The Topological Generating Rank of Solvable Lie Groups[*]

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Abstract. We define the topological generating rank $d(G)$ of a connected Lie group G as the minimal number of elements of G needed to generate a dense subgroup of G. We answer the following question posed by K. H. Hofmann and S.A. Morris [see: *Finitely generated connected locally compact groups*, J. Lie Theory (formerly Sem. Sophus Lie) 2(2) (1992) 123–134]: What is the topological generating rank of a connected solvable Lie group? If G is solvable we can reduce the question to the case that G is metabelian. We can furthermore reduce to the case that the natural representation of $Q:=G^{ab}:=G/\overline{G'}$ on $A := \overline{G'}$ is semisimple. Then $d(G)$ is the maximum of the following two numbers: $d(Q)$ and one plus the maximum of the multiplicities of the non-trivial isotypic components of the $\mathbb{R}Q$-module A.

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1. Introduction

If G is a finitely generated (abstract) group, the *generating rank* of G, is the smallest cardinality of a subset $X \subseteq G$ such that X generates G. The *generating rank problem* for a particular class of groups asks if there is an algorithm that, given a group G from the class, computes the generating rank of G. The rank problem "is one of the most enigmatic and little understood group-theoretic decision problems" [15].

In this paper we study the topological version of the rank problem. A subset X of a topological group G is said to be a *topological generating set* for G if the smallest closed subgroup containing X is G itself, or, equivalently, the group $\langle X \rangle$ generated by X is dense in G. The minimal cardinality of a topological generating set of G is called the *topological generating rank* of G and denoted by $d(G)$. The first result in this context is the theorem of Kronecker (1884), which asserts, in modern terms, that for every natural number n and every point $(t_1, \ldots, t_n) \in \mathbb{R}^n$ the image of this point in the torus $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ is a topological generator of $\mathbb{T}^n$ if and only if the numbers $1, t_1, \ldots, t_n$ are linearly independent over $\mathbb{Q}$.

In [14] Karl H. Hofmann and Sidney A. Morris point out: "An analysis of the topological rank of a connected Lie group requires an answer to the following question:

"Question B. *What is the topological generating rank of a solvable connected Lie group?*"

The aim of this paper is to give a complete answer to Question B.

If G is a connected solvable Lie group then we can reduce the question to the case that G belongs to a certain class $\mathcal{R}$ of Lie groups. Here $\mathcal{R}$ consists of all connected Lie groups G, such that

1. G is metabelian,

2. The (topological) commutator subgroup $A := \overline{G'}$ is a vector group, i.e. isomorphic to $\mathbb{R}^n$ for some n,

3. A contains no non-trivial vector fixed by Q, where $Q := G^{ab} := G/A$,

4. The natural representation of Q on A is semisimple.

Theorem 1.1 (reduction). *For every connected solvable Lie group G there is an (effectively constructed) Lie group quotient G/R belonging to the class $\mathcal{R}$ and such that $d(G) = d(G/R)$.*

The details of the reduction are as follows. Let G be a solvable connected Lie group. Let $G/\overline{G''}$ be its metabelianization. Then

$$d(G) = d\left(G/\overline{G''}\right),$$

see Section 4. Then we may assume that G is metabelian, i.e. G' is abelian. Set $A := \overline{G'}$ and $Q := G/A$. Both A and Q are abelian connected Lie groups. Then a subgroup H is dense in G if the image of H in G/K is dense where K is the maximal compact subgroup of A, by Lemmas 5.1 and 5.2 in Section 5. Thus we may assume that A contains no compact subgroup hence is a vector group. The action of G on A by inner automorphisms induces a linear representation α of Q on A. We may assume that A contains no non-trivial vector fixed by Q, by Corollary 5.3.

We may furthermore assume that the representation α of Q on A is semisimple, by Lemma 9.1. It turns out that the subspace of A where Q acts unipotently plays no role for our question. These steps taken together produce the required quotient G/R. Note that the reduction process can be more efficiently described in terms of the Frattini subgroup [2]. The work of Guivarc'h [10] contains many of our reduction steps and also contains the formula for the topological generating rank of the group of upper triangular matrices.

To complete the solution of the rank problem we give a precise formula for the rank of every group $G \in \mathcal{R}$. To be precise, A decomposes into weight subspaces A_χ corresponding to weights χ. We set $k_\chi = \mathbb{R}$ if $\chi(Q) \subseteq \mathbb{R}$ and $k_\chi = \mathbb{C}$ if $\chi(Q) \not\subseteq \mathbb{R}$. We consider A_χ as a vector space over k_χ. Let μ_χ be the multiplicity of the weight space A_χ, so

$$\mu_\chi = \dim_\mathbb{R} A_\chi \ \text{ if } \ \chi(Q) \subseteq \mathbb{R}, \quad \text{and} \quad \mu_\chi = \dim_\mathbb{C} A_\chi \ \text{ if } \ \chi(Q) \not\subseteq \mathbb{R}.$$

Let μ be the maximum of the μ_χ. Note that the trivial representation corresponding to the character $\chi \equiv 1$ is excluded by property 3 for the class $\mathcal{R}$.

Theorem 1.2 (main).   *If G is a connected metabelian Lie group belonging to the class $\mathcal{R}$ then*

$$d\left(G\right) = \max\left\{d\left(Q\right), 1+\mu\right\}. \tag{1}$$

The strategy of the proof is as follows. To prove the inequality $d\left(G\right) \geq 1+\dim_{k_\chi} A_\chi$ we first produce for every weight χ the isotypic quotient G_χ of G which is the extension of A_χ by Q. Let X be a set, which generates a dense subgroup of G_χ. Then the linear span $\operatorname{span}_{k_\chi}[X, X]$ of the commutator set $[X, X]$ coincides with A_χ. We show that A_χ is spanned over k_χ by a set of the form $[x_0, X - \{x_0\}]$ for some $x_0 \in X$. This implies the required estimate $\dim_{k_\chi} A_\chi \leq |X| - 1$, meaning that we need as many generators as claimed in the main theorem, see Section 10.

