Journal of Lie Theory
Volume **29** (2019) 473–492

The Algebraic Mackey-Higson Bijections

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Communicated by B. Krötz

Abstract. For a linear connected semisimple Lie group G, we construct an explicit collection of correspondences between the admissible dual of G and the admissible dual of the Cartan motion group associated with G. We conjecture that each of these correspondences induces an algebraic isomorphism between the admissible duals. The constructed correspondences are defined in terms of algebraic families of Harish-Chandra modules. We prove that the conjecture holds in the case of $SL_2(\mathbb{R})$, and, in that case, we characterize the bijections.

Mathematics Subject Classification: 22E47

Key Words: Mackey-Higson bijection, algebraic families of Harish-Chandra pairs, algebraic families of Harish-Chandra modules.

1. Introduction

Recently, Alexandre Afgoustidis constructed a bijection ("*The Mackey bijection*") between the tempered dual of a real connected semisimple Lie group and the tempered dual of its Cartan motion group [1, 2]. This generalized results of Higson for complex semisimple groups [9, 10]. Such a bijection was first suggested in the work of Mackey [15] and, in a special case, in [16, p. 10]. A nontrivial consequence of the Mackey bijection is a new proof for the Baum-Connes conjecture for a semisimple Lie group [9, 3].

In this paper, we construct algebraic correspondences between the admissible dual of a connected semisimple Lie group, and the admissible dual of its Cartan motion group. This is done using algebraic families of Harish-Chandra modules in the sense of [5, 6], and is a new application of these families (other applications include *contractions* [6], and the hydrogen atom system [17]). We conjecture that our correspondences are algebraic isomorphisms that extend the Mackey bijection between the tempered duals. For $SL_2(\mathbb{R})$, we show that the conjecture holds, and, more importantly, we characterize the Mackey bijections for the admissible dual.

We shall now state our results more carefully. Let G be a semisimple complex algebraic group with an algebraic involution Θ and a commuting antiholomorphic involution σ. We further assume that the real form G^σ is connected, having a maximal compact subgroup K^σ, which is a compact real form of $K = G^\Theta$. We denote the Lie algebras of G and K by $\mathfrak{g}$ and $\mathfrak{k}$, respectively, and their corresponding real

ISSN 0949–5932 / $2.50

forms by $\mathfrak{g}^\sigma$ and $\mathfrak{k}^\sigma$. Our aim is to understand the Mackey bijection for admissible irreducible representations of G^σ, viewed as irreducible $(\mathfrak{g}, K)$-modules.

Using a variant of the deformation to the normal cone construction, we obtain an algebraic family of Harish-Chandra pairs $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$ over the base $\mathbb{X} := \mathbb{CP}^1$. The fibers of $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$ satisfy

$$(\boldsymbol{\mathfrak{g}}|_z, \boldsymbol{K}|_z) \simeq \begin{cases} (\mathfrak{g}, K), & z \neq \infty \\ (\mathfrak{k} \ltimes \mathfrak{p}, K), & z = \infty, \end{cases}$$

where $\mathfrak{p}$ is the minus one eigenspace of Θ in $\mathfrak{g}$, and $\mathfrak{k} \ltimes \mathfrak{p}$ is the Lie algebra of the (complex) Cartan motion group, $G_C = K \ltimes \mathfrak{p}$. As a sheaf of K-equivariant $O_{\mathbb{X}}$-modules $\boldsymbol{\mathfrak{g}} \simeq (\mathcal{O}_{\mathbb{X}} \otimes_{\mathbb{C}} \mathfrak{k}) \oplus (\mathcal{O}_{\mathbb{X}}(-1) \otimes_{\mathbb{C}} \mathfrak{p})$.

Every Θ-stable Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{g}$, with a Cartan decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, can be lifted to a Cartan subfamily of $\boldsymbol{\mathfrak{g}}$, of the form $\boldsymbol{\mathfrak{h}} \simeq (\mathcal{O}_{\mathbb{X}} \otimes_{\mathbb{C}} \mathfrak{t}) \oplus (\mathcal{O}_{\mathbb{X}}(-1) \otimes_{\mathbb{C}} \mathfrak{a})$. By Vogan's classification of the admissible dual, $\widehat{G^\sigma}$, via minimal K^σ-types (see Section 5 and [20, 21, 22]), with any $\mu \in \widehat{K}$, we can associate a Θ-stable Cartan, $\mathfrak{h}_\mu$, and a standard tempered representation, I_μ, having a real infinitesimal character. In Subsection 3.3 we define a *generalized Harish-Chandra homomorphism* with respect to $\mathfrak{h}_\mu$, $\widetilde{\gamma}_{\mathfrak{h}_\mu} : \mathcal{Z}(\boldsymbol{\mathfrak{g}}) \longrightarrow \mathcal{U}(\boldsymbol{\mathfrak{h}}_\mu)$. It is a (rational) morphism of $O_{\mathbb{X}}$-algebras from the center of the sheaf of enveloping algebras of $\boldsymbol{\mathfrak{g}}$, to the sheaf of universal enveloping algebras of $\boldsymbol{\mathfrak{h}}_\mu$. An algebraic family of $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$-modules, on which the action of $\mathcal{Z}(\boldsymbol{\mathfrak{g}})$ factors through $\widetilde{\gamma}_{\mathfrak{h}_\mu}$, is said to have an *infinitesimal character with respect to* $\mathfrak{h}_\mu$, see Section 4. It is not the case that every generically irreducible family of Harish-Chandra modules has an infinitesimal character (see Example 6.9).

An algebraic family of $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$-modules, $\mathcal{F}$, has a canonical isotypic decomposition with respect to the action of K: $\mathcal{F} = \bigoplus_{\mu \in \widehat{K}} \mathcal{F}_\mu$. We define $\widetilde{\mathcal{M}}(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})_\mu$ to be the collection of all generically irreducible $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$-modules that are generated by their $\mathcal{F}_\mu$ part, having an infinitesimal character with respect to $\mathfrak{h}_\mu$, and their fiber at 0 is equivalent, in the Grothendieck group of $(\mathfrak{g}, K)$, to a submodule of I_μ. For $\mathcal{F} \in \widetilde{\mathcal{M}}(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})_\mu$ and $[\alpha : \beta] \in \mathbb{CP}^1$, we let $J_{\mu,[\alpha:\beta]}(\mathcal{F})$ be the unique composition factor of $\mathcal{F}|_{[\alpha:\beta]}$ that contains μ.

Conjecture 5.4 Fix $[\alpha : \beta] \in \mathbb{RP}^1$ with $\alpha\beta \neq 0$. As μ varies in $\widehat{K}$ and $\mathcal{F}$ varies in $\widetilde{\mathcal{M}}(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})_\mu$, the correspondence $J_{\mu,[\alpha:\beta]}(\mathcal{F}) \longleftrightarrow J_{\mu,[0:1]}(\mathcal{F})$ gives rise to a bijection between $\widehat{G^\sigma}^{\text{adms}}$ and $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}^{\text{adms}}$ such that:

1. The bijection is an algebraic isomorphism, namely: For each $\mu \in \widehat{K}$ the bijection restricts to an isomorphism of affine algebraic varieties between $\widehat{G^\sigma}_\mu$, the space of infinitesimal equivalence classes of irreducible admissible representations of G^σ having μ as a minimal K^σ-type, and the corresponding subspace of $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}$.

2. The bijection extends Vogan's bijection (Theorem 5.2) between the tempered dual of G^σ with real infinitesimal character, to $\widehat{K^\sigma} \subset \widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}$.

3. The bijection maps tempered representations to tempered representations.

In addition, we suggest a relatively simple way, to calculate the various $J_{\mu,[\alpha:\beta]}(\mathcal{F})$ using the Jantzen filtration. The real structure σ of G induces a real structure on the family $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$. With any family of of Harish-Chandra modules $\mathcal{F}$, one can associate a dual family, $\mathcal{F}^\sigma$ (*the* σ*-twisted dual*), that is defined with respect to σ, see Section 5. A non-zero rational intertwining operator between $\mathcal{F}$ and $\mathcal{F}^\sigma$ gives rise to a canonical filtration, the Jantzen filtration, on every $\mathcal{F}|_{[\alpha:\beta]}$. For $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$, we denote by $\widetilde{J}_{\mu,[\alpha:\beta]}(\mathcal{F})$ the unique Jantzen quotient of $\mathcal{F}|_{[\alpha:\beta]}$ that contains μ.

Conjecture 5.6 For any $[\alpha:\beta] \in \mathbb{RP}^1$ and any $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$ for which $\mathcal{F}|_{[\alpha:\beta]}$ is reducible, $\widetilde{J}_{\mu,[\alpha:\beta]}(\mathcal{F}) \simeq J_{\mu,[\alpha:\beta]}(\mathcal{F})$.

In Section 6 we consider the case of $SL_2(\mathbb{R})$ and we prove the following theorems.

Theorem 6.12 Conjecture 5.4 and 5.6 hold for $SL_2(\mathbb{R})$.

Theorem 6.13 Any bijection between $\widehat{SL_2(\mathbb{R})}^{\text{adms}}$ and $\widehat{SO(2) \ltimes \mathbb{R}^2}^{\text{adms}}$ satisfying the three conditions in Conjecture 5.4 above arises form the correspondence $J_{\mu,[\alpha:\beta]}(\mathcal{F}) \longleftrightarrow J_{\mu,[0:1]}(\mathcal{F})$, for some $[\alpha:\beta] \in \mathbb{RP}^1$ with $\alpha\beta \neq 0$.

The Mackey bijection for the tempered duals in the case of $SL_2(\mathbb{R})$ was studied before in [8]. Recently, it was reformulated in terms of twisted $\mathcal{D}$-modules on the flag variety of $SL_2(\mathbb{R})$ [18], see also [25]. Most of the formalism of algebraic families that we use here was developed in [5, 6]. We shall use Sections 2 and 4 to recall some facts from there. In addition, a lot of the calculations that we shall need were done, in some form or another, in those references. In Section 3 we define the generalized Harish-Chandra homomorphisms. Section 5 is used to state our conjectures and in Section 6 we deal with the case of $SL_2(\mathbb{R})$.

The author is grateful to Nigel Higson for introducing him to the subject of the *Mackey-bijection* and for many useful conversations. The author thanks Joseph Bernstein for fruitful discussions and the referee for his many comments, on an earlier version of the manuscript, which greatly improved the readability.

