A \overset{\mathrm{def}}{=} R^*$ its dual weakly complete coalgebra with comultiplication c such that $ab = c'(a \otimes b)$ for all $a, b \in R$. In these circumstances we have:

Proposition 6.5. *Let $g \in A$. Then the following statements are equivalent:*
(i) *$g \in A$ is grouplike in the coalgebra A.*
(ii) *$g \colon R \to \mathbb{K}$ is a character of R, that is, is an element of $\mathrm{Spec}(R)$.*

Proof. The dual of $A \otimes A$ is $R \otimes R$ in a canonical fashion such that for $r_1, r_2 \in R$ and $h_1, h_2 \in A$ we have

$$\langle r_1 \otimes r_2, h_1 \otimes h_2 \rangle = \langle r_1, h_1 \rangle \langle r_2, h_2 \rangle.$$

The set of linear combinations $L = \sum_{j=1}^{n} a_j \otimes b_j \in A \otimes A$ is dense in $A \otimes A$. So two elements $x, y \in R \otimes R$ agree if and only if for all such linear combinations L we have $\langle x, L \rangle = \langle y, L \rangle$, and this clearly holds if and only if for all $a, b \in A$ we have

$$\langle x, a \otimes b \rangle = \langle y, a \otimes b \rangle.$$

We apply this to $x = c(g)$ and $y = g \otimes g$ and observe that (i) holds if and only if

$$(\forall r, s \in R) \, \langle r \otimes s, c(g) \rangle = \langle r \otimes s, g \otimes g \rangle. \tag{10}$$

Now
$$g(rs) = \langle rs, g \rangle = \langle m(r \otimes s), g \rangle = \langle r \otimes s, c(g) \rangle, \tag{11}$$

$$g(r)g(s) = \langle r, g \rangle \langle s, g \rangle = \langle r \otimes s, g \otimes g \rangle. \tag{12}$$

So in view of (10), (11) and (12), assertion (i) holds if and only if $g(rs) = g(r)g(s)$ for all $r, s \in R$ is true. Since g is a linear form on A, this means exactly that a nonzero g is a morphism of weakly complete algebras, i.e., $g \in \mathrm{Spec}(R)$. ∎

A particularly relevant consequence of Proposition 6.5 will be the following conclusion for the group algebra $\mathbb{R}[G]$ of a *compact* group. Recall that after Theorem 5.4 we always identify a compact group G with a subgroup of $\mathbb{R}[G]$ in view of the embedding η_G. We shall see that the set $G \subseteq \mathbb{R}[G]$ contains already *all* grouplike elements (see Theorem 8.7 below).

Let us return to the primitive elements of a unital bialgebra. For this purpose assume that R is not only a unital algebra, but a bialgebra over $\mathbb{K}$, which implies that its dual $A \overset{\mathrm{def}}{=} R^*$ is not only a coalgebra but a bialgebra as well.

Proposition 6.6. *Let R be a unital bialgebra and $d \in A$. Then the following statements are equivalent:*

(i) *d is primitive in the bialgebra A.*

(ii) *$d \colon R \to \mathbb{K}$ a derivative of R with respect to the coidentity k, that is, an element in $\mathrm{Der}(R)$.*

The procedure of the proof of Proposition 6.5 allows us to leave the explicit proof of this proposition as an exercise.

Definition 6.7. Let A be a weakly complete Hopf algebra. Then we write $\Gamma(A)$ for the subset of all grouplike elements of A and $\Pi(A)$ for the subset of all primitive elements.

In view of Proposition 2.3, the following proposition is now readily verified:

Proposition 6.8. *In a weakly complete Hopf algebra A, the subset $\Gamma(A)$ is a closed subgroup of A^{-1} and $\Pi(A)$ is a closed Lie subalgebra of A_{Lie}. If $f \colon A \to B$ is a morphism of weakly complete Hopf algebras, then $f(\Gamma(A)) \subseteq \Gamma(B)$, and $f(\Pi(A)) \subseteq \Pi(B)$.*

Accordingly, if $\mathcal{WH}$ denotes the category of weakly complete Hopf algebras, then Γ defines a functor from $\mathcal{WH}$ to the category of topological groups $\mathcal{TG}$, also called *the grouplike grounding*, while Π defines a functor from $\mathcal{WH}$ to the category of topological Lie algebras over $\mathbb{K}$.

The functor $\mathbf{H} \overset{\mathrm{def}}{=} (G \mapsto \mathbb{R}[G]) : \mathcal{TG} \to \mathcal{WH}$ from topological groups to weakly complete Hopf algebras was known to us since Theorem 5.1 as a functor into the bigger category of all weakly complete unital algebras. That theorem will now be sharpened as follows:

Theorem 6.9.   (The Weakly Complete Group Hopf Algebra Adjunction Theorem)
The functor $\mathbf{H} \colon \mathcal{TG} \to \mathcal{WH}$ *from topological groups to weakly complete Hopf algebras
is left adjoint to the grouplike grounding* $\Gamma \colon \mathcal{WH} \to \mathcal{TG}$.

In other words, for a topological group G there is natural morphism of topological
groups $\eta_G \colon G \to \Gamma(\mathbf{H}(G)) = \Gamma(\mathbb{K}[G])$ such that for each morphism of topological
groups $f \colon G \to \Gamma(A)$ for a weakly complete Hopf algebra A there is a unique
morphism of weakly complete Hopf algebras $f' \colon \mathbf{H}(G) \to A$ such that we have
$f(g) = f'(\eta_G(g))$ for all $g \in G$:

$$
\begin{array}{ccc}
\mathcal{TG} & & \mathcal{WH} \\
\hline
\end{array}
$$

$$
\begin{array}{ccccc}
G & \xrightarrow{\eta_G} & \Gamma(\mathbf{H}(G)) & & \mathbf{H}(G) = \mathbb{K}[G] \\
{\scriptstyle \forall f}\downarrow & & \downarrow{\scriptstyle \Gamma(f')} & & \downarrow{\scriptstyle \exists! f'} \\
\Gamma(A) & \xrightarrow[\text{id}]{} & \Gamma(A) & & A
\end{array}
$$

Proof.   Let A be a weakly complete Hopf algebra and $f \colon G \to \Gamma(A)$ a continuous
group morphism. Since A is in particular a weakly complete associative unital
algebra and $\Gamma(A) \subseteq A^{-1}$, by the Weakly Complete Group Algebra Theorem 5.1
there is a unique morphism $f' \colon \mathbb{K}[G] \to A$ of weakly complete algebras such that
$f(g) = f'(\eta_G(g))$ for all $g \in G$. Since each $\eta_G(g)$ is grouplike by Remark 5.7 we have
$\eta_G(G) \subseteq \Gamma(\mathbf{H}(G))$. We shall see below that the morphism f' of weakly complete
algebras is indeed a morphism of weakly complete Hopf algebras and therefore maps
grouplike element into grouplike elements. Hence f' maps $\Gamma(\mathbf{H}(G))$ into $\Gamma(A)$.

We now have to show that f' is a morphism of Hopf algebras, that is,

(a) it respects comultiplication,
(b) coidentity, and
(c) symmetry.

For (a) we must show that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathbb{K}[G] & \xrightarrow{c_{\mathbb{K}[G]}} & \mathbb{K}[G] \otimes \mathbb{K}[G] & \cong & \mathbb{K}[G \times G] \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f' \otimes f'} & & \\
A & \xrightarrow[c_A]{} & A \otimes A. & &
\end{array}
$$

Since $\mathbb{K}[G]$ is generated as a topological algebra by $\eta_G(G)$ by Proposition 5.3, it
suffices to track all elements $x = \eta_G(g) \in \mathbb{K}[G]$ for $g \in G$. Every such element
is grouplike in $\mathbb{K}[G]$ by Remark 5.7, and so $(f' \otimes f')c_{\mathbb{K}[G]}(x) = (f' \otimes f')(x \otimes x) =
f'(x) \otimes f'(x)$ in $A \otimes A$, while on the other hand $f'(x) = f(g) \in \Gamma(A)$, whence
$c_A(f'(x)) = f'(x) \otimes f'(x)$ as well. This proves (a).

For (b) we must show that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathbb{K}[G] & \xrightarrow{k_{\mathbb{K}[G]}} & \mathbb{K} & \cong & \mathbb{K}[\{1_G\}] \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{K}}} & & \\
A & \xrightarrow[k_A]{} & \mathbb{K}. & &
\end{array}
$$

Again it suffices to check the elements $x = \eta_A(g)$. Since all grouplike elements are always mapped to 1, this is a trivial exercise.

