

Reduced and Nonreduced Presentations of Weyl Group Elements*

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Abstract. This paper is a sequel to work of Dynkin on subroot lattices of root lattices and to work of Carter on presentations of Weyl group elements as products of reflections.

The quotients L/L_1 are calculated for all irreducible root lattices L and all subroot lattices L_1 . The reduced (i.e. those with minimal number of reflections) presentations of Weyl group elements as products of arbitrary reflections are classified. Also nonreduced presentations are studied. Quasi-Coxeter elements and strict quasi-Coxeter elements are defined and classified. An application to extended affine root lattices is given. A side result is that any set of roots which generates the root lattice contains a \mathbb{Z} -basis of the root lattice.

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Key Words: Root system, subroot lattice, reduced presentation, quasi-Coxeter element, extended affine root system.

1. Introduction

This paper is a sequel to work of Dynkin [7, §5] on subroot lattices of root lattices and to work of Carter [6] on presentations of Weyl group elements as products of arbitrary reflections (that means, at all possible roots, not only at roots of a fixed root basis as in [8]).

A *root lattice* is a \mathbb{Z} -lattice L together with a scalar product $(\cdot, \cdot) : L_{\mathbb{R}} \times L_{\mathbb{R}} \rightarrow \mathbb{R}$ on the underlying real vector space $L_{\mathbb{R}}$ and a finite set $\Phi \subset L - \{0\}$ of *roots* such that Φ is a generating set of L as a \mathbb{Z} -lattice, the reflection $s_{\alpha} : L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}$ at a root α maps Φ to itself, $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for any two roots α and β , and $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for any root α (Definition 2.2). Then the group $W := \langle s_{\alpha} \mid \alpha \in \Phi \rangle \subset O(L, (\cdot, \cdot))$ is the Weyl group of the root lattice. The set Φ of roots is the *root system*.

Root lattices turn up in the theory of semisimple Lie algebras. A standard reference is [4, ch. VI]. The irreducible root lattices are classified and form the series A_n, B_n, C_n, D_n and the exceptional cases E_6, E_7, E_8, F_4 and G_2 . Section 2 recalls their classification, standard models and Dynkin diagrams. Because the condition $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for any root α is unnatural in the context of a generalization of root lattices, we will consider the slightly more general notion without this condition and call it *p.n. root lattice*. Here *p.n.* stands for *possibly nonreduced*. The classification

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of the irreducible p.n. root lattices contains besides the irreducible root lattices only the series BC_n .

Dynkin [7, §5] classified all isomorphism classes of pairs (L, L_1) where L is an irreducible root lattice and L_1 is a subroot lattice (Definition 3.1). Crucial is an algorithm which was discovered before by Borel and de Siebenthal [3] and which allows to construct by a sequence of two types of steps subroot lattices of a root lattice, by adding roots to and dropping roots from a given set of roots. We call the steps (BDdS1) and (BDdS2). In Theorem 3.3 we recall Dynkin's classification and give the easy extension to the series BC_n . In Theorem 3.8 we carry out the algorithm explicitly for the exceptional cases and construct thus for these cases for any isomorphism class of pairs (L, L_1) a representative. This allows to calculate also the quotient groups L/L_1 and to show the following. Define the numbers

$$k_1(L, L_1) := \min\{k \mid \text{the group } L/L_1 \text{ has } k \text{ generators}\}, \quad (1)$$

$$k_2(L, L_1) := \min\{k \mid \exists \alpha_1, \dots, \alpha_k \in \Phi \text{ s.t. } L = L_1 + \sum_{i=1}^k \mathbb{Z} \cdot \alpha_i\}, \quad (2)$$

$$k_3(L, L_1) := \min \left\{ k \mid \begin{array}{l} L_1 \text{ can be constructed with } k \text{ of} \\ \text{the steps (BDdS1) and (BDdS2)} \end{array} \right\}. \quad (3)$$

Theorem 1.1. (Part of Theorem 3.8)

$$k_1(L, L_1) = k_2(L, L_1) = k_3(L, L_1). \quad (4)$$

Here the inequalities $k_1(L, L_1) \leq k_2(L, L_1) \leq k_3(L, L_1)$ are quite obvious. The inverse inequalities require the knowledge of the groups L/L_1 and an explicit execution of the algorithm with the steps (BDdS1) and (BDdS2). The Tables 3.1–3.6 contain Dynkin's classification, the groups L/L_1 and the numbers $k_1(L, L_1)$. For the series, the execution of the algorithm is less important than for the exceptional cases, as for the series one can associate graphs $\mathcal{G}(A)$ to sets $A \subset \Phi$ of roots which are helpful for understanding such sets and the subroot lattices which they generate (Definition 3.5 and Lemma 3.6).

Section 4 gives a proof of the following basic fact which seems to have been unnoticed up to now and which may be of some independent interest.

Theorem 1.2. (Theorem 4.1) *Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice. Let $A \subset \Phi$ be any set of roots which generates the lattice L as a \mathbb{Z} -module. Then A contains a \mathbb{Z} -basis of L .*

The proof for the series is easy, it uses the graphs $\mathcal{G}(A)$ (Lemma 4.2). The proof for the exceptional cases is a case discussion. Thanks to the results in section 3, it can be reduced to a discussion of just a few cases, but they require some detailed work (Lemma 4.3 and Lemma 4.4). Theorem 1.2 is crucial for the proof of Theorem 6.2. By definition, any Weyl group element w can be written as a product of reflections at roots,

$$w = s_{\alpha_1} \circ \dots \circ s_{\alpha_k}, \quad \alpha_1, \dots, \alpha_k \in \Phi.$$

Then the tuple $(\alpha_1, \dots, \alpha_k)$ is called a *presentation* of w , k is its *length*, and the

subroot lattice $L_1 := \sum_{i=1}^k \mathbb{Z}\alpha_i \subset L$ is called the subroot lattice of this presentation (Definition 5.1). The *length* $l(w)$ is the minimum of the lengths of all presentations of w . A presentation is *reduced* if its length is $l(w)$. Carter [6, Lemma 2 and Lemma 3] proved the following.

Lemma 1.3. (Part of Lemma 5.2) *A presentation $(\alpha_1, \dots, \alpha_k)$ of a Weyl group element w is reduced if and only if $\alpha_1, \dots, \alpha_k$ are linearly independent. And then the subroot lattice L_1 of the presentation satisfies*

$$L_{1,\mathbb{C}} := \bigoplus_{j=1}^{l(w)} \mathbb{C}\alpha_j = \bigoplus_{\lambda \neq 1} \ker(w - \lambda \text{id} : L_{\mathbb{C}} \rightarrow L_{\mathbb{C}}). \quad (5)$$

Definition 1.4. (Definition 5.3) A Weyl group element w is a *quasi-Coxeter element* if it has a reduced presentation whose subroot lattice is the full lattice L . It is a *strict quasi-Coxeter element* if the subroot lattice of any reduced presentation is the full lattice L .

In the homogeneous cases, quasi-Coxeter elements and strict quasi-Coxeter elements agree. In these cases, the definition of quasi-Coxeter elements is due to Voigt [11, Def. 3.2.1]. In the inhomogeneous cases, Definition 1.4 is new. Theorem 5.6 gives the classification of the quasi-Coxeter elements and the strict quasi-Coxeter elements for all irreducible p.n. root lattices. In the homogeneous cases this is an easy consequence of the results of Carter [6]. But in the inhomogeneous cases and especially in the case F_4 , there is some additional work to do (Lemma 5.7).

Any Weyl group element can be written as a product of strict quasi-Coxeter elements for a suitable orthogonal sum of irreducible subroot lattices. The Tables 7-11 in [6] give for any conjugacy class only one subroot lattice L_1 and only one presentation as a strict quasi-Coxeter element for this subroot lattice.

Different presentations of one Weyl group element w may have different subroot lattices L_1 and L'_1 . Only $L_{1,\mathbb{Q}} = L'_{1,\mathbb{Q}}$ is clear, due to Lemma 1.3. Theorem 5.10 complements [6] and gives for the exceptional cases all subroot lattices of reduced presentations and all presentations as quasi-Coxeter elements for these subroot lattices. Here the case F_4 is more difficult than the cases E_6, E_7 and E_8 .

The Theorems 5.10, 5.6 and 3.1 together allow to recover the complete classification of all conjugacy classes of Weyl group elements for the irreducible p.n. root lattices in [6] and provide additional information.

Section 6 studies nonreduced presentations of Weyl group elements. Define for a Weyl group element w the numbers

$$k_4(L, w) := \min \left\{ k_2(L, L_1) \mid \begin{array}{l} \text{a reduced presentation of } w \\ \text{with subroot lattice } L_1 \text{ exists} \end{array} \right\}, \quad (6)$$

$$k_5(L, w) := \min \left\{ k \mid \begin{array}{l} \text{a presentation } (\alpha_1, \dots, \alpha_{l(w)+2k}) \text{ with} \\ \text{subroot lattice the full lattice exists} \end{array} \right\}. \quad (7)$$

It is easy to see $k_5(L, w) \leq k_4(L, w)$.

Theorem 1.5. (Theorem 6.2) $k_5(L, w) = k_4(L, w)$. (8)

The proof builds on Theorem 4.1 and some additional arguments especially for the cases C_n and F_4 .

Theorem 6.2 has an application to *extended affine root lattices* in section 7. They had been defined by K. Saito [10, (1.2) and (1.3)], see also [1][2] and Definition 7.1. One simply replaces in the definition of a p.n. root lattice the scalar product by a positive semidefinite bilinear form $(\cdot, \cdot) : L_{\mathbb{Q}} \times L_{\mathbb{Q}} \rightarrow \mathbb{Q}$ and drops the condition that Φ is finite. Then the quotient $L/\text{Rad}(L)$ becomes in a natural way a p.n. root lattice. Any element $w \in W(L)$ induces an element $\bar{w} \in W(L/\text{Rad}(L))$. Presentations and quasi-Coxeter elements in an extended affine root lattice are defined as in a p.n. root lattice. The simple Lemma 7.4 gives for a quasi-Coxeter element $w \in W(L)$ the inequalities

$$l(\bar{w}) \leq \text{rank } L - \text{rank Rad}(L), \tag{9}$$

$$l(\bar{w}) + 2k_5(L/\text{Rad}(L), \bar{w}) \leq \text{rank } L. \tag{10}$$

(10) gives a constraint on the elements \bar{w} which are induced by quasi-Coxeter elements. Theorem 6.2 says $k_5(L/\text{Rad}(L), \bar{w}) = k_4(L/\text{Rad}(L), \bar{w})$, and Theorem 5.10 allows to calculate this number.

2. Basic facts on (possibly nonreduced) root lattices

This section recalls some basic facts on root systems. A standard reference is [4, ch. VI]. Though we follow the more recent notation and call *root systems* what is called there *reduced root systems*. We call *p.n. root systems* (*p.n.* for *possibly nonreduced*) what is called there *root systems*. We include the p.n. root lattices because the condition (18) below, which distinguishes root systems, is not necessarily preserved if one goes from an *extended affine root lattice* (see section 7) to a quotient lattice.

Notation 2.1. A free \mathbb{Z} -module L of rank $n \in \mathbb{Z}_{>0}$ is called a *lattice*. Then $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$, $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ and $L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$. Let L be a lattice and (\cdot, \cdot) be a scalar product on $L_{\mathbb{R}}$. For $\alpha \in L - \{0\}$ and $\beta \in L$ define

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}. \tag{11}$$

Then $s_{\alpha} : L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}, \quad s_{\alpha}(x) := x - \langle x, \alpha \rangle \cdot \alpha \tag{12}$

is a reflection. Two reflections s_{α} and s_{β} satisfy

$$s_{\alpha}s_{\beta} = s_{\beta}s_{s_{\beta}(\alpha)} = s_{s_{\alpha}(\beta)}s_{\alpha}. \tag{13}$$

Definition 2.2. (a) A *p.n. root lattice* is a triple $(L, (\cdot, \cdot), \Phi)$ where L is a lattice, $(\cdot, \cdot) : L_{\mathbb{R}} \times L_{\mathbb{R}} \rightarrow \mathbb{R}$ is a scalar product, and $\Phi \subset L - \{0\}$ is a finite set such that the following properties hold.

(i) Φ is a generating set of L as a \mathbb{Z} -module. (14)

(ii) For any $\alpha \in \Phi$ $s_{\alpha}(\Phi) = \Phi$. (15)

(iii) $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for any $\alpha, \beta \in \Phi$. (16)

The elements of Φ are the *roots*, and Φ is a *p.n. root system*. The finite group

$$W := \langle s_\alpha \mid \alpha \in \Phi \rangle \subset O(L, (\cdot, \cdot)) \tag{17}$$

is the *Weyl group*.

- (b) A *root lattice* is a p.n. root lattice $(L, (\cdot, \cdot), \Phi)$ which satisfies additionally the condition:

$$\text{If } \alpha \in \Phi, \text{ then } \Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}. \tag{18}$$

Then Φ is a *root system*.

- (c) (Lemma) The orthogonal sum of several (p.n.) root lattices is (in a most natural way) a (p.n.) root lattice.
- (d) A (p.n.) root lattice is *irreducible* if it is not isomorphic to the orthogonal sum of several (p.n.) root lattices.

The classification of p.n. root lattices and of root lattices is as follows. Again, a standard reference is [4, ch. VI].

Theorem 2.3. (a) *Any (p.n.) root lattice is either irreducible or isomorphic to an orthogonal sum of several irreducible (p.n.) root lattices.*

- (b) *If $(L, (\cdot, \cdot), \Phi)$ is an irreducible (p.n.) root lattice then also $(L, c \cdot (\cdot, \cdot), \Phi)$ for any $c \in \mathbb{R}_{>0}$ is an irreducible (p.n.) root lattice. Two irreducible (p.n.) root lattices are of the same type if they differ up to isomorphism only by such a scalar c .*
- (c) *The types of irreducible p.n. root lattices are given by 5 series and 5 exceptional ones with the following names,*

$$\begin{aligned} &A_n \ (n \geq 1), \ B_n \ (n \geq 2), \ C_n \ (n \geq 3), \ BC_n \ (n \geq 1), \\ &D_n \ (n \geq 4), \ E_6, \ E_7, \ E_8, \ F_4, \ G_2. \end{aligned} \tag{19}$$

All except BC_n are root lattices.

- (d) *The following list presents one irreducible p.n. root lattice of each type. Always $L_{\mathbb{R}} \subset \mathbb{R}^m$ for some $m \in \{n, n+1, n+2\}$. Here (\cdot, \cdot) is the restriction to $L_{\mathbb{R}}$ of the standard scalar product on \mathbb{R}^m , and e_1, \dots, e_m is the standard ON-basis of \mathbb{R}^m .*

$$\mathbf{A}_n : m = n+1, \ \Phi = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}. \tag{20}$$

$$\mathbf{B}_n : m = n, \ \Phi = \{\pm e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \tag{21}$$

$$\begin{aligned} \mathbf{C}_n : m = n, \ \Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \\ \cup \{\pm 2e_i \mid 1 \leq i \leq n\}. \end{aligned} \tag{22}$$

$$\begin{aligned} \mathbf{BC}_n : m = n, \ \Phi = \{\pm e_i \mid 1 \leq i \leq n\} \\ \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}. \end{aligned} \tag{23}$$

$$\mathbf{D}_n : m = n, \ \Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}. \tag{24}$$

$$\begin{aligned} \mathbf{E}_6 : m = 8, \quad \Phi &= \{\pm e_i \pm e_j \mid 3 \leq i < j \leq 7\}. & (25) \\ &\cup \{\frac{1}{2} \sum_{i=1}^8 \varepsilon_i e_i \mid \varepsilon_i = \pm 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_8, \prod_{i=1}^8 \varepsilon_i = 1\}. \end{aligned}$$

$$\begin{aligned} \mathbf{E}_7 : m = 8, \quad \Phi &= \{\pm e_i \pm e_j \mid 2 \leq i < j \leq 7\} \cup \{\pm(e_1 + e_8)\} & (26) \\ &\cup \{\frac{1}{2} \sum_{i=1}^8 \varepsilon_i e_i \mid \varepsilon_i = \pm 1, \varepsilon_1 = \varepsilon_8, \prod_{i=1}^8 \varepsilon_i = 1\}. \end{aligned}$$

$$\begin{aligned} \mathbf{E}_8 : m = 8, \quad \Phi &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 8\} & (27) \\ &\cup \{\frac{1}{2} \sum_{i=1}^8 \varepsilon_i e_i \mid \varepsilon_i = \pm 1, \prod_{i=1}^8 \varepsilon_i = 1\}. \end{aligned}$$

$$\begin{aligned} \mathbf{F}_4 : m = 4, \quad \Phi &= \{\pm e_i \mid 1 \leq i \leq 4\} & (28) \\ &\cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}. \end{aligned}$$

$$\begin{aligned} \mathbf{G}_2 : m = 3, \quad \Phi &= \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\} & (29) \\ &\cup \{\pm(2e_{\pi(1)} - e_{\pi(2)} - e_{\pi(3)}) \mid \pi \in S_3\}. \end{aligned}$$

Remark 2.4. (i) The p.n. root lattices above have roots of the following lengths,

$$\begin{array}{c|c|c|c|c|c|c|c} A_n & D_n & E_n & B_n & F_4 & C_n & G_2 & BC_n \\ \hline \sqrt{2} & \sqrt{2} & \sqrt{2} & 1, \sqrt{2} & 1, \sqrt{2} & \sqrt{2}, 2 & \sqrt{2}, \sqrt{6} & 1, \sqrt{2}, 2 \end{array}$$

The root lattices of types A_n, D_n, E_n have only roots of one length and are therefore called *homogeneous*. The root lattices of types B_n, C_n, F_4 and G_2 have roots of two lengths, *short* and *long* roots. The p.n. root lattices BC_n have roots of three lengths, short, long and *extra long* roots.

(ii) In the Tables 3.1–3.4 the symbols A_n, \dots, G_2 will denote root lattices with roots of lengths as above. There we will also consider a few root systems with other lengths, and a few other names for some of the root lattices above:

- $A_0 = B_0 = BC_0 = \{0\}$ denotes the rank 0 lattice.
- $D_2 := 2A_1 := A_1 \perp A_1, \quad D_3 := A_3.$
- $\tilde{A}_1 = B_1$ denotes a root lattice of type A_1 with roots of length 1.
- \tilde{A}_2 denotes a root lattice of type A_2 with roots of length 1.
- C_1 denote a root lattice of type A_1 with roots of length 2.
- C_2 denotes a root lattice of type B_2 with roots of lengths $\sqrt{2}$ and 2.

In Table 3.5 the roots in the root systems of type C_3 have lengths 1 and $\sqrt{2}$. In Table 3.6, roots in A_2 and A_1 have length $\sqrt{6}$, roots in \tilde{A}_1 have length $\sqrt{2}$.

(iii) The Weyl group $W(A_n)$ of the root lattice above of type A_n acts on the basis e_1, \dots, e_{n+1} of $\mathbb{R}^{n+1} \supset L_{\mathbb{R}}$ by permutations, $W(A_n) \cong S_{n+1}$, and $\sigma \in S_{n+1}$ maps e_i to $e_{\sigma(i)}$.

The Weyl groups of the p.n. root lattices above of the types B_n, C_n and BC_n coincide and act on the basis e_1, \dots, e_n of $\mathbb{R}^n = L_{\mathbb{R}}$ by *signed permutations*,

$$W(B_n) = W(C_n) = W(BC_n) \cong \{\pm 1\}^n \rtimes S_n,$$

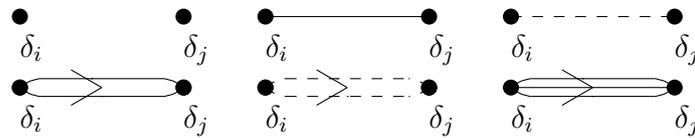
and $(\varepsilon_1, \dots, \varepsilon_n, \sigma) \in \{\pm 1\}^n \rtimes S_n$ maps e_i to $\varepsilon_i e_{\sigma(i)}$.

