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On Flag Curvature and Homogeneous Geodesics of Left Invariant Randers Metrics on the Semi-Direct Product $\mathfrak{a} \oplus_p \mathfrak{r}$

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Abstract. We study flag curvature and homogeneous geodesics of left invariant Randers metrics on the Lie group with Lie algebra $\mathfrak{a} \oplus_{\mathfrak{p}} \mathfrak{r}$, where $\mathfrak{a}$ and $\mathfrak{r}$ are abelian Lie algebra of dimension n and 1, respectively. We give their flag curvature formulas explicitly. We show that there is an $(n+1)-$dimensional Lie group with left invariant Randers metric which admits exactly one homogeneous geodesic.

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Key Words: Invariant metric, Randers metric, flag curvature, homogeneous geodesics, semi-direct product.

1. Introduction

The study of invariant structures on Lie groups is an interesting subject in differential geometry. In the last decade a generalization of these concepts from the Riemannian geometry into the Finsler geometry, specially Randers metrics, have been done [4, 5, 11, 15, 16]. A Randers metric on a manifold M is a Finsler metric defined in the following form:

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a $1-$form on M. Randers spaces were first introduced by Randers in 1941 [14], when he studied the metric problem in the 4-space of general relativity. They also occur naturally in many other physical applications, especially in electron optics [9].

Not only physicists, but also pure geometers started to show interest in the subject, because Randers metrics supply one of the most basic examples of Finsler manifolds. By adding a 1-form, their fundamental function perturbs the fundamental function of a Riemannian manifold. A lot of invariants in Finsler geometry were explicitly calculated for the first time for Randers manifolds. For a general survey of results and applications of Randers manifolds, we refer to [2, 3, 9].

A classical problem of differential geometry is to study geodesics of Riemannian and Finsler manifolds. Of particular interest are geodesics with some special properties,

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for example homogeneous geodesics. A geodesic γ of a Finsler space (M, F) ia called homogeneous if it is an orbit of a one-parameter group of isometries of M. For results on homogeneous geodesics in homogeneous Riemannian and Finsler manifolds we refer to [8, 10].

Homogeneous geodesics have important applications to mechanics. For example, the equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M.

A *Finsler metric* on a manifold M is a continuous function, $F\colon TM \longrightarrow [0, \infty)$ differentiable on $TM - \{0\}$ and satisfying three conditions:

1. $F(x, y) = 0$ if and only if $y = 0$;
2. $F(x, \lambda y) = \lambda F(x, y)$ for any $y \in T_xM$ and $\lambda > 0$;
3. For any non-zero $y \in T_xM$, the symmetric bilinear form $g_y : T_xM \times T_xM \longrightarrow R$ given by
$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]|s = t = 0,$$
is positive definite.

Definition 1.1. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, a *Finsler function* $F\colon TG \longrightarrow [0, \infty)$ will be called G-invariant if F is constant on all G-orbits in $TG = G \times \mathfrak{g}$; that is, $F(g, X) = F(e, X)$ for all $g \in G$ and $X \in \mathfrak{g}$.

Randers metrics are built from a Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$, and a 1-form $\beta = b_i dx^i$, both living globally on the smooth manifold M. For Randers metrics, strong convexity holds if and only if $\|\beta\| < 1$, see [3, 13]. The Riemannian metric $\tilde{a}$ induces a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field X on M such that

$$\tilde{a}(y, X_x) = \beta(x, y).$$

Also we have $\|\beta\| = \|X\|$ (for more details see [4, 5, 11]). Therefore we can write Randers metric as follows:

$$F(x, y) = \sqrt{\tilde{a}_x(y, y)} + \tilde{a}_x(X, y) \quad x \in M, y \in T_xM.$$

An important quantity which associates with a Finsler space is flag curvature. This quantity is a natural generalization of the concept of sectional curvature in Riemannian geometry which is computed by the following formula:

$$K(P; Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(U, U)g_Y(Y, Y)_Y - g_Y^2(U, Y)}. \tag{1}$$

where $R(U, Y)Y = \nabla_U \nabla_Y Y - \nabla_Y \nabla_U Y - \nabla_{[U,Y]} Y$ and ∇ is the Chern connection induced by F.