Finally consider (c), where again we follow all elements $x = \eta_A(g)$. On the one hand we have $f'(\sigma_{\mathbb{K}[G]}(x)) = f'(x^{-1}) = f'(x)^{-1}$ in A^{-1}. But again $f'(x)$ is grouplike, and thus $\sigma_A(f'(x)) = f'(x)^{-1}$, which takes care of case (c), and this completes the proof of the theorem. ∎

As with any adjoint pair of functors there is an alternative way to express the adjunction in the preceding theorem: see e.g. [19], Proposition A3.36, p. 777:

Corollary 6.10. *For each weakly complete Hopf algebra A there is a natural morphism of Hopf algebras $\varepsilon_A\colon \mathbf{H}(\Gamma(A)) \to A$ such that for any topological group G and any morphism of Hopf algebras $\varphi\colon \mathbf{H}(G) \to A$ there is a unique continuous group morphism $\varphi'\colon G \to \Gamma(A)$ such that for each $x \in \mathbf{H}(G) = \mathbb{K}[G]$ one has $\varphi(x) = \varepsilon_A(\mathbb{K}[\varphi](x))$, where $\mathbb{K}[\varphi] = \mathbf{H}(\varphi)$.*

The formalism we described in Corollary 5.2 we can formulate with the present adjunction as well:

Corollary 6.11. *For any weakly complete Hopf algebra A and any topological group G we have*

(i) $\qquad (\forall A) \left(\Gamma(A) \xrightarrow{\ \eta_{\Gamma(A)}\ } \Gamma(\mathbb{K}[\Gamma(A)]) \xrightarrow{\ \Gamma(\varepsilon_A)\ } \Gamma(A) \right) = \mathrm{id}_{\Gamma(A)},$ *and*

(ii) $\qquad (\forall G) \left(\mathbb{K}[G] \xrightarrow{\ \mathbb{K}[\eta_G]\ } \mathbb{K}[\Gamma(\mathbb{K}[G])] \xrightarrow{\ \varepsilon_{\mathbb{K}[G]}\ } \mathbb{K}[G] \right) = \mathrm{id}_{\mathbb{K}[G]}.$

Definition 6.12. For any weakly complete Hopf algebra A we let $\mathbf{S}(A)$ denote the closed linear span $\overline{\mathrm{span}}(\Gamma(A))$ in A. ∎

Since $\Gamma(A)$ is a multiplicative subgroup of $(A, \cdot)$, clearly $\mathbf{S}(A)$ is a weakly closed subalgebra of A. Similarly, one easily verifies $c(\Gamma(A)) \subseteq \Gamma(A \otimes A) \subseteq \mathbf{S}(A \otimes A)$ for the comultiplication of A, and so $c(\mathbf{S}(A)) \subseteq \mathbf{S}(A \otimes A)$. Therefore $\mathbf{S}(A)$ *is Hopf subalgebra of A.*

Lemma 6.13. *If $A = \mathbb{K}[G]$ is the weakly complete group algebra of a topological group, then $A = \mathbf{S}(A)$.*

Proof. By Remark 5.7, $\eta_G(G) \subseteq \Gamma(A)$. By Proposition 5.3, $\overline{\mathrm{span}}(\eta_G(G)) = A$. Hence $\mathbf{S}(A) = \overline{\mathrm{span}}(\Gamma(A)) = A$. ∎

Corollary 6.14. *In the circumstances of* Corollary 6.10,

$$\mathrm{im}(\varepsilon_A) = \varepsilon_A(\mathbf{H}(\Gamma(A))) = \mathbf{S}(A).$$

Proof. Set $B = \mathbf{H}(\Gamma(A))$; then $\varepsilon_A\colon B \to A$ is a morphism of Hopf algebras, and so, in particular, a morphism of weakly complete vector spaces. Hence $\mathrm{im}(\varepsilon_A) = \varepsilon_A(B)$ is a closed Hopf subalgebra of A. Since ε_A is a morphim of Hopf algebras, $\varepsilon_A(\Gamma(B)) \subseteq \Gamma(A)$ and thus $\varepsilon_A(\mathbf{S}(B)) \subseteq \mathbf{S}(A)$. By Lemma 6.13 we have $B = \mathbf{S}(B)$ and so $\mathrm{im}(\varepsilon_A) = \varepsilon_A(\mathbf{S}(B))$ and so $\mathrm{im}(\varepsilon_A) \subseteq \mathbf{S}(A)$.

On the other hand, by Corollary 6.11, we have $\Gamma(A) \subseteq \mathrm{im}(\varepsilon_A)$ and since $\mathrm{im}(\varepsilon_A)$ is closed, we conclude $\mathbf{S}(A) \subseteq \mathrm{im}(\varepsilon_A)$ which completes the proof. $\blacksquare$

In particular, ε_A *is quotient homomorphism if and only if* $A = \mathbf{S}(A)$.

After we identified the image of the Hopf algebra morphism ε_A, the question arises for its kernel. Thus let $J = \ker \varepsilon_A$. As a kernel of a morphism of Hopf algebras, J is, firstly, the kernal of a morphism of weakly complete algebras, and secondly, satisfies the condition

$$c(J) \subseteq A \otimes J + J \otimes A.$$

(See e.g. [16], pp. 50ff., notably, Proposition 10.6 with appropriate modifications.) We leave the significance of this kernel open for the time being and return to it in the context of compact groups.

Here we record what is known in general on the pro-Lie group $\Gamma(A) \subseteq A^{-1}$ for a weakly complete Hopf algebra A in general.

It was first observed in [2] that any weakly complete Hopf algebra A accommodates grouplike elements (and so the spectrum $\mathrm{Spec}(R)$ of the dual Hopf algebra of A) and its primitive elements (and hence the derivatives $\mathrm{Der}(R)$ of the dual) in a particularly satisfactory set-up for whose proof we refer to [2], [3], or [20]. In conjunction with the present results, we can formulate the following theorem:

Theorem 6.15.   *Let A be a weakly complete $\mathbb{K}$-Hopf-algebra and A^{-1} its group of units which is an almost connected pro-Lie group (Theorem 4.1) containing as a closed subgroup the group $\Gamma(A)$ of grouplike elements. In particular, $\Gamma(A)$ is a pro-Lie group. Correspondingly, the set $\Pi(A)$ of primitive elements of A is a closed Lie subalgebra of the pro-Lie algebra A_{Lie} (Theorem 3.12) and may be identified with the Lie algebra $\mathfrak{L}(\Gamma(A))$ of $\Gamma(A)$. The exponential function $\exp_A \colon A_{\mathrm{Lie}} \to A^{-1}$ restricts to the exponential function $\exp_{\Gamma(A)} \colon \mathfrak{L}(\Gamma(A)) \to \Gamma(A)$.*

Again, we shall pursue this line below by applying it specifically to the real group Hopf algebras $A \overset{\mathrm{def}}{=} \mathbb{R}[G]$ in the special case that G is a *compact* group by linking it with particular information we have on compact groups. But first we have to explore duality more explicitly on the Hopf algebra level.

7.   The dual of weakly complete Hopf algebras and of group algebras

In this section we will have a closer look at the dual space A' of a weakly complete Hopf algebra A. We let G denote the topological group of grouplike elements $g \in A$. The underlying weakly complete vector space of A is a topological left and right G-module A with the module operations

$$(g, a) \mapsto g{\cdot}a \colon \ G \times A \to A, \quad g{\cdot}a := ga, \ \ \text{and}$$
$$(a, g) \mapsto a{\cdot}g \colon \ G \times A \to A, \quad a{\cdot}g := ag.$$

Recall that $\mathbb{I}(A)$ is the filterbasis of closed two-sided ideals J of A such that A/J is a finite dimensional algebra and that $A \cong \lim_{J \in \mathbb{I}(A)} A/J$. We can clearly reformulate Corollary 3.3 in terms of G-modules as follows:

Lemma 7.1.   *For the topological group $G = \Gamma(A)$, the G-module A has a filter basis $\mathbb{I}(A)$ of closed two-sided submodules $J \subseteq A$ such that $\dim(A/J) < \infty$ and that $A = \lim_{J \in \mathbb{I}(A)} A/J$ is a strict projective limit of finite dimensional G-modules. The filter basis $\mathbb{I}(A)$ in A converges to $0 \in A$.*

For a $J \in \mathbb{I}(A)$ let $J^\perp = \{f \in A' : (\forall a \in J)\, \langle f, a \rangle = 0\}$ denote the annihilator of J in the dual V of A. We compare the "Annihilator Mechanism" from [19], Proposition 7.62 and observe the following configuration:

$$
\begin{array}{ccc}
A & \{0\} \\
| & | \quad \Big\} \cong (A/J)' \\
J & J^\perp \\
| & | \quad \Big\} \cong J' \\
\{0\} & A'.
\end{array}
$$

In particular we recall the fact that $J^\perp \cong (A/J)'$ showing that $J^\perp$ is a finite-dimensional G-module on either side. By simply dualizing Lemma 7.1 we obtain

Lemma 7.2.   *For the topological group $G = \Gamma(A)$, the dual G-module $R \stackrel{\text{def}}{=} A'$ of the weakly complete G-module A has an up-directed set $\mathbb{D}(R)$ of finite-dimensional two-sided G-submodules (and $\mathbb{K}$-coalgebras!) $F \subseteq R$ such that R is the direct limit*

$$
R = \operatorname{colim}_{F \in \mathbb{D}(R)} F = \bigcup_{F \in \mathbb{D}(R)} F.
$$

The colimit is taken in the category of (abstract) G-modules, i.e. modules without any topology.