The Weyl group of the root lattice above of type D_n is the subgroup of index 2 given by the condition $\prod_{i=1}^n \varepsilon_i = 1$.

(iv) Let $(L, (\cdot, \cdot), \Phi)$ be an irreducible root lattice. To any subset $A = \{\delta_1, \dots, \delta_l\} \subset \Phi$ with $A \cap (-A) = \emptyset$ we associate a *generalized Dynkin diagram* as follows. It is a graph with l vertices, labelled $\delta_1, \dots, \delta_l$. Between vertices δ_i and δ_j with $i \neq j$ there is no edge or an edge with additional information as follows.

- no edge if $(\delta_i, \delta_j) = 0$,
- a normal edge if $\|\delta_i\| = \|\delta_j\|$ and $\langle \delta_i, \delta_j \rangle = -1$,
- a dotted edge if $\|\delta_i\| = \|\delta_j\|$ and $\langle \delta_i, \delta_j \rangle = 1$,
- a double arrow from δ_i to δ_j if $\|\delta_i\| = \sqrt{2}\|\delta_j\|$ and $\langle \delta_i, \delta_j \rangle = -2$,
- a double dotted arrow from δ_i to δ_j if $\|\delta_i\| = \sqrt{2}\|\delta_j\|$ and $\langle \delta_i, \delta_j \rangle = 2$,
- a triple arrow from δ_i to δ_j if $\|\delta_i\| = \sqrt{3}\|\delta_j\|$ and $\langle \delta_i, \delta_j \rangle = -3$.

The corresponding pictures are depicted below.



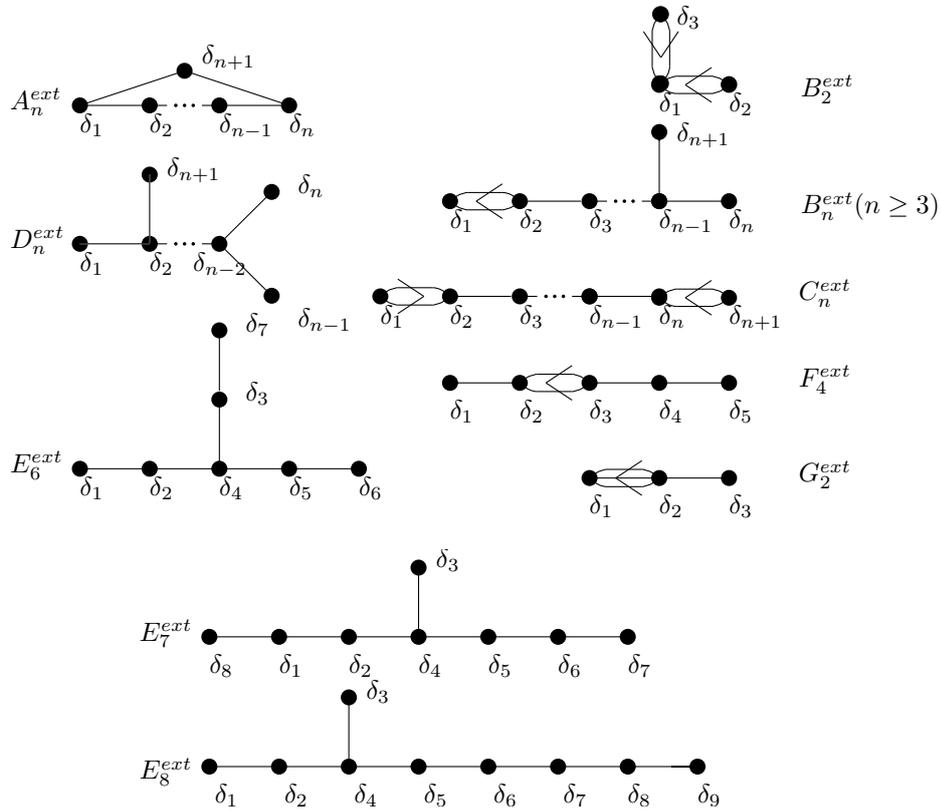
Other cases will not be considered. If A is a \mathbb{Z} -basis of L and the diagram is connected, then the diagram encodes up to a common scalar the intersection numbers (δ_i, δ_j) , and thus it determines the irreducible root system.

(v) The following list gives for each of the root lattices in theorem 2.3 (d) a *root basis* $\delta_1, \dots, \delta_n$ (a \mathbb{Z} -basis of L with additional properties [4]) and an additional root δ_{n+1} (which is minus the maximal root with respect to the root basis). The diagram for the root basis is called *Dynkin diagram*, the diagram for $\delta_1, \dots, \delta_{n+1}$ is called *extended Dynkin diagram*. The roots $\delta_1, \dots, \delta_{n+1}$ satisfy a linear relation. For the cases E_6, E_7, E_8, F_4, G_2 , it is given in Lemma 3.7 (c). In the case of E_6 , $\delta_7 = \frac{1}{2}(-\sum_{i=1,2,3,8} e_i + \sum_{i=4,5,6,7} e_i)$.

type	$\delta_1, \dots, \delta_n$	δ_{n+1}
A_n	$e_i - e_{i+1} (i = 1, \dots, n)$	$-e_1 + e_{n+1}$
B_n	$-e_1, e_i - e_{i+1} (i = 1, \dots, n - 1)$	$e_{n-1} + e_n$
C_n	$-2e_1, e_i - e_{i+1} (i = 1, \dots, n - 1)$	$2e_n$
D_n	$e_i - e_{i+1} (i = 1, \dots, n - 1), e_{n-1} + e_n$	$-e_1 - e_2$
E_6	$\frac{1}{2} \sum_{i=1}^8 e_i, -e_3 - e_4, e_i - e_{i+1} (i = 3, \dots, 6)$	δ_7
E_7	$\frac{1}{2} \sum_{i=1}^8 e_i, -e_2 - e_3, e_i - e_{i+1} (i = 2, \dots, 6)$	$-e_1 - e_8$
E_8	$\frac{1}{2} \sum_{i=1}^8 e_i, -e_1 - e_2, e_i - e_{i+1} (i = 1, \dots, 6)$	$e_7 - e_8$
F_4	$\frac{1}{2} \sum_{i=1}^4 e_i, -e_1, e_1 - e_2, e_2 - e_3$	$e_3 - e_4$
G_2	$e_1 - e_2, -e_1 + 2e_2 - e_3$	$-e_1 - e_2 + 2e_3$

The following table gives the extended Dynkin diagrams.

The Dynkin diagram encodes up to a common scalar the intersection numbers (δ_i, δ_j) of the basis $\delta_1, \dots, \delta_n$, and thus it determines the irreducible root system.



3. Subroot lattices and quotients

The subroot lattices of an irreducible root lattice can be determined up to isomorphism by a recipe due to Borel and de Siebenthal [3] and Dynkin [7]. In [7] also a list and additional information is given. In this section, we will review the recipe and go through it, and thus we will recover the list. We will extend the list in two ways. First, we consider also the p.n. subroot lattices of the p.n. root lattices of type BC_n . Second, we will calculate for any (isomorphism class of a) pair (L, L_1) where L_1 is a (p.n.) subroot lattice of an irreducible (p.n.) root lattice L the quotient group L/L_1 . This will be helpful in section 4 and crucial in the sections 6 and 7.

Definition 3.1. Let $(L, (\cdot, \cdot), \Phi)$ be a (p.n.) root lattice.

(a) A (p.n.) root lattice $(L_1, (\cdot, \cdot)_1, \Phi_1)$ is a (p.n.) subroot lattice of $(L, (\cdot, \cdot), \Phi)$ if $L_1 \subset L$ and $(\cdot, \cdot)_1$ is the restriction of (\cdot, \cdot) to L_1 and $\Phi_1 = L_1 \cap \Phi$.

A notation: Because $(\cdot, \cdot)_1$ and Φ_1 are determined by L_1 , we will talk of the subroot lattice L_1 .

(b) A (p.n.) root lattice $(L_1, (\cdot, \cdot)_1, \Phi_1)$ is the (p.n.) root lattice of a (p.n.) subroot system if $L_1 \subset L$ and $(\cdot, \cdot)_1$ is the restriction of (\cdot, \cdot) to L_1 and $\Phi_1 \subset L_1 \cap \Phi$.

(c) The index of a subroot lattice L_1 is $[L \cap L_{1, \mathbb{Q}} : L_1] \in \mathbb{Z}_{\geq 1}$.

Remark 3.2. Let $(L, (\cdot, \cdot), \Phi)$ be a (p.n.) root lattice.

(i) Let $L_1 \subset L$ be a \mathbb{Z} -sublattice. Define $(\cdot, \cdot)_1$ as the restriction of (\cdot, \cdot) to L_1 . Define $\Phi_1 := L_1 \cap \Phi$. Then $(L_1, (\cdot, \cdot)_1, \Phi_1)$ is a (p.n.) subroot lattice if and only if it is a (p.n.) root lattice, and this holds if and only if L_1 is generated by Φ_1 as a \mathbb{Z} -module: (16) holds for Φ_1 , and $s_\alpha(\Phi_1) \subset L_1$ holds for $\alpha \in \Phi_1$ because of (12) and (11). This gives (15).

(ii) If $A = \{\alpha_1, \dots, \alpha_l\} \subset \Phi$ is any nonempty subset, then the data

$$L_1 := \sum_{i=1}^l \mathbb{Z} \cdot \alpha_i, \quad (\cdot, \cdot)_1 := (\cdot, \cdot)|_{L_1}, \quad \Phi_1 := L_1 \cap \Phi \quad (30)$$

satisfy the conditions in (i) and are a (p.n.) subroot lattice.

(iii) Any (p.n.) subroot lattice is the root lattice of a subroot system. If $(L, (\cdot, \cdot), \Phi)$ is a homogeneous root lattice also the inverse holds. But if $(L, (\cdot, \cdot), \Phi)$ contains orthogonal summands which are of types B_k, C_k, BC_k, F_4 or G_2 , then there are subroot lattices $(L_1, (\cdot, \cdot), \Phi_1)$ such that the subsets $\Phi_2 \subsetneq \Phi_1$ of short roots give rise to root lattices $(L_1, (\cdot, \cdot), \Phi_2)$ of subroot systems Φ_2 , such that these root lattices are not subroot lattices. We will not work much with them, but in [6] they are used.

(iv) If one erases from any of the extended Dynkin diagrams one vertex, one obtains a disjoint union of Dynkin diagrams. This leads to the following recipe with two kinds of steps with which one obtains easily subroot lattices of a root lattice. It is due to [3] and [7, §5], therefore we call the steps (BDdS1) and (BDdS2). Start with a root lattice $(L, (\cdot, \cdot), \Phi)$. Choose a root basis $A \subset \Phi$, that is a \mathbb{Z} -basis of L consisting of roots such that its generalized Dynkin diagram (defined in Remark 2.4 (iv)) is a disjoint union of Dynkin diagrams. L decomposes uniquely into an orthogonal sum of irreducible subroot lattices, which are called the *summands* of L .

Step (BDdS1): Choose one summand L_1 of L , add to A the unique root $\tilde{\delta}$ in Φ_1 which gives together with the roots in $A \cap \Phi_1$ an extended Dynkin diagram ($\tilde{\delta}$ is a linear combination of the roots in $A \cap \Phi_1$) and delete from $A \cup \{\tilde{\delta}\}$ an arbitrary root in $A \cap \Phi_1$. The new set $\tilde{A} \subset \Phi$ defines via (ii) a subroot lattice \tilde{L} of L of the same rank as L .

Step (BDdS2): Choose one summand L_1 of L and delete from A an arbitrary root in $A \cap \Phi_1$. The new set $\tilde{A} \subset \Phi$ defines via (ii) a subroot lattice \tilde{L} of L with rank $L_1 = \text{rank } L - 1$.

In both cases \tilde{A} is a root basis of \tilde{L} . Therefore one can repeat the steps. The change in the Dynkin diagrams is easy to see. In the step (BDdS1) one extends one component to its extended version and then erases one vertex. In the step (BDdS2) one simply erases one vertex. ■

The following theorem is mainly due to Dynkin [7, §5], the recipe in part (a) is also in [3]. The only new (though rather trivial) part is the discussion of the cases BC_n . That will follow from Lemma 3.6 below.

Theorem 3.3. (a) *Let $(L, (\cdot, \cdot), \Phi)$ be a root lattice. Any subroot lattice is obtained by the choice of a suitable root basis of L and by a suitable sequence of the steps (BDdS1) and (BDdS2).*

(b) *The first column in Table 3.1 and the first columns in the Tables 3.2 – 3.6 list all isomorphism classes of pairs $((L, (\cdot, \cdot), \Phi), L_1)$ where $(L, (\cdot, \cdot), \Phi)$ is an irreducible (p.n.) root lattice with the lengths of the roots as in Theorem 2.3(d) and where L_1 is a subroot lattice.*

The tables give the name for the type of L_1 , where additionally the lengths of the roots of the summands of L_1 are taken into account. The symbols $A_0, B_0, BC_0, D_2, D_3, A_1, B_1, C_1, C_2$ from Remark 2.4(ii) are used. The new notations $[...]'$ and $[...]''$ are explained in (d) below.

(c) With one class of exceptions, the following holds. If $((L, (\cdot, \cdot), \Phi), L_1)$ and $((L, (\cdot, \cdot), \Phi), L_2)$ are isomorphic pairs as in (b), then a Weyl group element $w \in W$ with $w(L_1) = w(L_2)$ exists. The class of exceptions are the sublattices of D_n of types $A_{k_1} + \dots + A_{k_r}$ with all k_1, \dots, k_r odd. For each of those types there are two orbits with respect to W .

(d) The Tables 3.3 and 3.4 contain pairs $[H]'$ and $[H]''$ with

$$H \in \{A_5 + A_1, A_5, A_3 + 2A_1, A_3 + A_1, 4A_1, 3A_1\}$$

for E_7 and with $H \in \{A_7, A_5 + A_1, 2A_3, A_3 + 2A_1, 4A_1\}$ for E_8 . Here $[H]'$ and $[H]''$ denote (classes in the sense of (b) of) subroot lattices which are isomorphic if one forgets the embedding into L . But for a subroot lattice $L_1 \subset L$ of type $[H]'$ and a subroot lattice $L_2 \subset L$ of type $[H]''$, the pairs (L, L_1) and (L, L_2) are not isomorphic. This is an implication of the following properties: A subroot lattice $L_3 \subset L$ of type A_7 for E_7 and of type A_8 for E_8 with $L_1 \subset L_3 \subset L$ exists, but no subroot lattice $L_4 \subset L$ of type A_7 for E_7 and of type A_8 for E_8 with $L_2 \subset L_4 \subset L$ exists.

The information in the following Tables 3.1–3.6 is treated in Theorem 3.3, Lemma 3.7 and Theorem 3.8. Always L is one of the p.n. subroot lattices in Theorem 2.3(d), and L_1 is a p.n. subroot lattice of the type indicated. In the Tables 3.2–3.6 we have $L_1 := \sum_{i \in \{1, \dots, n\} \cup I - J} \mathbb{Z} \cdot \delta_i$. Here $I \subset \{n + 1, \dots, n + 5\}$ and $J \subset \{1, \dots, n\}$ are given in the tables and are explained in Lemma 3.7(b). The roots δ_k for $k \geq n + 2$ (in the cases E_7, E_8, F_4) are defined in Lemma 3.7. The quotient L/L_1 is given up to isomorphism. Here $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{Z}_{>0}$. For $k_1 = k_1(L, L_1)$ see Theorem 3.8. For the symbols $A_0, B_0, BC_0, D_2, D_3, \tilde{A}_1, B_1, \tilde{A}_2 C_1, C_2$ see Remark 2.4(ii).

Table 3.1 for A_n, B_n, C_n, BC_n, D_n : Here $r \geq 0, s \geq 0, a_i \geq 0, b_j \geq 1$ in the cases $C_{b_j}, b_j \geq 2$ in the cases $D_{b_j}, m = -1$ in the case $A_n, m = 0$ in the cases C_n and $D_n, m \geq 0$ in the cases B_n and $BC_n,$

$$\sum_{i=1}^r (a_i + 1) + \sum_{j=1}^s b_j + m = n. \tag{31}$$

L	L_1	L/L_1	$k_1(L, L_1)$
A_n	$\sum_{i=1}^r A_{a_i}$	\mathbb{Z}^{r-1}	$r - 1$
B_n	$\sum_{i=1}^r A_{a_i} + \sum_{j=1}^s D_{b_j} + B_m$	$\mathbb{Z}^r \times \mathbb{Z}_2^s$	$r + s$
C_n	$\sum_{i=1}^r A_{a_i} + \sum_{j=1}^s C_{b_j}$	$\mathbb{Z}^r \times \mathbb{Z}_2^{s-1}$	$r + s - 1$ if $s \geq 1$ r if $s = 0$
BC_n	$\sum_{i=1}^r A_{a_i} + \sum_{j=1}^s C_{b_j} + BC_m$	$\mathbb{Z}^r \times \mathbb{Z}_2^s$	$r + s$
D_n	$\sum_{i=1}^r A_{a_i} + \sum_{j=1}^s D_{b_j}$	$\mathbb{Z}^r \times \mathbb{Z}_2^{s-1}$	$r + s - 1$ if $s \geq 1$ r if $s = 0$

Table 3.2 for E_6 :

L_1	I	J	L/L_1	k_1	L_1	I	J	L/L_1	k_1
E_6	—	—	$\{0\}$	0	$2A_2$	—	3, 4	\mathbb{Z}^2	2
$A_5 + A_1$	7	2	\mathbb{Z}_2	1	$A_2 + 2A_1$	—	4, 5	\mathbb{Z}^2	2
$3A_2$	7	4	\mathbb{Z}_3	1	$4A_1$	7	2, 3, 5	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3
A_5	—	3	\mathbb{Z}	1	D_4	—	1, 6	\mathbb{Z}^2	2
$2A_2 + A_1$	—	4	\mathbb{Z}	1	A_3	—	1, 2, 3	\mathbb{Z}^3	3
$A_4 + A_1$	—	2	\mathbb{Z}	1	$A_2 + A_1$	—	2, 3, 6	\mathbb{Z}^3	3
D_5	—	1	\mathbb{Z}	1	$3A_1$	—	2, 3, 5	\mathbb{Z}^3	3
$A_3 + 2A_1$	7	2, 3	$\mathbb{Z} \times \mathbb{Z}_2$	2	A_2	—	1, 2, 3, 4	\mathbb{Z}^4	4
A_4	—	1, 2	\mathbb{Z}^2	2	$2A_1$	—	1, 2, 3, 5	\mathbb{Z}^4	4
$A_3 + A_1$	—	2, 3	\mathbb{Z}^2	2	A_1	—	1, ..., 5	\mathbb{Z}^5	5

Table 3.3 for E_7 :