On the other hand we show that the claimed number of generators suffices, as we show by giving a fairly explicit set of generators, see section 11.

2.   Abelian groups

For abelian groups the topological generating rank was well known, see [1, 8, 14].

Theorem 2.1.   *Let G be an abelian connected Lie group and let T be its maximal compact subgroup. Then*

$$d\left(G\right) = 1 + \dim\left(G/T\right).$$

Proof.   We reproduce a proof since later on we shall need information given in it. If $G = T$ is a compact connected abelian Lie group we may think of T as $\mathbb{R}^n/\mathbb{Z}^n$. Let $v = (v_1, \ldots, v_n)$ be a vector in $\mathbb{R}^n$. Then $\mathbb{N}v + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$ if (and only if) the real numbers $1, v_1, \ldots, v_n$ are linearly independent over $\mathbb{Q}$. This is a result of Kronecker [12] which also follows from Pontryagin duality.

If G is an arbitrary abelian connected Lie group it is isomorphic to $\mathbb{R}^n/\Gamma$ where Γ is the subgroup of $\mathbb{R}^n$ generated by the first m standard basis vectors. Then $T = \operatorname{span}_{\mathbb{R}}\left(\Gamma\right)/\Gamma$ is the maximal compact subgroup of G and $\dim(G/T) = n-m$. The set X consisting of the remaining $n-m$ standard basis vectors and a vector v as above has cardinality $n-m+1$. And the group H generated by Γ and X is dense in $\mathbb{R}^n$, hence $\{h + \Gamma; h \in H\} \subseteq \mathbb{R}^n/\Gamma = G$ is dense in G. This shows that $d\left(G\right) \leq 1+\dim\left(G/T\right)$. To show equality let Y be any subset of G of cardinality less than $1+n-m$. Then the subgroup of $\mathbb{R}^n$ generated by Γ and $\{y \in \mathbb{R}^n; y + \Gamma \in Y\}$ is generated by at most n vectors and so its closure is a proper subgroup of $\mathbb{R}^n$, which contains Γ, and hence yields modulo Γ a proper closed subgroup of G.   ∎

Corollary 2.2 (of proof).   *Let $d = d\left(G\right)$. Let $Gen\left(G\right)$ be the set of $(x_1, \ldots, x_d) \in G^d$ for which $\langle x_1, \ldots, x_d \rangle$ is dense in G. Then $Gen\left(G\right)$ is dense in G^d.*

Here and in what follows we denote for a subset X of a group G by $\langle X \rangle$ the subgroup of G generated by X.

3.　Nilpotent groups

The next result is also well known. For a topological group G we define its abelian-ization G^{ab} as $G^{ab} := G/\overline{G'}$, where G' is the commutator subgroup of G and the bar denotes the closure.

Theorem 3.1 (nilpotent).　　*Let G be a nilpotent connected Lie group. Then*

$$d(G) = d\left(G^{ab}\right).$$

This follows from the following lemma.

Lemma 3.2 (nilpotent).　　*Let H be a subgroup of a nilpotent topological group G. If $H \cdot G'$ is dense in G then H is dense in G.*

Proof.　　The proof is easy and elementary and is a model for similar proofs later on. So we give it here. Let $Z_n(G), n \geq 0$, be the n-th term of the descending central series. So $Z_0(G) = G$ and $Z_{n+1}(G) = [G, Z_n(G)]$. The degree of nilpotency of G is the least integer n such that $Z_n(G) = \{e\}$. We prove the claim by induction on n. The claim is trivial for $n = 0$ and $n = 1$. So suppose $n > 1$ and $H \cdot G'$ is dense in G. The inductive hypothesis applied to the image of H in $G/\overline{Z_{n-1}(G)}$ yields that $H \cdot Z_{n-1}(G)$ is dense in G. We have $H' = (H \cdot Z_{n-1}(G))'$ since $Z_{n-1}(G)$ is central in G. On the other hand $(H \cdot Z_{n-1}(G))'$ is dense in G'. So H' is dense in G'. It follows that the closure $\overline{H}$ of H contains G' and hence $\overline{H} = G$, as required.　■

4.　Reduction to the metabelian case

Corollary 4.1.　　*Let G be a solvable connected Lie group. Then*

$$d(G) = d\left(G/\overline{G''}\right).$$

The group G'' is the second commutator subgroup of G. So $G/\overline{G''}$ is metabelian and might be called the metabelianization of G. The corollary follows from the following statement.

Corollary 4.2.　　*Let H be a subgroup of the connected solvable Lie group G. If $H \cdot G''$ is dense in G then H is dense in G.*

Proof.　　It follows from Lie's theorem that the commutator subgroup G' of a solvable connected Lie group is nilpotent. If $H \cdot G''$ is dense in G, then $H' \cdot G''$ is dense in G', since $H' \cdot \overline{G''} = \pi^{-1}\left(\pi(H)'\right)$ where $\pi : G \to G/\overline{G''}$ is the natural map. So H' is dense in $\overline{G'}$ by lemma 3.2, since $\overline{G'}$ is a connected nilpotent Lie group. It follows that $\overline{H} = G$, since $\overline{H}$ contains $\overline{G'}$ and $H \cdot \overline{G'}$ is dense in G since it contains the dense subgroup $H \cdot G''$.　■

Remark 4.3.　　The following stronger statement is true. Let G be a connected Lie group and let U be a closed connected nilpotent normal subgroup of G. Suppose H is a subgroup of G such that $H \cdot U'$ is dense in G. Then H is dense in G. We give a proof in the broader context of Frattini subgroups, see the upcoming paper [2] of the authors.