This means that for the topological group $G = \Gamma(A)$, every element ω of the dual of A' is contained in a finite dimensional left- and right-G-module (and $\mathbb{K}$-subcoalgebra). We record this in the following form:

Lemma 7.3.   *Let $\omega \in A'$. Then the vector subspaces $\operatorname{span}(G \cdot \omega)$ and $\operatorname{span}(\omega \cdot G)$ of both the left orbit and the right orbit of ω are finite dimensional, and both are contained in a finite dimensional $\mathbb{K}$-subcoalgebra of A'.*

For any $\omega \in A'$ the restriction $f \stackrel{\text{def}}{=} \omega | G : G \to \mathbb{K}$ is a continuos function such that each of the sets of translates f_g, $f_g(h) = f(gh)$, respectively, $_g f$, $_g f(h) = f(hg)$ forms a finite dimensional vector subspace of the space $C(G, \mathbb{K})$ of the vector space of all continuous $\mathbb{K}$-valued functions f on G.

Definition 7.4.   For an arbitrary topological group G we define $\mathcal{R}(G, \mathbb{K}) \subseteq C(G, \mathbb{K})$ to be that set of continuous functions $f \colon G \to \mathbb{K}$ for which the linear span of the set of translations $_g f$, $_g f(h) = f(hg)$, is a finite dimensional vector subspace of $C(G, \mathbb{K})$. The functions in $\mathcal{R}(G, \mathbb{K})$ are called *representative functions*.

In Lemma 7.3 we saw that for a weakly complete Hopf algebra A and its dual A' (consisting of continuous linear forms) we have a natural linear map

$$
\tau_A \colon A' \to \mathcal{R}(\Gamma(A), \mathbb{K}), \ \ \tau_A(\omega)(g) = (\omega | \Gamma(A))(g).
$$

An element $\omega \in A'$ is in the kernel of τ_A if and only if $\omega(\Gamma(A)) = \{0\}$ if and only if $\omega(\mathbf{S}(A)) = \{0\}$ if and only if $\omega \in \mathbf{S}(A)^{\perp}$. We therefore observe:

Lemma 7.5. *There is an exact sequence of $\mathbb{K}$-vector spaces*

$$0 \to \mathbf{S}(A)^{\perp} \xrightarrow{\text{inc}} A' \xrightarrow{\tau_A} \mathcal{R}(\Gamma(A), \mathbb{K}).$$

Let us complete this exact sequence by determining the surjectivity of τ_A at least in the case that $A = \mathbb{K}[G]$ for a topological group G. Indeed, let us show that a function $f \in \mathcal{R}(G, \mathbb{K})$ for an arbitrary topological group G is in the image of τ_A.

So let $f \in \mathcal{R}(G, \mathbb{K})$, then $U \overset{\text{def}}{=} \text{span}\{{}_g f : g \in G\}$ is an n-dimensional $\mathbb{K}$-vector subspace of $C(G, \mathbb{K})$. Accordingly, $M_U \overset{\text{def}}{=} \text{Hom}(U, U)$ is an n^2-dimensional unital $\mathbb{K}$-algebra such that $\pi \colon G \to \text{Gl}(U)$, $\pi(g)(u) = {}_g u$, $u \in U$, is a continuous representation of G on U. Now we employ the universal property of the group algebra $\mathbb{K}[G]$ expressed in Theorem 5.1 and find a unique morphism of weakly complete algebras $\pi' \colon \mathbb{K}[G] \to M_U$ inducing a morphism of topological groups $\pi'|\mathbb{K}[G]^{-1} \colon \mathbb{K}[G]^{-1} \to \text{Gl}(U)$ such that $\pi = (\pi'|\mathbb{K}[G]^{-1}) \circ \eta_G$. If η_G happens to be an embedding and G is considered as a subgroup of $\mathbb{K}[G]^{-1}$, then the representation π is just the restriction of π' to G. Now we recall that $f \in U$ and note that for any endomorphism $\alpha \in M_U$ of U we have $\alpha(f) \in U \subseteq C(G, \mathbb{K})$. Then for the identity $1 \in G$ the element $\alpha(f)(1) \in \mathbb{K}$ is well defined and so we obtain a linear map $L \colon M_U \to \mathbb{K}$ by defining $L(\alpha) = \alpha(f)(1)$. So we finally define the linear form $\omega_\pi = L \circ \pi' \colon \mathbb{K}[G] \to \mathbb{K}$ and calculate for any $g \in G$ the field element

$$\omega_\pi(\eta_G(g)) = L(\pi'(\eta_G(g))) = L(\pi(g)) = \pi(g)(f)(1) = {}_g f(1) = f(g).$$

Thus we have in fact shown the following

Lemma 7.6. *For any topological group G the following statements are equivalent for a continuous function $f \in C(G, \mathbb{K})$:*

(i) $f \in \mathcal{R}(G, \mathbb{K})$.

(ii) *There is a continuous linear form $\omega \colon \mathbb{K}[G] \to \mathbb{K}$ such that $\omega \circ \eta_G = f$*

Note here that in case G is naturally embedded into $\mathbb{K}[G]$ condition (ii) just says

(ii′) $\omega|G = f$ *for some* $\omega \in \mathbb{K}[G]'$.

For special weakly complete Hopf algebras A which are close enough to $\mathbb{K}[G]$, these insights allow us to formulate a concrete identifiction of the dual A' of the group Hopf-algebra $\mathbb{K}$ over the ground fields $\mathbb{R}$ or $\mathbb{C}$, where again we consider the pair $(\mathcal{V}, \mathcal{W})$ of dual categories $\mathcal{V}$ of $\mathbb{K}$-vector spaces and $\mathcal{W}$ of weakly complete $\mathbb{K}$-vector spaces.

For an arbitrary topological group G, let again $\mathbb{K}[G]'$ denote the topological dual of the weakly complete group Hopf-algebra $\mathbb{K}[G]$ over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then the following identification of the topological dual of $\mathbb{K}[G]$ in the category $\mathcal{V}$ provides the background of the topological dual of any weakly complete Hopf algebra generated by its grouplike elements:

Theorem 7.7. (a) *For an arbitrary topological group G, the function*

$$F_G \colon \mathbb{K}[G]' \to \mathcal{R}(G, \mathbb{K}), \quad F(\omega) = \omega \circ \eta_G$$

is a natural isomorphism of Hopf algebras.

(b) *If A is a weakly complete Hopf algebra satisfying $\mathbf{S}(A) = A$, and if G is the group $\Gamma(A)$ of grouplike elements of A, then $\tau_A \colon A' \to \mathcal{R}(G, \mathbb{K})$, $\tau(\omega) = \omega|G$ is an injective morphism of Hopf algebras, embedding A' as Hopf subalgebra of $\mathcal{R}(G, \mathbb{K})$.*

Proof. (a) By Proposition 5.3 we have $\overline{\mathrm{span}}(\eta(G)) = \mathbb{K}[G]$ and so the relation $(\omega \circ \eta_G)(G) = \omega(\eta_G(G)) = \{0\}$ implies $\omega = 0$. Hence the natural linear function $F_G \colon \mathbb{K}[G]' \to \mathcal{R}(G, \mathbb{K})$, $F_G(\omega) = \omega_G \circ \eta_G$ is injective. By Lemma 7.6, it is surjective and so it is a natural isomorphism of vector spaces.