2. The deformation family

In this section, we recall and elaborate on the construction of the deformation family of Harish-Chandra pairs as given in [5, sec. 2.1.2]. Throughout this work, G will stand for a complex semisimple algebraic group, and we shall keep the notation and assumptions, mentioned in the introduction, for Θ, σ, K, and for the corresponding real forms and Lie algebras. The involutions Θ and σ shall also stand for the corresponding involutions of $\mathfrak{g}$. Whether we mean an involution of a group or its Lie algebra shall be clear from the context. The corresponding Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{g}^\sigma$ are given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{g}^\sigma = \mathfrak{k}^\sigma \oplus \mathfrak{p}^\sigma,$$

with $\mathfrak{p}$, and $\mathfrak{p}^\sigma$ the minus one eigenspace of Θ in $\mathfrak{g}$ and $\mathfrak{g}^\sigma$, respectively.

2.1. The family of Harish-Chandra pairs

Let $\mathfrak{g}_{\text{const}}$ be the constant family of Lie algebras over $\mathbb{X} := \mathbb{CP}^1$ with fiber $\mathfrak{g}$. Explicitly, $\mathfrak{g}_{\text{const}}$ is the sheaf of (algebraic) sections of the bundle $\mathbb{X} \times \mathfrak{g} \longrightarrow \mathbb{X}$. Thinking of $\mathbb{X}$ as $\mathbb{C} \cup \{\infty\}$, we define $\boldsymbol{\mathfrak{g}}$ to be the smallest subsheaf of $\mathfrak{g}_{\text{const}}$ containing all sections having their values at infinity lie in $\mathfrak{k}$.

Explicitly, for any open $U \subset \mathbb{X}$,

$$\Gamma(U, \mathfrak{g}) = \begin{cases} \Gamma(U, \mathfrak{g}_{\text{const}}), & \infty \notin U \\ \{\xi \in \Gamma(U, \mathfrak{g}_{\text{const}}) | \xi(\infty) \in \mathfrak{k}\}, & \infty \in U. \end{cases}$$

The involution Θ naturally induces an involution of $\boldsymbol{\mathfrak{g}}$, which we also denote by Θ. The corresponding decomposition into eigen-subfamilies is denoted by $\boldsymbol{\mathfrak{g}} = \boldsymbol{\mathfrak{k}} \oplus \boldsymbol{\mathfrak{p}}$ with $\boldsymbol{\mathfrak{k}} \simeq \mathcal{O}_{\mathbb{X}} \otimes_{\mathbb{C}} \mathfrak{k}$ and $\boldsymbol{\mathfrak{p}} \simeq \mathcal{O}_{\mathbb{X}}(-1) \otimes_{\mathbb{C}} \mathfrak{p}$. Hence, this decomposition of $\boldsymbol{\mathfrak{g}}$ is a decomposition of K-equivariant sheaves of $\mathcal{O}_{\mathbb{X}}$-modules. The family $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$, with $\boldsymbol{K}$ being the constant group scheme over $\mathbb{X}$ with fiber K, is an algebraic family of Harish-Chandra pairs in the sense of [5, sec. 2.3], and its fibers satisfy

$$(\boldsymbol{\mathfrak{g}}|_z, \boldsymbol{K}|_z) \simeq \begin{cases} (\mathfrak{g}, K), & z \neq \infty \\ (\mathfrak{k} \ltimes \mathfrak{p}, K), & z = \infty, \end{cases}$$

where z is the natural coordinate on $\mathbb{X}$. In terms of the standard coordinates $r = \frac{\beta}{\alpha}$ on $\mathbb{X}_0 := \{[\alpha : \beta] \in \mathbb{X} | \alpha \neq 0\}$, and $R = \frac{\alpha}{\beta}$ on $\mathbb{X}_\infty := \{[\alpha : \beta] \in \mathbb{X} | \beta \neq 0\}$ we have $\Gamma(\mathbb{X}_0, \boldsymbol{\mathfrak{g}}) = \mathbb{C}[r] \otimes_{\mathbb{C}} \mathfrak{g}$, and $\Gamma(\mathbb{X}_\infty, \boldsymbol{\mathfrak{g}}) = (\mathbb{C}[R] \otimes_{\mathbb{C}} \mathfrak{k}) \oplus (R\mathbb{C}[R] \otimes_{\mathbb{C}} \mathfrak{p})$.

2.2. The real structure

The morphisms

$$\sigma_{\mathbb{X}} : \mathbb{X} \longrightarrow \overline{\mathbb{X}}, \quad [\alpha : \beta] \longmapsto [\overline{\alpha} : \overline{\beta}],$$
$$\sigma_{\boldsymbol{\mathfrak{g}}} : \boldsymbol{\mathfrak{g}} \longrightarrow \sigma_{\mathbb{X}}^* \overline{\boldsymbol{\mathfrak{g}}}, \quad \xi \longmapsto \sigma(\xi \circ \sigma_{\mathbb{X}}),$$
$$\sigma_{\boldsymbol{K}} : \boldsymbol{K} \longrightarrow \sigma_{\boldsymbol{K}}^* \overline{\boldsymbol{K}}, \quad \tau \longmapsto \sigma(\tau \circ \sigma_{\mathbb{X}}),$$

determine a real form of the family $(\boldsymbol{\mathfrak{g}}, \boldsymbol{K})$, in the sense of [5, sec. 2.5]. The fixed point set of $\sigma_{\mathbb{X}}$ is $\mathbb{X}_{\mathbb{R}} := \mathbb{RP}^1$. The corresponding family of real Harish-Chandra pairs over $\mathbb{X}_{\mathbb{R}}$, denoted by $(\boldsymbol{\mathfrak{g}}^\sigma, \boldsymbol{K}^\sigma)$, has fibers satisfying

$$(\boldsymbol{\mathfrak{g}}^\sigma|_x, \boldsymbol{K}^\sigma|_x) \simeq \begin{cases} (\mathfrak{g}^\sigma, K^\sigma), & x \neq \infty \\ (\mathfrak{k}^\sigma \ltimes \mathfrak{p}^\sigma, K^\sigma), & x = \infty. \end{cases} \tag{1}$$

By a family of real Harish-Chandra pairs we mean that we have a collection of real Harish-Chandra pairs, parameterized by the topological space $\mathbb{X}_{\mathbb{R}}$ (as a subset of $\mathbb{X}$ equipped with its analytic topology).

2.3. The family of real groups

Using the deformation to the normal cone construction [7, sec 2.6], one can construct a corresponding family of complex algebraic groups over $\mathbb{X}$. Then, using the real structure, one obtains a family of Lie groups over $\mathbb{X}_{\mathbb{R}}$. The focus in this paper, is on the admissible dual of a real semisimple Lie group and the admissible dual of its Cartan motion group. These duals are defined in terms of the corresponding complex Harish-Chandra pairs, and in this sense the groups play no role. We note that each of the real Harish-Chandra pairs in (1) determines a unique Lie group, up to an isomorphism. In this way we obtain a Lie group for each $x \in \mathbb{RP}^1$.

Remark 2.1. The family of complex algebraic groups that one obtains via the deformation to the normal cone construction is related by a change of base operation to the one obtained in [4]. On the other hand the real structures are very different in nature.

3. The generalized Harish-Chandra homomorphism

In this section we show how the classical Harish-Chandra homomorphism of $\mathfrak{g}$ can be lifted to $\boldsymbol{\mathfrak{g}}$.

3.1. Subfamilies generated by a set

Any vector ξ of $\mathfrak{g}$ gives rise to a section $1 \otimes \xi$ in $\Gamma(\mathbb{X}_0, \boldsymbol{\mathfrak{g}})$. For a subset A of $\mathfrak{g}$, we denote by S_A, the subsheaf of $\boldsymbol{\mathfrak{g}}$ that is generated by the sections $\{1 \otimes \xi | \xi \in A\} \subset \Gamma(\mathbb{X}_0, \boldsymbol{\mathfrak{g}})$. In other words, S_A is the smallest subsheaf of $\boldsymbol{\mathfrak{g}}$ (as a quasi-coherent sheaf of $O_{\mathbb{X}}$-modules) that contains the local sections $\{1 \otimes \xi | \xi \in A\}$.

Lemma 3.1. *Let V be a vector subspace of $\mathfrak{g}$. Then S_V consists of all sections of $\boldsymbol{\mathfrak{g}}$ with values in V and such that their value at ∞ lies inside $V \cap \mathfrak{k}$. In particular, if $l = \dim(V \cap \mathfrak{k})$ and $m = \dim(V) - l$ then*

$$S_V \simeq (\underbrace{O_{\mathbb{X}} \oplus ... \oplus O_{\mathbb{X}}}_{l-times}) \oplus (\underbrace{O_{\mathbb{X}}(-1) \oplus ... \oplus O_{\mathbb{X}}(-1)}_{m-times}). \qquad \blacksquare$$

3.2. Families of Cartan subalgebras

From now on, we assume that $\mathfrak{h}$ is a Θ-stable Cartan subalgebra of $\mathfrak{g}$, with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. The Cartan subalgebra gives rise to $\boldsymbol{\mathfrak{h}} = S_{\mathfrak{h}}$, an abelian subfamily of $\boldsymbol{\mathfrak{g}}$. We shall call $\boldsymbol{\mathfrak{h}}$ the Cartan subfamily of $\boldsymbol{\mathfrak{g}}$ corresponding to $\mathfrak{h}$, or simply a Cartan subfamily. As already mentioned in Lemma 3.1, $\boldsymbol{\mathfrak{h}} = S_{\mathfrak{h}} \simeq (O_{\mathbb{X}} \otimes_{\mathbb{C}} \mathfrak{t}) \oplus (O_{\mathbb{X}}(-1) \otimes_{\mathbb{C}} \mathfrak{a})$. Later on, we shall be interested in the dual of $\boldsymbol{\mathfrak{h}}$, it is useful to note now that

$$\boldsymbol{\mathfrak{h}}^* := \mathrm{Hom}_{O_{\mathbb{X}}}(\boldsymbol{\mathfrak{h}}, O_{\mathbb{X}}) \simeq (O_{\mathbb{X}} \otimes_{\mathbb{C}} \mathfrak{t}^*) \oplus (O_{\mathbb{X}}(1) \otimes_{\mathbb{C}} \mathfrak{a}^*).$$

We note that we can promote each root $\alpha \in \Delta(\mathfrak{h}, \mathfrak{g})$ to an element $\widetilde{\alpha} \in \boldsymbol{\mathfrak{h}}^*$, by first extending α, using extension of scalars, to be a morphism from $O_{\mathbb{X}} \otimes \mathfrak{h}$ into $O_{\mathbb{X}}$, and then use restriction to $\boldsymbol{\mathfrak{h}}$.