L_1	I	J	L/L_1	k_1	L_1	I	J	L/L_1	k_1
E_7	—	—	$\{0\}$	0	$A_3 + A_2 + A_1$	—	4	\mathbb{Z}	1
$D_6 + A_1$	8	6	\mathbb{Z}_2	1	$[A_5 + A_1]'$	8	1, 3	$\mathbb{Z} \times \mathbb{Z}_2$	2
$A_5 + A_2$	8	5	\mathbb{Z}_3	1	$[A_5 + A_1]''$	—	2	\mathbb{Z}	1
$2A_3 + A_1$	8	4	\mathbb{Z}_4	1	D_6	—	1	\mathbb{Z}	1
A_7	8	3	\mathbb{Z}_2	1	$D_4 + 2A_1$	8	1, 6	$\mathbb{Z} \times \mathbb{Z}_2$	2
$D_4 + 3A_1$	8, 9	1, 6	\mathbb{Z}_2^2	2	$A_3 + 3A_1$	8	1, 4	$\mathbb{Z} \times \mathbb{Z}_2$	2
$7A_1$	8, 9, 10	1, 6, 4	\mathbb{Z}_2^3	3	$3A_2$	8	2, 5	$\mathbb{Z} \times \mathbb{Z}_3$	2
E_6	—	7	\mathbb{Z}	1	$2A_3$	8	3, 4	$\mathbb{Z} \times \mathbb{Z}_2$	2
$D_5 + A_1$	—	6	\mathbb{Z}	1	A_6	—	3	\mathbb{Z}	1
$A_4 + A_2$	—	5	\mathbb{Z}	1	$6A_1$	8, 9	1, 4, 6	$\mathbb{Z} \times \mathbb{Z}_2^2$	3

D_5	—	1, 7	\mathbb{Z}^2	2	$[A_3 + A_1]'$	—	1, 3, 4	\mathbb{Z}^3	3
$A_4 + A_1$	—	3, 6	\mathbb{Z}^2	2	$[A_3 + A_1]''$	—	1, 2, 4	\mathbb{Z}^3	3
$2A_2 + A_1$	—	2, 5	\mathbb{Z}^2	2	$2A_2$	—	1, 2, 5	\mathbb{Z}^3	3
$[A_5]'$	—	1, 3	\mathbb{Z}^2	2	$A_2 + 2A_1$	—	2, 4, 5	\mathbb{Z}^3	3
$[A_5]''$	—	1, 2	\mathbb{Z}^2	2	$[4A_1]'$	8	1, 3, 4, 6	$\mathbb{Z}^3 \times \mathbb{Z}_2$	4
$D_4 + A_1$	—	1, 6	\mathbb{Z}^2	2	$[4A_1]''$	—	2, 4, 6	\mathbb{Z}^3	3
$A_3 + A_2$	—	3, 5	\mathbb{Z}^2	2	A_3	—	1, 2, 3, 4	\mathbb{Z}^4	4
$5A_1$	8	1, 4, 6	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3	$A_2 + A_1$	—	1, 2, 3, 5	\mathbb{Z}^4	4
$A_2 + 3A_1$	—	4, 6	\mathbb{Z}^2	2	$[3A_1]'$	—	1, 3, 4, 6	\mathbb{Z}^4	4
$[A_3 + 2A_1]'$	8	1, 3, 4	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3	$[3A_1]''$	—	1, 2, 4, 6	\mathbb{Z}^4	4
$[A_3 + 2A_1]''$	—	1, 4	\mathbb{Z}^2	2	A_2	—	1, ..., 5	\mathbb{Z}^5	5
D_4	—	1, 6, 7	\mathbb{Z}^3	3	$2A_1$	—	1, ..., 4, 6	\mathbb{Z}^5	5
A_4	—	1, 2, 3	\mathbb{Z}^3	3	A_1	—	1, ..., 6	\mathbb{Z}^6	6

Table 3.4 for E_8 :

L_1	I	J	L/L_1	k_1	L_1	I	J	L/L_1	k_1
E_8	—	—	$\{0\}$	0	$E_6 + A_2$	9	7	\mathbb{Z}_3	1
A_8	9	3	\mathbb{Z}_3	1	$E_7 + A_1$	9	8	\mathbb{Z}_2	1
D_8	9	1	\mathbb{Z}_2	1	$D_6 + 2A_1$	9, 10	1, 8	\mathbb{Z}_2^2	2
$A_7 + A_1$	9	2	\mathbb{Z}_4	1	$D_5 + A_3$	9	6	\mathbb{Z}_4	1
$A_5 + A_2 + A_1$	9	4	\mathbb{Z}_6	1	$2D_4$	9, 10	1, 6	\mathbb{Z}_2^2	2
$2A_4$	9	5	\mathbb{Z}_5	1	$D_4 + 4A_1$	9, 10, 12	1, 6, 4	\mathbb{Z}_2^3	3
$4A_2$	9, 13	7, 4	\mathbb{Z}_3^2	2	$2A_3 + 2A_1$	9, 10	8, 4	$\mathbb{Z}_2 \times \mathbb{Z}_4$	2
					$8A_1$	9, 10, 11, 12	1, 6, 8, 4	\mathbb{Z}_2^4	4

$A_6 + A_1$	—	2	\mathbb{Z}	1	$7A_1$	9, 10, 11	1, 6, 8, 4	$\mathbb{Z} \times \mathbb{Z}_2^3$	4
$A_4 + A_2 + A_1$	—	4	\mathbb{Z}	1	$D_6 + A_1$	9	1, 8	$\mathbb{Z} \times \mathbb{Z}_2$	2
$A_5 + A_2$	9	3, 4	$\mathbb{Z} \times \mathbb{Z}_3$	2	$D_5 + A_2$	—	6	\mathbb{Z}	1
$3A_2 + A_1$	9	4, 7	$\mathbb{Z} \times \mathbb{Z}_3$	2	$A_3 + A_2 + 2A_1$	9	4, 6	$\mathbb{Z} \times \mathbb{Z}_2$	2
$E_6 + A_1$	—	7	\mathbb{Z}	1	$D_4 + A_3$	9	1, 6	$\mathbb{Z} \times \mathbb{Z}_2$	2
E_7	—	8	\mathbb{Z}	1	$A_3 + 4A_1$	9, 10	1, 4, 8	$\mathbb{Z} \times \mathbb{Z}_2^2$	3
D_7	—	1	\mathbb{Z}	1	$A_4 + A_3$	—	5	\mathbb{Z}	1
$D_5 + 2A_1$	9	6, 8	$\mathbb{Z} \times \mathbb{Z}_2$	2	$A_5 + 2A_1$	9	1, 4	$\mathbb{Z} \times \mathbb{Z}_2$	2
$D_4 + 3A_1$	9, 10	1, 4, 6	$\mathbb{Z} \times \mathbb{Z}_2^2$	3	$[A_7]'$	—	3	\mathbb{Z}	1
$2A_3 + A_1$	9	2, 6	$\mathbb{Z} \times \mathbb{Z}_4$	2	$[A_7]''$	9	1, 2	$\mathbb{Z} \times \mathbb{Z}_2$	2

$3A_2$	9	3, 4, 7	$\mathbb{Z}^2 \times \mathbb{Z}_3$	3	$6A_1$	9, 10	1, 4, 6, 8	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$	4
E_6	—	7, 8	\mathbb{Z}^2	2	$A_2 + 4A_1$	9	4, 6, 8	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3
D_6	—	1, 8	\mathbb{Z}^2	2	$A_4 + 2A_1$	—	1, 4	\mathbb{Z}^2	2
$D_4 + 2A_1$	9	1, 6, 8	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3	A_6	—	1, 3	\mathbb{Z}^2	2
$[2A_3]'$	—	3, 5	\mathbb{Z}^2	2	$A_3 + A_2 + A_1$	—	4, 8	\mathbb{Z}^2	2
$[2A_3]''$	9	1, 2, 6	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3	$[A_5 + A_1]'$	—	2, 3	\mathbb{Z}^2	2
$D_5 + A_1$	—	1, 7	\mathbb{Z}^2	2	$[A_5 + A_1]''$	9	1, 2, 4	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3
$A_3 + 3A_1$	9	1, 4, 6	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3	$A_4 + A_2$	—	3, 4	\mathbb{Z}^2	2
$D_4 + A_2$	—	1, 6	\mathbb{Z}^2	2	$2A_2 + 2A_1$	—	4, 6	\mathbb{Z}^2	2

D_5	—	1, 7, 8	\mathbb{Z}^3	3	D_4	—	1, 6, 7, 8	\mathbb{Z}^4	4
$[A_3 + 2A_1]'$	—	2, 3, 5	\mathbb{Z}^3	3	$[4A_1]'$	—	2, 3, 5, 7	\mathbb{Z}^4	4
$[A_3 + 2A_1]''$	9	1, 2, 4, 6	$\mathbb{Z}^3 \times \mathbb{Z}_2$	4	$[4A_1]''$	9	1, 2, 4, 6, 8	$\mathbb{Z}^4 \times \mathbb{Z}_2$	5
$A_3 + A_2$	—	3, 5, 8	\mathbb{Z}^3	3	$A_2 + 2A_1$	—	1, 2, 4, 6	\mathbb{Z}^4	4
A_5	—	1, 2, 8	\mathbb{Z}^3	3	$2A_2$	—	1, 2, 3, 6	\mathbb{Z}^4	4
$5A_1$	9	1, 4, 6, 8	$\mathbb{Z}^3 \times \mathbb{Z}_2$	4	$A_3 + A_1$	—	1, 2, 3, 5	\mathbb{Z}^4	4
$A_4 + A_1$	—	1, 3, 4	\mathbb{Z}^3	3	A_4	—	1, 2, 3, 4	\mathbb{Z}^4	4
$D_4 + A_1$	—	1, 6, 7	\mathbb{Z}^3	3	A_3	—	1, 2, 3, 4, 5	\mathbb{Z}^5	5
$A_2 + 3A_1$	—	1, 4, 6	\mathbb{Z}^3	3	$A_2 + A_1$	—	1, 2, 3, 4, 6	\mathbb{Z}^5	5
$2A_2 + A_1$	—	2, 3, 6	\mathbb{Z}^3	3	$3A_1$	—	1, 2, 3, 5, 7	\mathbb{Z}^5	5

A_2	—	1, ..., 6	\mathbb{Z}^6	6
$2A_1$	—	1, ..., 5, 7	\mathbb{Z}^6	6

$$A_1 \mid - \mid 1, \dots, 7 \mid \mathbb{Z}^7 \mid 7$$

Table 3.5 for F_4 :

L_1	I	J	L/L_1	k_1	A_3	5	1, 2	$\mathbb{Z} \times \mathbb{Z}_2$	2
F_4	—	—	$\{0\}$	0	$2A_1 + \tilde{A}_1$	5	2, 4	$\mathbb{Z} \times \mathbb{Z}_2$	2
B_4	5	1	\mathbb{Z}_2	1	$A_1 + \tilde{A}_2$	—	3	\mathbb{Z}	1
$A_3 + \tilde{A}_1$	5	2	\mathbb{Z}_4	1	C_3	—	4	\mathbb{Z}	1
$A_2 + \tilde{A}_2$	5	3	\mathbb{Z}_3	1	$3A_1$	5, 6	1, 2, 4	$\mathbb{Z} \times \mathbb{Z}_2^2$	3
$C_3 + A_1$	5	4	\mathbb{Z}_2	1	A_2	—	1, 2	\mathbb{Z}^2	2
D_4	5, 6	1, 2	\mathbb{Z}_2^2	2	B_2	—	1, 4	\mathbb{Z}^2	2
$B_2 + 2A_1$	5, 6	1, 4	\mathbb{Z}_2^2	2	$A_1 + \tilde{A}_1$	—	2, 3	\mathbb{Z}^2	2
$4A_1$	5, 6, 7	1, 2, 4	\mathbb{Z}_2^3	3	$2A_1$	5	1, 2, 4	$\mathbb{Z}^2 \times \mathbb{Z}_2$	3
B_3	—	1	\mathbb{Z}	1	\tilde{A}_2	—	3, 4	\mathbb{Z}^2	2
$B_2 + A_1$	5	1, 4	$\mathbb{Z} \times \mathbb{Z}_2$	2	\tilde{A}_1	—	2, 3, 4	\mathbb{Z}^3	3
$A_2 + \tilde{A}_1$	—	2	\mathbb{Z}	1	A_1	—	1, 2, 3	\mathbb{Z}^3	3

Table 3.6 for G_2 :

L_1	I	J	L/L_1	k_1	$A_1 + \tilde{A}_1$	3	2	\mathbb{Z}_2	1
G_2	—	—	$\{0\}$	0	\tilde{A}_1	—	2	\mathbb{Z}	1
A_2	3	1	\mathbb{Z}_3	1	A_1	—	1	\mathbb{Z}	1

Remark 3.4. (i) Let $(L, (\cdot, \cdot), \Phi)$ be an irreducible root lattice. Theorem 3.3 (a)+(b)+(d) tells the following. There is an almost 1:1 correspondence between the set of isomorphism classes of pairs $((L, (\cdot, \cdot), \Phi), L_1)$ with L_1 a subroot lattice and the set of unions of Dynkin diagrams which are obtained by iterations of the graphical versions of the steps (BDdS1) and (BDdS2) in remark 3.2 (iv), namely

(BDdS1): Go from one Dynkin diagram to the extended Dynkin diagram and erase an arbitrary vertex.

(BDdS2): Erase an arbitrary vertex.

The only exceptions are the pairs $[H]'$ and $[H]''$ discussed in Theorem 3.3 (d). They have the same Dynkin diagrams.

(ii) In the Table 11 in [7, ch. II, §5] there are two misprints. $A_6 + A_2$ has to be replaced by $E_6 + A_2$. And one of the two $A_7 + A_1$ has to be replaced by $E_7 + A_1$. ■

In the cases of the series A_n, B_n, C_n and D_n , one can see the subroot lattices also in a different way, by associating a graph $\mathcal{G}(A)$ to a generating set $A \subset \Phi_1$ of a subroot lattice L_1 . This works also in the case of the series BC_n and will give the proof of the statements in Theorem 3.3 for BC_n . The graphs are defined as follows. They are related to graphs in [5], especially if A is a \mathbb{Z} -basis of the root lattice L_1 .

Definition 3.5. Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice in Theorem 2.3(d) of one of the types A_n, B_n, C_n, BC_n, D_n . Let $A = \{\alpha_1, \dots, \alpha_l\} \subset \Phi$ be a nonempty subset. It defines a p.n. subroot lattice $L_1 = \sum_{i=1}^l \mathbb{Z} \cdot \alpha_i$. A graph $\mathcal{G}(A)$ with or without markings of the vertices and with one or two types of edges is defined as follows.

(a) L of type A_n : The graph $\mathcal{G}(A)$ has $n+1$ vertices which are labelled $1, \dots, n+1$. It has l edges. A root $\alpha \in A$ with $\alpha = \pm(e_i - e_j)$ gives an edge between the vertices i and j . So, if $e_i - e_j$ and $e_j - e_i$ are in A , there are two edges between the vertices i and j . The same applies in the cases (b) and (c).

(b) L of type BC_n : The graph $\mathcal{G}(A)$ has n vertices which are labelled $1, \dots, n$. Any root $\pm e_i$ in A leads to a marking of the vertex i which is called a *short* marking (and which may be represented by a circle around the vertex). Any root $\pm 2e_i$ in A leads to a marking of the vertex i which is called a *long* marking (and which may be represented by a square around the vertex). So, depending on how many of the roots $\pm e_i$ and $\pm 2e_i$ are in A , the vertex i has between 0 and 4 markings. Any root $\pm(e_i - e_j)$ gives a normal edge between the vertices i and j . Any root $\pm(e_i + e_j)$ gives a dotted edge between the vertices i and j . So, between the vertices i and j there are between 0 and 4 edges.

(c) L of type B_n or C_n or D_n : The graph is defined as in the case of type BC_n . (In the case of B_n there are no long markings, in the case of C_n there are no short markings, in the case of D_n there are no markings at all).

The following lemma is obvious.

Lemma 3.6. *Consider the same data as in Definition 3.5. The orthogonal irreducible summands of the subroot lattice L_1 can be read off from the graph $\mathcal{G}(A)$ as follows. Each of the following subgraphs yields a summand, which is generated by the roots which contribute via markings or edges to this subgraph.*

A_k: *A component of $\mathcal{G}(A)$ which has no markings and in which any cycle has an even number of dotted edges yields a summand of type A_k . Here $k+1$ is the number of vertices of the component. (An isolated vertex with no markings yields thus the summand $A_0 = \{0\}$).*

B_k or **BC_k**: *The union of all components of $\mathcal{G}(A)$ which contain a vertex with a short marking yields a summand of type B_k if L is of type B_n and a summand of type BC_k if L is of type BC_n . Here k is the number of vertices of the union of these components. If this union is empty, we write $B_0 (= \{0\})$ if L is of type B_n and $BC_0 (= \{0\})$ if L is of type BC_n .*

C_k: *A component of $\mathcal{G}(A)$ which does not contain a vertex with a short marking, but which contains a cycle with an odd number of dotted edges or which contains a vertex with a long marking yields a summand of type C_k if L is of type C_n or BC_n . Here k is the number of vertices of this component.*

D_k: *A component of $\mathcal{G}(A)$ which does not contain a vertex with a marking, but which contains a cycle with an odd number of dotted edges yields a summand of type D_k if L is of type B_n or D_n . Here k is the number of vertices of this component.*

The statements in Theorem 3.3 for the cases A_n, B_n, C_n, BC_n and D_n follow easily from this lemma and from the structure of the Weyl group, which was described in Remark 2.4 (iii). As Theorem 3.3 for A_n, B_n, C_n and D_n had been proved by Dynkin [7, §5], we give here details only for the case BC_n .

Proof of Theorem 3.3(b) and (c) for BC_n :

Let $(L, (\cdot, \cdot), \Phi)$ be the p.n. root lattice of type BC_n in Theorem 2.3 (d).

(b) Let $L_1 \subset L$ be a subroot lattice, and let $A = \{\alpha_1, \dots, \alpha_l\} \subset \Phi$ be a subset of Φ with $L_1 = \sum_{i=1}^l \mathbb{Z} \cdot \alpha_i$. Consider the decomposition in Lemma 3.6 of the graph $\mathcal{G}(A)$ into subgraphs:

There are $r(\geq 0)$ components of type A_{a_i} ($a_i \geq 0$). And there are $s(\geq 0)$ components of type C_{b_j} ($b_j \geq 1$). The roots in A which give rise to one such component of type A_{a_i} or C_{b_j} generate a sublattice of type A_{a_i} or C_{b_j} in L .

Each other component of $\mathcal{G}(A)$ contains a vertex with a short marking. The roots of A which give rise to the union of these components generate a sublattice of type BC_m ($m = 0$ if no such component exists, $m \geq 1$ else). Counting the vertices in all subgraphs of $\mathcal{G}(A)$ gives the relation (31).

(c) Let $L_1 \subset L$ and $L_2 \subset L$ be p.n. subroot lattices such that $L_1 \cong L_2$ as p.n. root lattices. Choose the maximal sets $A_1 := L_1 \cap \Phi$ and $A_2 := L_2 \cap \Phi$.

In $\mathcal{G}(A_1)$ as well as in $\mathcal{G}(A_2)$, a component of type A_{a_i} has the following shape: Between each two vertices, it has either two normal edges or two dotted edges, and each cycle has an even number of dotted edges.

In $\mathcal{G}(A_1)$ as well as in $\mathcal{G}(A_2)$, a component of type C_{b_j} has the following shape: Between each two vertices, it has two normal edges and two dotted edges, and each vertex has two long markings.

In $\mathcal{G}(A_1)$ as well as in $\mathcal{G}(A_2)$, there is nothing left if $m = 0$, and if $m \geq 1$ the rest of the graph is one component, such that between each two vertices it has two normal edges and two dotted edges, and each vertex has two short and two long markings.