5. Reduction: No central elements in G'

From now on let G be a connected metabelian Lie group. Let us put $A := \overline{G'}$ and $Q = G/A$. Both A and Q are connected abelian groups. The action of G on A by inner automorphisms is trivial on A hence induces a continuous homomorphism

$$\alpha : Q \to \mathrm{Aut}\,(A).$$

Let $\mathfrak{a}$ be the Lie algebra of A. So α induces a continuous representation of Q on $\mathfrak{a}$ which we also denote by α. In this section we will show that we may assume that A contains no compact subgroup hence is a vector group and that $\mathfrak{a}$ contains no non-trivial vector fixed by Q.

Lemma 5.1. *Let H be a subgroup of G and let Z be the center of G. If $H \cdot (Z \cap A)$ is dense in G then H is dense in G.*

Proof. The commutator subgroup H' of H coincides with $(H \cdot (Z \cap A))'$ since Z is centralized by H and hence H' is dense in G', by hypothesis. So the closure of H contains $\overline{G'} = A$ and the image of H in $Q = G/A$ is dense in Q, thus H is dense in G. ∎

Lemma 5.2. *The group $Z \cap A$ is the Lie group corresponding to the Lie subalgebra $\mathfrak{a}^0$ of vectors of $\mathfrak{a}$ fixed by Q. The group $Z \cap A$ contains the maximal compact subgroup of A.*

Proof. Let $\exp : \mathfrak{a} \to A$ be the exponential map and let $\Gamma \subseteq \mathfrak{a}$ be its kernel. The group Γ is invariant under $\alpha\,(Q)$ hence fixed by $\alpha\,(Q)$ since Q is connected and Γ is discrete. We may think of an element of A as a coset $a + \Gamma$ in $\mathfrak{a}$. Then $a + \Gamma$ is central in G if and only if $a + \Gamma$ is fixed by $\alpha\,(Q)$ if and only if $\alpha\,(Q)\,(a + \gamma) \subseteq a + \Gamma$ for every $\gamma \in \Gamma$ which again implies that $\alpha\,(Q)\,a = a$, since Q is connected and Γ is discrete. This shows the first claim of the lemma. The second claim follows from the fact that the maximal compact subgroup of the abelian Lie group A is $\exp\,(\mathrm{span}_{\mathbb{R}}\Gamma)$. ∎

Corollary 5.3. *Let $\mathfrak{a}_0$ be the set of elements $a \in \mathfrak{a}$ such that $(\alpha\,(q) - \mathbb{1})^n\,a = 0$ for some $n \in \mathbb{N}$. A subgroup H of G is dense in G if $H \cdot \exp \mathfrak{a}_0$ is dense in G.*

Proof. This follows by a repeated application of Lemma 5.1 since if $\mathfrak{a}_0 \neq 0$ then $\mathfrak{a}^0 \neq 0$. ∎

Here is a related result.

Lemma 5.4. *A subgroup H of G is dense if (and only if) the image of H in both $G/\overline{G'}$ and $\mathrm{Ad}\,(G)$ is dense.*

Proof. The following proof is false: If H has dense image in $\mathrm{Ad}\,(G) = G/Z$, where Z is the center of G, then $H' = (H \cdot Z)'$ is dense in G' and thus H is dense in G if additionally $H \cdot G'$ is dense in G. The reason why this "proof" is wrong is the following. The natural map $G/Z \to \mathrm{Ad}\,(G)$ is an isomorphism of abstract groups and continuous but not a homeomorphism in general, since $\mathrm{Ad}\,(G)$ may not be closed in $\mathrm{GL}\,(\mathfrak{g})$. A correct proof is as follows. Let $\rho : G \to \mathrm{GL}\,(\mathfrak{g})$ be the adjoint representation of G. The group $\rho\,(G')$ is an analytic unipotent subgroup of $\mathrm{GL}\,(\mathfrak{g})$, by Lie's theorem, hence closed. It follows that

$$\rho\left(G'\right) = \rho\left(G\right)' = \overline{\rho\left(G\right)}' = \overline{\rho\left(G\right)'}.$$

If $\rho\left(H\right)$ is dense in $\rho\left(G\right)$ then $\rho\left(H'\right)$ is dense in $\rho\left(G'\right)$. The natural map

$$G'/\left(G' \cap Z\right) \to \rho\left(G'\right)$$

induced by ρ is an isomorphism of Lie groups, since $\rho\left(G'\right)$ is closed. It follows that $H' \cdot \left(G' \cap Z\right)$ is dense in G'. If furthermore $H \cdot G'$ is dense in G then $H \cdot \left(H' \cdot \left(G' \cap Z\right)\right) = H \cdot \left(G' \cap Z\right)$ is dense in G. We have $H' = \left(H \cdot \left(G' \cap Z\right)\right)'$ since $Z \cap G'$ is central in G and hence H' is dense in G'. It follows that then H is dense in G, since $H \cdot G'$ is dense in G, by hypothesis. ∎

The proof shows that the following conditions suffice.

Corollary 5.5.   *A subgroup H of G is dense if the following two conditions hold.*
(1) *The image of H in $G/\overline{G'}$ is dense.*
(2) *Let $\rho : G \to \mathrm{GL}\left(\mathfrak{g}\right)$ be the adjoint representation of G. Then $\rho\left(H \cap \overline{G'}\right)$ is dense in $\rho\left(G'\right) = \overline{\rho\left(G'\right)}$.*

6. Invariant subgroups

Let Q be an abelian group, let V be a finite dimensional real vector space and let $\alpha : Q \to \mathrm{GL}\left(V\right)$ be a representation of Q on V. We will have to determine the closed Q-invariant subgroups of V in case $\alpha\left(Q\right)$ is connected. The offshot is that they are actually real vector spaces except for the case that we have unipotent or trivial subrepresentation, a case we can ignore by Corollary 5.3. The result is contained in the lemma at the end of this section.