3.3. The Harish-Chandra homomorphism

Let $\mathfrak{h}$ be a Θ-stable Cartan subalgebra of $\mathfrak{g}$ and $\Delta^+(\mathfrak{h}, \mathfrak{g})$ a fixed positive system. Let $\gamma_{\mathfrak{h}} : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ be the corresponding normalized Harish-Chandra morphism. We shall show that $\gamma_{\mathfrak{h}}$ can be lifted to a rational morphism of sheaves of commutative $O_{\mathbb{X}}$-algebras $\widetilde{\gamma}_{\mathfrak{h}} : \mathcal{Z}(\boldsymbol{\mathfrak{g}}) \longrightarrow \mathcal{U}(\boldsymbol{\mathfrak{h}})$.

By extension of scalars, $\gamma_{\mathfrak{h}}$ can be lifted to a morphism of sheaves of commutative $O_{\mathbb{X}}$-algebras of the constant families:

$$\widetilde{\gamma}_{\mathfrak{h}} : O_{\mathbb{X}} \otimes_{\mathbb{C}} \mathcal{Z}(\mathfrak{g}) \longrightarrow O_{\mathbb{X}} \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}), \quad f \otimes \xi \longmapsto f \otimes \gamma_{\mathfrak{h}}(\xi).$$

The following lemma shows that $\mathcal{Z}(\boldsymbol{\mathfrak{g}})$ is a subsheaf of the center of the constant family $\mathcal{Z}(\boldsymbol{\mathfrak{g}}_{\mathrm{const}}) = O_{\mathbb{X}} \otimes_{\mathbb{C}} \mathcal{Z}(\mathfrak{g})$.

Lemma 3.2. $\mathcal{Z}(\boldsymbol{\mathfrak{g}}) = \mathcal{Z}(\boldsymbol{\mathfrak{g}}_{const}) \cap \mathcal{U}(\boldsymbol{\mathfrak{g}})$.

Proof. It is enough to show that $\Gamma(\mathbb{X}_\infty, \mathcal{Z}(\boldsymbol{\mathfrak{g}})) = \Gamma(\mathbb{X}_\infty, \mathcal{Z}(\boldsymbol{\mathfrak{g}}_{\mathrm{const}}) \cap \mathcal{U}(\boldsymbol{\mathfrak{g}}))$. Let $s \in \Gamma(\mathbb{X}_\infty, \mathcal{Z}(\boldsymbol{\mathfrak{g}}))$, then

$$s|_{\mathbb{X}_0 \cap \mathbb{X}_\infty} \in \Gamma(\mathbb{X}_0 \cap \mathbb{X}_\infty, \mathcal{Z}(\boldsymbol{\mathfrak{g}})) = \Gamma(\mathbb{X}_0 \cap \mathbb{X}_\infty, \mathcal{Z}(\boldsymbol{\mathfrak{g}}_{\mathrm{const}}) \cap \mathcal{U}(\boldsymbol{\mathfrak{g}})).$$

If $\tau \in \Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{g}_{\mathrm{const}}))$ then over $\mathbb{X}_0 \cap \mathbb{X}_\infty$ we have $[s|_{\mathbb{X}_0\cap\mathbb{X}_\infty}, \tau|_{\mathbb{X}_0\cap\mathbb{X}_\infty}] = 0$ and from the continuity of s, τ, and the brackets, it follows that $[s,\tau] = 0$. Hence, $s \in \Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g}_{\mathrm{const}}) \cap \mathcal{U}(\mathfrak{g}))$. The inclusion $\mathcal{Z}(\mathfrak{g}) \supset (\mathcal{Z}(\mathfrak{g}_{\mathrm{const}}) \cap \mathcal{U}(\mathfrak{g}))$ is obvious. ■

By restriction, we obtain a morphism $\widetilde{\gamma}_{\mathfrak{h}} : \mathcal{Z}(\mathfrak{g}) \longrightarrow O_{\mathbb{X}} \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h})$. This morphism is a morphism of sheaves of commutative $O_{\mathbb{X}}$-algebras. We can interpret it as a rational morphism $\widetilde{\gamma}_{\mathfrak{h}} : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$. In section 6, we will show, in the case of $SL_2(\mathbb{R})$, that for any Θ-stable Cartan subalgebra, the morphism $\widetilde{\gamma}_{\mathfrak{h}} : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ is a morphism of $O_{\mathbb{X}}$-modules. In general, one can show that for most compact Θ-stable Cartans $\widetilde{\gamma}_{\mathfrak{h}}$ is regular.

Remark 3.3. One can not expect $\widetilde{\gamma}_{\mathfrak{h}}|_{\mathcal{Z}(\mathfrak{g})}$ to be an isomorphism, since, in general (and already for $\mathfrak{sl}_2(\mathbb{C})$), there might be two Θ-stable Cartan subalgebras, $\mathfrak{h}_1$ and $\mathfrak{h}_2$, with $\mathfrak{h}_1$ not isomorphic to $\mathfrak{h}_2$. See Example 6.9.

4. Algebraic families of Harish-Chandra modules

In this section we set notations and recall some properties of algebraic families of Harish-Chandra modules, as they were defined in [5].

Let $(\mathfrak{g}, \boldsymbol{K})$ be an algebraic family of Harish-Chandra pairs over a quasi-projective complex algebraic variety X. An *algebraic family of Harish-Chandra modules* for $(\mathfrak{g}, \boldsymbol{K})$ is a flat, quasicoherent O_X-module $\mathcal{F}$, that is equipped with compatible actions of $\boldsymbol{K}$ and $\mathfrak{g}$. For a constant reductive $\boldsymbol{K}$ (which is the case considered in this paper) with fiber K, any quasi-admissible $\mathcal{F}$ has a canonical isotypic decomposition into K-equivariant locally free sheaves of O_X-modules,

$$\mathcal{F} = \bigoplus_{\tau \in \widehat{K}} \mathcal{F}_\tau .$$

We shall denote the category of quasi-admissible families of Harish-Chandra modules for $(\mathfrak{g}, \boldsymbol{K})$ by $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})$. When X is a curve, $\mathcal{F}$ is *generically irreducible* if, up to at most a countable number of fibers, all its fibers are irreducible. The family $\mathcal{F}$ is *quasisimple* if the action of $\mathcal{Z}(\mathfrak{g})$ on $\mathcal{F}$ factors through a morphism of sheaves of O_X-algebras $\chi_{\mathcal{F}} : \mathcal{Z}(\mathfrak{g}) \longrightarrow O_X$. A generically irreducible family of Harish-Chandra modules is quasisimple. In the classical case, thanks to the Harish-Chandra isomorphism, characters of the center of the enveloping algebras are the same thing as Weyl-group-invariant linear functionals of $\mathfrak{h}$. In the context of families, this is not true. Below we define in which sense a family of Harish-Chandra modules can have an infinitesimal character, with respect to a given Θ-stable Cartan subfamily. We shall keep our assumptions on $\mathfrak{g}, \mathfrak{k}$ and Θ as before, and denote by $(\mathfrak{g}, \boldsymbol{K})$ the corresponding deformation family of Harish-Chandra pairs over $\mathbb{X}$.

Definition 4.1. Let $\mathfrak{h}$ be a Θ-stable Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{F}$ a quasisimple family of Harish-Chandra modules for $(\mathfrak{g}, \boldsymbol{K})$. The family $\mathcal{F}$ *has an infinitesimal character (with respect to* $\mathfrak{h}$ *)*, if there is a morphism of sheaves of $O_{\mathbb{X}}$-algebras $\psi_{\mathcal{F}} : \mathcal{U}(\mathfrak{h}) \longrightarrow O_{\mathbb{X}}$, such that, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xrightarrow{\chi_{\mathcal{F}}} & O_{\mathbb{X}} \\ {\scriptstyle \widetilde{\gamma}_{\mathfrak{h}}}\big\downarrow & \nearrow{\scriptstyle \psi_{\mathcal{F}}} & \\ \mathcal{U}(\mathfrak{h}) & & \end{array}$$

A family $\mathcal{F}$ may have an infinitesimal character with respect to one Θ-stable Cartan and not with respect to another. See Example 6.9.

5. The correspondences between $\widehat{G^\sigma}$ and $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}$

In the first part of this section we recall the algebraic description of $\widehat{G^\sigma}$ by minimal K-types due to Vogan. Then, we define certain classes of algebraic families of Harish-Candra modules. In the last part, we construct correspondences between $\widehat{G^\sigma}$ and $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}$ and conjecture that each correspondence is an algebraic bijection.

5.1. Vogan's classification

In this section, we shall briefly recall the algebraic description of $\widehat{G^\sigma}$, which is due to Vogan [20, 21, 22]. Since G^σ is connected so is K^σ, and the highest weight theory of Cartan and Weyl can be used, e.g., see [12, Chapter IV]. By choosing a maximal torus T^σ of K^σ, any $\mu \in \widehat{K^\sigma}$ is uniquely determined by a dominant integral linear functional (a highest weight) of $\mathfrak{t}$. The Killing form gives rise to a norm on $\mathfrak{t}^*$, which in turn defines a "norm", $\|\_\| : \widehat{K^\sigma} \longrightarrow [0,\infty)$. A *minimal K^σ-type* of $\pi \in \widehat{G^\sigma}$, is $\mu \in \widehat{K^\sigma}$, that appears in $\pi|_{K^\sigma}$, and minimizes the restriction of $\|\_\|$ to the K^σ-types of $\pi|_{K^\sigma}$, e.g., see [14, chapter X]. For any $\mu \in \widehat{K^\sigma}$, we denote by $\widehat{G^\sigma}_\mu$, the subset of $\widehat{G^\sigma}$ consisting of those infinitesimal equivalence classes having μ as a minimal K^σ-type. It is known that $\widehat{G^\sigma} = \bigcup_{\mu \in \widehat{K^\sigma}} \widehat{G^\sigma}_\mu$.

Let P be a parabolic subgroup of G^σ with a Langlands decomposition $P = MAN$ (e.g., see [13, chapter VII]). Denote the linear dual of the complexified Lie algebra of A by $\mathfrak{a}^*$. For any $\delta \in \widehat{M}$ and $\nu \in \mathfrak{a}^*$, we denote the parabolically (normalized) induced representation $\operatorname{Ind}_P^{G^\sigma} \delta \otimes \nu \otimes 1$ by $I_P(\delta,\nu)$. The following theorem gives a complete description of $\widehat{G^\sigma}$.