One sees that a suitable signed permutation of the set $\{\pm e_1, \dots, \pm e_n\}$ maps $\mathcal{G}(A_1)$ to $\mathcal{G}(A_2)$. The corresponding Weyl group element (see Remark 2.4 (iii)) maps L_1 to L_2 . ■

For the calculation of the quotients L/L_1 in theorem 3.8, we need in the cases E_6, E_7, E_8, F_4, G_2 a concrete subroot lattice L_1 for each isomorphism class of pairs (L, L_1) . This is found in Lemma 3.7 by carrying out the recipe with the steps (BDdS1) and (BDdS2).

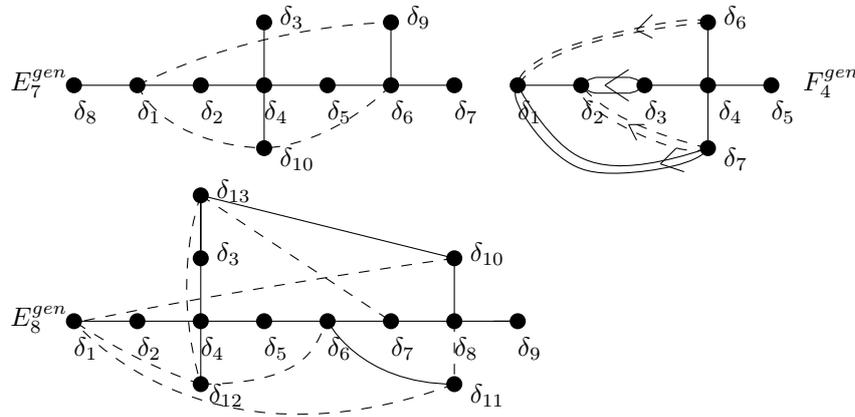
Lemma 3.7. *Let L be an irreducible root lattice in Theorem 2.3(d) of one of the types E_6, E_7, E_8, F_4, G_2 . Additionally to the roots $\delta_1, \dots, \delta_{n+1}$ which are defined in Remark 2.4(v), the following roots are considered.*

$$\begin{array}{l|l}
 E_7 & \delta_9 = e_6 + e_7, \delta_{10} = e_4 + e_5. \\
 E_8 & \delta_{10} = e_7 + e_8, \delta_{11} = e_5 + e_6, \delta_{12} = e_3 + e_4, \\
 & \delta_{13} = \frac{1}{2}(-e_1 + \sum_{i=2}^5 e_i - \sum_{i=6}^8 e_i). \\
 F_4 & \delta_6 = e_3 + e_4, \delta_7 = -e_1 - e_2.
 \end{array} \tag{32}$$

(a) *In the cases E_7, E_8 and F_4 , the generalized Dynkin diagrams which take into account the roots $\delta_1, \dots, \delta_{n+1}$ and the roots above look as shown below.*

(The edges which are not horizontal or vertical will be irrelevant except for the dotted edge between δ_1 and δ_{10} in the Dynkin diagram of E_8 . It will be used once, in the construction of a subroot lattice of type $2A_3 + 2A_1$.)

(b) *The second and third column in the Tables 3.2–3.6 encode a realization of the recipe in Remark 3.2(iv), in the following way.*



Suppose in the Tables 3.2–3.6 in the line for one subroot lattice, I is given as the sequence i_1, \dots, i_r of numbers and J is given as the sequence j_1, \dots, j_s of numbers, with $0 \leq r \leq s$. One carries out r steps (BDdS1): In the k -th step one adds the root δ_{i_k} and then erases the root δ_{j_k} . Afterwards one carries out $s - r$ steps (BDdS2): One erases the roots $\delta_{j_{s-r+1}}, \dots, \delta_{j_s}$. This leads to a subroot lattice of the type indicated in the first column.

(c) The roots δ_k for $k \geq n + 1$ are linear combinations of the roots $\delta_1, \dots, \delta_n$. The linear relations are as follows.

$$\begin{aligned}
 \mathbf{E}_6 : \quad & 0 = \delta_1 + 2\delta_2 + 2\delta_3 + 3\delta_4 + 2\delta_5 + \delta_6 + \delta_7. \\
 \mathbf{E}_7 : \quad & 0 = 2\delta_1 + 3\delta_2 + 2\delta_3 + 4\delta_4 + 3\delta_5 + 2\delta_6 + \delta_7 + \delta_8, \\
 & 0 = \delta_2 + \delta_3 + 2\delta_4 + 2\delta_5 + 2\delta_6 + \delta_7 + \delta_9, \\
 & 0 = \delta_2 + \delta_3 + 2\delta_4 + \delta_5 + \delta_{10}. \\
 \mathbf{E}_8 : \quad & 0 = 2\delta_1 + 4\delta_2 + 3\delta_3 + 6\delta_4 + 5\delta_5 + 4\delta_6 + 3\delta_7 + 2\delta_8 + \delta_9, \\
 & 0 = 2\delta_1 + 3\delta_2 + 2\delta_3 + 4\delta_4 + 3\delta_5 + 2\delta_6 + \delta_7 - \delta_{10}, \\
 & 0 = \delta_2 + \delta_3 + 2\delta_4 + 2\delta_5 + 2\delta_6 + \delta_7 + \delta_{11}, \\
 & 0 = \delta_2 + \delta_3 + 2\delta_4 + \delta_5 + \delta_{12}, \\
 & 0 = \delta_1 + 2\delta_2 + 2\delta_3 + 3\delta_4 + 2\delta_5 + \delta_6 + \delta_{13}. \\
 \mathbf{F}_4 : \quad & 0 = 2\delta_1 + 4\delta_2 + 3\delta_3 + 2\delta_4 + \delta_5, \\
 & 0 = 2\delta_1 + 2\delta_2 + \delta_3 - \delta_6, \\
 & 0 = 2\delta_2 + \delta_3 - \delta_7. \\
 \mathbf{G}_2 : \quad & 0 = 3\delta_1 + 2\delta_2 + \delta_3.
 \end{aligned}$$

Proof: The proof of the lemma is tedious as there are many cases, but the parts (a) and (c) and most of part (b) are completely elementary. In part (b) one has to check not only that the result has the correct Dynkin diagram, but also that the steps (BDdS1) work, i.e. that one has after adding a root an extended Dynkin diagram and that then a root of this extended Dynkin diagram is erased. The details are left to the reader.

The only nontrivial part concerns the subroot lattices of types $[H]'$ and $[H]''$ in the cases E_7 and E_8 (see Theorem 3.3 (d) for $[H]'$ and $[H]''$). One sees that in the cases of E_k , $k = 7, 8$, the constructed subroot lattices of types $[H]'$ are contained in the

subroot lattice $\bigoplus_{i \in \{1, \dots, k+1\} - \{3\}} \mathbb{Z}\delta_i$ of type A_k . The constructed subroot lattices of type $[H]''$ contain in the case E_7 the roots $\delta_3, \delta_5, \delta_7$ and in the case E_8 the roots $\delta_3, \delta_5, \delta_7, \delta_9$. The following claim shows that the constructed subroot lattices $[H]''$ are not contained in subroot lattices of type A_7 respectively A_8 .

Claim: (i) Let L be the lattice of type E_7 in Theorem 2.3(d). There is no subroot lattice L_1 of type A_7 with $\delta_3, \delta_5, \delta_7 \in L_1$.

(ii) Let L be the lattice of type E_8 in Theorem 2.3(d). There is no subroot lattice L_1 of type A_8 with $\delta_3, \delta_5, \delta_7, \delta_9 \in L_1$.

Proof of the claim: (i) Suppose that L_1 is a subroot lattice of type A_7 with $\delta_3, \delta_5, \delta_7 \in L_1$. These three roots generate a subroot lattice $L_2 \subset L_1$ of type $3A_1$. By Theorem 3.3, up to isomorphism there is only one pair of type $(A_7, 3A_1)$. Therefore a root $\alpha \in L_1$ with $(\alpha, \delta_3) = -1, (\alpha, \delta_5) = 0, (\alpha, \delta_7) = 0$ exists. But now observe $\delta_3 = e_2 - e_3, \delta_5 = e_4 - e_5, \delta_7 = e_6 - e_7$. Neither the roots in $\Phi(L)$ of type $\pm e_i \pm e_j$ nor the roots in $\Phi(L)$ of type $\frac{1}{2} \sum_{i=1}^8 \varepsilon_i e_i$ can serve as a root α . Contradiction.

(ii) Analogous to (i). ■

This finishes the proof of Lemma 3.7. ■

Theorem 3.8. (a) Let $(L, (\cdot, \cdot), \Phi)$ be an irreducible $p.n.$ root lattice, and let L_1 be a $p.n.$ subroot lattice. The third column in Table 3.1 and the fourth column in the Tables 3.2–3.6 gives the isomorphism class of the quotient group L/L_1 .

(b) Let $(L, (\cdot, \cdot), \Phi)$ be a $p.n.$ root lattice, and let L_1 be a $p.n.$ subroot lattice. Define the numbers

$$k_1(L, L_1) := \min\{k \mid \text{the group } L/L_1 \text{ has } k \text{ generators}\}, \tag{33}$$

$$k_2(L, L_1) := \min\{k \mid \exists \alpha_1, \dots, \alpha_k \in \Phi \text{ s.t. } L = L_1 + \sum_{i=1}^k \mathbb{Z} \cdot \alpha_i\}, \tag{34}$$

$$k_3(L, L_1) := \min \left\{ k \mid \begin{array}{l} L_1 \text{ can be constructed with } k \text{ of} \\ \text{the steps (BDdS1) and (BDdS2)} \end{array} \right\}. \tag{35}$$

Then
$$k_1(L, L_1) = k_2(L, L_1) = k_3(L, L_1). \tag{36}$$

The numbers are additive, i.e. if $L = L_2 + L_3$ and $L_1 = L_4 + L_5$ and $L_2 \supset L_4, L_3 \supset L_5$

then
$$k_1(L, L_1) = k_1(L_2, L_4) + k_1(L_3, L_5). \tag{37}$$

The last column of the Tables 3.1–3.6 gives the numbers $k_1(L, L_1)$ for the pairs with L irreducible. Minimal sequences of the steps (BDdS1) and (BDdS2) for the cases with L of type E_6, E_7, E_8, F_4, G_2 are given in the second and third column of the Tables 3.2–3.6 (see Lemma 3.7).

Proof: (a) First we treat the cases A_n, B_n, C_n, BC_n and D_n . Let A and $\mathcal{G}(A)$ be as in Definition 3.5 and Lemma 3.6, and let $L_1 \subset L$ be the corresponding subroot lattice. Let $\mathcal{G}(A) = \bigcup_{k \in K} \mathcal{G}_k$ (we choose the index set K such that $1 \notin K$ if $|K| \geq 2$ and $K = \{1\}$ if $|K| = 1$) be the decomposition into subgraphs \mathcal{G}_k with the properties in Lemma 3.6, let L_k be the subroot lattice which corresponds to the subgraph \mathcal{G}_k , and let

$$V_k := \bigoplus_{i \text{ is a vertex in } \mathcal{G}_k} \mathbb{Z} \cdot e_i.$$

Then
$$\bigoplus_{k \in K} V_k = \bigoplus_{i=1}^m \mathbb{Z} \cdot e_i \supset L \supset L_1 = \bigoplus_{k \in K} L_k$$

(with $m = n + 1$ for A_n and $m = n$ else) and

$$L/L_1 \subset \left(\bigoplus_{i=1}^m \mathbb{Z} \cdot e_i \right) / L_1 \cong \bigoplus_{k \in K} V_k / L_k.$$

The following table lists in the second and fourth line the isomorphism classes of the quotients in the first column.

L	A_n	B_n	C_n	BC_n	D_n
$\bigoplus_{i=1}^m \mathbb{Z}e_i/L$	\mathbb{Z}	$\{0\}$	\mathbb{Z}_2	$\{0\}$	\mathbb{Z}_2
L_k	A_l	B_l	C_l	BC_l	D_l
V_k/L_k	\mathbb{Z}	$\{0\}$	\mathbb{Z}_2	$\{0\}$	\mathbb{Z}_2

Finally, in the cases C_n and D_n , for any $k \in K$

$$L \not\subset L_k \oplus \bigoplus_{j \in K - \{k\}} V_j.$$

Therefore L/L_1 has the isomorphism type claimed in Table 3.1.

Now we treat the cases E_6, E_7, E_8, F_4, G_2 . Let $L_1 \subset L$ be one of the subroot lattices constructed in Lemma 3.7 using the data in the Tables 3.2 – 3.6. Let $\delta_{i_1}, \dots, \delta_{i_r}$ respectively $\delta_{j_1}, \dots, \delta_{j_s}$ with $0 \leq r \leq s$ be the roots in one line in the second respectively third column of these tables. Then

$$L/L_1 = \frac{\left(\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k} \right) + L_1}{L_1} \cong \frac{\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k}}{\left(\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k} \right) \cap L_1}.$$

The denominator of the right hand side is a \mathbb{Z} -lattice of rank r (because $\text{rank } L_1 = n - s + r$) and is generated by parts of those relations in Lemma 3.7 (c) which express the roots $\delta_{i_1}, \dots, \delta_{i_r}$ as linear combinations of the roots $\delta_1, \dots, \delta_n$.

We give one example: (L, L_1) of type $(E_8, D_4 + 3A_1)$, then $(\delta_{i_1}, \dots, \delta_{i_r}) = (\delta_9, \delta_{10})$ and $(\delta_{j_1}, \dots, \delta_{j_s}) = (\delta_1, \delta_4, \delta_6)$. The relation for δ_9 gives the element $2\delta_1 + 6\delta_4 + 4\delta_6$ of $\left(\bigoplus_{k=1}^s \mathbb{Z} \cdot \delta_{j_k} \right) \cap L_1$, the relation for δ_{10} gives the element $2\delta_1 + 4\delta_4 + 2\delta_6$. Therefore here

$$\frac{L}{L_1} \cong \frac{\mathbb{Z} \cdot \delta_1 \oplus \mathbb{Z} \cdot \delta_4 \oplus \mathbb{Z} \cdot \delta_6}{\mathbb{Z} \cdot (2\delta_1 + 6\delta_4 + 4\delta_6) \oplus \mathbb{Z} \cdot (2\delta_1 + 4\delta_4 + 2\delta_6)} \cong \mathbb{Z} \times \mathbb{Z}_2^2.$$

The calculations for all other cases (L, L_1) are analogous. They are tedious as there are many cases, but elementary.

(b) The additivity of the numbers $k_1(L, L_1), k_2(L, L_1), k_3(L, L_1)$ is obvious. Therefore it is sufficient to prove (36) for irreducible L . The last column of the Tables 3.1–3.6 can be read off from the second to the last column immediately. The first of the inequalities

$$k_1(L, L_1) \leq k_2(L, L_1) \leq k_3(L, L_1) \tag{38}$$

is obvious. The second inequality follows simply from the fact that in each step (BDdS1) or (BDdS2), one root is erased. In the cases A_n, B_n, C_n, BC_n and D_n , one easily constructs the p.n. subroot lattices in $k_1(L, L_1)$ steps. Therefore then $k_3(L, L_1) \leq k_1(L, L_1)$, and equalities hold in (38).

In the cases E_6, E_7, E_8, F_4, G_2 one observes $|J| = k_1(L, L_1)$ in the Tables 3.2–3.6. In Lemma 3.7 $|J|$ steps of type (BDsD1) and (BDsD2) are used. Therefore $k_3(L, L_1) \leq k_1(L, L_1)$, and equalities hold in (38). ■

4. Any generating set of roots contains a \mathbb{Z} -basis

The purpose of this section is to prove the following theorem. It is crucial in the proof of Theorem 6.2. But it may be also of some independent interest. The proof in the cases A_n, B_n, C_n, BC_n, D_n is an almost trivial application of the graphs in Definition 3.5 and Lemma 3.6. The proof in the cases E_6, E_7, E_8, F_4, G_2 is more involved.

Theorem 4.1. *Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice. Let $A \subset \Phi$ be any set of roots which generates the lattice L as a \mathbb{Z} -module. Then A contains a \mathbb{Z} -basis of L .*

In the case of vector spaces instead of \mathbb{Z} -modules, the analogous statement is trivial. For \mathbb{Z} -modules it is not true in general, that any generating set contains a basis. For example the lattice \mathbb{Z}^2 with standard basis $(1, 0), (0, 1)$ has the set $\{(1, 0), (1, 2), (0, 3)\}$ as generating set, but any two of these elements generate a proper sublattice.

The rest of this section is devoted to the proof of Theorem 4.1. It is obviously sufficient to prove it in the cases where L is an irreducible p.n. root lattice.

In the cases A_n, B_n, C_n, BC_n, D_n , Theorem 4.1 is an immediate consequence of the following lemma. The lemma follows directly from Lemma 3.6.

Lemma 4.2. *Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice in Theorem 2.3(d) of one of the types A_n, B_n, C_n, BC_n, D_n . Let $A = \{\alpha_1, \dots, \alpha_l\} \subset \Phi$ be a nonempty subset. The properties whether A is a generating set of L or a \mathbb{Z} -basis of L , will be characterized by properties of the graph $\mathcal{G}(A)$ from Definition 3.5.*

(a) L of type A_n :

- (i) A generates L as a \mathbb{Z} -module $\iff \mathcal{G}(A)$ is connected.
- (ii) A is a \mathbb{Z} -basis of L $\iff \mathcal{G}(A)$ is a tree.

(b) L of type B_n :

- (i) A generates L as a \mathbb{Z} -module \iff each component of $\mathcal{G}(A)$ contains at least one vertex with a (automatically short) marking.
- (ii) A is a \mathbb{Z} -basis of L \iff each component of $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex, and the vertex has only one (automatically short) marking.

(c) L of type C_n :

- (i) A generates L as a \mathbb{Z} -module $\iff \mathcal{G}(A)$ is connected, and it contains a marking (automatically long) or a cycle with an odd number of dotted edges.

- (ii) A is a \mathbb{Z} -basis of $L \iff$ either $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex and the marking is simple, or $\mathcal{G}(A)$ is connected and contains no marking, but it contains exactly one cycle and the cycle has an odd number of dotted lines.
- (d) L of type BC_n :
- (i) A generates L as a \mathbb{Z} -module \iff each component of $\mathcal{G}(A)$ contains at least one vertex with a short marking.
- (ii) A is a \mathbb{Z} -basis of $L \iff$ each component of $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex, and there is only one marking, and the marking is short.
- (e) L of type D_n :
- (i) A generates L as a \mathbb{Z} -module $\iff \mathcal{G}(A)$ is connected, and it contains a cycle with an odd number of dotted edges.
- (ii) A is a \mathbb{Z} -basis of $L \iff \mathcal{G}(A)$ is connected and contains exactly one cycle, and the cycle has an odd number of dotted lines.

It remains to prove Theorem 4.1 in the cases G_2, F_4, E_6, E_7, E_8 . Let L be the root lattice of one of these types in Theorem 2.3(d). In each of these cases it is sufficient to consider a generating set $A = \{\alpha_1, \dots, \alpha_{n+1}\} \subset \Phi$ with $n+1$ elements (where n is the rank of the root lattice). The cases of bigger generating sets can be reduced to the case of such a set by an easy inductive argument.

There is an up to the sign unique linear combination

$$0 = \sum_{i=1}^{n+1} \lambda_i \alpha_i \quad \text{with } \lambda_i \in \mathbb{Z}, \gcd(\lambda_1, \dots, \lambda_{n+1}) = 1, \text{ not all } \lambda_i = 0. \quad (39)$$

It has to be shown that an index j with $\lambda_j = \pm 1$ exists.

Denote by $L_i \subset L$, $i \in \{1, \dots, n+1\}$, the subroot lattice generated by $A - \{\alpha_i\}$.