We start by recalling some linear algebra and establishing notation. For every homomorphism $\chi : Q \to \mathbb{C}^*$ let $V_{\mathbb{C},\chi}$ be the subspace of elements $v \in V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ such that for every $q \in Q$ we have

$$\left(\alpha\left(q\right) - \chi\left(q\right)\mathbb{1}\right)^n v = 0$$

for each $n \geq \dim_{\mathbb{R}} V$. Every subspace $V_{\mathbb{C},\chi}$ is Q-invariant and

$$V_{\mathbb{C}} = \oplus_{\chi} V_{\mathbb{C},\chi}.$$

Let σ be the $\mathbb{R}$-linear map $V_{\mathbb{C}} \to V_{\mathbb{C}}$ induced by complex conjugation, so that we have $\sigma\left(v \otimes z\right) = v \otimes \bar{z}$ for every $v \in V$ and $z \in \mathbb{C}$. Then $\sigma \circ \alpha\left(q\right) = \alpha\left(q\right) \circ \sigma$ for every $q \in Q$ and hence

$$\sigma\left(V_{\mathbb{C},\chi}\right) = V_{\mathbb{C},\bar{\chi}}$$

where $\bar{\chi}\left(q\right) = \overline{\chi\left(q\right)}$. There are two cases

1. $\chi\left(Q\right) \subset \mathbb{R}$ equivalently $\sigma\left(V_{\mathbb{C},\chi}\right) = V_{\mathbb{C},\bar{\chi}}$

2. $\chi\left(Q\right) \not\subset \mathbb{R}$ equivalently $\sigma\left(V_{\mathbb{C},\chi}\right) \neq V_{\mathbb{C},\bar{\chi}}$.

In the first case $V_{\mathbb{C},\chi}$ is the complexification of $V_{\mathbb{C},\chi} \cap V$. In this case we define

$$V_{\chi} := V_{\mathbb{C},\chi} \cap V.$$

In the second case we define $V_{\chi,\bar{\chi}} := \left(V_{\mathbb{C},\chi} \oplus V_{\mathbb{C},\bar{\chi}}\right) \cap V.$

This real vector space has a unique structure as complex vector space for which $\alpha(q)|V_{\chi,\bar{\chi}}$ is $\mathbb{C}$-linear for every $q \in Q$ and $\alpha(q) - \chi(q) \cdot \mathbb{1}$ is nilpotent on $V_{\chi,\bar{\chi}}$ for every $q \in Q$. This structure can be defined by transport of structure with respect to the $\mathbb{R}$-linear isomorphism $V_{\chi,\bar{\chi}} \to V_{\mathbb{C},\chi}$ given by projection to the first component. There is also a unique structure as complex vector space on $V_{\chi,\bar{\chi}}$ for which $\alpha(q)|V_{\chi,\bar{\chi}}$ is $\mathbb{C}$-linear and $\alpha(q) - \bar{\chi}(q) \cdot \mathbb{1}$ is nilpotent on $V_{\chi,\bar{\chi}}$ for every $q \in Q$. This is the complex structure which is the complex conjugate of the first one. From now on we fix a complex structure on $V_{\chi,\bar{\chi}}$ with the above properties for every pair $\chi, \bar{\chi}$ of homomorphisms $\chi : Q \to \mathbb{C}^*$ with $\chi(Q) \not\subset \mathbb{R}$.

We put $k_\chi = \mathbb{R}$ if $\chi(Q) \subset \mathbb{R}$ and $k_\chi = \mathbb{C}$ if $\chi(Q) \not\subset \mathbb{R}$. We set $V_\chi = V_{\chi,\bar{\chi}}$ in the latter case. So

$$V = \oplus V_\chi,$$

every V_χ is a vector space over k_χ, $\alpha(q)|V_\chi$ is k_χ-linear and $\alpha(q) - \chi(q) \cdot \mathbb{1}$ is nilpotent on V_χ for every $q \in Q$. We call the subspaces V_χ the *weight components* of V. Now let

$$\alpha(q) = q_s \cdot q_u = q_u \cdot q_s$$

be the multiplicative Jordan decomposition of $\alpha(q)$. We define the mappings $\alpha_s \colon Q \to GL(V)$ and $\alpha_u \colon Q \to GL(V)$ by $\alpha_s(q) = q_s$ and $\alpha_u(q) = q_u$. Both q_u and q_s commute with $\alpha(q')$ for every $q' \in Q$ and with both q'_s and q'_u for every $q' \in Q$, since both the semisimple part q_s and the unipotent part q_u of an operator $q \in \mathrm{GL}(V)$ can be written as polynomials in q. So the maps $\alpha_s \colon Q \to GL(V)$ and $\alpha_u \colon Q \to GL(V)$ are group homomorphisms. The group $\alpha_s(Q)$ resp. $\alpha_u(Q)$ consists of commuting semisimple, resp. unipotent automorphisms of V.

Every q_u maps every V_χ to itself, again since the projection of V to V_χ can be written as polynomial in the $\alpha(q)$. So $\alpha_u(Q)$ restricts to a commutative group of unipotent k_χ-linear automorphisms of V_χ. The representation α can be put into triangular form over $\mathbb{C}$. The weights $\chi : Q \to \mathbb{C}^*$ of Q on V are then the projections to the diagonal entries. So

$$\alpha_s(q)|V_\chi = \chi(q) \cdot \mathbb{1}_{V_\chi},$$

It follows that if Q is a topological group and α is continuous then α_s and hence α_u are continuous homomorphisms from Q to $GL(V)$.