We recall that $\mathbb{K}[G] \otimes \mathbb{K}[G] \cong \mathbb{K}[G \times G]$ in the category of weakly complete vector spaces by Theorem 5.5. By Proposition 1.5 we derive that $\mathbb{K}[G]' \otimes \mathbb{K}[G]'$ is naturally isomorphic to $\mathbb{K}[G \times G]'$, and so $G \mapsto \mathbb{K}[G]'$ is a functor mapping topological groups to group objects in the category of vector spaces, that is, to Hopf algebras.

Analogously, $\mathcal{R}(G, \mathbb{K}) \otimes \mathcal{R}(H, \mathbb{K}) \cong \mathcal{R}(G \times H, \mathbb{K})$ in the category of vector spaces. Hence the functor $G \mapsto \mathcal{R}(G, \mathbb{K})$ is a functor mapping topological groups into the category of Hopf algebras (in the category of $\mathbb{K}$-vector spaces). The naturality of F_G then implies that it is a morphism of Hopf algebras.

(b) The relation $S(A) = A$ is equivalent to $S(A)^\perp = \{0\}$ in A', and so by Lemma 7.5, the linear function τ_A is injective. By (a) we identify $\mathcal{R}(G, \mathbb{K})$ and $\mathbb{K}[G]'$ as Hopf algebras. Then the injection $\tau_A \colon A' \to \mathcal{R}(G, \mathbb{K}) = \mathbb{K}[G]'$, $G = \Gamma(A)$ is none other than the dual ε_A' of the surjective back adjunction $\varepsilon_A \colon \mathbb{K}[G] = \mathbb{K}[\Gamma(A)] \to A$ of Corollary 6.10, and so it is a morphism of Hopf algebras and the assertion follows. ∎

In the sense of (a) in the preceding theorem, $\mathbb{K}[G]'$ may be identified with the vector space $\mathcal{R}(G, \mathbb{K})$ of continuous functions $f \colon G \to \mathbb{K}$ for which that the translates $_gf$ (where $_gf(h) = f(gh)$) (equivalently, the translates f_g (where $f_g(h) = f(gh)$) span a finite dimensional $\mathbb{K}$-vector space. The vector space $\mathcal{R}(G, \mathbb{K})$ is familiar in the literature as the vector space of representative functions on G, where it is most frequently formulated for compact groups G and where it is also considered as a Hopf-algebra. We are choosing here the covariant group algebra $\mathbb{K}[G]$ to be at the center of attention and obtain $\mathcal{R}(G, \mathbb{K})$ via vector space duality from $\mathbb{K}[G]$. Conversely, if one asks for a "concrete" description of $\mathbb{K}[G]$, then the answer may now be that, in terms of topological vector spaces, as a topological vector space, $\mathbb{K}[G]$ is the algebraic dual (consisting of all linear forms) of the (abstract) vector subspace $\mathcal{R}(G, \mathbb{K})$ of the vector space $C(G, \mathbb{K})$ of continuous functions $G \to \mathbb{K}$. If G is a compact group, $C(G, \mathbb{K})$ is a familiar Banach space.

At this point one realizes that in case of a finite group G the $\mathbb{K}$-vector spaces $\mathbb{K}[G]$ and $\mathcal{R}(G, \mathbb{K})$ are finite dimensional of the same dimension and are, therefore, isomorphic. In particular, we have a class of special examples of $\mathbb{K}$-Hopf algebras:

Example 7.8. Let G be a finite group. Then $A = \mathcal{R}(G, \mathbb{K}) = \mathbb{K}^G$ is a finite dimensional real Hopf algebra which may be considered as the dual of the group algebra $\mathbb{K}[G]$. A grouplike element $f \in \Gamma(A)$ then is a character $f \colon \mathbb{K}[G] \to \mathbb{K}$. The group G may be considered as the subgroup $\Gamma(\mathbb{K}[G])$, and then $f|G \colon G \to \mathbb{K}^{-1}$ is a

morphism of groups. Conversely, every group morphism $G \to \mathbb{K}^{-1}$ yields a grouplike element of A. In view of the finiteness of G we have

$$\Gamma\big(\mathcal{R}(G, \mathbb{K})\big) \cong \mathrm{Hom}(G, \mathbb{S}^1), \quad \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

If G is a finite simple group, then the $|G|$-dimensional Hopf algebra $A = \mathcal{R}(G, \mathbb{K})$ has no nontrivial grouplike elements, that is, $\Gamma(A) = \{1\}$ and $\mathbf{S}(A) = \mathbb{K}{\cdot}1$. The smallest example of this kind arises from the group $G = A_5$, the group of 120 even permutations of a set of 5 elements. ∎

8. $\mathbb{C}[G]$ as an involutive weakly complete algebra

A complex unital algebra A is called *involutive* if it is endowed with a real vector space automorphism $a \mapsto a^*$ such that for all $a, b \in A$ with have $a^{**} = a$, $(c{\cdot}a)^* = \overline{c}{\cdot}a^*$, and $(ab)^* = b^*a^*$. A complex weakly complete unital topological algebra whose underlying algebra is involutive with respect to an involution $*$ is a weakly complete unital involutive algebra, if $a \mapsto a^*$ is an isomorphism of the underlying weakly complete topological vector space.

Every C*-Algebra is an involutive algebra; a simple example is $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, i.e, an algebra isomorphic to the algebra of $n \times n$ complex matrices with the conjugate transpose as involution. An element $a \in A$ is *unitary* if $aa^* = a^*a = 1$, that is if $a^* = a^{-1}$, and it is called *hermitian* if $a = a^*$. If A is an involutive topological unital algebra with an exponential function $\exp$, then for each hermitian element h the element $\exp(i{\cdot}h)$ is unitary.

Note that in $A = \mathbb{C}$ the unitary elements are the elements on the unit circle $\mathbf{S}^1$ and the hermitian elements are the ones on the real line $\mathbb{R}$. The theory of the function $h \mapsto \exp(ih)$ is called *trigonometry*.

The involutive algebras form a category with respect to morphisms $f : A \to B$ satisfying $f(a^*) = f(a)^*$.

Lemma 8.1.   *If A is a complex weakly complete unital topological coalgebra, then the complex vector space dual A' is an involutive unital algebra for the involution $f \mapsto f^*$ defined by $f^*(a) = \overline{f(a^*)}$.*

The proof is a straightforward exercise.

Assume now that G is a topological group. We shall introduce an involution on $\mathbb{C}[G]$ making it an involutive algebra.

(a) For every complex vector space V we introduce a complex vector space $\widetilde{V}$ by endowing the real vector space underlying V with a complex scalar multiplication $\bullet$ defined by $c \bullet v = \overline{c}{\cdot}v$.

(b) The composition $|\mathbb{C}[G]| \xrightarrow{\sigma_G} |\mathbb{C}[G]| \xrightarrow{\mathrm{id}_{\mathbb{C}[G]}} |\mathbb{C}[G]|^{\widetilde{}}$ of the Hopf algebra symmetry $\sigma_G : |\mathbb{C}[G]| \to |\mathbb{C}[G]|$ and the identity map $\mathbb{C}[G] \to \mathbb{C}[G]^{\widetilde{}}$ yield an involution

$$*: |\mathbb{C}[G]| \to |\mathbb{C}[G]|, \ such \ that \ (\forall c \in \mathbb{C}, \ a \in \mathbb{C}[G]) \, a^* = \sigma(a) \ and \ (c{\cdot}a)^* = \overline{c}{\cdot}a^*.$$

Moreover, by definition of σ_G, we have $\eta_G(g)^* = \eta_G(g^{-1})$ for all $g \in G$.

The proof of the following remarks may again be safely left as an exercise.

Lemma 8.2. *For each topological group G, the complex algebra $\mathbb{C}[G]$ is an involutive algebra with respect to $*$, and the comultiplication $c\colon \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ and the coidentity $k\colon \mathbb{C}[G] \to \mathbb{C}$ are morphisms of involutive algebras. All elements in $\eta_G(G) \subseteq \mathbb{C}[G]$ are unitary grouplike elements.*

From Lemma 8.1 we derive directly the following observation:

Lemma 8.3. *The dual $\mathcal{R}(G,\mathbb{C})$ of the weakly complete involutive algebra $\mathbb{C}[G]$ is an involutive algebra.*

Definition 8.4. Let R be a unital algebra over $\mathbb{C}$. Then the set of hermitian characters of R is denoted by $\mathrm{Spec}_h(R)$ is called the *hermitian spectrum* of R.