Then $\text{rank } L_i < n \iff \lambda_i = 0$. (40)

If this holds for some i then by induction on the rank of the lattice one can conclude that $A - \{\alpha_i\}$ contains a \mathbb{Z} -basis of this subroot lattice. Then this \mathbb{Z} -basis together with α_i forms a \mathbb{Z} -basis of L . Thus suppose that all $\lambda_i \notin \{0, \pm 1\}$. Then

$$[L : L_i] = |\lambda_i|, \quad L_i + \mathbb{Z}\alpha_i = L, \quad k_2(L, L_i) = 1.$$

A priori, there are 20 possible cases in the Tables 3.2–3.6, summarized in the Table of the next page.

In the cases G_2, F_4, E_6, E_7 , the only possible values for $|\lambda_i|$ are in $\{2, 3, 4\}$. The condition $\gcd(\lambda_1, \dots, \lambda_{n+1}) = 1$ tells that at least one $j \in \{1, \dots, n+1\}$ with $|\lambda_j| = 3$ exists.

The following more complicated argument gives the same conclusion in the case E_8 . Assume in the case E_8 that all $|\lambda_i|$ are in $\{2, 4, 5, 6\}$. Define the decomposition

$$I_1 := \{i \mid 2 \mid \lambda_i\}, \quad I_2 := \{i \mid \lambda_i = \pm 5\}$$

of $\{1, \dots, n+1\}$ into two disjoint subsets.

L	L_i	$ \lambda_i $	L	L_i	$ \lambda_i $
F_4	B_4	2	G_2	$A_1 + \tilde{A}_1$	2
	$C_3 + A_1$	2		A_2	3
	$A_2 + \tilde{A}_2$	3	E_8	D_8	2
	$A_3 + \tilde{A}_1$	4		$E_7 + A_1$	2
E_6	$A_5 + A_1$	2		$E_6 + A_2$	3
	$3A_2$	3		A_8	3
E_7	$D_6 + A_1$	2	$D_5 + A_3$	4	
	A_7	2	$A_7 + A_1$	4	
	$A_5 + A_2$	3	$2A_4$	5	
	$2A_3 + A_1$	4	$A_5 + A_2 + A_1$	6	

Because of $\gcd(\lambda_1, \dots, \lambda_{n+1}) = 1$, both are nonempty. Define the subroot lattices

$$\tilde{L}_1 := \sum_{i \in I_1} \mathbb{Z}\alpha_i, \quad \tilde{L}_2 := \sum_{i \in I_2} \mathbb{Z}\alpha_i.$$

Then $c := \gcd(\lambda_i \mid i \in I_1) \in \{2, 4, 6\}$, and $c^{-1} \sum_{i \in I_1} \lambda_i \alpha_i$ is a primitive vector in \tilde{L}_1 , and $5^{-1} \sum_{i \in I_2} \lambda_i \alpha_i$ is a primitive vector in \tilde{L}_2 . But

$$\sum_{i \in I_1} \lambda_i \alpha_i = - \sum_{i \in I_2} \lambda_i \alpha_i.$$

Therefore the order of the torsion part of L/\tilde{L}_1 is divisible by 5. But Table 3.4 contains only one type of subroot lattices with this property, the type $2A_4$. Therefore $|I_1| = 8, |I_2| = 1, I_2 = \{j_0\}$ for some index j_0 , and $\frac{5}{2}\alpha_{j_0} \in \tilde{L}_1 \subset L$, which is impossible. Therefore the assumption above that all $|\lambda_i|$ are in $\{2, 4, 5, 6\}$ was wrong.

In the cases G_2, F_4, E_6, E_7 the type of the subroot lattices L_j with $[L : L_j] = 3$ is unique, in the case E_8 there are two possibilities,

L	G_2	F_4	E_6	E_7	E_8
L_i	A_2	$A_2 + \tilde{A}_2$	$3A_2$	$A_5 + A_2$	$E_6 + A_2, A_8$

By renumbering the roots, we can assume $[L : L_{n+1}] = 3$.

The case **G₂**: The roots α_1 and α_2 generate an A_2 lattice and thus are long. Therefore α_3 is short. At least one of α_1 and α_2 is not orthogonal to α_3 . That root and α_3 form a \mathbb{Z} -basis of L .

The cases **F₄, E₆, E₇** and the case **(E₈, E₆ + A₂)**: The sublattice $L_{n+1} \subset L$ (with \mathbb{Z} -basis $\alpha_1, \dots, \alpha_n$) contains one orthogonal summand \tilde{L}_1 of type A_2 . Suppose that α_1 and α_2 form a \mathbb{Z} -basis of this lattice \tilde{L}_1 . Then $\tilde{L}_2 := \sum_{i \in \{1, 2, n+1\}} \mathbb{Z}\alpha_i$ is a subroot lattice with $\tilde{L}_1 \subset \tilde{L}_2 \subset L$ and $\tilde{L}_2 \not\subset L_{n+1}$ and $L_{n+1} \not\subset \tilde{L}_2$. Because of $0 = \sum_{i=1}^{n+1} \lambda_i \alpha_i$ and all $\lambda_i \neq 0$ and $n + 1 > 3$, the root α_{n+1} is not in $\tilde{L}_{1, \mathbb{Q}}$, so the lattice \tilde{L}_2 has rank 3. The sum

$$\sum_{i \in \{1, 2, n+1\}} \lambda_i \alpha_i = - \sum_{i=3}^n \lambda_i \alpha_i$$

is in the sum $\tilde{L}_{1,\mathbb{R}}^\perp \cap L_{n+1}$ of the other orthogonal summands of L_{n+1} and in the rank one \mathbb{Z} -lattice $\tilde{L}_{1,\mathbb{R}}^\perp \cap \tilde{L}_2$. In fact, it is a generator of this rank one lattice: This is equivalent to $c_1 = 1$ where

$$c_1 := \gcd(\lambda_1, \lambda_2, \lambda_{n+1}).$$

If $c_1 > 1$, then $c_1^{-1} \sum_{i \in \{1,2,n+1\}} \lambda_i \alpha_i$ were in the root lattice \tilde{L}_2 . But then also $c_1^{-1} \sum_{i=3}^n \lambda_i \alpha_i$ were in the root lattice L , thus c_1 would divide $\lambda_3, \dots, \lambda_n$. Then $\gcd(\lambda_1, \dots, \lambda_{n+1}) > 1$ which is not true.

If L is of type E_6, E_7 or E_8 , the root lattice \tilde{L}_2 is of type A_3 . If L is of type F_4 , the root lattice \tilde{L}_2 cannot be of type C_3 , because the roots in \tilde{L}_1 are long. Therefore then \tilde{L}_2 is either of type A_3 or of type B_3 . In all cases, the following lemma gives the claim.

Lemma 4.3. *In both cases, at least one of λ_1 and λ_2 is equal to ± 1 .*

Proof: (a) The case \tilde{L}_2 of type A_3 : Embed $\tilde{L}_{2,\mathbb{R}}$ as usual into a Euclidean space \mathbb{R}_4 with ON-basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$ such that $\Phi(\tilde{L}_2) = \{\pm(\tilde{e}_i - \tilde{e}_j) \mid 1 \leq i < j \leq 4\}$ and $\Phi(\tilde{L}_1) = \{\pm(\tilde{e}_i - \tilde{e}_j), \mid 1 \leq i < j \leq 3\}$. A generator of the rank one \mathbb{Z} -lattice $\tilde{L}_{1,\mathbb{R}}^\perp \cap \tilde{L}_2$ is obviously $\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 - 3\tilde{e}_4$. Thus

$$\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 - 3\tilde{e}_4 = \pm \sum_{i \in \{1,2,n+1\}} \lambda_i \alpha_i.$$

Lemma 4.2(a) applies to the graph $\mathcal{G}(\{\alpha_1, \alpha_2, \alpha_{n+1}\})$. The graph is a tree, as $\alpha_1, \alpha_2, \alpha_{n+1}$ is a \mathbb{Z} -basis of \tilde{L}_2 . At least two of the four vertices $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$ are leaves, so at least one of the three vertices $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ is a leaf. Let α_j with $j \in \{1, 2\}$ give the edge which contains this vertex. The coefficient 1 of this vertex must be equal to $\pm \lambda_j$ because the other two terms in the sum $\pm \sum_{i \in \{1,2,n+1\}} \lambda_i \alpha_i$ have no contribution to the coefficient of this vertex.

(b) The case \tilde{L}_2 of type B_3 : Embed $\tilde{L}_{2,\mathbb{R}}$ as usual into a Euclidean space \mathbb{R}^3 with ON-basis $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ such that $\Phi(\tilde{L}_2) = \{\pm\tilde{e}_i \pm \tilde{e}_j\} \cup \{\pm\tilde{e}_i\}$ and $\Phi(\tilde{L}_1) = \{\pm(\tilde{e}_i - \tilde{e}_j), \mid 1 \leq i < j \leq 3\}$. A generator of the rank one \mathbb{Z} -lattice $\tilde{L}_{1,\mathbb{R}}^\perp \cap \tilde{L}_2$ is obviously $\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3$.

Thus
$$\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 = \pm \sum_{i \in \{1,2,n+1\}} \lambda_i \alpha_i. \tag{41}$$

Lemma 4.2 applies to the graph $\mathcal{G}(\{\alpha_1, \alpha_2, \alpha_{n+1}\})$. As α_1 and α_2 are long roots, the graph is a tree with one marked vertex. Then at least one vertex \tilde{e}_j of the 3 vertices $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ has no marking and is a leaf. Then $\lambda_j = \pm 1$ as in (a). ■

The case **(E₈, A₈)**: We can choose a root basis $\delta_1, \dots, \delta_8$ of L and an additional root δ_9 such that they give rise to the extended Dynkin diagram in Remark 2.4 (v) and such that the subroot lattice L_{n+1} is generated by $\delta_1, \delta_2, \delta_4, \dots, \delta_9$. Further, we can embed $L_{\mathbb{R}} = L_{n+1,\mathbb{R}}$ into a Euclidean space \mathbb{R}^9 with ON-basis $\tilde{e}_1, \dots, \tilde{e}_9$ such that

$$(\delta_1, \delta_2, \delta_4, \dots, \delta_9) = (\tilde{e}_1 - \tilde{e}_2, \tilde{e}_2 - \tilde{e}_3, \tilde{e}_3 - \tilde{e}_4, \dots, \tilde{e}_8 - \tilde{e}_9). \tag{42}$$

Part (a) of the following lemma tells how $\Phi(L)$ can be expressed using the $\tilde{e}_1, \dots, \tilde{e}_9$. Part (b) solves the case (E_8, A_8) and finishes the proof of Theorem 4.1.

Lemma 4.4.

$$(a) \quad \delta_3 = \frac{1}{3} \left(-2 \sum_{i=1}^3 \tilde{e}_i + \sum_{i=4}^9 \tilde{e}_i \right), \tag{43}$$

$$\Phi(A_8) = \{ \pm(\tilde{e}_i - \tilde{e}_j) \mid 1 \leq i < j \leq 9 \},$$

$$\begin{aligned} \Phi(E_8) &= \Phi(A_8) \cup \left\{ \pm \frac{1}{3} \left(-2 \sum_{i \in I_1} \tilde{e}_i + \sum_{i \in I_2} \tilde{e}_i \right), \right. \\ &\quad \left. |I_1 \cup I_2 = \{1, \dots, 9\}, |I_1| = 3, |I_2| = 6 \right\}. \end{aligned} \tag{44}$$

(b) *Above, at least one of the λ_j with $1 \leq j \leq 8$ is equal to ± 1 .*

Proof: (a) (43) follows from (42) and the relation

$$0 = 2\delta_1 + 4\delta_2 + 3\delta_3 + 6\delta_4 + 5\delta_5 + 4\delta_6 + 3\delta_7 + 2\delta_8 + \delta_9,$$

see Lemma 3.7 (c). As $\Phi(E_8)$ contains δ_3 , it contains the combination of \tilde{e}_i on the right hand side of (43). As the Weyl group $W(A_8) \subset W(E_8)$ consists of all permutations of $\tilde{e}_1, \dots, \tilde{e}_9$, the root system $\Phi(E_8)$ contains the right hand side of (44). Counting the size of the right hand side, one finds

$$2 \binom{9}{2} + 2 \binom{9}{3} = 72 + 168 = 240 = |\Phi(E_8)|,$$

thus equality holds in (44).

(b) The roots $\alpha_1, \dots, \alpha_8$ form a \mathbb{Z} -basis of L_{n+1} . The root α_9 must be a root in $\Phi(L) - \Phi(L_{n+1})$, so it must be

$$\alpha_9 = \pm \frac{1}{3} \left(-2 \sum_{i=1}^3 \tilde{e}_{\pi(i)} + \sum_{i=4}^9 \tilde{e}_{\pi(i)} \right) \quad \text{for some } \pi \in S_9.$$

The graph $\mathcal{G}(\{\alpha_1, \dots, \alpha_8\})$ is a tree by Lemma 4.2 (a).

1st Case. At least one of the roots $\tilde{e}_{\pi(4)}, \dots, \tilde{e}_{\pi(9)}$ is a leaf in this graph: Let α_j be the only edge which contains this leaf. Then $\lambda_j \alpha_j$ contains the only contribution to the leaf, in the right hand side of the following formula,

$$-2 \sum_{i=1}^3 \tilde{e}_{\pi(i)} + \sum_{i=4}^9 \tilde{e}_{\pi(i)} = \pm 3\alpha_9 = \pm \sum_{j=1}^8 \lambda_j \alpha_j.$$

Thus, $\lambda_j = \pm 1$.

2nd Case. None of the roots $\tilde{e}_{\pi(4)}, \dots, \tilde{e}_{\pi(9)}$ is a leaf in the graph: Then there are two or three leaves, and they form a subset of the set $\{\tilde{e}_{\pi(1)}, \tilde{e}_{\pi(2)}, \tilde{e}_{\pi(3)}\}$. For one of these leaves, the number of vertices on the path from this leaf to the branching vertex (in the case of three leaves) or to the unique inner vertex which is in $\{\tilde{e}_{\pi(1)}, \tilde{e}_{\pi(2)}, \tilde{e}_{\pi(3)}\}$, is maximal. Then the first two edges within this path, which starts at the leaf, have the coefficients λ_j with values $\pm 2, \pm 1$. So ± 1 arises. ■

This finishes the proof of Theorem 4.1. ■

5. Reduced presentations of Weyl group elements

Carter studied and classified the conjugacy classes of the elements of the Weyl groups of the irreducible root lattices. Here we will review a part of his results and extend them. Crucial are the (in the inhomogeneous cases new) notions of quasi-Coxeter elements and strict quasi-Coxeter elements. The control of these elements reduces the classification of conjugacy classes of Weyl group elements to the control of subroot lattices in section 3. But first some definitions will be given.

Definition 5.1. Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice with Weyl group W .

(a) For any element $w \in W$ any tuple $(\alpha_1, \dots, \alpha_k) \in \Phi^k$ with $k \in \mathbb{Z}_{\geq 0}$ and

$$w = s_{\alpha_1} \circ \dots \circ s_{\alpha_k} \quad (45)$$

is a *presentation of w* . Its *length* is $k \in \mathbb{Z}_{\geq 0}$. The length $l(w) \in \mathbb{Z}_{\geq 0}$ of w is the minimum of the lengths of all presentations. A presentation with $k = l(w)$ is called *reduced*. The subroot lattice of a presentation $(\alpha_1, \dots, \alpha_k)$ is $L_1 := \sum_{i=1}^k \mathbb{Z} \cdot \alpha_i$. The *index* of the presentation is the index $[L \cap L_{1, \mathbb{Q}} : L_1] \in \mathbb{Z}_{\geq 1}$ of the subroot lattice L_1 .

(b) An element w is of *maximal length* if $l(w) = n :=$ the rank of the root lattice.

(c) For any element $w \in W$ and any $\lambda \in S^1$ define

$$V_\lambda(w) := \ker(w - \lambda \cdot \text{id}) \subset L_{\mathbb{C}}, \quad (46)$$

$$V_{\neq 1}(w) := \bigoplus_{\lambda \neq 1} V_\lambda(w) \supset V_{\neq 1, \mathbb{R}} := L_{\mathbb{R}} \cap V_{\neq 1}(w), \quad (47)$$

and analogously $V_{\neq 1, \mathbb{Q}}(w)$, $V_{\neq 1, \mathbb{Z}}(w)$, $V_{1, \mathbb{R}}(w)$, $V_{1, \mathbb{Q}}(w)$, $V_{1, \mathbb{Z}}(w)$.
Of course $V_{\neq 1, \mathbb{R}} = V_{1, \mathbb{R}}^\perp$.

Lemma 5.2. Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice with Weyl group W .

(a) [6, Lemmata 2 and 3] A presentation $(\alpha_1, \dots, \alpha_k)$ of an element $w \in W$ is reduced if and only if $\alpha_1, \dots, \alpha_k$ are linearly independent (in $L_{\mathbb{Q}}$). The subroot lattice $L_1 \subset L$ of a reduced presentation satisfies

$$\left(\bigoplus_{i=1}^{l(w)} \mathbb{Q} \cdot \alpha_i = \right) L_{1, \mathbb{Q}} = V_{\neq 1, \mathbb{Q}}(w) \quad (48)$$

$$\text{and especially } l(w) = \dim V_{\neq 1, \mathbb{Q}}(w). \quad (49)$$

So, the subroot lattices of all reduced presentations of w generate the same subspace of $L_{\mathbb{C}}$, and it is $V_{\neq 1}(w)$.

(b) [9, Satz 3.2][11, Satz 3.2.3] If $(L, (\cdot, \cdot), \Phi)$ is a homogeneous root lattice, then all reduced presentations of one element $w \in W$ have the same index.

The following definition of a quasi-Coxeter element is in the homogeneous cases due to Voigt [11, Def. 3.2.1] and in the inhomogeneous cases new.

Definition 5.3. Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice of rank $n \in \mathbb{Z}_{> 0}$ with Weyl group W .

- (a) An element $w \in W$ is a *quasi-Coxeter element* if a reduced presentation of w exists whose subroot lattice is the full root lattice L . Of course then it is of *maximal length* $l(w) = n$.
- (b) An element $w \in W$ is a *strict quasi-Coxeter element* if the subroot lattice of any reduced presentation is the full root lattice L . Of course then it is a quasi-Coxeter element.

Remark 5.4. (i) An element w in the Weyl group of a p.n. root lattice has many presentations. Often there are several presentations such that the isomorphism classes of their subroot lattices are different. In the homogeneous cases at least their indices are equal. But in the inhomogeneous cases, even their indices can differ.

(ii) In a homogeneous root lattice, Lemma 5.2 (b) implies that there the notions of quasi-Coxeter element and of strict quasi-Coxeter element coincide. But in any irreducible inhomogeneous root lattice, there are quasi-Coxeter elements which are not strict quasi-Coxeter elements. See theorem 5.6.

(iii) Of course, if $(\alpha_1, \dots, \alpha_k)$ is a reduced presentation of a Weyl group element w , then w is a quasi-Coxeter element in the subroot lattice L_1 of this presentation. And of course, any Weyl group element has a reduced presentation such that it is a strict quasi-Coxeter element in the subroot lattice L_1 of this presentation.

(iv) Let $L = \bigoplus_{k \in K} L_k$ be the decomposition of a p.n. root lattice into an orthogonal sum of irreducible p.n. root lattices, and let $w \in W$ be a (strict) quasi-Coxeter element. Then it decomposes into a product $\prod_{k \in K} w_k$ of commuting elements $w \in W(L_k)$, and w_k is a (strict) quasi-Coxeter element in L_k .