Lemma 6.1. *Suppose that Q is a connected abelian topological group and that $\alpha : Q \to \mathrm{GL}(V)$ is a continuous homomorphism. Let B be a closed Q-invariant subgroup of V and let B_0 be the connected component of 0 in B. Then*

(1) *B_0 is a Q-invariant real vector subspace of V. For every character χ the subgroup $B_0 \cap V_\chi$ is a vector space over k_χ and $B_0 = \oplus_\chi (B_0 \cap V_\chi)$,*

(2) *B is connected if V contains no non-zero vector fixed by Q.*

Proof. Every closed subgroup B of V is of the following form. There is a basis $v_1, \ldots, v_m$ of V and integers $0 \le n \le k \le m$ such that $B_0 = \mathrm{span}_{\mathbb{R}} \{v_1, \ldots, v_n\}$ and B is the sum of B_0 and the discrete subgroup Γ generated by $\{v_{n+1}, \ldots, v_k\}$. In our case the vector subspace B_0 over $\mathbb{R}$ is Q-invariant since Q is connected. Then B/B_0 is a discrete subgroup of the vector space V/B_0 which carries an induced continuous representation of Q. The discrete group B/B_0 is fixed by Q. Thus if B is not connected, V/B_0 contains a non-zero vector fixed by Q, in particular the subspace

of V/B_0 corresponding to the trivial character $\chi \equiv 1$ is non-zero, hence so is V_χ for $\chi \equiv 1$ since $\alpha_s(Q)$ is semisimple. It follows that V contains a non-zero Q-fixed vector since $\alpha(Q)|V_\chi = \alpha_u(Q)|V_\chi$ is a commutative unipotent group and $V_\chi \neq 0$ for $\chi \equiv 1$. This shows claim (2).

To prove (1) we may assume that B is connected. Then B is a real Q-invariant subspace of V. It follows that $B \cap V_\chi \subset B$ for every χ since the projections of V to its weight components can be written as polynomials in $\alpha(q), q \in Q$, with real coefficients. For the same reason $B \cap V_\chi$ is invariant under $\alpha_s(q)$ and $\alpha_u(q)$ and hence $\chi(q)(B \cap V_\chi) \subset (B \cap V_\chi)$ which shows that $B \cap V_\chi$ is a vector space over k_χ, a non-trivial fact only if $\chi(Q) \not\subset \mathbb{R}$. ∎

7. The semisimplification

Let Q be an abelian Lie group and let V be a finite dimensional real vector space with a continuous representation $\alpha : Q \to \mathrm{GL}(V)$ of Q on V. The discussion of the last section now yields

Remark 7.1. $Q_s := \alpha_s(Q)$ and $Q_u := \alpha_u(Q)$ are both connected abelian Lie groups, the projections of $\alpha(Q)$ to Q_s and Q_u are continuous and $\alpha(Q) \subseteq Q_s \cdot Q_u$.

Note that in general $\alpha(Q)$ may be strictly contained in $Q_s \cdot Q_u$, see the example in Section 13. Note also that if Q is not abelian then the constituents q_s and q_u do not depend continuously on q, e.g. if a sequence of triangular hyperbolic hence semisimple matrices in $\mathrm{SL}_2(\mathbb{R})$ converges to a unipotent matrix $\neq 1$.

Let V_u be the span of $\{(q_u - \mathbb{1})v; q \in Q, v \in V\}$. Then V_u is a Q-invariant vector subspace of V. Let us define $V_s = V/V_u$ and let π_s be the natural map of V onto V_s. The representation α of Q on V induces a representation α_s of Q on V_s. We have

$$\alpha_s(q)(v + V_u) = \alpha(q)v + V_u = q_s v + V_u$$

since $qv = q_s \cdot q_u v = q_s v + q_s(q_u - 1)v \in q_s v + V_u$. The representation α_s is semisimple. We call the Q-module V_s the *semisimplification* of V. Note that every Q-invariant vector subspace W of V is invariant under both Q_s and Q_u, again since q_s and q_u are polynomials with real coefficients in q for every $q \in Q$.

In the next section we will prove the following lemma.

Lemma 7.2. *Let W be a Q-invariant vector subspace of V. Then $W = V$ if (and only if) $W + V_u = V$.*

We thus obtain the following consequence. Let $\pi_s^\chi : V \to V_s^\chi$ be the composition of the natural map $\pi_s : V \to V_s$ to the semisimplification of V and the projection $V \to V_s^\chi$ to its χ-weight component.

Corollary 7.3. *Suppose V contains no non-zero Q-fixed vector. Let X be a subset of V. Then the smallest Q-invariant subgroup of V containing X is dense in V if (and only if) $\pi_s^\chi(X)$ spans V_s^χ as a vector space over k_χ for every weight $\chi : Q \to \mathbb{C}^*$.*

Proof.   Let B be the closure of the smallest Q-invariant subgroup of V containing X. Then B is a real vector subspace of V, by Lemma 6.1(2) of Section 6. So $B = V$ if $B_s = \pi_s(B)$ by Lemma 7.2 above. Let us set $B_s := \pi_s(B)$. Then $B_s = \oplus B_s^\chi$ where $B_s^\chi = B_s \cap V_s^\chi$, by Lemma 6.1(1) and every B_s^χ is a vector subspace V_s^χ over k_χ. But B is an $\mathbb{R}Q$-module, hence so is B_s, and hence $\pi_s^\chi(B_s) = B_s \cap V_s^\chi = B_s^\chi$ for every weight χ. This proves the lemma.   ∎

For every character χ of Q define the *multiplicity* $\mu_\chi := \dim_{k_\chi} V_s^\chi$.