Theorem 5.1 (Vogan [21] Theorem 1.1 and 1.2)**.** *Let $\mu \in \widehat{K^\sigma}$. There exists a cuspidal parabolic subgroup of G^σ, $P_\mu = M_\mu A_\mu N_\mu$, and a discrete series representation δ_μ of M_μ, such that:*

1. *For any $\nu \in \mathfrak{a}^*$, $I_{P_\mu}(\delta_\mu,\nu)$ has a unique irreducible subquotient, $J_{P_\mu}(\delta_\mu,\nu)$, that contains μ (as a minimal K^σ-type).*
2. *For any $\pi \in \widehat{G^\sigma}_\mu$, there exists $\nu \in \mathfrak{a}^*$ such that π is infinitesimally equivalent to $J_{P_\mu}(\delta_\mu,\nu)$.*
3. *$J_{P_\mu}(\delta_\mu,\nu) \simeq J_{P_{\mu'}}(\delta'_{\mu'},\nu')$ implies that $(M_\mu A_\mu, \delta_\mu, \otimes\nu)$ is conjugate to $(M_{\mu'}A_{\mu'}$, $\delta_{\mu'}$, $\otimes\nu')$. Furthermore, assuming $P_\mu = P_{\mu'}$ and $\delta_\mu = \delta_{\mu'}$, then $J_{P_\mu}(\delta_\mu,\nu) \simeq J_{P_\mu}(\delta_\mu,\nu')$ if and only if ν' is obtained from ν under the action of the corresponding R group* (*for more details see* [21])*.*

Later on, we shall need the following related result of Vogan, identifying a certain part of $\widehat{G^\sigma}$ with $\widehat{K^\sigma}$.

Theorem 5.2 (Vogan [23] Theorem 8.1)**.** *For a reductive Lie group, G^σ, the map that sends an irreducible tempered Harish-Chandra module with a real infinitesimal character to its (unique) lowest K^σ-type, establishes a bijection between equivalence classes of irreducible tempered Harish-Chandra modules with a real infinitesimal character and $\widehat{K^\sigma}$.*

It is useful to note that the inverse of the map in Theorem 5.2 is given by

$$\mu \longmapsto J_{P_\mu}(\delta_\mu, 0).$$

5.2. Classes of families of modules

In what follows, for each $\mu \in \widehat{K^\sigma}$, we shall define a class of families in $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})$.

By Weyl's unitarity trick (e.g., [12, Proposition 5.7]), there is no difference between locally finite continuous representations of a compact group, and algebraic representations of its complexification. From now on, whenever K is a complex algebraic reductive group with a compact real form K^σ, we shall identify $\widehat{K^\sigma}$ with $\widehat{K}$, the set of equivalence classes of irreducible algebraic representations of K. Fix $\mu \in \widehat{K}$. By Theorem 5.1, there is a corresponding cuspidal parabolic subgroup, $P_\mu = M_\mu A_\mu N_\mu$, of G^σ. To P_μ one can attach a Θ-stable Cartan subalgebra $\mathfrak{h}_\mu$ of $\mathfrak{g}$, namely, one chooses a compact Cartan subalgebra $\mathfrak{t}_\mu$ of $\mathfrak{m}_\mu := \mathrm{Lie}(M_\mu)$, there is such a Cartan since P_μ is cuspidal. Then a corresponding Θ-stable Cartan subalgebra $\mathfrak{h}_\mu$ is given by the complexification of $\mathfrak{t}_\mu \oplus \mathfrak{a}_\mu$ in $\mathfrak{g}$, where $\mathfrak{a}_\mu = \mathrm{Lie}(A_\mu)$.

Definition 5.3. Fix $\mu \in \widehat{K}$. We define $\widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$ to be the collection of all $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})$ that satisfies the following conditions.

1. $\mathcal{F}$ is generically irreducible.
2. $\mathcal{F}$ is generated by $\mathcal{F}_\mu$.
3. $\mathcal{F}$ has an infinitesimal character with respect to $\mathfrak{h}_\mu$.
4. The collection of composition factors of $\mathcal{F}|_{[1:0]}$ (with their multiplicities) is contained in the collection of composition factors of $I_{P_\mu}(\delta_\mu, 0)$ (with their multiplicities).

Note that $J_{P_\mu}(\delta_\mu, 0)$ is one of the composition factors of $\mathcal{F}|_{[1:0]}$. Heuristically, a family $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$, should be thought of as a compactification of a (restriction to a line of a) subfamily of $\{I_{P_\mu}(\delta_\mu, \nu)\}_{\nu \in \mathfrak{a}^*}$.

5.3. The correspondence using composition factors

If $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$, the K-types of the fibers $\mathcal{F}|_{[\alpha:\beta]}$ are independent of $[\alpha : \beta] \in \mathbb{CP}^1$. Moreover, the multiplicity of μ is one, and hence, there is a unique irreducible composition factor of $\mathcal{F}|_{[\alpha:\beta]}$ that contains μ. We shall denote it by $J_{\mu,[\alpha:\beta]}(\mathcal{F})$.

The correspondence: Fix $[\alpha : \beta] \in \mathbb{RP}^1$ different from "0" $= [1 : 0]$ and "∞" $= [0 : 1]$. For any $\mu \in \widehat{K}$, we say that $\pi \in \widehat{G^\sigma}_\mu$ and $\eta \in \widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}_\mu$ are in correspondence, if there exists $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$ with $\pi \simeq J_{\mu,[\alpha:\beta]}(\mathcal{F})$ and $\eta \simeq J_{\mu,[0:1]}(\mathcal{F})$.

Conjecture 5.4. As μ varies in $\widehat{K}$ and $\mathcal{F}$ varies in $\widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$, the correspondence $J_{\mu,[\alpha:\beta]}(\mathcal{F}) \longleftrightarrow J_{\mu,[0:1]}(\mathcal{F})$ gives rise to a bijection between $\widehat{G^\sigma}$ and $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}$, such that:

1. The bijection is an algebraic isomorphism, in the following sense. For each $\mu \in \widehat{K}$, the bijection restricts to an isomorphism of affine algebraic varieties between $\widehat{G^\sigma}_\mu$ and $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}_\mu$.

2. The bijection extends Vogan's bijection (Theorem 5.2) between the tempered dual of G^σ with real infinitesimal character to $\widehat{K^\sigma} \subset \widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}$.
3. The bijection maps tempered representations to tempered representations.

Remark 5.5. It follows from [21, 24] that $\widehat{G^\sigma}_\mu$ can be identified with the spectrum of a certain finitely generated abelian algebra of regular functions on $\mathfrak{a}^*_\mu$, and hence, a complex affine algebraic variety. Similarly, $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}_\mu$ is an affine algebraic variety. The structure of $\widehat{G^\sigma}_\mu$ and $\widehat{K^\sigma \ltimes \mathfrak{p}^\sigma}_\mu$ as affine algebraic varieties, for complex semisimple G^σ, is described in [10].

5.4. The Jantzen filtration and the correspondence

Here we reformulate an additional conjecture expressing the various $J_{\mu,[\alpha:\beta]}(\mathcal{F})$, in term of the Jantzen filtration.

Recall that for $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})$, the σ-twisted dual, $\mathcal{F}^\sigma$, is also in $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})$, see [6, sec. 2.4]. Let $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$. For any nonzero rational intertwining operator $\varphi \colon \mathcal{F} \longrightarrow \mathcal{F}^\sigma$, and any $[\alpha : \beta] \in \mathbb{RP}^1$, there is a corresponding decreasing filtration, $\{\mathcal{F}|^n_{[\alpha:\beta]}\}_{n\in\mathbb{Z}}$, of $F|_{[\alpha:\beta]}$ (the Jantzen filtration). The filtration is independent of φ (up to a shift of the filtration parameter), see [6, sec. 4.1]. We shall denote by $\widetilde{J}_{\mu,[\alpha:\beta]}(\mathcal{F})$, the unique Jantzen quotient of $F|_{[\alpha:\beta]}$ containing μ. For some $\mathcal{F}$, there are no nonzero rational intertwining operators from $\mathcal{F}$ to $\mathcal{F}^\sigma$. For example, in the case of $SL_2(\mathbb{R})$, it exists if and only if the function by which the Casimir section (see Section 6) acts is real-valued on $\mathbb{RP}^1$ (see [6, prop. 3.5]). The following conjecture suggests a relatively easy method to calculate $J_{\mu,[\alpha:\beta]}(\mathcal{F})$ in terms of the Jantzen quotients.

Conjecture 5.6. For any $[\alpha : \beta] \in \mathbb{RP}^1$, and any $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_\mu$ for which $\mathcal{F}|_{[\alpha:\beta]}$ is reducible, $\widetilde{J}_{\mu,[\alpha:\beta]}(\mathcal{F}) \simeq J_{\mu,[\alpha:\beta]}(\mathcal{F})$.

It should be stressed that, in principle, the Jantzen quotients are much easier to calculate than the composition factors. In addition to that, Conjecture 5.6, if it holds of course, implies that the unique Jantzen quotient of $\mathcal{F}|_{[\alpha:\beta]}$ that contains μ, is irreducible. In the next section we shall see that both Conjecture 5.4 and Conjecture 5.6 hold for $SL_2(\mathbb{R})$. We note that if $\widetilde{J}_{\mu,[\alpha:\beta]}(\mathcal{F})$ is irreducible then $\widetilde{J}_{\mu,[\alpha:\beta]}(\mathcal{F}) \simeq J_{\mu,[\alpha:\beta]}(\mathcal{F})$.

6. The $SL_2(\mathbb{R})$-case

In this section, for the case of $SL_2(\mathbb{R})$, we explicitly describe all relevant families, we calculate the various Jantzen quotients $\widetilde{J}_{\mu,[\alpha:\beta]}(\mathcal{F})$ and show that they coincide with $J_{\mu,[\alpha:\beta]}(\mathcal{F})$. We then prove that Conjectures 5.4 and 5.6 hold.