(v) Recall that a Coxeter element in an irreducible root lattice is an element $w \in W$ which has a presentation $(\alpha_1, \dots, \alpha_n)$ such that $\alpha_1, \dots, \alpha_n$ form a root basis. Because their Dynkin diagram is a tree, Lemma 1 in [4, Ch. V §6] implies that the products of $s_{\alpha_1}, \dots, s_{\alpha_n}$ in any order are conjugate. As all root bases are conjugate, all Coxeter elements are conjugate. Obviously the Coxeter elements are quasi-Coxeter elements. It turns out that they are even strict quasi-Coxeter elements, see Theorem 5.6.

(vi) Carter's work [6] on the classification of Weyl group elements gives in a direct way the classification of the quasi-Coxeter elements in the irreducible homogeneous root lattices and in a less direct way the classification of the strict quasi-Coxeter elements in the irreducible inhomogeneous root lattices. In Theorem 5.6 these classifications will be given, and also the classification of the quasi-Coxeter elements in the irreducible inhomogeneous root lattices.

(vii) Recall the description of the Weyl group W in Remark 2.4(iii) for the root lattices of the types A_n, B_n, C_n, D_n in Theorem 2.3(d):

$$W(A_n) \cong S_{n+1}, \quad W(B_n) = W(C_n) \cong \{\pm 1\}^n \rtimes S_n.$$

A signed permutation in $\{\pm 1\}^n \rtimes S_n$ will be called *positive* if the number of sign changes in it is even, it will be called *negative* if the number of sign changes in it is odd. The subgroup $W(D_n) \subset W(B_n) = W(C_n)$ consists of the positive signed permutations.

A signed cycle will be written as $(\varepsilon_1 a_1 \varepsilon_2 a_2 \dots \varepsilon_k a_k)$ with $k \geq 1$ and $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$, $a_1, \dots, a_k \in \{1, \dots, n\}$ with $a_i \neq a_j$ for $i \neq j$. It maps $\pm a_i$ to $\pm \varepsilon_{i+1} a_{i+1}$ for

$1 \leq i \leq k-1$ and $\pm a_k$ to $\pm \varepsilon_1 a_1$. It is positive if $\prod_j \varepsilon_j = 1$ and negative if $\prod_j \varepsilon_j = -1$. Its *support* is defined to be $\{a_1, \dots, a_k\}$.

Any signed permutation is up to the order a unique product of signed cycles (=cyclic permutations) such that their supports are disjoint and the union of the supports is $\{1, \dots, n\}$. They are called the signed cycles of the signed permutation. Here cycles of length one are used. For example $\text{id} = (1)(2)\dots(n)$ and $-\text{id} = (-1)(-2)\dots(-n)$.

Remark 5.5. (i) Carter classified in [6] the conjugacy classes of Weyl group elements for all irreducible root lattices. A crucial point was the proof that any element w can be written as a product $w = w_1 w_2$ where w_1 and w_2 are involutions with $V_{-1}(w_1) \cap V_{-1}(w_2) = \{0\}$ (Proposition 38 and Corollary (ii) in [6]).

By [6, Lemma 5], any involution has a reduced presentation $(\alpha_1, \dots, \alpha_k)$ which consists of pairwise orthogonal roots. The composition of two such reduced presentations of two involutions w_1 and w_2 with $V_{-1}(w_1) \cap V_{-1}(w_2) = \{0\}$ is a reduced presentation of $w = w_1 w_2$. Its generalized Dynkin diagram is a graph whose cycles (if any exist) have all even length. In [6, Theorem A] all graphs are classified which have the following properties: The graph contains cycles, all cycles have even length, the graph is a generalized Dynkin diagram of a presentation of an element $w = w_1 w_2$ with w_1 and w_2 as above, the subroot lattice of the presentation is the full lattice, and w is not contained in the Weyl group of a subroot system. The graphs are labelled $D_n(a_k)$, $E_6(a_k)$, $E_7(a_k)$, $E_8(a_k)$, $F_4(a_1)$ and also $D_n(b_{n/2-1})$ if n is even and $E_7(b_2)$, $E_8(b_3)$, $E_8(b_5)$.

In fact, the graphs in [6] are simplified by not distinguishing normal and dotted edges. The generalized Dynkin diagrams are obtained from the graphs in [6] by replacing some edges by dotted edges such that any cycle obtains an odd number of dotted edges. This is possible.

It turns out that the graphs $D_n(a_k)$, $E_6(a_k)$, $E_7(a_k)$, $E_8(a_k)$, $F_4(a_1)$ correspond to conjugacy classes of Weyl group elements, and that these include the elements with presentations giving rise to the graphs $D_n(b_{n/2-1})$ (n even) and $E_7(b_2)$, $E_8(b_3)$, $E_8(b_5)$. These Weyl group elements are strict quasi-Coxeter elements, because they are not contained in the Weyl group of a subroot lattice. They are not Coxeter elements [6]. They and the Coxeter elements are the only strict quasi-Coxeter elements (Theorem 5.6 below).

(ii) Recall that the Coxeter elements in $W(A_n)$ are the cycles of length $n+1$ in S_{n+1} , the Coxeter elements in $W(B_n) = W(C_n) = W(BC_n)$ are the negative cycles of length n in $\{\pm 1\}^n \rtimes S_n$, and the Coxeter elements in $W(D_n)$ are the products of two negative cycles of lengths 1 and $n-1$. The products of two negative cycles of lengths k and $n-k$ for $2 \leq k \leq [n/2]$ form the conjugacy class $D_n(a_{k-1})$ in $W(D_n)$.

In the second column in the Tables 5.1 and 5.2, $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ and G_2 denote the conjugacy classes of the Coxeter elements. The root lattice of type F_4 contains subroot lattices of types B_4 , $C_3 + A_1$ and D_4 . In $W(F_4)$ the symbols $B_4, C_3 + A_1, D_4(a_1)$ denote the conjugacy classes in $W(F_4)$ of the Coxeter elements in $W(B_4)$ and $W(C_3 + A_1)$ and of the quasi-Coxeter elements of type $D_4(a_1)$ in $W(D_4)$. The Coxeter elements of the subroot lattice of type A_2 in G_2 give rise to a conjugacy class in $W(G_2)$ denoted by A_2 .

Theorem 5.6 gives the classification of the quasi-Coxeter elements and the strict quasi-Coxeter elements for the irreducible p.n. root lattices. A good part of it is due to [6].

Theorem 5.6. *Let $(L, (\cdot, \cdot), \Phi)$ be one of the irreducible p.n. root lattices in Theorem 2.3(d). The following Tables 5.1 and 5.2 list the conjugacy classes of the strict quasi-Coxeter elements and in the inhomogeneous cases the conjugacy classes of the quasi-Coxeter elements. See the Remarks 5.5 for the notations.*

Table 5.1:

	strict quasi-Coxeter el. = quasi-Coxeter el.
A_n	A_n
D_n	$D_n, D_n(a_1), \dots, D_n(a_{[n/2-1]})$
E_6	$E_6, E_6(a_1), E_6(a_2)$
E_7	$E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4)$
E_8	$E_8, E_8(a_1), \dots, E_8(a_8)$

Table 5.2:

	strict quasi-Coxeter el.	quasi-Coxeter el.
B_n	B_n	products of negative cycles s.t. the sum of their lengths is n
BC_n	–	products of negative cycles s.t. the sum of their lengths is n
C_n	C_n	$C_n, D_n, D_n(a_1), \dots, D_n(a_{[n/2-1]})$
F_4	$F_4, F_4(a_1)$	$F_4, F_4(a_1), B_4, C_3 + A_1, D_4(a_1)$
G_2	G_2	G_2, A_2

Thus the quasi-Coxeter elements of C_n are the products of one or two negative cycles such that the sum of their lengths is n .

Proof: It is well known that the Coxeter elements are not elements of some proper Weyl subgroup. Therefore they are strict quasi-Coxeter elements. The other elements listed in the second columns are strict quasi-Coxeter elements because of the results of Carter [6] discussed in the Remarks 5.5 (i).

By the same results, any other element $w \in W$ is in some proper Weyl subgroup. In the homogeneous cases, a proper Weyl subgroup is the Weyl group of a proper subroot lattice. Therefore then w is not a strict quasi-Coxeter element. This completes the proof of Table 5.1.

In the inhomogeneous cases, the fact that the second column of Table 5.2 lists all strict quasi-Coxeter elements is a consequence of the third column of Table 5.2, in the following way. In the cases of the root lattices of types F_4 and G_2 it is obvious that the quasi-Coxeter elements of types $B_4, C_3 + A_1, D_4(a_1)$ and A_2 are not strict quasi-Coxeter elements. In the cases of the p.n. root lattices B_n, C_n and BC_n , observe

$$\begin{aligned}
 s_{e_i} s_{e_j} &= s_{e_i+e_j} s_{e_i-e_j} = s_{2e_i} s_{2e_j} \sim (-i)(-j) && \text{and} && (50) \\
 \mathbb{Z}e_i + \mathbb{Z}e_j &\not\supseteq \mathbb{Z}(e_i + e_j) + \mathbb{Z}(e_i - e_j) \not\supseteq \mathbb{Z}2e_i + \mathbb{Z}2e_j && \text{for } i \neq j.
 \end{aligned}$$

This shows that all permutations whose signed cycles contain at least two negative cycles are not strict quasi-Coxeter elements. BC_n has no strict quasi-Coxeter elements because of $s_{e_i} = s_{2e_i}$. Therefore the only elements in the third column of Table 5.2 which are strict quasi-Coxeter elements are those in the second column.

It remains to prove the third column of Table 5.2.

The root lattice of type \mathbf{C}_n : Because of $L(C_n) = L(D_n)$ the quasi-Coxeter elements of D_n are also quasi-Coxeter elements of C_n . Let $w \in W(C_n)$ be a quasi-Coxeter element of C_n which is not a quasi-Coxeter element of D_n . Then it has a presentation $(\alpha_1, \dots, \alpha_n)$ such that its subroot lattice is the full root lattice $L(C_n)$ and $A := \{\alpha_1, \dots, \alpha_n\} \not\subset \Phi(D_n)$. By Lemma 4.2 (c)(ii) and (e)(ii) then the graph $\mathcal{G}(A)$ is a tree and contains exactly one marked vertex. Then w is a negative cycle, so a Coxeter element of C_n .

The p.n. root lattices of types \mathbf{B}_n and \mathbf{BC}_n : Let $w \in W(B_n) = W(BC_n)$ be a quasi-Coxeter element of B_n or BC_n , and let $(\alpha_1, \dots, \alpha_n)$ be a presentation whose subroot lattice is the full root lattice $L(B_n) = L(BC_n)$. By Lemma 4.2 (b)(ii) and (d)(ii), the graph $\mathcal{G}(A)$ for $A = \{\alpha_1, \dots, \alpha_n\}$ is a union of trees which have each exactly one marking and which is short. Thus w is a product of negative cycles. Vice versa, any product w of negative cycles has a presentation $(\alpha_1, \dots, \alpha_n)$ such that the graph $\mathcal{G}(\{\alpha_1, \dots, \alpha_n\})$ is a union of trees which have each exactly one marking and which is short. Thus w is a quasi-Coxeter element.

The root lattice of type \mathbf{G}_2 : Obviously its quasi-Coxeter elements are the products $s_\alpha s_\beta$ with α short and β long and $\alpha \not\perp \beta$ and the products $s_{\alpha_1} s_{\alpha_2}$ with α_1 and α_2 short and $\alpha_2 \neq \pm\alpha_1$. The elements of the first type are the Coxeter elements of G_2 , the elements of the second type can also be written as products $s_{\beta_1} s_{\beta_2}$ with β_1 and β_2 long roots and $\beta_2 \neq \pm\beta_1$. They are the Coxeter elements of the subroot system of long roots, which is of type A_2 .

The root lattice of type \mathbf{F}_4 : See Lemma 5.7 (b). The restriction there that in the presentation $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ first the short roots come and then the long roots, is not serious. One can obtain a presentation with this property from an arbitrary presentation using (13). \blacksquare

Lemma 5.7. *Let $(L, (\cdot, \cdot), \Phi)$ be the root lattice of type F_4 in Theorem 2.3(d). Obviously the short roots form a root system of type D_4 , which is called \tilde{D}_4 , and the long roots form a root system of type D_4 , which is called D_4 .*

- (a) *Let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \Phi(F_4)$ be a \mathbb{Z} -basis of $L(F_4)$ such that first the short roots come and then the long roots. Then one of the following cases holds.*
- (i) *All four roots are short.*
 - (ii) *α_1, α_2 and α_3 are short and α_4 is long. Then an element $w \in W(F_4)$ exist such that $w(\alpha_1), w(\alpha_2), w(\alpha_3)$ generate the subroot system of type \tilde{A}_3 which is also generated by $e_1, \frac{1}{2} \sum_{i=1}^4 e_i, e_2$, and then $w(\alpha_4) = \pm e_i \pm e_j$ with $i \in \{1, 2\}$ and $j \in \{3, 4\}$.*
 - (iii) *α_1 and α_2 are short and α_3 and α_4 are long. Then $\langle \alpha_1, \alpha_2 \rangle = \pm 1$ and $\langle \alpha_3, \alpha_4 \rangle = \pm 1$ and $\mathbb{R}\alpha_1 + \mathbb{R}\alpha_2 \not\perp \mathbb{R}\alpha_3 + \mathbb{R}\alpha_4$.*
- (b) *Let $w \in W(F_4)$ be a quasi-Coxeter element, and let $(\alpha_1, \dots, \alpha_4)$ be a presentation of w whose subroot lattice is the full lattice $L(F_4)$ and such that first the short*

roots come and then the long roots. Then the cases in (a) hold, and w is in each case as follows.

- (i) w is in $W(C_3 + A_1)$ and is a Coxeter element there, or it is in $W(D_4)$ and is a quasi-Coxeter element of type $D_4(a_1)$ there.
- (ii) w is in $W(L_1)$ for some subroot lattice L_1 of type B_4 and is a Coxeter element there.
- (iii) w is a Coxeter element of type F_4 or a strict quasi-Coxeter element of type $F_4(a_1)$.

Proof: (a) The following obvious statements will be used:

- (A) The root lattice $L(F_4)$ contains the subroot lattice L_2 of type B_4 with root system

$$\Phi(L_2) = \{\pm e_i \mid i \in \{1, 2, 3, 4\}\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}.$$

It has the same long roots as $\Phi(F_4)$, but less short roots.

- (B) For any short root β_1 a Weyl group element w exists such that $w(\beta_1) = e_1$. If β_2 is a short root with $\beta_1 \perp \beta_2$ then $w(\beta_2) \in \{\pm e_2, \pm e_3, \pm e_4\}$, and w can be chosen such that $w(\beta_2) = e_2$.

The case that all four roots $\alpha_1, \dots, \alpha_4$ are long is impossible because they would only generate the subroot lattice $L(D_4)$. The case that α_1 is short and the three roots $\alpha_2, \alpha_3, \alpha_4$ are long, is also impossible, because by (B) a Weyl group element exists such that $w(\alpha_1) = e_1$, and then all four images $w(\alpha_i)$ are in L_2 . Thus either two roots are short and two roots are long, or three roots are short and one root is long, or all four roots are short.

Consider the case that $\alpha_1, \alpha_2, \alpha_3$ are short and α_4 is long. Then α_1, α_2 and α_3 generate a subroot system of rank 3 of $\Phi(\tilde{D}_4)$. Only the two types \tilde{A}_3 and $3\tilde{A}_1$ are possible a priori. Here the type $3\tilde{A}_1$ is not possible, because then by (B) an element $w \in W(F_4)$ exists such that $w(\alpha_i) = e_i$ for $i \in \{1, 2, 3\}$, and these roots and any long root $w(\alpha_4)$ are in L_2 .

A subroot system of type \tilde{A}_3 in \tilde{D}_4 is the set of short roots of a subroot system of type C_3 in L (which is of type F_4). By Theorem 3.3 (b), any two subroot systems of type C_3 in L are conjugate by an element of $W(F_4)$. Therefore any two subroot systems of type \tilde{A}_3 in \tilde{D}_4 are conjugate by an element of $W(F_4)$. This shows the first half of part (ii). Obviously $w(\alpha_4) = \pm e_i \pm e_j$ with $i \in \{1, 2\}$ and $j \in \{3, 4\}$. This gives part (ii).

Consider the case that α_1 and α_2 are short and α_3 and α_4 are long. If $\alpha_1 \perp \alpha_2$ then by (B) $\alpha_1, \dots, \alpha_4$ are mapped by a suitable element $w \in W(F_4)$ into the subroot lattice L_2 . Therefore $\langle \alpha_1, \alpha_2 \rangle = \pm 1$. Furthermore

$$\mathbb{R}\alpha_1 + \mathbb{R}\alpha_2 \not\subset \mathbb{R}\alpha_3 + \mathbb{R}\alpha_4$$

because else the four roots would generate a reducible subroot lattice. An element $w \in W(F_4)$ exists such that

$$w(\alpha_1) = \pm e_1, w(\alpha_2) = \frac{1}{2} \sum_{i=1}^4 e_i.$$

If $\alpha_3 \perp \alpha_4$ then either

$$w(\alpha_3) = \varepsilon_1(e_i + \varepsilon_2 e_j) \text{ and } w(\alpha_4) = \varepsilon_3(e_i - \varepsilon_2 e_j)$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$ and some i, j with $1 \leq i < j \leq 4$, or

$$w(\alpha_3) = \varepsilon_1(e_i + \varepsilon_2 e_j) \text{ and } w(\alpha_4) = \varepsilon_3(e_k - \varepsilon_4 e_l)$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{\pm 1\}$ and some i, j, k, l with $\{i, j, k, l\} = \{1, 2, 3, 4\}$. One sees easily with some case discussion that in both cases $w(\alpha_1), \dots, w(\alpha_4)$ do not generate $L(F_4)$.

(b) Of course, the cases in (a) hold.

The case (i): w is in $W(\tilde{D}_4)$ and is either a Coxeter element there or a quasi-Coxeter element of type $\tilde{D}_4(a_1)$. In the first case, w is conjugate to

$$s_{\frac{1}{2} \sum_i e_i} s_{e_1} s_{e_2} s_{e_3} = s_{\frac{1}{2} \sum_i e_i} s_{e_1} s_{e_2+e_3} s_{e_2-e_3},$$

which is in $W(C_3 + A_1)$ and which is a Coxeter element there. In the second case, w is conjugate to

$$s_{\frac{1}{2} \sum_i e_i} s_{\frac{1}{2}(e_1+e_2-e_3-e_4)} s_{e_1} s_{e_3} = s_{e_1+e_2} s_{e_3+e_4} s_{e_1+e_3} s_{e_1-e_3},$$

which is in $W(D_4)$ and which is a quasi-Coxeter element of type $D_4(a_1)$ there.

The case (ii): The element w is conjugate to

$$s_{\frac{1}{2} \sum_a e_a} s_{e_1} s_{e_2} s_{\beta} = s_{\frac{1}{2} \sum_a e_a} s_{e_1-e_2} s_{e_1+e_2} s_{\beta}$$

for some $\beta = e_i + \varepsilon e_k$ with $i \in \{1, 2\}$ and $k \in \{3, 4\}$ and $\varepsilon \in \{\pm 1\}$. This is in $W(L_3)$ for a subroot lattice L_3 of type B_4 . In the case $\varepsilon = -1$ the generalized Dynkin diagram of the four roots on the right hand side is (up to the distinction between dotted and normal edges) the B_4 Dynkin diagram, so then the element is a Coxeter element in $W(L_3)$. In the case $\varepsilon = 1$, the right hand side is equal to

$$s_{\frac{1}{2} \sum_a e_a} s_{e_1-e_2} s_{-e_j+e_k} s_{e_1+e_2},$$

where j is determined by $\{i, j\} = \{1, 2\}$. This is again a Coxeter element in $W(L_3)$.