For a *compact* group G, however, the commutative unital algebra $\mathcal{R}(G,\mathbb{K})$ is a *dense* subalgebra of the commutative unital Banach $\mathbb{K}$–algebra $C(G,\mathbb{K})$ (see e.g. [19], Theorem 3.7.)

We note that for every $g \in G$, the *point evaluation* $f \mapsto f(g) : \mathcal{R}(G,\mathbb{C}) \to \mathbb{C}$ belongs to $\mathrm{Spec}_h(\mathcal{R}(G,\mathbb{C}))$. In this particular situation, the literature provides the following result in which we consider $\mathbb{C}[G]$ as the dual $\mathcal{R}(G,\mathbb{C})^*$ of the involutive algebra $\mathcal{R}(G,\mathbb{C})$.

Theorem 8.5. *For a compact group G, the hermitian spectrum*

$$\mathrm{Spec}_h(\mathcal{R}(G,\mathbb{C})) \subseteq \mathbb{C}[G]$$

is precisely the set of point evaluations.

Proof. See e.g. [16], Proposition 12.26 or [15], p. 28. Cf. also [8], no. **1**.3.7., p. 7.   ∎

In the case $\mathbb{K} = \mathbb{R}$, any morphism $f\colon \mathcal{R}(G,\mathbb{R}) \to \mathbb{R}$ of real algebras, that is, any (real) character of $\mathcal{R}(G,\mathbb{R})$ can uniquely be extended to a (complex) character $\widetilde{f}\colon \mathbb{C} \otimes_{\mathbb{R}} \mathcal{R}(G,\mathbb{R}) \to \mathbb{C}$, where $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{R}(G,\mathbb{R}) \cong \mathcal{R}(G,\mathbb{C})$, such that $\overline{\widetilde{f}} = \widetilde{\overline{f}}$.

Trivially, f is a point evaluation of $\mathcal{R}(G,\mathbb{R})$, if and only if $\widetilde{f}$ is a point evaluation of $\mathcal{R}(G,\mathbb{C})$, and f is continuous if and only if $\widetilde{f}$ is continuous. Hence Theorem 8.5 implies the analogous result over $\mathbb{R}$:

Corollary 8.6. *For a compact group G, the set $\mathrm{Spec}(\mathcal{R}(G,\mathbb{R})) \subseteq \mathcal{R}(G,\mathbb{R})^*$ of characters of $\mathcal{R}(G,\mathbb{R})$ is precisely the set of point evaluations $f \mapsto f(g)$, $G \in G$.*

Recall that $\mathcal{R}(G,\mathbb{R})^*$ and $\mathbb{R}[G]$ are natually isomorphic by duality since $\mathbb{R}[G]'$ and $\mathcal{R}(G,\mathbb{R})$ are naturally isomorphic by Theorem 7.7.

After Theorem 5.4 we may identify a compact group G with its isomorphic image via η_G in $\mathbb{R}[G]$. From Remark 5.7 we know $G \subseteq \Gamma(\mathbb{R}[G])$ (see Definition 6.7). Now finally we have the full information on one half of the adjunction of **H** and Γ of Theorem 6.9 in the case of *compact* groups G:

Theorem 8.7. (Main Theorem for Compact Groups A) *For every compact group G the natural morphism of topological groups $\eta_G \colon G \to \Gamma\big(\mathbf{H}(G)\big) = \Gamma(\mathbb{R}[G])$ is an isomorphism of compact groups.*

In other words, if for a compact group G we identify G with a subgroup of $(\mathbb{R}[G], \cdot)$ for its weakly complete group algebra $\mathbb{R}[G]$, then $G = \Gamma(\mathbb{R}[G])$, i.e. every grouplike element of the Hopf algebra $\mathbb{R}[G]$ is a group element of G.

Proof. We apply Proposition 6.5 with $R = \mathcal{R}(G, \mathbb{R})$ and $A = \mathbb{R}[G]$. Then the set of grouplike elements of $\mathbb{R}[G]$ is $\mathrm{Spec}(\mathcal{R}(G, \mathbb{R}))$. Let $\mathrm{ev} \colon G \to \mathrm{Spec}(\mathcal{R}(G, \mathbb{R}))$ denote the function given by $\mathrm{ev}(g)(f) = f(g)$ for all $g \in G$, $f \in \mathcal{R}(G, \mathbb{R})$. Then $\mathrm{ev}(G) \subseteq \mathrm{Spec}(\mathcal{R}(G, \mathbb{R}))$ by Remark 5.7 and Proposition 6.5. However, Theorem 8.5 and Corollary 8.6 show that in fact equality holds, and that is the assertion of the theorem. ∎

It appears that a direct proof of the assertion that every grouplike element is a group element in $\mathbb{R}[G]$ would be difficult to come by even in the case of an infinite compact group G. (If G happens to be finite then the proof is elementary linear algebra.).

As it stands, after Theorem 8.7, the category theoretical background on the adjunction of the functors Γ and $\mathbf{H}$ expressed in Corollary 6.11(i) and (ii) gives some additional insight for compact G, respectively, compact $\Gamma(A)$. Indeed Corollary 6.11(i) yields at once:

Corollary 8.8. *Let A be a weakly complete real Hopf algebra and assume that the group $G \overset{\text{def}}{=} \Gamma(A)$ of its grouplike elements is compact. Then the natural morphism of topological groups $\Gamma(\varepsilon_A) \colon \Gamma\big(\mathbf{H}(G)\big) \to G$ is an isomorphism.*

We recall that every compact group G is a pro-Lie group with a Lie algebra $\mathfrak{L}(G)$ and an exponential function $\exp_G \colon \mathfrak{L}(G) \to G$ according to [19], Definition 9.44ff.

Theorem 8.9. (Main Theorem for Compact Groups B) *Consider the compact group G as a subgroup of the multiplicative monoid $(\mathbb{R}[G], \cdot)$ of its weakly complete group Hopf algebra. Then $G = \Gamma(\mathbb{R}[G])$, that is, G is the set of grouplike elements of the Hopf algebra $\mathbb{R}[G]$, while $\mathfrak{L}(G) \cong \Pi(\mathbb{R}[G])$, that is, the Lie algebra of G is the set of primitive elements of $\mathbb{R}[G]$. Hence the Lie algebra of G is isomorphic to the subalgebra $\Pi(\mathbb{R}[G])$ of the Lie algebra $\mathbb{R}[G]_{\mathrm{Lie}}$. Moreover, the global exponential function $\exp_{\mathbb{R}[G]} \colon \mathbb{R}[G]_{\mathrm{Lie}} \to \mathbb{R}[G]^{-1}$ of $\mathbb{R}[G]$ restricts, up to isomorphy, to the exponential function $\exp_G \colon \mathfrak{L}(G) \to G$ of G.*

We have now seen that a compact group and its Lie theory are completely incorporated into its weakly complete group Hopf algebra. Under such circumstances it is natural to ask whether a similar assertion could be made for the ample theory of Radon measures on a compact group including its Haar measure. We shall address this question and give a largely affirmative answer in the next section.

What we have to address at this point is the question, whether for a weakly complete real Hopf algebra A in which the group $\Gamma(A)$ is compact and algebraically and topologically generates A, the natural morphism $\varepsilon_A \colon \mathbf{H}\big(\Gamma(A)\big) = \mathbb{R}[\Gamma(A)] \to A$ is in fact an isomorphism.

For the investigation of this question we need some preparation. Assume that G is a compact group and $\mathcal{R}(G,R) \subseteq C(G,\mathbb{R})$ is the Hopf algebra of all functions $f \in C(G,\mathbb{R})$ whose translates span a finite dimensional vector subspace.