Corollary 7.4.   *Suppose V contains no non-zero vector fixed by Q. Then let $m = \max_\chi \{\mu_\chi\}$. Then there are m elements in V such that the $\mathbb{Z}Q$-module generated by them is dense in V. Every $\mathbb{Z}Q$-module generated by less than m elements is not dense in V.*

Proof.   Every vector v of V can be uniquely written as $v = \sum_\eta v^\eta$ with $v^\eta \in V^\eta$. If $X = \{v_1, \ldots, v_m\}$ is a set of m vectors of V then the smallest Q-invariant subgroup B of V containing X is the direct sum of $B \cap V^\eta$ and $B \cap V^\eta$ is the span of $\{v_1^\eta, \ldots, v_m^\eta\}$ over k_η. This implies the corollary.   ∎

Corollary 7.5.   *Suppose V contains no non-zero vector fixed by Q. Let $Gen_Q(V)$ be the set of m-tuples of vectors of V which generate a dense $\mathbb{Z}Q$-module in V. Then $Gen_Q(V)$ is an open Zariski dense subset of V^m.*

8.   A Nakayama type lemma

In this section we use the following hypothesis and notation. Let Q be an abelian Lie group and let V be a finite dimensional real vector space with a representation $\alpha : Q \to \mathrm{GL}(V)$ of Q on V. We suppose that $\alpha(q)$ is unipotent for every $q \in Q$. For every Q-invariant vector subspace W of V we define

$$W_u = \sum_{q \in Q} (\alpha(q) - \mathbb{1})W.$$

Let W be a Q-invariant vector subspace of V. Then W_u is a Q-invariant vector subspace of W. We have the following Nakayama type lemma.

Lemma 8.1.   *Let W be a Q-invariant vector subspace of V. Then $W = V$ if (and only if) $W + V_u = V$.*

Proof.   We prove our claim by induction on $\dim V$. If $\alpha(q) = 1$ for every $q \in Q$ then our claim is trivial. This settles the case that $\dim V \leq 1$. Suppose now that $W + V_u = V$. If $\beta : V \to V$ is an $\mathbb{R}Q$-module homomorphism then $(\beta V)_u = \beta V_u$ and hence $\beta W = \beta V$ if $\dim \beta V < \dim V$ by the inductive hypothesis. We apply this for the $\mathbb{R}Q$-module homomorphism $\beta = \alpha(q) - \mathbb{1}$ with $q \in Q$. The $\dim \beta(V) < \dim V$ and we obtain that $(\alpha(q) - \mathbb{1})V = (\alpha(q) - \mathbb{1})W \subset W$ and hence

$$V_u = \sum_{q \in Q} (\alpha(q) - \mathbb{1})V \subset W$$

which implies our claim.   ∎

For a more general statement see Remark 12.2. The lemma of the previous section follows by applying our Nakayama type lemma to the abelian group Q_u of unipotent elements of $\mathrm{GL}(V)$. Note that Q_u is connected if Q is connected, by Remark 7.1 in Section 7.

9. Synopsis of the reduction steps

Let G be a connected solvable Lie group. Then

$$d(G) = d\left(G/\overline{G''}\right)$$

by Corollary 4.1 in Section 4. So we may assume that G is metabelian. We will keep our earlier notation, $A := \overline{G'}$ and $Q = G/A$. Then

$$d(G) = d(G/(Z \cap A)) = d(G/\exp \mathfrak{a}_0)$$

by Lemmas 5.1 and 5.2 in Section 5. Then by Lemma 5.2 in Section 5 we may assume that A is a vector group and that Q acts without fixed points on the set of all nonzero vectors. Then every closed Q-invariant subgroup B of A is a vector subspace of A over $\mathbb{R}$, by lemma 6.1. We will thus identify A with its Lie algebra. Let $A_s = A/A_u$ be the semisimplification of A.

Lemma 9.1. *Let H be a subgroup of G. Then H is dense in G if (and only if) the image of H in G/A_u is dense. In particular*

$$d(G) = d(G/A_u).$$

In short (and less precise) terms, the topological generating rank is invariant under semisimplification.

Proof. Let $\pi : G \to G/A_u =: G_s$ be the natural homomorphism. Suppose $\pi(H)$ is dense in G_s. Then the image of H in $Q = G/A$ is dense. Thus $\bar{H} \cap A := W$ is a Q-invariant closed subgroup of A, hence a vector subspace over $\mathbb{R}$. By hypothesis $H \cdot A_u$ is dense in G, so $H' \cdot A_u$ is dense in $\overline{G'} = A$, in particular $W + A_u = A$, since $W = \bar{H} \cap A \supseteq H'$. So $W = A$, by Lemma 7.2. It follows that $\bar{H} = G$, since $\bar{H} \supseteq A$ and $\bar{H}/A$ is a closed dense subgroup of Q. ∎

10. The lower bound for the generating rank

Let G be a connected metabelian Lie group in class $\mathcal{R}$. We will keep our earlier notation, $A := \overline{G'}$ and $Q = G/A$. The weight components A_χ of this representation are vector spaces over k_χ, where $k_\chi = \mathbb{R}$ if $\chi(Q) \subset \mathbb{R}$ and $k_\chi = \mathbb{C}$ if $\chi(Q) \not\subset \mathbb{R}$, by lemma 6.1, since by condition 3 the weight component A_1 corresponding to the weight $\chi \equiv 1$ is the zero space.