The case (iii): Using (13) for α_1 and α_2 , one can suppose $\mathbb{R}\alpha_2 \not\subset \mathbb{R}\alpha_3 + \mathbb{R}\alpha_4$. After conjugation, one can suppose

$$\alpha_1 = \frac{1}{2} \sum_{i=1}^4 \pm e_i, \alpha_2 = \pm e_1, \{\alpha_3, \alpha_4\} \subset \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\}.$$

Using (13) for α_3, α_4 and changing possibly some signs and conjugating possibly again, one can suppose

$$\alpha_1 = \frac{1}{2}(e_1 + \varepsilon_2 e_2 + \varepsilon_3 e_3 + e_4), \alpha_2 = e_1, \alpha_3 = e_2 - e_3, \alpha_4 = e_1 - e_2$$

for some $\varepsilon_2, \varepsilon_3 \in \{\pm 1\}$.

In the case $(\varepsilon_2, \varepsilon_3) = (1, 1)$, the generalized Dynkin diagram of the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is (up to the distinction between dotted and normal edges) a Dynkin diagram of type F_4 . Thus w is a Coxeter element in $W(F_4)$.

In the case $(\varepsilon_2, \varepsilon_3) = (1, -1)$, the element w is conjugate to the product of the two involutions $s_{\alpha_4}s_{\alpha_1}$ and $s_{\alpha_2}s_{\alpha_3}$ with admissible diagram of type $F_4(a_1)$. Thus it is a quasi-Coxeter element in $W(F_4)$ of type $F_4(a_1)$.

In the case $(\varepsilon_2, \varepsilon_3) = (-1, -1)$, the element w is

$$w = s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4} = s_{\alpha_2}s_{s_{\alpha_2}(\alpha_1)}s_{s_{\alpha_3}(\alpha_4)}s_{\alpha_3} = s_{e_1}s_{\frac{1}{2}(-e_1-e_2-e_3+e_4)}s_{e_1-e_3}s_{e_2-e_3}.$$

The generalized Dynkin diagram of the roots on the right hand side is (up to the distinction between dotted and normal edges) a Dynkin diagram of type F_4 . Thus w is a Coxeter element in $W(F_4)$.

In the case $(\varepsilon_2, \varepsilon_3) = (-1, 1)$, the element w is

$$w = s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4} = s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4} = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{s_{\alpha_4}(\alpha_2)} = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{e_2}.$$

This is conjugate to the element

$$s_{e_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_4} = s_{s_{e_2}(\alpha_1)}s_{e_2}s_{s_{\alpha_3}(\alpha_4)}s_{\alpha_3} = s_{\frac{1}{2}\sum_i e_i}s_{e_2}s_{e_1-e_3}s_{e_2-e_3}.$$

The generalized Dynkin diagram of the roots on the right hand side is (up to the distinction between dotted and normal edges) a Dynkin diagram of type F_4 . Thus w is a Coxeter element in $W(F_4)$. ■

Remark 5.8. (i) In the Tables 7–11 in [6] all conjugacy classes of elements of the Weyl groups of the root lattices of types G_2, F_4, E_6, E_7 and E_8 are listed in the following form. For any conjugacy class one element and one presentation of it as a strict quasi-Coxeter element is chosen. The tables show the isomorphism class of the pair of full lattice and subroot lattice and the type of the strict quasi-Coxeter element.

(ii) Theorem 5.10 below gives more information for the root lattices of types G_2, F_4, E_6, E_7 and E_8 . For F_4 it lists for any conjugacy class all isomorphism classes of pairs of full lattice and subroot lattice, which turns up as subroot lattice of a presentation as a quasi-Coxeter element, and the type of the quasi-Coxeter element. This gives all types of reduced presentations for any element. For G_2, E_6, E_7 and E_8 one can extract the same information from Theorem 5.10 and the Tables 3.6, 3.2, 3.3 and 3.4.

(iii) For the tables in Theorem 5.10, the notations in Table 5.2 have to be refined: There are three conjugacy classes of quasi-Coxeter elements in F_4 which have also presentations as Coxeter elements in subroot lattices of type B_4 and $C_3 + A_1$ respectively as a quasi-Coxeter element of type $D_4(a_1)$ in the subroot lattice of type D_4 . The presentations of these elements as quasi-Coxeter elements in F_4 are now called $F_4(a_2), F_4(a_3)$ and $F_4(a_4)$.

Analogously, the presentations as quasi-Coxeter elements in $W(G_2)$ of those elements which have also presentations as Coxeter elements in A_2 (the subroot lattice of long roots) are denoted by \tilde{A}_2 .

For $2 \leq k \leq 4$, the presentations as quasi-Coxeter elements in $W(B_k)$ of those elements in $W(B_k)$ which are products of negative cycles of lengths l_1, \dots, l_r with

$l_1 + \dots + l_r = k$ are denoted by $B_k(l_1, \dots, l_r)$. The case $B_k(k)$ is also denoted by B_k . This will be used in the Table 5.4 for F_4 . Similarly, $C_3(2, 1)$ is used there.

(iv) The complete control in Theorem 5.10 on the reduced presentations of the Weyl group elements of the irreducible root lattices allows to determine the number $k_4(L, w)$ in Definition 5.9.

Definition 5.9. Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice, and let w be a Weyl group element. Recall $k_2(L, L_1)$ from Theorem 3.8. Define the number

$$k_4(L, w) := \min \left\{ k_2(L, L_1) \mid \begin{array}{l} \text{a reduced presentation of } w \\ \text{with subroot lattice } L_1 \text{ exists} \end{array} \right\}. \tag{51}$$

This number will be important in Section 6. Because of $k_1(L, L_1) = k_2(L, L_1)$ (Theorem 3.8(b)),

$$k_4(L, w) \geq \dim L_{\mathbb{Q}}/V_{\neq 1, \mathbb{Q}}(w) = n - l(w). \tag{52}$$

Equality holds if and only if a reduced presentation with subroot lattice $L_1 = V_{\neq 1, \mathbb{Q}} \cap L$ exists. This is the unique primitive subroot lattice L_1 with $L_{1, \mathbb{Q}} = V_{\neq 1, \mathbb{Q}}$. Often equality holds, often not.

Theorem 5.10. *Let $(L, (\cdot, \cdot), \Phi)$ be one of the irreducible p.n. root lattices in Theorem 2.3(d).*

(a) *In the cases A_n, B_n and BC_n ,*

$$k_4(L, w) = n - l(w). \tag{53}$$

(b) *Consider in the cases C_n and D_n a Weyl group element w which is a product of r positive cycles and s negative cycles with disjoint supports whose union is $\{1, \dots, n\}$ (Remark 5.4(vii)). (In the case of D_n , s is even.) Then*

$$r = n - l(w),$$

and any reduced presentation with subroot lattice L_1 with minimal $k_2(L, L_1)$ satisfies

$$L/L_1 \cong \mathbb{Z}^r \times \mathbb{Z}_2^{\lfloor (s+1)/2 \rfloor}, \tag{54}$$

$$k_4(L, w) = k_2(L, L_1) = n - l(w) + \left\lfloor \frac{s+1}{2} \right\rfloor. \tag{55}$$

(c) *In the cases G_2, E_6, E_7 and E_8 , for the big majority of the Weyl group elements there is only one type of reduced presentations. That means, the pairs (L, L_1) are isomorphic where L_1 runs through the subroot lattices of all reduced presentations.*

Table 5.3 lists for the (conjugacy classes of the) exceptions the different ways to write them as quasi-Coxeter elements of subroot lattices L_1 , and it lists the numbers $k_4(L, w)$.

For the other elements, $k_4(L, w) = k_2(L, L_1)$ for the unique isomorphism class (L, L_1) . All these other elements can be found by replacing in the Tables 3.6, 3.2, 3.3 and 3.4 L_1 by the possible quasi-Coxeter elements with subroot lattice of type L_1 . See Table 5.1 for the possibilities. (E.g. $D_5 + A_3$ has to be replaced by the two possibilities $D_5 + A_3$ and $D_5(a_1) + A_3$.)

Table 5.3:

L	presentation of w as quasi-Coxeter element in $W(L_1)$ for some subroot lattice L_1	$k_4(L, w)$
G_2	$\tilde{A}_2 \sim A_2$	0
E_7	$D_4(a_1) + 2A_1 \sim 2A_3$	2
E_7	$D_4(a_1) + 3A_1 \sim 2A_3 + A_1$	1
E_7	$D_6(a_1) + A_1 \sim A_7$	1
E_8	$D_4(a_1) + 2A_1 \sim [2A_3]''$	3
E_8	$D_4(a_1) + 3A_1 \sim 2A_3 + A_1$	2
E_8	$D_5(a_1) + 2A_1 \sim D_4 + A_3$	2
E_8	$D_6(a_1) + A_1 \sim [A_7]''$	2
E_8	$D_4(a_1) + 4A_1 \sim 2A_3 + 2A_1$	2
E_8	$D_4 + D_4(a_1) \sim D_5(a_1) + A_3$	1
E_8	$D_5 + A_3 \sim A_7 + A_1 \sim D_6(a_1) + 2A_1$	1
E_8	$D_6(a_2) + 2A_1 \sim 2D_4$	2
E_8	$E_6(a_1) + A_2 \sim A_8$	1
E_8	$E_7(a_1) + A_1 \sim D_8$	1
E_8	$E_7(a_3) + A_1 \sim D_8(a_2)$	1

(d) In the case of F_4 , the following Table 5.4 lists for (the conjugacy classes of) all Weyl group elements all ways to write them as quasi-Coxeter elements of subroot lattices. See Remark 5.8(iii) for the notations. It also lists the numbers $k_4(L, w)$.

Table 5.4:

Presentation of w as quasi-Coxeter element in $W(L_1)$ for some subroot lattice L_1		$k_4(L, w)$
F_4		0
$F_4(a_1)$		0
$F_4(a_2) \sim B_4$		0
$F_4(a_3) \sim C_3 + A_1$		0
$F_4(a_4) \sim D_4(a_1) \sim B_4(2, 2)$		0
$B_4(3, 1) \sim D_4$		1
$B_4(2, 1, 1) \sim A_3 + \tilde{A}_1 \sim C_3(2, 1) + A_1 \sim B_2 + 2A_1$		1
$B_4(1, 1, 1, 1) \sim B_2(1, 1) + 2A_1 \sim 4A_1$		1
$A_2 + \tilde{A}_2$		1

w	$k_4(L, w)$	w	$k_4(L, w)$
B_3	1	A_2	2
$B_3(2, 1) \sim A_3$	1	B_2	2
$B_3(1, 1, 1) \sim 2A_1 + \tilde{A}_1$	1	$B_2(1, 1) \sim 2A_1$	2
$B_2 + A_1 \sim C_3(2, 1)$	1	$A_1 + \tilde{A}_1$	2
$B_2(1, 1) + A_1 \sim 3A_1$	2	\tilde{A}_2	2
$A_2 + \tilde{A}_1$	1	\tilde{A}_1	3
$A_1 + \tilde{A}_2$	1	A_1	3
C_3	1	\emptyset	4

Proof: (a) In the cases A_n, B_n, BC_n, C_n and D_n , any positive cycle in the Weyl group can be written as a product $s_{\alpha_1} \circ \dots \circ s_{\alpha_r}$ where $\alpha_1, \dots, \alpha_r$ are roots of the type $\pm e_i \pm e_j$ whose graph $\mathcal{G}(\{\alpha_1, \dots, \alpha_r\})$ is a tree. The subroot lattice $L_1 = \sum_{i=1}^r \mathbb{Z}\alpha_i$ is a primitive sublattice.

Any element of $W(A_n)$ is a product of positive cycles with disjoint supports whose union is $\{1, \dots, n\}$. Because the supports are disjoint, the sum of the subroot lattices of the presentations above of the positive cycles is also a primitive sublattice. Therefore there (53) holds.

In the cases B_n and BC_n , any negative cycle can be written as a product in the form $s_{\alpha_1} \circ \dots \circ s_{\alpha_r} \circ s_{\alpha_{r+1}}$ such that $s_{\alpha_1} \circ \dots \circ s_{\alpha_r}$ is a positive cycle with graph a tree, and such that α_{r+1} is a short root which gives a marking of one vertex of the tree. The subroot lattice $L_1 = \sum_{i=1}^{r+1} \mathbb{Z}\alpha_i$ is the primitive sublattice, which is generated by all the short roots which correspond to the vertices of the tree.

Any element of $W(B_n) = W(BC_n)$ is a product of positive cycles and/or negative cycles with disjoint supports whose union is $\{1, \dots, n\}$. Because the supports are disjoint, the sum of the subroot lattices of the presentations above of the positive and/or negative cycles is also a primitive sublattice. Therefore there (53) holds.

(b) In the cases C_n and D_n , any pair of negative cycles can be written as a product $s_{\alpha_1} \circ \dots \circ s_{\alpha_a} \circ s_{\beta_1} \circ \dots \circ s_{\beta_b} \circ s_{\alpha_{a+1}} \circ s_{\beta_{b+1}}$ such that $s_{\alpha_1} \circ \dots \circ s_{\alpha_a}$ and $s_{\beta_1} \circ \dots \circ s_{\beta_b}$ are positive cycles whose graphs are disjoint trees and such that $\alpha_{a+1} = e_i - e_j$ and $\beta_{b+1} = e_i + e_j$ with i a vertex of one tree and j a vertex of the other tree. The subroot lattice $L_1 = \sum_{i=1}^{a+1} \mathbb{Z}\alpha_i + \sum_{j=1}^{b+1} \mathbb{Z}\beta_j$ is of type C_{a+b+2} respectively D_{a+b+2} .

In the case C_n , any single negative cycle can be written as a product $s_{\alpha_1} \circ \dots \circ s_{\alpha_a} \circ s_{\alpha_{a+1}}$ such that $s_{\alpha_1} \circ \dots \circ s_{\alpha_a}$ is a positive cycle with graph a tree, and such that α_{a+1} is a long root of the type $2e_i$ which gives a marking of one vertex of the tree. The subroot lattice $L_1 = \sum_{i=1}^{a+1} \mathbb{Z}\alpha_i$ is of type C_{a+1} .

Let w be a Weyl group element which is a product of r positive cycles and s negative cycles with disjoint supports whose union is $\{1, \dots, n\}$. One presents the positive cycles as above (in the proof of (a)), and one presents as many pairs of negative cycles as above. At most one (none in the case D_n) single negative cycle is left and is also presented as above. Let L_1 be the subroot lattice of the presentation. Table 3.1 shows (54). One sees easily that no reduced presentation with smaller $k_1(L, L_1)$ exists. (55) holds.

(c) The Tables 7, 9, 10 and 11 in [6] list all conjugacy classes of elements of the Weyl groups of root lattices of the types G_2, E_6, E_7 and E_8 . They give in each case one type of presentation as a strict quasi-Coxeter element. It is easy to find all presentations as quasi-Coxeter elements which are not in the list. One has to find out which elements in the list are given also by these presentations. In most cases it is sufficient to compare the characteristic polynomials. A table of characteristic polynomials is Table 3 in [6].

The only cases where this is not sufficient arise for the E_8 root lattice and there for the presentations as quasi-Coxeter elements of types $D_4(a_1) + 2A_1$ and $D_6(a_1) + A_1$. In the first case the presentations as strict quasi-Coxeter elements of types $[2A_3]'$ and $[2A_3]''$ have the same characteristic polynomial, in the second case the presentations of types $[A_7]'$ and $[A_7]''$. Because of

$$\begin{aligned} \text{index}(D_4 + 2A_1) &= 2 = \text{index}([2A_3]'') \neq \text{index}([2A_3]') = 1, \\ \text{index}(D_6 + A_1) &= 2 = \text{index}([A_7]'') \neq \text{index}([A_7]') = 1, \end{aligned}$$

Lemma 5.2 (b) tells that $D_4(a_1) + 2A_1$ gives the same conjugacy class as $[2A_3]''$ and that $D_6(a_1) + A_1$ gives the same conjugacy class as $[A_7]''$.

(d) Table 8 in [6] lists 9, 8, 5, 2 and 1 conjugacy classes of elements of the Weyl group of type F_4 of lengths 4, 3, 2, 1 respectively 0. On the other hand there are 19, 12, 6, 2 and 1 types of presentations of elements as quasi-Coxeter elements of lengths 4, 3, 2, 1 respectively 0:

length	type of presentation as a quasi-Coxeter element
4	$F_4, F_4(a_1), F_4(a_2), F_4(a_3), F_4(a_4), B_4, B_4(3, 1), B_4(2, 2),$ $B_4(2, 1, 1), B_4(1, 1, 1, 1), A_3 + \tilde{A}_1, A_2 + \tilde{A}_2, C_3 + A_1,$ $C_3(2, 1) + A_1, D_4, D_4(a_1), B_2 + 2A_1, B_2(1, 1) + 2A_1, 4A_1$
3	$B_3, B_3(2, 1), B_3(1, 1, 1), B_2 + A_1, B_2(1, 1) + A_1, A_2 + \tilde{A}_1,$ $A_3, 2A_1 + \tilde{A}_1, A_1 + \tilde{A}_2, C_3, C_3(2, 1), 3A_1$
2	$B_2, B_2(1, 1), \tilde{A}_2, A_1 + \tilde{A}_1, A_2, 2A_1$
1	\tilde{A}_1, A_1
0	\emptyset

For those types of presentations as quasi-Coxeter elements in the table above which are not in the Table 8 in [6], one has to find out which conjugacy classes they give. In many cases this is determined by the characteristic polynomials. The cases where the characteristic polynomials is not sufficient, can be drawn from Lemma 26 in [6]. It lists the presentations as strict quasi-Coxeter elements which give different conjugacy classes, but with the same characteristic polynomials. Of the 8 pairs in Lemma 26 in [6], only those 4 are relevant here, for which presentations as quasi-Coxeter elements exist which are not in Table 8 in [6] and which have the same characteristic polynomials. These 4 pairs and their characteristic polynomials are as follows:

D_4	A_3	$3A_1$	$2A_1$
$C_3 + A_1$	$B_2 + A_1$	$2A_1 + \tilde{A}_1$	$A_1 + \tilde{A}_1$
$(t^3 + 1)(t + 1)$	$t^3 + t^2 + t + 1$	$(t + 1)^3(t - 1)$	$(t + 1)^2(t - 1)^2$

The equality

$$s_{e_3} s_{e_4} = s_{e_3+e_4} s_{e_3-e_4}$$

tells that in Table 5.4

$$B_2(1, 1) \sim 2A_1, B_2(1, 1) + A_1 \sim 3A_1, B_3(1, 1, 1) \sim 2A_1 + \tilde{A}_1.$$

The equality $s_{e_1} s_{\frac{1}{2}(e_1+e_2-e_3-e_4)} s_{\frac{1}{2}(e_1+e_2+e_3+e_4)} = s_{e_1} s_{e_1+e_2} s_{e_3+e_4}$

tells that in Table 5.4 $C_3(2, 1) \sim B_2 + A_1.$

The equalities $s_{e_2-e_3} s_{e_3} s_{e_4} = s_{e_2-e_3} s_{e_3+e_4} s_{e_3-e_4}$

and $s_{e_1-e_2} s_{e_2-e_3} s_{e_3} s_{e_4} = s_{e_1-e_2} s_{e_2-e_3} s_{e_3+e_4} s_{e_3-e_4}$

tell that in Table 5.4 $B_3(2, 1) \sim A_3$ and $B_4(3, 1) \sim D_4.$

The equivalence $F_4(a_3) \sim C_3 + A_1$ in Table 5.4 holds by definition of $F_4(a_3)$. All other equivalences in Table 5.4 follow from comparison of characteristic polynomials. ■

Remark 5.11. (i) From the Theorems 5.10, 5.6 and 3.3 (respectively the first columns of the Tables 3.1–3.6), one can recover the classification of conjugacy classes of the Weyl group elements of the root lattices of types G_2, F_4, E_6, E_7 and E_8 which is given in the Tables 7–11 in [6].