6.1. The family of Harish-Chandra pairs

The analysis described below is parallel to the one in [5, Sec. 4.2-4.3]). Throughout this section $G = SL_2(\mathbb{C})$, the Cartan involution (of $SL_2(\mathbb{R})$) is given by $\Theta(A) = (A^t)^{-1}$ and its fixed point subgroup is $K = SO(2, \mathbb{C})$. A corresponding antiholomorphic involution is given by $\sigma(A) = \overline{A}$ and its fixed point subgroup is

$G^\sigma = SL_2(\mathbb{R})$. We shall fix bases $\{H\}$ of $\mathfrak{k}$ and $\{X, Y\}$ of $\mathfrak{p}$, satisfying

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H. \tag{2}$$

For concreteness, we take

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Each one of the vectors H, X, and Y generates a subsheaf of $\mathfrak{g}$ (in the sense of Subsection 3.1) that is a K-equivariant invertible sheaf. We denote these sheaves by $\mathfrak{g}_0$, $\mathfrak{g}_2$, and $\mathfrak{g}_{-2}$, respectively. For $n \in \{-2, 0, 2\}$, the action of $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in SO(2, \mathbb{C})$ on $\mathfrak{g}_n$ is given by multiplication by $(\alpha + i\beta)^n$. The sheaf of Lie algebras corresponding to $\boldsymbol{K}$ coincides with $\mathfrak{g}_0$, and shall be denoted by $\mathfrak{k}$ as well. As $O_{\mathbb{X}}$-modules $\mathfrak{g}_0 = \mathfrak{k} \simeq \mathcal{O}_{\mathbb{X}}$, $\mathfrak{g}_{\pm 2} \simeq \mathcal{O}_{\mathbb{X}}(-1)$. The isotypic decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$. More explicitly, the section $H_0 := 1 \otimes H$ is a nowhere vanishing regular section of $\mathfrak{g}_0$, the sections $X_0 := 1 \otimes X$ and $Y_0 := 1 \otimes Y$ are nowhere vanishing rational sections of $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$, respectively. Moreover, the only pole of X_0 (and Y_0) is a simple pole at $\infty = [0 : 1]$. The sections H_0, X_0, Y_0 form an $\mathfrak{sl}_2$-triplet and satisfy the same commutation relations as in (2). This completes the description of $\mathfrak{g}$. The family $\boldsymbol{K}$ is the constant group scheme over $\mathbb{X}$ with fiber $K = SO(2, \mathbb{C})$.

6.2. Generically irreducible families of Harish-Chandra modules

In the next few paragraphs we describe the classification of certain generically irreducible families of Harish-Chandra modules for the deformation family of $SL_2(\mathbb{R})$. The analysis is parallel to the case of the *contraction family* of $SL_2(\mathbb{R})$ that is given in [5, sec. 4]).

The Casimir section. Recall that the *Casimir sheaf* $\mathcal{C}$ of $\mathfrak{g}$, is the subsheaf of the center of the sheaf of enveloping algebras of $\mathfrak{g}$ consisting of all sections of order 2 (with respect to the PBW filtration) that act trivially on the trivial family of Harish-Chandra modules [5, sec. 4.4.1]). As a sheaf of $O_{\mathbb{X}}$-modules, $\mathcal{C} \simeq O_{\mathbb{X}}(-2)$. The canonical Casimir section (a rational section) of $\mathcal{C}$ [5, sec. 4.4.2]) is given by

$$\Omega_0 = H_0^2 + 2H_0 + 4Y_0X_0. \tag{3}$$

This section satisfies $\mathrm{ord}_\infty(\Omega_0) = -2$ and $\mathrm{ord}_z(\Omega_0) = 0$ elsewhere on $\mathbb{X}$.

Lemma 6.1. *If $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})$ is quasi-simple then Ω_0 acts on $\mathcal{F}$ via multiplication by a regular function of the form $c(r) = c_2r^2 + c_1r + c_0 \in \mathbb{C}[r]$.*

Proof. Since Ω_0 is a regular section over $\mathbb{X}_0$ it must act via multiplication by some regular function $c(r) \in \mathbb{C}[r]$. Since $R^2\Omega_0$ is a regular section over $\mathbb{X}_\infty$, $R^2c(R^{-1}) \in \mathbb{C}[R]$ and hence, $c(r) = c_2r^2 + c_1r + c_0$, for some $c_2, c_1, c_0 \in \mathbb{C}$. ∎

Remark 6.2. In general, for the deformation family arising from a symmetric pair (G, K), the Casimir section generates a subsheaf that is isomorphic to $O_{\mathbb{X}}(-2)$, and Lemma 6.1 holds.

The isotypic subsheaves. As mentioned in Section 4, each $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})$, has an isotypic decomposition $\mathcal{F} = \bigoplus_{n \in \widehat{K}} \mathcal{F}_n$. Here we identify the algebraic dual of K with $\mathbb{Z}$, and we denote the parameter by n instead of μ. The sequence $\{\mathcal{F}_n\}_{n \in \mathbb{Z}}$ is an invariant of an equivalence class of an algebraic family of Harish-Chandra modules. If $\mathcal{F}$ is generically irreducible, then each non zero $\mathcal{F}_n$ is an invertible sheaf.

Generically irreducible families generated by a minimal K-type. For each $m \in \mathbb{Z}$, we denote by $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_m$, the set of equivalence classes of generically irreducible families in $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})$, that have m as a minimal K-type and are being generated by their isotypic subsheaf corresponding to m. For $SL_2(\mathbb{R})$ and its Cartan motion group, an irreducible Harish-Chandra module is completely determined by its K-types and the action of the center of the universal enveloping algebras. For generically irreducible families of Harish-Chandra modules the following analogous statement holds.

Theorem 6.3. *Let $m \in \mathbb{Z}$ and $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})_m$. Then, up to an isomorphism, $\mathcal{F}$ is completely determined by $\mathcal{F}_m$ and the action of the Casimir section.*

The proof is analogous to the one of Theorem 4.9.3 of [5]. The main change is in the action of the canonical Casimir section. In our case, it must act via multiplication by a polynomial of degree bounded by two, while in [5], it must act via multiplication by $c_1 r + c_0 + c_{-1} r^{-1} \in \mathbb{C}[r, r^{-1}]$. It should be noted that, $\mathcal{F}_m$ is isomorphic to $O_{\mathbb{X}}(l) \otimes_{\mathbb{C}} V_m$, for some $l \in \mathbb{Z}$, where V_m is an irreducible representation of K corresponding to m. Up to the action of the Picard group of $\mathbb{X}$, a family $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})_m$, is determined by the action of the Casimir section. We list all possibilities in Table 1.

	K-types	**Casimir** $c(r)$	**Parameters' domain**
$\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_0$	$2\mathbb{Z}$	$c_2 r^2 + c_1 r + c_0$	$c(r) \not\equiv k(k+2)$ for $0 \leq k \in \mathbb{Z}$ and even
	$-k, -k+2, \ldots, k$	$k(k+2)$	$0 \leq k \in \mathbb{Z}$ and even
$\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_1$	$2\mathbb{Z}+1$	$c_2 r^2 + c_1 r + c_0$	$c(r) \not\equiv k(k+2)$ for $-1 \leq k \in \mathbb{Z}$ odd
	$-k, -k+2, \ldots, k$	$k(k+2)$	$0 \leq k \in \mathbb{Z}$ and odd
	$1, 3, 5, \ldots$	-1	
$\mathcal{M}(\mathfrak{g}, \boldsymbol{K}))_{-1}$	$2\mathbb{Z}+1$	$c_2 r^2 + c_1 r + c_0$	$c(r) \not\equiv k(k+2)$ for $-1 \leq k \in \mathbb{Z}$ odd
	$-k, -k+2, \ldots, k$	$k(k+2)$	$0 \leq k \in \mathbb{Z}$ and odd
	$-1, -3, -5, \ldots$	-1	
$\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_d$	$d, d+2, d+4, \ldots$	$d(d-2)$	$1 < d \in \mathbb{Z}$
$\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_d$	$d, d-2, d-4, \ldots$	$d(d+2)$	$-1 > d \in \mathbb{Z}$

Table 1: K-types and Casimir action for $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})_m$

Concrete bases. Throughout this paragraph, let $\mathcal{F}$ be in $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_m$, with Casimir action that is given by $c(r)$ and set of K-types, $I \subset \mathbb{Z}$. As we already mentioned $\mathcal{F}$ decomposes with respect to the action of K as $\mathcal{F} = \bigoplus_{n\in I}\mathcal{F}_n$ and each of the nonzero isotypic subsheavs $\mathcal{F}_n$ is an invertible sheaf. For a nonzero $\mathcal{F}_n$, we can choose a rational section, f_n, that is regular, except perhaps at $r=\infty$, and nowhere vanishing, except perhaps at $r=\infty$. Moreover, $\mathrm{ord}_\infty(f_n) = \deg(\mathcal{F}_n)$. The section f_n is unique, up to a nonzero multiplicative factor. For all of these facts see [5, sec 4.6]. The commutation relations of the rational sections X_0, H_0, and Y_0, introduced in Subsection 6.1, imply that we can rescale the sections f_n, such that,

$$H_0 f_n = n f_n, \tag{4}$$

$$X_0 f_n = \begin{cases} f_{n+2}, & n+2\in I, m\le n \\ \frac{1}{4}\left(c(r) - n(n+2)\right) f_{n+2}, & n+2 \in I, m>n, \end{cases} \tag{5}$$

$$Y_0 f_n = \begin{cases} \frac{1}{4}\left(c(r) - n(n-2)\right) f_{n-2}, & n-2\in I, m< n \\ f_{n-2}, & n-2 \in I, m\ge n, \end{cases} \tag{6}$$

hold, as long as $n\pm 2 \in I$.

6.3. The classes $\widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_m$

Below we explicitly describe $\widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_m$. We shall start by proving that the generalized Harish-Chandra homomorphisms are regular. Then, for each $m \in \mathbb{Z} = \widehat{K}$, we shall specify a corresponding cuspidal parabolic $P_m = M_m A_m N_m$, the Θ-stable Cartan $\mathfrak{h}_m$, the discrete series representation $\delta_m \in \widehat{M_m}$, and the modules $I_{P_m}(\delta_m, 0)$ and $J_{P_m}(\delta_m, 0)$.

The generalized Harish-Chandra homomorphism. Recall that the center of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ is a polynomial algebra in the Casimir element. Similarly, for the family $\mathfrak{g}$, $\mathcal{Z}(\mathfrak{g})$ is generated, as a sheaf of $O_{\mathbb{X}}$-algebras, by the Casimir section. More precisely, the following lemma hold.

Lemma 6.4. *Let Ω be the Casimir of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then*
1. $\Gamma(\mathbb{X}_0, \mathcal{Z}(\mathfrak{g})) = \mathbb{C}[r]\otimes_{\mathbb{C}} \mathbb{C}[\Omega] = \mathbb{C}[r, \Omega_0]$,
2. $\Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g})) = \mathbb{C}[R, R^2\Omega_0]$.