Let now $A_{\neq \chi} = \oplus_{\chi' \neq \chi} A_{\chi'}$. Then $G_\chi := G/A_{\neq \chi}$ is an isotypic group. Clearly

$$d(G) \geq d(G_\chi).$$

Lemma 10.1. *For every $G \in \mathcal{R}$ and every weight χ the following inequality holds*

$$d(G_\chi) \geq 1 + \dim_{k_\chi} A_\chi.$$

Proof.   Let X be a set, generating a dense subgroup H of G_χ. The commutator subgroup H' is dense in $\overline{G'_\chi} = A_\chi$ and generated as a normal subgroup of H by the set of commutators $[X, X] = \{[x, y] \, ; x, y \in X\}$ where

$$[x, y] = x^{-1} y^{-1} x y.$$

So the linear span $\mathrm{span}_{k_\chi}([X, X])$ coincides with A_χ. We have to show that $|X| \geq 1 + \dim_{k_\chi} A_\chi$. We will show that A_χ is spanned over k_χ by a set of the form $[x_0, X - \{x_0\}]$ for some $x_0 \in X$. This will imply that $\dim_{k_\chi} A_\chi \leq |X| - 1$ and hence the lemma. There is at least one element $x_0 \in X$ such that $\chi(x_0) \neq 1$ since H is dense in G_χ. Then we make use of the antisymmetry of the commutator

$$[y, x] = -[x, y]$$

and the following version of the P.Hall-Witt identity

$$[x^y, [y, z]] \cdot [y^z, [z, x]] \cdot [z^x, [x, y]] = 1.$$

It implies for our case $x, y, z \in G_\chi$ and commutators in the abelian group A_χ

$$\left(1 - \chi\left(\bar{x}^{-1}\right)\right) [y, z] + \left(1 - \chi\left(\bar{y}^{-1}\right)\right) [z, x] + \left(1 - \chi\left(\bar{z}^{-1}\right)\right) [x, y] = 0$$

for $x \in G_\chi$ and $a \in A_\chi$, where $\bar{x} = x \cdot A_\chi \in G_\chi/A_\chi = Q$. Thus, for $x = x_0$, the commutator $[y, z]$ is contained in the span of $[x_0, y]$ and $[x_0, z]$ over k_χ.   ∎

11.   The upper bound for the generating rank

Let G be a connected metabelian Lie group contained in the class $\mathcal{R}$. We will exhibit a dense subgroup H of G and keep control of the number of its generators. We keep our earlier notation: π for the natural projection of G onto $Q := G/\overline{G'}$ and $A := \overline{G'}$ for the closure of the commutator subgroup of G. Recall that A is a vector group and we identify A with its Lie algebra $\mathfrak{a}$ via the exponential map. The adjoint action of G is trivial on A and hence induces a semisimple representation $\alpha : Q \to \mathrm{GL}(A)$. For every weight χ of Q define the multiplicity

$$\mu_\chi := \dim_{k_\chi} A_\chi,$$

where the field k_χ is $\mathbb{R}$ if $\chi(Q) \subset \mathbb{R}$ and $k_\chi = \mathbb{C}$ if $\chi(Q) \not\subset \mathbb{R}$. Note that for a non-trivial weight χ every closed Q-invariant subgroup of A_χ is a vector space over k_χ, by Lemma 6.1 in Section 6. There is a subset X of G with $d = d(Q)$ elements such that the cosets $xA, x \in X$ generate a dense subgroup of Q. We have much freedom in choosing X. First of all we may multiply each $x \in X$ with an arbitrary element $a_x \in A$. Secondly, the set of $X \in \pi^{-1} Gen(Q)$ for which $\langle xA ; x \in X \rangle$ is dense in Q forms a dense subset of G^d, by Corollary 2.2 in Section 2. We will show that by   appropriately choosing X and taking a subset Y of $1 + \mu - d$ elements of A, if $1 + \mu - d > 0$, we obtain a set $X \cup Y$ which generates a dense subgroup of G. We will show specifically that the $\mathbb{Z}Q$-module generated by Y and the set of commutators $[X, X] = \{[x, y] \, ; x, y \in X\}$ is dense in A. Let us compute how changing x in its coset changes the commutator.

Set
$$[x, y] = x^{-1}y^{-1}xy, \quad x^y = y^{-1}xy.$$

Then
$$[x, yz] = [x, z][x, y]^z,$$

which gives for $z = a \in A$ $[x, ya] = [x, a] + [x, y] = (1 - \alpha(\bar{x}^{-1})) a + [x, y]$

since
$$[x, a] = x^{-1}a^{-1}xa = (1 - \alpha(\bar{x}^{-1})) a$$

and $[x, y] \in A$. Hence we may assume that there is an element $x_0 \in X$ such that

(1) $\chi(x_0 A) \neq 1$ for every weight χ of Q on A by density of $\pi^{-1}Gen(Q)$.

(2) The $d-1$ commutators $[x_0, x], x \in X, x \neq x_0$, have arbitrarily prescribed values in A by replacing x by xa for an appropriate $a \in A$.

It follows from the Corollary 7.5 at the end of section 7 that G contains a dense subgroup generated by $\max\{d, \mu + 1\}$ elements.

12. Remarks

Remark 12.1.   (1) The semisimplification V_s is the universal quotient of V as an $\mathbb{R}Q$-module with the property that it is semisimple as an $\mathbb{R}Q$-module.

(2) If Q is an abelian subgroup of $GL(V)$ then any Q-invariant vector subspace W of V is also invariant under the Zariski closure Q^Z of Q. Now Q^Z is the direct product of its torus Q_s^Z of semisimple elements and its subgroup Q_u^Z of unipotent elements. Obviously $Q_u \subseteq Q_u^Z$ and $Q_s \subseteq Q_s^Z$. This gives a different approach to V_s and V_u.

Remark 12.2.   Lemma 8.1 and Theorem 3.1 are similar in spirit and have a father in common as follows.