(ii) The proof above of Theorem 5.10 had used these tables, but not in a very crucial way. Those few cases where different conjugacy classes have the same characteristic polynomials, can be dealt with by hand. In fact, informations on them are given in the Lemmata 26 and 27 in [6]. But Theorem 5.6 on the (strict) quasi-Coxeter elements depends in a crucial way on the results in [6].

(iii) The characteristic polynomials of the strict quasi-Coxeter elements in all irreducible root lattices are given in Table 3 in [6].

(iv) In [11, (2.3.4)] a table similar to Table 5.3 for E_7 and E_8 is given. But one of the cases for E_7 and four of the cases for E_8 are missing there. The case for E_7 which is missing in [11, (2.3.4)], is also missing in [11, (3.2.9)].

Remark 5.12. There is a strange correspondence. Define for any irreducible root lattice $(L, (\cdot, \cdot), \Phi)$ the two numbers

$$k_6(L) := |\{\text{conjugacy classes of quasi-Coxeter elements}\}| - 1,$$

$$k_7(L) := \left| \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (L, L_1) \text{ with } L_1 \\ \text{a subroot lattice of full rank with } k_1(L, L_1) = 1 \end{array} \right\} \right|.$$

Then $k_6(L) = k_7(L)$ for $A_n, C_n, D_n, F_4, E_6, E_7, E_8$ and B_2 , but not for B_n ($n \geq 3$) and G_2 , as the following table shows.

	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
$k_6(L)$	0	$p(n) - 1$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor - 1$	1	4	2	4	8
$k_7(L)$	0	$n - 1$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor - 1$	2	4	2	4	8

Here $p(n)$ is the number of partitions of n .

6. Nonreduced presentations of Weyl group elements

Definition 6.1. Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice, and let w be a Weyl group element. Define the number

$$k_5(L, w) := \min \left\{ k \mid \begin{array}{l} \text{a presentation } (\alpha_1, \dots, \alpha_{l(w)+2k}) \text{ with} \\ \text{subroot lattice the full lattice exists} \end{array} \right\}. \tag{56}$$

Recall the Definition (51) of the number $k_4(L, w)$ in the same situation. Let $(\alpha_1, \dots, \alpha_{l(w)})$ be a reduced presentation of an element w with subroot lattice L_1 such that $k_2(L, L_1)$ is minimal, i.e. $k_2(L, L_1) = k_4(L, w) =: k$. Let β_1, \dots, β_k be roots such that $L_1 + \sum_{j=1}^k \mathbb{Z}\beta_j = L$. Then obviously $(\alpha_1, \dots, \alpha_{l(w)}, \beta_1, \beta_1, \beta_2, \beta_2, \dots, \beta_k, \beta_k)$ is a presentation with root lattice the full root lattice L . Therefore

$$k_5(L, w) \leq k_4(L, w). \tag{57}$$

Theorem 6.2. *Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice, and let w be a Weyl group element. Then*

$$k_5(L, w) = k_4(L, w). \tag{58}$$

The proof consists of a reduction to the special case in the following lemma and of the proof of the following lemma. The proof of the lemma is given first.

Lemma 6.3. *Let $(L, (\cdot, \cdot), \Phi)$ be a p.n. root lattice of some rank n , and let w be a Weyl group element of length $n - 1$. Then*

$$k_5(L, w) = 1 \iff k_4(L, w) = 1. \tag{59}$$

Proof of Lemma 6.3: If $(L, (\cdot, \cdot), \Phi)$ is reducible with orthogonal summands $\bigoplus_{k \in K} L_k$, then w decomposes accordingly into a product of commuting elements $w_k \in W(L_k)$, and the numbers $k_4(L, w)$ and $k_5(L, w)$ are additive,

$$k_4(L, w) = \sum_{k \in K} k_4(L_k, w_k), \quad k_5(L, w) = \sum_{k \in K} k_5(L_k, w_k).$$

Therefore it is sufficient to prove the lemma and also Theorem 6.2 for the irreducible p.n. root lattices.

Let $(L, (\cdot, \cdot), \Phi)$ be an irreducible p.n. root lattice of rank n , and let w be a Weyl group element with $l(w) = n - 1$. Then $k_5(L, w) \geq 1$. If $k_4(L, w) = 1$ then by (57) also $k_5(L, w) = 1$. Thus it is sufficient to prove $k_5(L, w) = 1 \Rightarrow k_4(L, w) = 1$.

The cases **A_n**, **D_n**, **E₆**, **E₇**, **E₈**: Suppose $k_5(L, w) = 1$, and let $(\alpha_1, \dots, \alpha_{n+1})$ be a presentation of w whose subroot lattice is the full lattice. By Theorem 4.1, the set $\{\alpha_1, \dots, \alpha_{n+1}\}$ contains a \mathbb{Z} -basis of the full lattice L . Using (13), we can suppose that $\alpha_1, \dots, \alpha_n$ is a \mathbb{Z} -basis of L . Let $(\beta_1, \dots, \beta_{n-1})$ be an arbitrary reduced presentation of w . Then

$$s_{\alpha_1} \circ \dots \circ s_{\alpha_n} = s_{\beta_1} \circ \dots \circ s_{\beta_{n-1}} \circ s_{\alpha_{n+1}}.$$

The subroot lattice of the presentation on the left hand side is the full lattice, so it has index one. By Lemma 5.2 (b), the index of the subroot lattice of the presentation on the right hand side is the same, so it is also one. Thus

$$\sum_{j=1}^{n-1} \mathbb{Z}\beta_j + \mathbb{Z}\alpha_{n+1} = L.$$

This shows here $k_4(L, w) = 1$.

The cases **B_n** and **BC_n**: Because of Theorem 5.10 (a), $k_4(L, w) = n - l(w) = 1$ holds anyway.

The cases **C_n**: $k_4(L, w) = 1$ holds if and only if a reduced presentation with subroot lattice of type A_{n-1} or of type $A_{k-1} + C_{n-k}$ for some $k \in \{1, 2, \dots, n - 1\}$ exists. This follows from Table 3.1. In the case A_{n-1} , w is a positive cycle of length n . In the case $A_{k-1} + C_{n-k}$, w is a product of a positive cycle of length k and of one or two negative cycles such that the sum of their lengths is $n - k$.

It remains to show that w is such an element if $k_5(L, w) = 1$. Thus suppose $k_5(L, w) = 1$. Let $(\alpha_1, \dots, \alpha_{n+1})$ be a presentation of w whose subroot lattice is

the full lattice L . By Theorem 4.1, the set $\{\alpha_1, \dots, \alpha_{n+1}\}$ contains a \mathbb{Z} -basis of the full lattice L . Using (13), we can suppose that $\alpha_2, \dots, \alpha_{n+1}$ is a \mathbb{Z} -basis of L . Thus $s_{\alpha_2} \circ \dots \circ s_{\alpha_{n+1}} =: v$ is a quasi-Coxeter element, so either one negative cycle or the product of two negative cycles.

If α_1 is a long root, multiplying v from the left with s_{α_1} will turn one of the (one or two) negative cycles into a positive cycle.

If α_1 is a short root, so $\alpha_1 = \pm e_i \pm e_j$, then the type of $s_{\alpha_1} \circ v$ depends on the position of the vertices i and j in the supports of the (one or two) negative cycles. If i and j are in the support of the same negative cycle, then it splits into two cycles, one positive and one negative. If i and j are in the supports of different negative cycles, then $s_{\alpha_1} \circ v$ is a positive cycle of length n .

In any case, w is of one of the types which satisfy $k_4(L, w) = 1$.

The case \mathbf{G}_2 : By Table 3.6, all subroot lattices of rank 1 are primitive sublattices. Therefore $k_4(L, w) = n - 1 = 1$ holds anyway.

The case \mathbf{F}_4 : By Table 5.4, the only elements w with $l(w) = 3$ and $k_4(L, w) \geq 2$ are those of type $B_2(1, 1) + A_1 \sim 3A_1$, and the elements of this type satisfy $k_4(L, w) = 2$. It remains to show for them $k_5(L, w) \geq 2$.

Suppose that such an element w satisfies $k_5(L, w) = 1$, and let $(\alpha_1, \dots, \alpha_5)$ be a presentation of w whose subroot lattice is the full lattice. By Theorem 4.1, the set $\{\alpha_1, \dots, \alpha_5\}$ contains a \mathbb{Z} -basis of the full lattice L . Using (13), we can suppose that $\alpha_1, \dots, \alpha_4$ is a \mathbb{Z} -basis of L . We may suppose $w = s_{e_1 - e_2} s_{e_1 + e_2} s_{e_3 - e_4}$. Then

$$s_{e_1 - e_2} s_{e_1 + e_2} s_{e_3 - e_4} s_{\alpha_5} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}.$$

Because of the right hand side, this is a quasi-Coxeter element in $W(F_4)$.

First case, α_5 is a long root $\alpha_5 = \pm e_i \pm e_j$: Then $\{i, j\} = \{1, 2\}$ is impossible because else the four roots on the left hand side were linearly dependent. $|\{i, j\} \cap \{1, 2\}| = 1$ is impossible because else the element on the left hand side were a Coxeter element in $W(D_4)$, and this is not a quasi-Coxeter element in $W(F_4)$. Also $\{i, j\} = \{3, 4\}$ is impossible because else the left hand side were an element of type $4A_1$, and this is not a quasi-Coxeter element in $W(F_4)$, or the four roots on the left hand side were linearly dependent. The first case is impossible.

Second case, α_5 is a short root: By conjugation and renumbering of the e_j we can suppose $\alpha_5 = \pm e_i$ for some i . Then $i \in \{1, 2\}$ is impossible because else the four roots on the left hand side were linearly dependent. $i \in \{3, 4\}$ is impossible because else the left hand side were an element of type $B_2 + 2A_1$, and this is not a quasi-Coxeter element in $W(F_4)$. The second case is impossible.

Thus $k_5(L, w) \neq 1$, so $k_5(L, w) \geq 2$. This finishes the proof of the case F_4 and the whole proof of Lemma 6.3. ■

Proof of Theorem 6.2: Let $(L, (\cdot, \cdot), \Phi)$ be an irreducible p.n. root lattice of some rank n . At the beginning of the proof of Lemma 6.3 it was shown that it is sufficient to prove Theorem 6.2 in this case.

Let w be a Weyl group element, and let $(\alpha_1, \dots, \alpha_{l(w)+2k})$ be a presentation with subroot lattice the full lattice L and with $k = k_5(L, w)$ minimal with this property. By Theorem 4.1, the set $\{\alpha_1, \dots, \alpha_{l(w)+2k}\}$ contains a \mathbb{Z} -basis of the full lattice L . Using (13), we can suppose that $\alpha_1, \dots, \alpha_n$ is a \mathbb{Z} -basis of L . The element $s_{\alpha_1} \circ \dots \circ s_{\alpha_n}$

has length n . Thus the element

$$v := s_{\alpha_1} \circ \dots \circ s_{\alpha_{n+1}}$$

has length $l(v) = n - 1$. And it satisfies $k_5(L, v) = 1$. Lemma 6.3 applies. Therefore a reduced presentation $(\gamma_1, \dots, \gamma_{n-1})$ of v and a root γ_0 exist such that $\sum_{i=0}^{n-1} \mathbb{Z}\gamma_i = L$. Let L_1 be the subroot lattice of the presentation

$$(\gamma_1, \dots, \gamma_{n-1}, \alpha_{n+2}, \dots, \alpha_{l(w)+2k}) \tag{60}$$

of w . As k is minimal, $L_1 \subsetneq L$. Because of $L_1 + \mathbb{Z}\gamma_0 = L$, $k_1(L, L_1) = 1$. The presentation of w in (60) shows $k_5(L_1, w) \leq k - 1$. If $k_5(L_1, w) < k - 1$ then by adding two times γ_0 to a shortest presentation of w with subroot lattice L_1 , one obtains also $k_5(L, w) < k$, which contradicts the minimality of k . Thus $k_5(L_1, w) = k - 1$. Induction on k gives $k_4(L_1, w) = k - 1$. Now

$$k_4(L, w) \leq k_2(L, L_1) + k_4(L_1, w) = 1 + (k - 1) = k = k_5(L, w). \tag{61}$$

Together with (57) this gives (58). ■

7. An application to extended affine root lattices

The number $k_5(L, w)$ in Definition 6.1 and Theorem 6.2 controls the existence of quasi-Coxeter elements in *extended affine root systems*. These had been defined by K. Saito in [10, (1.2) and (1.3)]. In [2] the equivalence with an alternative definition in [1] was shown.

The inequalities in Lemma 7.4 below give constraints on a quasi-Coxeter element w in an extended affine root system in terms of conditions for a nonreduced presentation of the induced element \bar{w} in the Weyl group of the associated p.n. root lattice $L/\text{Rad}(L)$.

Definition 7.1. An *extended affine root lattice* is a triple $(L, (.,.), \Phi)$ where L is a lattice, $(.,.) : L_{\mathbb{Q}} \times L_{\mathbb{Q}} \rightarrow \mathbb{Q}$ is a symmetric positive semidefinite bilinear form, and $\Phi \subset L - \{\alpha \in L \mid (\alpha, \alpha) = 0\}$ is a subset such that the following properties hold. Here $\langle \beta, \alpha \rangle$ and s_{α} are defined as in (11) and (12).

$$\Phi \text{ is a generating set of } L \text{ as a } \mathbb{Z}\text{-module.} \tag{62}$$

$$\text{For any } \alpha \in \Phi \text{ } s_{\alpha}(\Phi) = \Phi. \tag{63}$$

$$\langle \beta, \alpha \rangle \in \mathbb{Z} \text{ for any } \alpha, \beta \in \Phi. \tag{64}$$

The elements of Φ are the *roots*, and Φ is an *extended affine root system*.

$$W := \langle s_{\alpha} \mid \alpha \in \Phi \rangle \subset O(L, (.,.)) \tag{65}$$

is the *Weyl group of the extended affine root lattice*.

Remark 7.2. (i) In [10] the definition of an extended affine root system contains additionally the following irreducibility property:

$$\Phi = \Phi_1 \cup \Phi_2 \text{ with } \Phi_1 \perp \Phi_2 \Rightarrow \Phi_1 = \emptyset \text{ or } \Phi_2 = \emptyset. \tag{66}$$

(ii) Let $(L, (\cdot, \cdot), \Phi)$ be an extended affine root lattice. Because (\cdot, \cdot) is positive semidefinite, the radical of $(L_{\mathbb{R}}, (\cdot, \cdot))$ is

$$\begin{aligned} \text{Rad}(L_{\mathbb{R}}) &:= \{ \alpha \in L_{\mathbb{R}} \mid (\alpha, \beta) = 0 \text{ for all } \beta \in L_{\mathbb{R}} \} \\ &= \{ \alpha \in L_{\mathbb{R}} \mid (\alpha, \alpha) = 0 \}. \end{aligned}$$

Define the radicals $\text{Rad}(L) := \text{Rad}(L_{\mathbb{R}}) \cap L$ and $\text{Rad}(L_{\mathbb{Q}}) := \text{Rad}(L_{\mathbb{R}}) \cap L_{\mathbb{Q}}$. The quotient $L/\text{Rad}(L)$ with the induced bilinear form $(\cdot, \cdot)_{\text{quot}}$ and the induced set of roots

$$\Phi_{\text{quot}} := (\Phi + \text{Rad}(L))/\text{Rad}(L)$$

is obviously a p.n. root lattice. It is called the *quotient p.n. root lattice*. Any element $w \in O(L, (\cdot, \cdot))$ induces an element $\bar{w} \in O(L/\text{Rad}(L), (\cdot, \cdot)_{\text{quot}})$. If $w \in W(L)$, then $\bar{w} \in W(L/\text{Rad}(L))$. If $\alpha \in \Phi$ induces $\bar{\alpha} := [\alpha] \in L/\text{Rad}(L)$, then $s_{\bar{\alpha}} = s_{\alpha} \in W(L/\text{Rad}(L))$.

(iii) The reducedness property (18) is not required here. Even if $(L, (\cdot, \cdot), \Phi)$ satisfies it, it does not necessarily hold for the quotient p.n. root lattice. That is the reason why in this paper *p.n.* root lattices and not only root lattices are considered.

Definition 7.3. Let $(L, (\cdot, \cdot), \Phi)$ be an extended affine root lattice of rank n .

- (a) For any element w of its Weyl group, a *presentation* $(\alpha_1, \dots, \alpha_k)$, the *length of a presentation*, the *subroot lattice of a presentation*, and the *length* $l(w)$ of the element are defined as in definition 5.1 (a).
- (b) An element $w \in W$ is a *quasi-Coxeter element* if a presentation of length n exists whose subroot lattice is the full lattice (this generalizes Definition 5.3(a)).

The following simple lemma connects the existence of quasi-Coxeter elements with the numbers $k_5(L/\text{Rad}(L), \bar{w})$ from section 6. Theorem 6.2 says $k_5(L/\text{Rad}(L), \bar{w}) = k_4(L/\text{Rad}(L), \bar{w})$, and Theorem 5.10 allows to calculate this number.

Lemma 7.4. *Let $(L, (\cdot, \cdot), \Phi)$ be an extended affine root lattice of rank n with a radical $\text{Rad}(L)$ of rank $r \geq 1$. Let $w \in W$ be a quasi-Coxeter element, and let \bar{w} be the induced element in the Weyl group $W(L/\text{Rad}(L))$ of the quotient p.n. root lattice. Then*

$$l(\bar{w}) \leq n - r \quad \text{and} \tag{67}$$

$$l(\bar{w}) + 2k_5(L/\text{Rad}(L), \bar{w}) \leq n. \tag{68}$$

Proof: (67) is a trivial consequence of (49), i.e. $l(\bar{w}) = \dim V_{\neq 1}(\bar{w})$. A presentation of length n of w whose subroot lattice is the full lattice L induces a presentation of length n of \bar{w} whose subroot lattice is the full lattice $L/\text{Rad}(L)$. This shows (68). ■

Examples 7.5. (i) The classification of the extended affine root lattices whose quotient root lattices are inhomogeneous p.n. root lattices is nontrivial, see [2] and references therein. If, however, $L/\text{Rad}(L)$ is a homogeneous root lattice, then there is a sublattice $L_1 \subset L$ such that $(L_1, (\cdot, \cdot)|_{L_1}, \Phi \cap L_1)$ is isomorphic to the quotient root lattice $L/\text{Rad}(L)$ and $(L, (\cdot, \cdot), \Phi)$ is equal to

$$(L_1 \oplus \text{Rad}(L), (\cdot, \cdot), \Phi \cap L_1 + \text{Rad}(L)).$$

Thus up to isomorphism, $(L, (\cdot, \cdot), \Phi)$ is determined by the isomorphism class of the (homogeneous) quotient root lattice and by the rank r of the radical.

(ii) Let $(L, (\cdot, \cdot), \Phi)$ be an extended affine root lattice of rank n with radical $\text{Rad}(L)$ of rank r . Let $w \in W$ be a quasi-Coxeter element such that \bar{w} has maximal length $l(\bar{w}) = n - r$. Then Lemma 7.4 and Theorem 6.2 give

$$r \geq 2k_5(L/\text{Rad}(L), \bar{w}) = 2k_4(L/\text{Rad}(L), \bar{w}). \quad (69)$$

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