Proof. By Lemma 3.2, $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{Z}(\mathfrak{g}_{\text{const}}) = O_{\mathbb{X}} \otimes_{\mathbb{C}} \mathcal{Z}(\mathfrak{g}) = O_{\mathbb{X}}\otimes_{\mathbb{C}} \mathbb{C}[\Omega]$. Hence, the first part of the lemma is clear. For the second part, observe that $\Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g}))$ is a submodule of the free $\mathbb{C}[R]$-module, $\mathbb{C}[R]\otimes_{\mathbb{C}}\mathbb{C}[\Omega]$. By a direct calculation $R^2\Omega_0 = R^2\otimes\Omega \in \Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g}))$ and hence $\Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g})) \supset \mathbb{C}[R, R^2\Omega_0]$. For the reversed inclusion, first observe that a section ξ of $\Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{g}_{\text{const}}))$ is in $\Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{g}))$ iff it can be written as

$$\xi(R) = \Sigma_{\ell=0}^{M} R^\ell \otimes \xi_\ell, \quad \text{with } \xi_\ell \in \mathcal{U}(\mathfrak{g})_\ell \mathcal{U}(\mathfrak{k}), \tag{7}$$

and where $\mathcal{U}(\mathfrak{g})_\ell$ is the ℓ-th piece of the PBW filtration. Such a decomposition of ξ is unique. Now assume that $\xi \in \Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g})) \subset \Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g}_{\text{const}})) = \mathbb{C}[R]\otimes_{\mathbb{C}}\mathbb{C}[\Omega]$. There are $f_0, f_1, ..., f_n \in \mathbb{C}[R]$ such that

$$\xi(R) = \Sigma_{i=0}^{n} f_i(R)\otimes \Omega^i.$$

Each f_i can be expanded as $f_i(R) = \Sigma_{j=0}^{m_i} f_{ij} R^j$ where $f_{ij} \in \mathbb{C}$. Let M be the maximum of $\{n, m_0, m_1, ..., m_n\}$. We can write

$$\xi(R) = \Sigma_{i=0}^{M} \Sigma_{j=0}^{M} f_{ij} R^j \otimes \Omega^i = \Sigma_{j=0}^{M} R^j \otimes \left(\Sigma_{i=0}^{M} f_{ij} \Omega^i \right).$$

Hence by the unique expansion for ξ in the form of (7), $\Sigma_{i=0}^{M} f_{ij} \Omega^i$ must be in $\mathcal{U}(\mathfrak{g})_j \mathcal{U}(\mathfrak{k})$. Since $\Omega^i \in \mathcal{U}(\mathfrak{g})_{2i} \mathcal{U}(\mathfrak{k}) \setminus \mathcal{U}(\mathfrak{g})_{2i-1} \mathcal{U}(\mathfrak{k})$ we must have $f_{ij} = 0$ for $j < 2i$. That is $f_i(R) = \Sigma_{j=2i}^{m_i} f_{ij} R^j$ and in particular each f_i can be written as $f_i(R) = g_i(R) R^{2i}$ for some $g_i(R) \in \mathbb{C}[R]$. All together we can write

$$\xi(R) = \Sigma_{i=0}^{n} g_i(R) R^{2i} \otimes \Omega^i = \Sigma_{i=0}^{n} g_i(R) (R^2 \Omega_0)^i \in \mathbb{C}[R, R^2 \Omega_0]. \qquad \blacksquare$$

Proposition 6.5. *For any Θ-stable Cartan, $\mathfrak{h}$, of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the generalized Harish-Chandra homomorphism maps $\mathcal{Z}(\mathfrak{g})$ into $\mathcal{U}(\mathfrak{h})$.*

Proof. By Lemma 6.4, it is enough to show that $\widetilde{\gamma}_{\mathfrak{h}}(\Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g})) \subset \Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{h}))$. Since $\Gamma(\mathbb{X}_\infty, \mathcal{Z}(\mathfrak{g})) = \mathbb{C}[R, R^2 \Omega_0]$, it is enough to show that

$$\widetilde{\gamma}_{\mathfrak{h}}(R^2 \otimes \Omega) = R^2 \otimes \gamma_{\mathfrak{h}}(\Omega) \in \Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{h})).$$

Since $\gamma_{\mathfrak{h}}(\Omega) \in \mathcal{U}(\mathfrak{g})_2$ we get $R^2 \otimes \gamma_{\mathfrak{h}}(\Omega) \in \Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{h}))$. ∎

The case of $|m| > 1$ (the discrete series case). For $m \in \mathbb{Z}$ with $|m| > 1$, a corresponding cuspidal parabolic and its Levi part can be taken to be $P_m = M_m = SL_2(\mathbb{R})$. The subgroups A_m and N_m are equal to the trivial group. The representation δ_m is the unique discrete series representation of $M_m = SL_2(\mathbb{R})$ having m as a minimal K-type. In this case, $I_{P_m}(\delta_m, 0) = J_{P_m}(\delta_m, 0) = \delta_m$. The Θ-stable Cartan, $\mathfrak{h}_m$, is $\mathfrak{so}(2, \mathbb{C}) = \mathfrak{k} = \text{span}\{H\}$, and $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ can be chosen to be $\mathbb{C}X$ and $\mathbb{C}Y$, respectively. We shall also use the notation $\mathfrak{h}_c$ (c for compact), for $\mathfrak{h}_m$. The corresponding $\rho_c \in \mathfrak{h}_c^*$ is given by $\rho_c(H) = 1$ and the generalized Harish-Chandra homomorphism is determined by

$$\gamma_c(\Omega_0) = (H_0 - 1)^2 + 2(H_0 - 1) = H_0^2 - 1 \tag{8}$$

In particular, $\gamma_c(R^2 \Omega_0) = R^2 \otimes (H_0^2 - 1) \in \Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{h}_m))$. Note that since $\mathfrak{h}_c = \mathfrak{k}$ we have $\boldsymbol{\mathfrak{h}}_c \simeq O_{\mathbb{X}}$. Therefore, any morphism of sheaves of commutative $O_{\mathbb{X}}$-algebras $\psi \colon \mathcal{U}(\boldsymbol{\mathfrak{h}}_c) \longrightarrow O_{\mathbb{X}}$ is completely determined by one complex number; $\psi(H_0) =: \alpha \in \mathbb{C}$. We shall denote this morphism by ψ_α.

Proposition 6.6. *For every $d \in \mathbb{Z}$ with $|d| > 1$, $\widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_d = \mathcal{M}(\mathfrak{g}, \boldsymbol{K})_d$.*

Proof. For any such d, up to the action of the Picard group of $\mathbb{X}$, there is exactly one equivalence class in $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_d$. For simplicity, we shall assume that $1 < d$, the other case is proven similarly. Let $\mathcal{F} \in \mathcal{M}(\mathfrak{g}, \boldsymbol{K})_d$, then the K-types of $\mathcal{F}$ are d, $d+2, \ldots$ and Ω_0 acts via multiplication by $d(d-2)$. Since $I_{P_d}(\delta_d, 0)$ is irreducible (discrete series), it is isomorphic to $\mathcal{F}|_0$. The only thing that is left to check is that $\mathcal{F}$ has an infinitesimal character with respect to $\mathfrak{h}_c$. That is, we ask whether there there is $\alpha \in \mathbb{C}$ such that,

$$\psi_\alpha(\gamma_c(\Omega_0)) = \chi(\Omega_0) \Longleftrightarrow \alpha^2 - 1 = d(d-2)$$

This obviously holds since we can take $\alpha = \pm(d-1)$. ∎

The case of $|m| \leq 1$. For $m \in \mathbb{Z}$ with $|m| \leq 1$, the cuspidal parabolic P_m can be taken to be the subgroup of $SL_2(\mathbb{R})$ consisting of upper triangular matrices. Its Langlands decomposition is given by $M_m = \{\pm\mathbb{I}\}$, A_m is the subgroup of diagonal matrices in $SL_2(\mathbb{R})$ with positive entries, and N_m the subgroup of P_m with all diagonal entries equal to 1. The representation δ_0 is the trivial representation and both δ_1 and δ_{-1} are the sign representation (the irreducible non-trivial representation of the two elements group). The standard representation $I_{P_0}(\delta_0, 0)$ is an irreducible spherical principal series representation (on which $\Omega_0|_{[1:0]}$ acts by multiplication by -1), and hence, $I_{P_0}(\delta_0, 0) = J_{P_0}(\delta_0, 0)$. The representations $I_{P_1}(\delta_1, 0)$ and $I_{P_{-1}}(\delta_{-1}, 0)$ coincide, and are equal to the nonspherical principal series representation which is the direct sum of the two limit discrete series. $J_{P_1}(\delta_1, 0)$ is the limit discrete series with minimal K-type 1 and $J_{P_{-1}}(\delta_{-1}, 0)$ is the limit discrete series with minimal K-type -1. The Θ-stable Cartan $\mathfrak{h}_m$ is the Lie algebra of diagonal matrices in $\mathfrak{sl}_2(\mathbb{C})$. We denote $\mathfrak{h}_m$ also by $\mathfrak{h}_s$ (s for split). Similarly, we shall use the notation $P_s = M_s A_s N_s$ for $P_m = M_m A_m N_m$, with $|m| \leq 1$. The subalgebras $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ can be taken to be the upper and lower triangular matrices (respectively) with zeros along the diagonal. Explicitly,

$$\mathfrak{h}_s = \mathbb{C}H^s, \ \mathfrak{n} = \mathbb{C}X^s, \ \overline{\mathfrak{n}} = \mathbb{C}Y^s,$$

where

$$H^s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X^s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ Y^s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The canonical Casimir section can be written as

$$\Omega_0 = 1 \otimes \left((H^s)^2 + 2H^s + 4Y^s X^s\right).$$

The corresponding $\rho_s \in \mathfrak{h}_s^*$ is given by $\rho_s(H^s) = 1$. The generalized Harish-Chandra homomorphism is determined by

$$\gamma_s(\Omega_0) = 1 \otimes ((H^s)^2 - 1).$$

In particular, $\gamma_s(R^2\Omega_0) = R^2 \otimes ((H^s)^2 - 1) \in \Gamma(\mathbb{X}_\infty, \mathcal{U}(\mathfrak{h}_m))$. Note that since $\mathfrak{h}_s \subset \mathfrak{p}$, $\boldsymbol{\mathfrak{h}}_s \simeq O_{\mathbb{X}}(-1)$. Hence, any morphism of sheaves of commutative $O_{\mathbb{X}}$-algebras $\psi : \mathcal{U}(\boldsymbol{\mathfrak{h}}_s) \longrightarrow O_{\mathbb{X}}$ is completely determined by two complex numbers α_0, α_1 defined by $\psi(H_0^s) = \alpha_1 r + \alpha_0$, where as before, $H_0^s := 1 \otimes H^s \in \Gamma(\mathbb{X}_0, \boldsymbol{\mathfrak{h}}_s)$. We shall denote this morphism by ψ_{α_0,α_1}.