We now let M be a Hopf subalgebra and a G-submodule of $\mathcal{R}(G,\mathbb{R})$. Recall that $\operatorname{Spec} M$ denotes the set of all algebra morphisms $M \to \mathbb{R}$. Trivially we have a morphism $\omega \mapsto \omega|M : \operatorname{Spec} \mathcal{R}(G,\mathbb{R}) \to \operatorname{Spec}(M)$. From Corollary 8.6 we know that $G \cong \operatorname{Spec} \mathcal{R}(G,\mathbb{R})$ via point evaluation. In the case that $M = A'$ and $G = \Gamma(A)$ such that $\Gamma(\varepsilon_A)\colon \Gamma(\mathbb{R}[G]) \to G$ an isomorphism as in Corollary 8.8 we know that

$$(E) \qquad\qquad g \mapsto (f \mapsto f(g)) : G \to \operatorname{Spec}(M) \quad \text{is an isomorphism.}$$

From [19], Definition 1.20, p. 13 we recall that $M \subseteq C(G,\mathbb{R})$ is said to *separate points* if for two points $g_1 \neq g_2$ in G there is an $f \in M$ such that $f(g_1) \neq f(g_2)$. In other words, different points in G can be distinguished by different point evaluations of functions from M. So condition (E) secures that the functions of M separate the points of G. This has the following consequence:

Lemma 8.10. *If the unital subalgebra M of $\mathcal{R}(G,\mathbb{R})$ satisfies (E), then M is dense in $C(G,\mathbb{R})$ with respect to the sup norm and is dense in $L^2(G,\mathbb{R})$ with respect to the L^2-norm.*

Proof. Since M is a unital subalgebra of $\mathcal{R}(G,\mathbb{R})$, it contains the scalar multiples of the constant functions of value 1, that is, M contains all the constant functions. Moreover, by Hypothesis (E), the algebra $M \subseteq C(G,\mathbb{R})$ separates the points of G. Therefore the Stone-Weierstraß Theorem applies and shows that M is dense in $C(G,\mathbb{R})$ in the sup norm topology of $C(G,\mathbb{R})$. Since $L^2(G,\mathbb{R})$ is the L^2-norm completion of $C(G,\mathbb{R})$ and M is uniformly dense in $C(G,R)$ it follows that M is dense in $L^2(G,\mathbb{R})$ in the L^2-norm. (Cf. e.g. [19] Theorem 3.7 and its proof.) $\blacksquare$

Lemma 8.11. *If a G-submodule M of $\mathcal{R}(G,\mathbb{R})$ is L^2-dense in $L^2(G,\mathbb{R})$ then it agrees with $\mathcal{R}(G,\mathbb{R})$.*

Proof. Let $\widehat{G}$ denote the set of isomorphy classes of irreducible G-modules. By the Fine Structure Theorem of $\mathcal{R}(G,\mathbb{R})$ (see [19], Theorem 3.28) $\mathcal{R}(G,R) = \sum_{\varepsilon \in \widehat{G}} \mathcal{R}(G,\mathbb{R})_\varepsilon$ where $\sum$ denotes the algebraic direct sum of (finite dimensional) vector subspaces and where $\mathcal{R}(G,\mathbb{R})_\varepsilon$ is a finite direct sum of simple modules for each $\varepsilon \in \widehat{G}$. In particular, each $\mathcal{R}(G,\mathbb{R})_\varepsilon$ is finite dimensional. Further $L^2(G,\mathbb{R}) = \bigoplus_{\varepsilon \in \widehat{G}} \mathcal{R}(G,\mathbb{R})_\varepsilon$ where $\bigoplus$ denotes the Hilbert space direct sum.

The submodule M of $\mathcal{R}(G,\mathbb{R})$ adjusts to the canonical decomposition of $\mathcal{R}(G,\mathbb{R})$ since M_ε is necessarily a submodule of $\mathcal{R}(G,\mathbb{R})_\varepsilon$. Hence $M = \sum_{\varepsilon \in \widehat{G}} M_\varepsilon$ and the L_2-closure of M in $L^2(G,\mathbb{R})$ is the Hilbert space sum $\bigoplus_{\varepsilon \in \widehat{G}} M_\varepsilon$.

By way of contradiction suppose now that that $M \neq \mathcal{R}(G,\mathbb{R})$. Then there is an $\varepsilon' \in \widehat{G}$ such that $M_{\varepsilon'} \neq \mathcal{R}(G,\mathbb{R})_{\varepsilon'}$. Since $\mathcal{R}(G,\mathbb{R})_{\varepsilon'}$ is finite dimensional,

$$\bigoplus_{\varepsilon \in \widehat{G}} M_\varepsilon = M_{\varepsilon'} \oplus \bigoplus_{\varepsilon \neq \varepsilon'} M_\varepsilon$$

is properly smaller than the Hilbert space sum

$$\bigoplus_{\varepsilon \in \widehat{G}} \mathcal{R}(G,\mathbb{R})_\varepsilon = \mathcal{R}(G,\mathbb{R})_{\varepsilon'} \oplus \bigoplus_{\varepsilon \neq \varepsilon'} \mathcal{R}(G,R)_\varepsilon = L^2(G,\mathbb{R}),$$

contradicting the hypothesis that M is L^2-dense in $L^2(G, \mathbb{R})$. This contradiction proves the lemma. ∎

Now we are ready for the third main result on compact groups in the present context: the statement which parallels Theorem 8.7.

Theorem 8.12. (Main Theorem on Compact Groups C) *Let A be a weakly complete real Hopf Algebra satisfying the following two conditions:*
(i) *The subgroup $\Gamma(A)$ of grouplike elements of A is compact,*
(ii) *$\Gamma(A)$ generates A algebraically and topologically, that is, $\mathbf{S}(A) = A$.*
Then $\varepsilon_A \colon \mathbb{R}[\Gamma(A)] = \mathbf{H}\big(\Gamma(A)\big) \to A$ *is a natural isomorphism.*

Proof.   We set $G = \Gamma(A)$. By Corollary 6.14 the morphism $\varepsilon_A \colon \mathbb{R}[G] \to A$ is a quotient homomorphism of weakly complete Hopf algebras which by Corollary 8.8 induces an isomorphism $\Gamma(\varepsilon_A) \colon \Gamma(\mathbb{R}[G]) \to G$. So $\Gamma(\mathbb{R}[G])$ is identified with G if we consider G as included in $\mathbb{R}[G]$ according to Theorem 8.7.

By the duality between real Hopf algebras in $\mathcal{V}$ and weakly complete real Hopf algebras in $\mathcal{W}$, the dual morphism $\varepsilon'_A \colon A' \to \mathbb{R}[G]'$ is injective. Theorem 7.7(b) then gives us an inclusion $A' \subseteq \mathcal{R}(G, \mathbb{R})$ of real Hopf algebras as well as of G-modules such that the natural map $\mathrm{Spec}(A') \to \mathrm{Spec}\big(\mathcal{R}(G, \mathbb{R})\big)$ is the identity.

That is, Condition (E) above (preceding Lemma 8.10) holds and Lemmas 8.10 and 8.11 apply. Therefore $A' = \mathcal{R}(G, \mathbb{R})$. This in turn shows that ε_A is an isomorphism. ∎

For a concise formulation of the consequences let us use the following notation:

Definition 8.13.   A real weakly complete Hopf algebra A will be called *compactlike* if it is such that the subgroup $\Gamma(A)$ of grouplike elements is compact and $\mathbf{S}(A) = \overline{\mathrm{span}}\big(\Gamma(A)\big) = A$.

Theorem 8.14.   (The Equivalence Theorem of the Category of Compact Groups and compactlike Hopf algebras) *The categories of compact groups and of weakly complete compactlike Hopf algebras are equivalent.*

Proof.   This follows immediately from Theorems 8.7 and 8.12. ∎

Corollary 8.15.   (Tannaka Duality) *The category of compact groups is dual to the full category of real abstract Hopf algebras of the form $\mathcal{R}(G, \mathbb{R})$ with a compact group G.*

9.   The Radon measures within the group algebra of a compact group

We shall invoke measure theory in the form pioneered for arbitrarly locally compact groups in [4]. For a *compact* group G it is less technical and adapts reasonably to the formalism of its real group algebra $\mathbb{R}[G]$. This discussion will help us to understand the power ot the group algebras $\mathbb{R}[G]$ for a compact group.

Indeed any compact Hausdorff topological group provides us with a real Banach algebra $C(G, \mathbb{R})$ endowed with the sup-norm which makes it into a Hopf-algebra in the categoy of Banach spaces. Accordingly, its topological dual $C(G, \mathbb{R})'$ yields the Banach algebra and indeed Banach Hopf algebra $M(G, \mathbb{R})$ (see e.g. [16]). Its elements μ are the so called *Radon measures* on G and the general source books of

this orientation of measure and probability theory is Bourbaki's book [4] and for the foundations of harmonic analysis the book of Hewitt and Ross [12]. For a measure theory in the context of compact groups see also [19], Appendix 5: "Measures on Compact Groups".