Lemma 12.3.   *Let Q be a connected Lie group which acts continuously by automorphisms on a connected Lie group A. Suppose that the induced action α of Q on the Lie algebra $\mathfrak{a}$ of A is unipotent and that $\alpha(Q)$ contains $Ad(A)$. Put $\mathfrak{a}_u = \sum_{q \in Q}(\alpha(q) - 1)\mathfrak{a}$ and $A_u = \exp \mathfrak{a}_u$. Then $\mathfrak{a}_u$ is a Q-invariant ideal of $\mathfrak{a}$ and A_u is a Q-invariant normal subgroup of A. If B is a Q-invariant subgroup of A such that $B \cdot A_u$ is dense in A then B is dense in A.*

13. Example

Consider the group $\qquad H = \left\{ \begin{pmatrix} 2^t & 2^t t & a \\ 0 & 2^t & b \\ 0 & 0 & 1 \end{pmatrix} ; t, a, b \in \mathbb{R} \right\}.$

Put $\qquad\qquad\qquad Q = \left\{ \begin{pmatrix} 2^t & 2^t t & 0 \\ 0 & 2^t & 0 \\ 0 & 0 & 1 \end{pmatrix} ; t \in \mathbb{R} \right\}$

and $\qquad\qquad\qquad A = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; a, b \in \mathbb{R} \right\}.$

Then Q is 1-dimensional abelian, A is 2-dimensional abelian and the representation α of Q on A is given by the upper left hand corner of the matrices in Q. Then

$$Q_s \simeq \left\{ \begin{pmatrix} 2^t & 0 \\ 0 & 2^t \end{pmatrix}; \ t \in \mathbb{R} \right\}, \quad \text{and} \quad Q_u \simeq \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \ t \in \mathbb{R} \right\}.$$

Note that in this case Q_s and Q_u are not contained in Q, but $Q \subseteq Q_s \cdot Q_u$. We have $d(Q) = 2$ since $Q \simeq \mathbb{R}$. We have $\mu_\chi = 1$ for the only non-trivial weight χ, since $V_u = \sum_q (q_u - 1) V = \mathbb{R}^2/\mathbb{R}e_1$, so the dimension of the semisimplification is 1. So $d(G) = 2$.

14. Historical remarks

Koksma's book [17] contains an exhaustive bibliography of work in the theory of diophantine approximations (up to 1936) inspired by Kronecker's theorem. Generalizing the case of $\mathbb{T}^n$, van Dantzig called a topological group G *monothetic* if its (topological) rank equals 1 [21]. In the work of Halmos and Samelson [11] the monothetic groups are thoroughly studied. Every monothetic group is abelian, and, if locally compact and not isomorphic to $\mathbb{Z}$, must be compact.

It was shown that a compact abelian group G is monothetic if and only if:

(a) G is *separable* (there exists a dense countable subset of G) and

(b) the group of characters of finite order of G is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$.

The abelian groups of finite topological generating rank were considered in [20] under the name of *polythetic groups*. Relying on the main structure theorem for LCA groups it is shown in [20] that every polythetic locally compact abelian group G has the form $G = K \times \mathbb{R}^l \times \mathbb{Z}^k$, where K is a compact abelian group, $k, l \geq 0$. This result is related to Pontryagin duality. Furthermore, if K is a compact abelian separable group and is the direct sum of a connected group and a totally disconnected group, then $d(G) = \max\{d(K), l + 1 + k\}$.

Unfortunately, at present we do not know of any such formula for a general compact abelian group K. It is proved in [13] that a connected compact group G is topologically finitely generated if and only if the weight $w(G)$ of G does not exceed $\mathfrak{c}$, the cardinality of the continuum. Recall that the *weight* of a space X, denoted $w(X)$, is the smallest cardinal number which arises as the cardinality of some base of the topology of X. Cleary and Morris [8] proved that for every compact connected (not necessarily abelian) group K with weight not exceeding $\mathfrak{c}$ and for every positive integer n, one has $d(K \times \mathbb{R}^n) = n + 1$. In [14] Hofmann and Morris extended the above result significantly. Namely, if K is a compact connected group of weight $\leq \mathfrak{c}$, and L is a nonsingleton connected Lie group, then $d(K \times L) = d(L)$.

As far as Lie groups are concerned, Kuranishi showed in 1949, extending earlier work of Auerbach [3], that $d(G) = 2$ for every connected semisimple Lie group [18], [4, Chapter VIII, §2, Exercise 8]. It is easy to deduce the inequality $d(G) \leq 2 \dim(G)$ for every connected Lie group G. It follows that every connected Lie group is topologically finitely generated. As was mentioned in the introduction the work of Guivarc'h [10] contains a formula for the topological generating rank of the group

of upper triangular matrices. In [9] Gelander and Le Maître study an interesting variation of the topological generating rank, namely *infinitesimal generating rank.*

Emmanuel Breuillard, Ben Green, Robert Guralnick and Terence Tao obtained the following result concerning the topological generating rank problem with respect to the Zariski topology (see [6]): *Let $G(k)$ be a semisimple algebraic group over an uncountable algebraically closed field k. Assume that $G(k)$ is not C_2 with k of characteristic 3. Then the generating rank of $G(k)$ with respect to the Zariski topology is equal 2.*

The basic tool to study a Lie group G is its tangent Lie algebra $\mathfrak{g}$, so the natural question occurs what is the relation between $d(G)$ and $d(\mathfrak{g})$, where $d(\mathfrak{g})$ is the minimal number of generators of $\mathfrak{g}$. It follows from the results of Breuillard and Gelander [5] that $d(G) \leq d(\mathfrak{g})$ for every connected real Lie group G which is topologically perfect (recall that G is *topologically perfect* if the commutator subgroup $G' = [G, G]$ is dense in G, or, equivalently, if G has no continuous surjective homomorphism to the circle). We will come back to this question in a future paper.

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