Proposition 6.7. *$\widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_0$ consists of all $\mathcal{F}$ in $\mathcal{M}(\mathfrak{g}, \boldsymbol{K})_0$ that have $2\mathbb{Z}$ as their set of K-types and on which Ω_0 acts via multiplication by $c(r) = c_2 r^2 - 1$ with $c_2 \in \mathbb{C}$.*

Proof. Let $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_0$. Since $\mathcal{F}|_0 \simeq I_{P_0}(\delta_0, 0) = J_{P_0}(\delta_0, 0)$, the set of K-types of $\mathcal{F}$ must be $2\mathbb{Z}$, and $c(r)$ must be of the form $c_2 r^2 + c_1 r - 1$. Since $\mathcal{F}$ has an infinitesimal character with respect to $\boldsymbol{\mathfrak{h}}_s$, there must be $\alpha_0, \alpha_1 \in \mathbb{C}$ such that $\psi_{\alpha_0,\alpha_1}(\gamma_s(\Omega_0)) = \chi(\Omega_0)$. Hence,

$$(\alpha_1 r + \alpha_0)^2 - 1 = c_2 r^2 + c_1 r - 1 \iff \alpha_0 = 0, (\alpha_1)^2 = c_2, c_1 = 0.$$

Hence, $c(r) = c_2 r^2 - 1$. ∎

Proposition 6.8. (1) *$\widetilde{\mathcal{M}}(\mathfrak{g},\boldsymbol{K})_1$ consists of all $\mathcal{F}$ in $\mathcal{M}(\mathfrak{g},\boldsymbol{K})_1$, on which the Casimir acts via multiplication by $c(r)=c_2r^2-1$, with $c_2\in\mathbb{C}$, along with the following constrains on their K-types. If $c_2\neq 0$, the set of K-types is $2\mathbb{Z}+1$, and if $c_2=0$, then the set of K-types consists of all odd positive integers.*

(2) *$\widetilde{\mathcal{M}}(\mathfrak{g},\boldsymbol{K})_{-1}$ consists of all $\mathcal{F}$ in $\mathcal{M}(\mathfrak{g},\boldsymbol{K})_{-1}$, on which the Casimir acts by multiplication by $c(r)=c_2r^2-1$, with $c_2\in\mathbb{C}$, along with the following constrains on their K-types. If $c_2\neq 0$, the set of K-types is $2\mathbb{Z}+1$, and if $c_2=0$, then the set of K-types consists of all odd negative integers.*

Proof. Let $\mathcal{F}\in\widetilde{\mathcal{M}}(\mathfrak{g},\boldsymbol{K})_1$. Since the collection of composition factors of $\mathcal{F}|_0$ is contained in $\{J_{P_{-1}}(\delta_{-1},0),J_{P_1}(\delta_1,0)\}$, $c(0)=-1$ and the same calculation as in Proposition 6.7 shows that $c(r)=c_2r^2-1$. If $\mathcal{F}|_0\simeq J_{P_1}(\delta_1,0)$ then, as in the discrete series case, c_2 must be zero. If $\mathcal{F}|_0\not\simeq J_{P_1}(\delta_1,0)$ then the set of K-types of $\mathcal{F}|_0$ (which is also the set of K-types of $\mathcal{F}$) must be $2\mathbb{Z}+1$, since $I_{P_1}(\delta_1,0)=J_{P_{-1}}(\delta_{-1},0)\oplus J_{P_1}(\delta_1,0)$. From the explicit description of the action of $\mathfrak{g}$ (equations 4-6), we see that generic irreducibility forces c_2 to be nonzero. The second part of the proposition is proven similarly. ■

6.4. Examples of families with and without infinitesimal character

Example 6.9. Consider a family of Harish-Chandra modules $\mathcal{F}\in\mathcal{M}(\mathfrak{g},\boldsymbol{K})_0$. By Theorem 6.3 such a family is completely determined by two invariants; a (trivial) K-equivariant invertible sheaf $\mathcal{F}_0$, and a polynomial function $c(r)=c_2r^2+c_1r+c_0$, which determines the action of the Casimir section. In fact, for any choice of $c(r)=c_2r^2+c_1r+c_0$ and $\mathcal{F}_0$ there is a unique generically irreducible family $\mathcal{F}$ in $\mathcal{M}(\mathfrak{g},\boldsymbol{K})_0$ with those invariants. This follows from Section 4 of [5] and can be verified directly using (4)-(6) and (10)-(12). There are exactly two non-isomorphic Θ-stable Cartan subalgebras of $\mathfrak{sl}_2(\mathbb{C})$ and accordingly, exactly two Θ-stable Cartan subfamilies of $\mathfrak{g}$. Explicitly these are $\mathfrak{h}_c$ and $\mathfrak{h}_s$. Since $\mathfrak{h}_c\cong O_{\mathbb{X}}$ and $\mathfrak{h}_s\cong O_{\mathbb{X}}(-1)$, $\mathfrak{h}_c\not\cong\mathfrak{h}_s$. We note that $\mathcal{F}$ has an infinitesimal character with respect to $\mathfrak{h}_c$ iff $\psi_\alpha(\gamma_c(\Omega_0))=\chi(\Omega_0)$ for some $\alpha\in\mathbb{C}$. This holds iff

$$\alpha^2-1=c(r),$$

that is, iff $c_2=c_1=0$. On the other hand $\mathcal{F}$ has an infinitesimal character with respect to $\mathfrak{h}_s$, iff $\psi_{\alpha_0,\alpha_1}(\gamma_s(\Omega_0))=\chi(\Omega_0)$ for some $\alpha_0,\alpha_1\in\mathbb{C}$. This holds iff

$$(\alpha_1r+\alpha_0)^2-1=c(r),$$

that is, iff $c_2=\alpha_1^2,c_1=2\alpha_0\alpha_1,c_0=\alpha_0^2-1$ for some $\alpha_0,\alpha_1\in\mathbb{C}$. In particular we see that if $\mathcal{F}$ has an infinitesimal character with respect to $\mathfrak{h}_c$ it also has an infinitesimal character with respect to $\mathfrak{h}_s$. On the other, any family $\mathcal{F}$ that have an infinitesimal character with respect to $\mathfrak{h}_s$ with $(\alpha_1r+\alpha_0)^2-1=c(r)$ and $\alpha_0\alpha_1\neq 0$ does not have an infinitesimal character with respect to $\mathfrak{h}_c$. Finally, the family $\mathcal{F}$ with $c(r)=r^2+r+1$ has no infinitesimal character with respect to $\mathfrak{h}_s$ or $\mathfrak{h}_c$.

6.5. The admissible duals

In the next paragraphs we recall a classification of the admissible dual of $SL_2(\mathbb{R})$ and its Cartan motion group $SO(2)\ltimes\mathbb{R}^2$. The sections $H_\infty=1\otimes H$, $X_\infty=RX_0:=R\otimes X$

and $Y_\infty = RY_0 := R \otimes Y$ form a basis for the Lie algebra $\Gamma(\mathbb{X}_\infty, \mathfrak{g})$, over $\mathbb{C}[R]$. They satisfy the commutation relations

$$[H_\infty, X_\infty] = 2X_\infty, [H_\infty, Y_\infty] = -2Y_\infty, [X_\infty, Y_\infty] = R^2 H_\infty. \tag{9}$$

The section $$\Omega_\infty := R^2\Omega_0 = R^2(H_\infty^2 + 2H_\infty) + 4Y_\infty X_\infty,$$

is a nonvanishing regular section of the Casimir sheaf $\mathcal{C}$ over $\mathbb{X}_\infty$. In particular, for every $R \in \mathbb{C}$, the image of Ω_∞ in the fiber $\mathcal{U}(\mathfrak{g}|_R)$, $\Omega_\infty|_R$, is central. In fact, $\mathcal{Z}(\mathfrak{g}|_R)$ is equal to the polynomial algebra $\mathbb{C}[\Omega_\infty|_R]$. By essentially linear algebra, it can be shown that an irreducible admissible $(\mathfrak{g}|_R, K)$-module is completely determined by the action of $\Omega_\infty|_R$ (which is given by scalar multiplication) and a minimal K-type. Such calculations can be found, for example, in [22, ch.1], [11, sec. II. 1] and [19, ch. 8]. For $R \neq 0$, it is more common to look at the action of the canonical Casimir $\Omega_0|_R = R^{-2}\Omega_\infty|_R$, instead of the action of $\Omega_\infty|_R$.

	K-types	**Parameters** $(R^{-2}\Omega_\infty\|_R, m)_R$, $R \neq 0$	**Parameters' domain**
$\mathcal{M}(\mathfrak{g}\|_R, K)_0$	$2\mathbb{Z}$	$(\omega, 0)_R$	$\omega \neq k(k+2)$ for $0 \leq k \in \mathbb{Z}$ and even
	$-k, -k+2, \dots, k$	$(k(k+2), 0)_R$	$0 \leq k \in \mathbb{Z}$ and even
$\mathcal{M}(\mathfrak{g}\|_R, K)_1$	$2\mathbb{Z}+1$	$(\omega, 1)_R$	$\omega \neq k(k+2)$ for $-1 \leq k \in \mathbb{Z}$ odd
	$-k, -k+2, \dots, k$	$(k(k+2), 1)_R$	$0 \leq k \in \mathbb{Z}$ and odd
	$1, 3, 5, \dots$	$(-1, 1)_R$	
$\mathcal{M}(\mathfrak{g}\|_R, K)_{-1}$	$2\mathbb{Z}+1$	$(\omega, -1)_R$	$\omega \neq k(k+2)$ for $-1 \leq k \in \mathbb{Z}$ odd
	$-k, -k+2, \dots, k$	$(k(k+2), -1)_R$	$0 \leq k \in \mathbb{Z}$ and odd
	$-1, -3, -5, \dots$	$(-1, -1)_R$	
$\mathcal{M}(\mathfrak{g}\|_R, K)_d$	$d, d+2, d+4, \dots$	$(d(d-2), d)_R$	$1 < d \in \mathbb{Z}$
$\mathcal{M}(\mathfrak{g}\|_R, K)_d$	$d, d-2, d-4, \dots$	$(d(d+2), d)_R$	$-1 > d \in \mathbb{Z}$

Table 2: The admissible dual of $SL_2(\mathbb{R})$ by minimal K-types

We denote by $\mathcal{M}(\mathfrak{g}|_R, K)$ the space of equivalence classes of irreducible admissible $(\mathfrak{g}|_R, K)$-modules. For every $m \in \mathbb{Z}$, we denote by $\mathcal{M}(\mathfrak{g}|_R, K)_m$ the subspace of $\mathcal{M}(\mathfrak{g}|_R, K)$ consisting of all modules having m as a minimal K-type. We introduce parameters for $\mathcal{M}(\mathfrak{g}|_R, K)_m$ as follows. For $R \neq 0$, we shall denote by $(\omega, m)_R$ the (equivalence class of) irreducible admissible $(\mathfrak{g}|_R, K)$-modules on which $\Omega_0|_R = R^{-2}\Omega_\infty|_R$ acts via multiplication by ω, having minimal K-type m. For $R = 0$, we shall denote by $(\omega, m)_0$ the (equivalence class of) irreducible admissible $(\mathfrak{g}|_0, K)$-module on which $\Omega_\infty|_0$ acts via multiplication by ω, which has minimal K-type m.