Let $\mathfrak{a}$ and $\mathfrak{r}$ be abelian Lie algebras of dimension n and 1, respectively. Let $P = (p_{ij}) \in \mathfrak{gl}(\mathfrak{n}, \mathfrak{R})$ be any real $(n \times n)$-matrix. A homomorphism $\varphi\colon \mathfrak{r} \longrightarrow End(\mathfrak{a})$ can be defined for $\alpha \in \mathfrak{r}$ and $x \in \mathfrak{a}$ by

$$\varphi(\alpha)(x) = \alpha P x.$$

One can form a semi-direct product of the Lie algebra $\mathfrak{a}$ by $\mathfrak{r}$ as follows: The underlying linear space is the direct sum $\mathfrak{a} \oplus \mathfrak{r}$, and the bracket operation is given by

$$[(a, \alpha), (b, \beta)] = (\varphi(\alpha)b - \varphi(\beta)a, [\alpha, \beta]) = (\varphi(\alpha)b - \varphi(\beta)a, 0).$$

We denote this new Lie algebra by $\mathfrak{a} \oplus_p \mathfrak{r}$.

Clearly, if the matrix P is nilpotent, then $\mathfrak{a} \oplus_p \mathfrak{r}$ is also nilpotent. If P is the zero matrix, then $\mathfrak{a} \oplus_p \mathfrak{r}$ is abelian. If P has trace 0, then $\mathfrak{a} \oplus_p \mathfrak{r}$ is unimodular. Moreover, if the matrix P is the identity matrix, then the associated simply connected Lie group is isometric to the n+1-dimensional hyperbolic space H^{n+1} [18].

2. Flag curvature of left invariant Randers metric on Lie group of $\mathfrak{a} \oplus_p \mathfrak{r}$

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{a} \oplus_p \mathfrak{r}$ defined in the introduction, $E_i = (0, ..., 1, ..., 0) \in R^{n+1}$ and let $\{E_1, ..., E_{n+1}\}$ be an orthonormal basis for $\mathfrak{g}$ and equip the left invariant metric on the associated Lie group with the Lie algebra $\mathfrak{g}$. Then we have [18]:

Proposition 2.1. *For $1 \leq i, j \leq n$, we have*

$$(1)\ \ [E_i, E_j] = 0, \quad (2)\ \ [E_{n+1}, E_i] = \Sigma_{j=1}^{n} p_{ji} E_j, \quad (3)\ \ [E_{n+1}, E_{n+1}] = 0.$$

Let α_{ijk} be defined by $[E_i, E_j] = \Sigma_{k=1}^{n+1} \alpha_{ijk} E_k$. Then we have the following

Proposition 2.2. *For $1 \leq i, j, k \leq n$, we have*

$$(1)\ \ \alpha_{ijk} = 0, \quad (2)\ \ \alpha_{(n+1)jk} = -\alpha_{j(n+1)k} = p_{kj}, \quad (3)\ \ \alpha_{(n+1)j(n+1)} = \alpha_{(n+1)(n+1)k} = 0.$$

Theorem 2.3. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{a} \oplus_p \mathfrak{r}$ and let $\{E_1, ..., E_{n+1}\}$ be an orthonormal basis for $\mathfrak{g}$ and equip the left invariant metric $\tilde{a}$ on the associated Lie group G with Lie algebra $\mathfrak{g}$. Let F be a Randers metric defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{r=1}^{n+1} x_r E_r$ then F is of Berwald type if and only if*

(1) *Assume that the matrix (p_{ij}) is invertible*
   (a) *If the matrix (p_{ij}) is skew-symmetric, then $X = cE_{n+1}$.*
   (b) *If the matrix (p_{ij}) is not skew-symmetric, then $X = 0$ and F is Riemannian.*

(2) *Assume that the matrix (p_{ij}) is not invertible.*
   (a) *If the i-th row and i-th column of the matrix (p_{ij}) is zero then $X = x_i E_i$.*
   (b) *If the i-th row and i-th column of the matrix (p_{ij}) is zero and the matrix is skew-symmetric then $X = x_i E_i + x_{n+1} E_{n+1}$.*