So let W be a weakly complete real vector space. Then W may be identified with W'^{*} (see Theorem 1.4). For $F \in C(G, W)$ and $\mu \in M(G, \mathbb{R})$ we obtain a unique element $\int_G F \, d\mu \in W$ such that we have

$$(*) \qquad (\forall \omega \in W') \quad \left\langle \omega, \int_G F \, d\mu \right\rangle = \int_{g \in G} \langle \omega, F(g) \rangle d\mu(g).$$

(See [4], Chap. III, §3, $\mathbf{n}^{\mathrm{o}1}$, Définition 1.) Let $\mathrm{supp}(\mu)$ denote the support of μ. (See [4], Chap. III, §2, $\mathbf{n}^{\mathrm{o}2}$, Définition 1.)

Lemma 9.1. *Let $T \colon W_1 \to W_2$ be a morphism of weakly complete vector spaces, G a compact Hausdorff space and μ a measure on G. If $F \in C(G, W_1)$, then $T(\int_G F \, d\mu) = \int_G (T \circ F) d\mu$.*

(See e.g. [4], Chap. III, §3, $\mathbf{n}^{\mathrm{o}2}$, Proposition 2.)

In [4] it is shown that the vector space $M(G, \mathbb{R})$ is also a complete lattice w.r.t. a natural partial order (see [4], Chap. III, §1, $\mathbf{n}^{\mathrm{o}}$ 6) so that each $\mu \in M(G)$ is uniquely of the form $\mu = \mu^+ - \mu^-$ for the two positive measures $\mu^+ = \mu \vee 0$ and $\mu^- = -\mu \vee 0$. One defines $|\mu| = \mu^+ + \mu^-$. If $M^+(G)$ denotes the cone of all positive measures, we have $M(G) = M^+(G) - M^+(G)$ ([4], Chap. III, §1, $\mathbf{n}^{\mathrm{o}}$ 5, Théorème 2). Moreover, $\|\mu\| = |\mu|(1) = \int d|\mu|$. A measure is called a *probability measure* if it is positive and $\mu(1) = 1$. We write $P(G)$ for the set of all probability measures on G and we note $M^+(G) = \mathbb{R}_+ \cdot P(G)$ where $\mathbb{R}_+ = [0, \infty[\subseteq \mathbb{R}$.

We denote by $M_p(G)$ the vector space $M(G, \mathbb{R})$ with the topology of pointwise convergence and recall that $P(G)$ has the structure of a compact submonoid of $M_p(G)^{\times}$; some aspects are discussed in [19], Appendix 5. On $M^+(G)$ the topologies of $M_p(G)$ and the compact open topology of $M(G, \mathbb{R})$ agree ([4], Chap. III, §1, $\mathbf{n}^{\mathrm{o}}$ 10, Proposition 18), Also $M_p^+(G)$ is a locally compact convex pointed *cone* with the closed convex hull $P(G)$ of the set of point measures as *basis*. We also recall, that any positive linear form on $C(G, \mathbb{R})$ is in $M^+(G)$ (i.e., is continuous) (see [4], Chap. III, §1, $\mathbf{n}^{\mathrm{o}}$ 5, Théorème 1).

9.1. Measures and group algebras

Now we allow this machinery and Theorems 6.15 and 5.4 to come together to elucidate the structure of $\mathbb{R}[G]$ for compact groups G.

We let G be a compact group. By Theorem 5.4 it is no loss of generality to assume that G is a compact subgroup of $\mathbb{R}[G]^{-1}$, where $\mathbb{R}[G]$ is the weakly complete group Hopf algebra of G and $\eta_G \colon G \to \mathbb{R}[G]^{-1}$ is the inclusion morphism. By Theorem 7.7 there is an isomorphism $\omega \mapsto f_\omega \colon \mathbb{R}[G]' \to \mathcal{R}(G, \mathbb{R})$ such that

$$(\forall \omega \in \mathbb{R}[G]', \, g \in G) \, \langle \omega, \eta_G(g) \rangle = f_\omega(g),$$

and, in the reverse direction, the function $\omega \mapsto \omega|G \colon \mathbb{R}[G]' \to C(G, \mathbb{R})$ induces an isomorphism of vector spaces $\mathbb{R}[G]' \to \mathcal{R}(G, \mathbb{R})$.

Therefore, in the spirit of relation $(*)$, we are led to the following definition:

Definition 9.2. Let G be a compact group. Then each $\mu \in M(G, \mathbb{R})$ gives rise to an element

$$\rho_G(\mu) \overset{\text{def}}{=} \int_G \eta_G \, d\mu \in \mathbb{R}[G]$$

such that for all $\omega \in \mathbb{R}[G]'$ we have

$$(**) \qquad \langle \omega, \rho_G(\mu) \rangle = \int_{g \in G} \langle \omega, \eta_G \rangle \, d\mu(g) = \int_{g \in G} f_\omega(g) \, d\mu(g) = \mu(f_\omega).$$

Therefore we have a morphism of vector spaces $\rho_G \colon M(G, \mathbb{R}) \to \mathbb{R}[G]$. We let $\tau_{\mathcal{R}(G,\mathbb{R})}$ denote the weakest topology making the functions $\mu \mapsto \mu(f) \colon M(G, \mathbb{R}) \to \mathbb{R}$ continuous for all $f \in \mathcal{R}(G, \mathbb{R})$. ∎

On any compact subspace of $M_p(G)$ such as $P(G)$ the topology $\tau_{\mathcal{R}(G,\mathbb{R})}$ agrees with the topology of $M_p(G)$.

Lemma 9.3. *The morphism ρ_G is injective and has dense image.*

Proof. We observe $\mu \in \ker \rho_G$ if for all $f \in \mathcal{R}(G, \mathbb{R})$ we have $\int_{g \in G} f(g) \, d\mu(g) = 0$. Since μ is continuous on $C(G, \mathbb{R})$ in the norm topology and $\mathcal{R}(G, \mathbb{R})$ is dense in $C(G, \mathbb{R})$ by the Theorem of Peter and Weyl (see e.g. [19], Theorem 3.7), it follows that $\mu = 0$. So ρ_G is injective.

If $\mu = \delta_x$ is a measure with support $\{x\}$ for some $x \in G$, then $\rho_G(\mu) = \int_G \eta_G d\delta_x = x$. Thus $G \subseteq \rho_G(M(G))$. Since $\mathbb{R}[G]$ is the closed linear span of G by Proposition 5.3, it follows that ρ_G has a dense image. ∎

We note that in some sense ρ_G is dual to the inclusion morphism of vector spaces $\sigma_G \colon \mathcal{R}(G, \mathbb{R}) \to C(G, \mathbb{R})$.

Returning to $(**)$ in Definition 9.2, for a compact group G, we observe

Lemma 9.4. *The morphism $\rho_G : (M(G, \mathbb{R}), \tau_{\mathcal{R}(G,\mathbb{R})}) \to \mathbb{R}[G]$ is a topological embedding.*

If μ is a probabililty measure, then the element $\rho_G(\mu) = \int_G \eta_G \, d\mu$ is contained in the compact closed convex hull $\overline{\operatorname{conv}}(G) \subseteq \mathbb{R}[G]$. Intuitively, $\int_G \eta_G d\mu \in \overline{\operatorname{conv}}(G)$ is the center of gravity of the "mass" distribution μ contained in $G \subseteq \mathbb{R}[G]$. In particular, if $\gamma \in M(G, \mathbb{R})$ denotes normalized Haar measure on G, then

$$\rho_G(\gamma) = \int_G \eta_G \, d\gamma = \int_{g \in G} g \, dg$$

is the center of gravity of G itself with respect to Haar measure.

We note that in the weakly complete vector space $\mathbb{R}[G]$ the closed convex hull

$$B(G) \overset{\text{def}}{=} \overline{\operatorname{conv}}(G) \subseteq \mathbb{R}[G]$$

is compact. (See e.g. [19], Exercise E3.13.)

Lemma 9.5. *The restriction $\rho_G|P(G) : P(G) \to B(G)$ is an affine homeomorphism.*

Proof.   (i) Affinity is clear and injectivity we know from Lemma 9.3.

(ii) Since $P(G)$ is compact in the weak topology and ρ_G is injective and continuous, $\rho_G|P(G)$ is a homeomorphism onto its image. But $G \subseteq \rho_G(P(G))$, and $B(G)$ is the closed convex hull of G in $\mathbb{R}[G]$, it follows that $B(G) \subseteq \rho_G(p(G))$.   ∎

If $k\colon \mathbb{R}[G] \to \mathbb{R}$ is the augmentation map (i.e., the coidentity morphism), then $k(G) = \{1\}$ and so $k(B(G)) = \{1\}$ as well. From $GG \subseteq G$ we deduce that $\operatorname{conv}(G)\operatorname{conv}(G) \subseteq \operatorname{conv}(G)$ and from there, by the continuity of the multiplication in $\mathbb{R}[G]$ and $1 \in G \subseteq B(G)$, it follows that $B(G)$ is a compact submonoid of $\mathbb{R}[G]^\times$ contained in the submonoid $k^{-1}(1)$.