We list all possibilities for the case of $R \neq 0$ corresponding to $SL_2(\mathbb{R})$ in Table 2, and the case of $R = 0$ corresponding to the Cartan motion group $SO(2) \ltimes \mathbb{R}^2$ in Table 3. Note that the same $(\mathfrak{g}|_R, K)$-module can correspond to two different parameters. Two different parameters $(\alpha, m)_R$, $(\beta, m')_R$ correspond to isomorphic

$(\mathfrak{g}|_R, K)$-modules if and only if $\{m, m'\} = \{1, -1\}$ and $\alpha = \beta$. These are the only equivalences.

For any $R \in \mathbb{C}$ and any $m \in \mathbb{Z}$, we can use the parameter ω in $(\omega, m)_R$ to identify $\mathcal{M}(\mathfrak{g}|_R, K)_m$ with a subset of $\mathbb{C}$. Explicitly,

$$\mathcal{M}(\mathfrak{g}|_R, K)_m \simeq \begin{cases} \mathbb{C}, & |m| \le 1 \\ \{\text{point}\}, & |m| > 1. \end{cases}$$

Using this identification, we can equip each $\mathcal{M}(\mathfrak{g}|_R, K)_m$ with the structure of an affine algebraic variety (see Remark 5.5 above). It should be stressed that $\mathcal{M}(\mathfrak{g}|_R, K)_m$ has it own intrinsic structure as an affine algebraic variety and this structure coincides with the one we just obtained. We shall not prove this here.

	K-types	**Parameters** $(\Omega_\infty|_0, m)_0$	**Parameters' domain**
$\mathcal{M}(\mathfrak{g}|_0, K)_0$	$2\mathbb{Z}$	$(c, 0)_0$	$c \neq 0$
	0	$(0, 0)_0$	
$\mathcal{M}(\mathfrak{g}|_0, K)_1$	$2\mathbb{Z}+1$	$(c, 1)_0$	$c \neq 0$
	1	$(0, 1)_0$	
$\mathcal{M}(\mathfrak{g}|_0, K)_{-1}$	$2\mathbb{Z}+1$	$(c, -1)_0$	$c \neq 0$
	-1	$(0, -1)_0$	
$\mathcal{M}(\mathfrak{g}|_0, K)_d$	d	$(0, d)_0$	$1 < d \in \mathbb{Z}$
$\mathcal{M}(\mathfrak{g}|_0, K)_d$	d	$(0, d)_0$	$-1 > d \in \mathbb{Z}$

Table 3: The admissible dual of $SO(2) \ltimes \mathbb{R}^2$ by minimal K-types

6.6. The Jantzen quotients

In this section, following the calculations given in [6, sec. 5.2]), we list all the Jantzen quotients for $\mathcal{F}$ in the various $\widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_m$. Let $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_m$ be a family on which Ω_0 acts via multiplication by $c(r)$. For any K-type n of $\mathcal{F}$, we can find a section e_n, such that, $\Gamma(\mathbb{X}_\infty, \mathcal{F}_n) = \mathbb{C}[R]e_n$. And moreover, the action of $\Gamma(\mathbb{X}_\infty, \mathfrak{g})$ on $\Gamma(\mathbb{X}_\infty, \mathcal{F})$ is given by,

$$H_\infty e_n = n e_n, \tag{10}$$

$$X_\infty e_n = \begin{cases} e_{n+2}, & n+2 \in I, m \le n \\ \frac{R^2}{4}\left(c(R^{-1}) - n(n+2)\right) e_{n+2}, & n+2 \in I, m > n, \end{cases} \tag{11}$$

$$Y_\infty e_n = \begin{cases} \frac{R^2}{4}\left(c(R^{-1}) - n(n-2)\right) e_{n-2}, & n-2 \in I, m < n \\ e_{n-2}, & n-2 \in I, m \ge n. \end{cases} \tag{12}$$

From these equations and the concrete expression for $c(r)$ (which is either $c_2 r^2 - 1$ or $m(m \pm 2)$ according to the value of m), we see that $\mathcal{F}|_R$ is reducible for some $R \in \mathbb{R}$, if and only if $c(r)$ takes only real values on $\mathbb{R}$. From these formulas, doing the same calculations as in [6, sec. 5.2]), we obtain the following.

Theorem 6.10. *Let $m \in \mathbb{Z}$, and $\mathcal{F} \in \widetilde{\mathcal{M}}(\mathfrak{g}, \boldsymbol{K})_m$ such that Ω_0 acts on $\mathcal{F}$ via multiplication by $c(r) = c_2 r^2 + c_0$. For any $[\alpha : \beta] \in \mathbb{X}_\infty$, $R := \frac{\alpha}{\beta}$, we have*

$$\widetilde{J}_{m,[\alpha:\beta]}(\mathcal{F}) \simeq \begin{cases} (\frac{c_2}{R^2} + c_0, m)_R & R \neq 0 \\ (c_2, m)_0 & R = 0. \end{cases}$$

Note that in particular the Jantzen quotients are irreducible and as a result we obtain the following corollary.

Corollary 6.11. *Conjecture* 5.6 *holds for $SL_2(\mathbb{R})$.*

6.7. The Mackey-Higson-bijection for $SL_2(\mathbb{R})$

Theorem 6.12. *Conjecture* 5.4 *holds for $SL_2(\mathbb{R})$.*

Proof. Since $\boldsymbol{K}$ is a constant family and since the Casimir of $SL_2(\mathbb{R})$ is canonically defined, for any nonzero $R_1, R_2 \in \mathbb{X}_\infty$ and any $m \in \mathbb{Z}$, there is a canonical identification $\mathcal{M}(\mathfrak{g}|_{R_1}, \boldsymbol{K})_m \simeq \mathcal{M}(\mathfrak{g}|_{R_2}, \boldsymbol{K})_m$. For each $0 \neq R \in \mathbb{X}_\infty$ and $m \in \mathbb{Z}$, the correspondence arising from Theorem 6.10 gives rise to a well-defined isomorphism of affine algebraic varieties. We denote it by $\eta_m^R : \mathcal{M}(\mathfrak{g}|_0, K)_m \longrightarrow \mathcal{M}(\mathfrak{g}|_R, \boldsymbol{K})_m$, and it is given by $\eta_m^R((z,m)_0) = (\frac{z}{R^2} + c_0, m)_R$. We define a map η^R from $\mathcal{M}(\mathfrak{g}|_0, K)$ into $\mathcal{M}(\mathfrak{g}|_R, \boldsymbol{K})$ by sending $(z,m)_0$ to $\eta_m^R((z,m)_0) = (\frac{z}{R^2} + c_0, m)_R$. It is well-defined since $(z,m)_0 \simeq (z',m')_0$ if and only if either $z = z'$ and $m = m'$, or $z = z'$ and $\{m, m'\} = \{-1, 1\}$. In both cases $\eta_m^R((z,m)_0) \simeq \eta_{m'}^R((z',m')_0)$. Since

$$\eta_m^R((0,m)_0) = \begin{cases} (-1,0)_R, & m = 0 \\ (-1,1)_R, & m = 1 \\ (-1,-1)_R, & m = -1 \\ (m(m-2), m)_R, & m > 1 \\ (m(m+2), m)_R, & m < -1, \end{cases}$$

it extends Vogan's bijection. Since the tempered dual of $(\mathfrak{g}|_0, K)$ is given by

$$\{(0,m)_0 | m \in \mathbb{Z}\} \cup \{(c, \pm 1)_0 | c < 0\} \cup \{(c,0)_0 | c < 0\},$$

and the tempered dual of $(\mathfrak{g}|_R, K)$ is given by

$$\{(d(d-2), d)_R | 0 < d \in \mathbb{Z}\} \cup \{(d(d+2), d)_R | 0 > d \in \mathbb{Z}\}$$
$$\cup \{(\omega, \pm 1)_0 | \omega < -1\} \cup \{(\omega, 0)_0 | \omega \leq -1\},$$

η^R takes tempered to tempered. ∎

Theorem 6.13. *Any bijection between $\widehat{SL_2(\mathbb{R})}^{adms}$ and $\widehat{SO(2) \ltimes \mathbb{R}^2}^{adms}$ satisfying the three conditions in Conjecture* 5.4, *arises form one of the correspondences $J_{\mu,[\alpha:\beta]}(\mathcal{F}) \longleftrightarrow J_{\mu,[0:1]}(\mathcal{F})$, for some $[\alpha : \beta] \in \mathbb{RP}^1$ with $\alpha\beta \neq 0$.*

Proof. Let Ψ be a bijection from $\mathcal{M}(\mathfrak{g}|_0, K)$ onto $\mathcal{M}(\mathfrak{g}|_1, \boldsymbol{K})$ (the choice of $R = 1$ is not important) satisfying the three conditions in Conjecture 5.4. For any $d \in \mathbb{Z}$ with $|d| > 1$, $\mathcal{M}(\mathfrak{g}|_0, K)_d$ and $\mathcal{M}(\mathfrak{g}|_1, \boldsymbol{K})_d$ consist of exactly one equivalence class and Ψ must agree with any of the η^R when restricted to $\mathcal{M}(\mathfrak{g}|_0, K)_d$. Now, since Ψ restricts to an algebraic isomorphism from $\mathcal{M}(\mathfrak{g}|_0, \boldsymbol{K})_0$ onto $\mathcal{M}(\mathfrak{g}|_1, \boldsymbol{K})_0$, it must be of the form $\Psi((z,0)_0) = (\alpha z + \beta, 0)_1$, for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. Since Ψ extends Vogan's bijection, $\Psi((0,0)_0) = (-1,0)_1$ and hence $\beta = -1$. Since Ψ takes tempered modules to tempered modules, it must map $\{(z,0)_0 | z \in (-\infty, 0]\}$ into $\{(z,0)_1 | z \in (-\infty, -1]\}$. Hence, $\alpha > 0$ and Ψ coincides with $\eta^{\frac{1}{\sqrt{\alpha}}}$ on $\mathcal{M}(\mathfrak{g}|_0, \boldsymbol{K})_0$. Similarly, it also coincides with it on $\mathcal{M}(\mathfrak{g}|_0, \boldsymbol{K})_{\pm 1}$. ■

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Received September 26, 2017
and in final form February 13, 2019