Then the cone $\mathbb{R}_+[G] \stackrel{\text{def}}{=} \mathbb{R}_+ \cdot B(G)$, due to the compactness of $B(G)$, is a locally compact submonoid as well. The set

$$k^{-1}(1) \cap \mathbb{R}_+[G] = \{x \in \mathbb{R}_+[G] : k(x) = 1\} = B(G)$$

is a compact basis of the cone $\mathbb{R}_+[G]$.

Corollary 9.6.   *The function $\rho_G|M^+(G) : M^+(G) \to \mathbb{R}_+[G]$ is an isomorphism of convex cones and $\rho_G(M(G)) = \mathbb{R}_+[G] - \mathbb{R}_+[G]$*

Proof.   Since $M^+(G) = \mathbb{R}_+ \cdot P(G)$ and $\mathbb{R}_+[G] = \mathbb{R}_+ \cdot B(G)$, Lemma 9.5 shows that $\rho_G|M^+(G)$ is an affine homeomorphism. Since $M(G) = M^+(G) - M^+(G)$, the corollary follows.   ∎

Among other things this means that every element of $\mathbb{R}_+[G] - \mathbb{R}_+[G]$ is a an integral $\int_G \eta \, d\mu$ in $\mathbb{R}[G]$ for some Radon measure $\mu \in M(G)$ on G.

In order to summarize our findings we firstly list the required conventions:

Let G be a compact group viewed as a subgroup of the group $\mathbb{R}[G]^{-1}$ of units of the weakly complete group algebra $\mathbb{R}[G]$. Let $B(G) = \overline{\operatorname{conv}}(G)$ denote the closed convex hull of G in $\mathbb{R}[G]$ and define $\mathbb{R}_+[G] = \mathbb{R}_+ \cdot B(G)$. Let $k\colon \mathbb{R}[G] \to \mathbb{R}$ denote the augmentation morphism and $I = \ker k$ the augmentation ideal. We let $\eta_G\colon G \to \mathbb{R}[G]$ denote the inclusion map and consider $\rho_G\colon M(G, \mathbb{R}) \to \mathbb{R}[G]$ with $\rho_G(\mu) = \int_G \eta_G \, d\mu$.

Theorem 9.7.   *For a compact group G we have the following conclusions:*

(a) *$B(G) \supseteq G$ is a compact submonoid of $1 + I \subseteq (\mathbb{R}[G], \cdot)$ with Haar measure γ of G as zero element.*

(b) *$\mathbb{R}_+[G]$ is a locally compact pointed cone with basis $B(G)$, and is a submonoid of $(\mathbb{R}[G], \cdot)$.*

(c) *The function $\rho_G\colon (M(G), \tau_{\mathcal{R}(G, \mathbb{R})}) \to \mathbb{R}[G]$ is an injective morphism of topological vector spaces with dense image $\mathbb{R}_+[G] - \mathbb{R}_+[G]$. It induces a homeomorphism onto its image.*

(d) *The function $\rho_G|M^+(G)\colon M_p^+(G) \to \mathbb{R}_+[G]$ is an affine homeomorphism from the locally compact convex cone of positive Radon measures $M_p^+(G)$ on G onto $\mathbb{R}_+[G] \supseteq B(G) \supseteq G$.*

Remark 9.8.   The Haar measure γ is mapped by ρ_G onto the center of gravity $\int_G \eta_G \, d\gamma$ of G, $\gamma \in B(G) \subseteq 1 + I$.   ∎

It should be noted that $\rho_G\colon M(G,\mathbb{R}) \to \mathbb{R}[G]$ is far from surjective if G is infinite: If we identify $\mathbb{R}[G]$ with $\mathcal{R}(G,\mathbb{R})^*$ according to Theorem 7.7, then any element $u \in \mathbb{R}[G]$ representing a linear form on $\mathcal{R}(G,\mathbb{R})$ which is discontinuous in the norm topology induced by $C(G,\mathbb{R})$ fails to be an element of $\rho_G(M(G))$.

Theorem 9.7 shows that for a compact group G, the weakly complete real group algebra $\mathbb{R}[G]$ does not only contain G and the entire pro-Lie group theory encapsulated in the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ but also the measure theory, notably, that of the monoid of probability measures $P(G) \cong B(G)$.

Recall the hyperplane ideal $I = \ker k$ for the augmentation $k\colon \mathbb{R}[G] \to \mathbb{R}$.

Corollary 9.9. *Let G be a compact group, $\mathbb{R}[G]$ its real group algebra, and $\gamma \in \mathbb{R}[G]$ its normalized Haar measure. Then $J \stackrel{\text{def}}{=} \mathbb{R}\cdot\gamma$ is a one-dimensional ideal, and $\mathbb{R}[G] = I \oplus J$ is the ideal direct sum of I and J. The vector subspace J is a minimal nonzero ideal. In particular, $J \cong \mathbb{R}[G]/I \cong \mathbb{R}$ and $I \cong \mathbb{R}[G]/J$.*

Proof. In the multiplicative monoid $B(G) \subseteq \mathbb{R}[G]$, the idempotent element λ is a zero of the monoid $(I,\cdot)$, that is, $\lambda B(G) = B(G)\lambda = \{\lambda\}$. (See [19], Corollary A5.12.) As a consequence,

$$JB(G) = B(G)J \subseteq J.$$

The vector space span $B(G)$ contains span G which is dense in $\mathbb{R}[G]$ by Proposition 5.3. Hence $J\mathbb{R}[G] = \mathbb{R}[G]J \subseteq J$ and so J is a two-sided ideal. Since $k(B(G)) = \{1\}$ by Theorem 9.7(a) we know $J \not\subseteq I$, and since I is a hyperplane, $\mathbb{R}[G] = I \oplus J$ follows. ∎

We note that $\mathbb{R}[G]/J$ is a weakly complete topological algebra containing a copy of G and indeed of $P(G)$ with Haar measure in the copy of $P(G)$ being the zero of the algebra. It is not a group algebra nor a Hopf algebra in general as the example of $G = \mathbb{Z}(3)$ shows.

While the group $\Gamma(\mathbb{R}[G]) \cong G$ of grouplike elements of $\mathbb{R}[G]$ (and its closed convex hull $B(G)$) is contained in the affine hyperplane $1 + I$, in the light of Theorem 8.9 it is appropriate to observe that in the circumstances of Corollary 9.9, the Lie algebra of primitive elements $\Pi(\mathbb{R}[G]) \cong \mathfrak{L}(G)$ is contained in $I = \ker k$.

Indeed the ground field $\mathbb{R}$ is itself a Hopf algebra with the natural isomorphism $c_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ satisfying $c_{\mathbb{R}}(r) = r\cdot(1 \otimes 1) = r \otimes 1 = 1 \otimes r$. Now the coidentity k of any coalgebra A is a morphism of coalgebras so that we have a commutative diagram for $A \stackrel{\text{def}}{=} \mathbb{R}[G]$:

$$
\begin{array}{ccc}
A & \xrightarrow{\,c_A\,} & A \otimes A \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k \otimes k} \\
\mathbb{R} & \xrightarrow{\,c_{\mathbb{R}}\,} & \mathbb{R} \otimes \mathbb{R},
\end{array}
\tag{13}
$$

If $a \in A$ is primitive, then $c(a) = a \otimes 1 + 1 \otimes a$. The commutativity of (13) provides $k(a) \otimes 1 = \alpha(k(a))(k \otimes k)(c(a)) = (k \otimes k)(a \otimes 1 + 1 \otimes a) = k(a) \otimes 1 + 1 \otimes k(a)$, yielding $1 \otimes k(a) = 0$, that is $k(a) = 0$ which indeed means $a \in \ker k = I$. We note that these matters are also compatible with the Main Theorem for Compact Groups B 8.9 insofar as, trivially, $\exp(I) \subseteq 1 + I$.

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Rafael Dahmen
Karlsruher Inst. für Technologie (KIT)
76131 Karlsruhe
Germany
rafael.dahmen@kit.edu

Karl Heinrich Hofmann
Fachbereich Mathematik
Technische Universität
Schlossgartenstraße 7
64289 Darmstadt
Germany
hofmann@mathematik.tu-darmstadt.de