Proof. The Randers metric F is of Berwald type if and only if X is parallel with respect to $\tilde{a}$. By using the equation

$$\nabla_{E_i} E_j = \sum_{k=1}^{n+1} \frac{1}{2} (\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) E_k,$$

we have the following

$$\begin{aligned}
\nabla_{E_i}E_j &= \frac{1}{2}(p_{ij}+p_{ji})E_{n+1} && (2)\\
\nabla_{E_i}E_{n+1} &= -\frac{1}{2}\sum_{k=1}^{n}(p_{ki}+p_{ik})E_k && (3)\\
\nabla_{E_{n+1}}E_i &= \frac{1}{2}\sum_{k=1}^{n}(p_{ki}-p_{ik})E_k && (4)\\
\nabla_{E_{n+1}}E_{n+1} &= 0. && (5)
\end{aligned}$$

So we have

$$\begin{aligned}
\nabla X &= \sum_{i=1}^{n+1} x_i \nabla E_i\\
&= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n}\frac{1}{2}(p_{ij}+p_{ji})E_{n+1}\otimes\theta_j + \sum_{j=1}^{n}\frac{1}{2}(p_{ji}-p_{ij})E_j\otimes\theta_{n+1}\right)\\
&\quad -x_{n+1}\sum_{i=1}^{n}\frac{1}{2}\left(\sum_{j=1}^{n}(p_{ij}+p_{ji})E_j\times\theta_i\right)\\
&= \frac{1}{2}\sum_{i,j=1}^{n} x_i(p_{ij}+p_{ji})E_{n+1}\otimes\theta_j + \frac{1}{2}\sum_{i,j=1}^{n} x_i(-p_{ij}+p_{ji})E_j\otimes\theta_{n+1}\\
&\quad -\frac{1}{2}x_{n+1}\sum_{i,j=1}^{n}(p_{ij}+p_{ji})E_j\otimes\theta_i.
\end{aligned}$$

This proves the theorem. ■

Theorem 2.4. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{a}\oplus_p\mathfrak{r}$ and let $\{E_1,...,E_{n+1}\}$ be an orthonormal basis for $\mathfrak{g}$ and equip the left invariant metric $\tilde{a}$ on the associated Lie group G with Lie algebra $\mathfrak{g}$. Let F be a Randers metric defined by the Riemannian metric $\tilde{a}$ and the vector field $X=\sum_{r=1}^{n+1}x_rE_r$ which is of Berwald type. Then the flag curvature of F is, for $j=1,...,n$, given by*

$$K(P,E_j)=\frac{\left(\frac{1}{4}\left(p_{ij}+p_{ji}\right)^2-p_{ii}p_{jj}\right)(1+x_j)+x_j\tilde{a}(X,A)}{(1+x_j)^3},$$

$$\begin{aligned}
K(P,E_{n+1}) = \frac{1}{(1+x_{n+1})^3}\Big\{&(1+x_{n+1})\sum_{j=1}^{n}\Big(\frac{1}{4}(p_{ji}+p_{ij})(p_{ij}-p_{ji})-\frac{1}{2}p_{ji}(p_{ij}+p_{ji})\Big)\\
&+x_i\Big(\tilde{a}(X,B)+\sum_{j=1}^{n}\frac{1}{4}(p_{ji}+p_{ij})(p_{(n+1)j}-p_{j(n+1)})-\frac{1}{2}p_{ji}(p_{(n+1)j}+p_{j(n+1)})\Big)\Big\},
\end{aligned}$$

where $$A=\sum_{k=1}^{n}\Big(-\frac{1}{2}p_{jj}(p_{ki}+p_{ik})+\frac{1}{4}(p_{ij}+p_{ji})(p_{kj}+p_{jk})\Big)E_k,$$

and $$B=\sum_{r=1}^{n}\Big(\sum_{j=1}^{n}\frac{1}{4}(p_{ji}+p_{ij})(p_{rj}-p_{jr})-\frac{1}{2}p_{ji}(p_{rj}+p_{jr})\Big)E_r.$$

Proof. Since F is of Berwald type therefore the Chern connection of F and the Levi-Civita connection of $\tilde{a}$ coincide, by using the the equations (2), (3), (4), (5) for the curvature tensor we have:

$$R(E_i, E_j)E_j = \sum_{k=1}^{n} \left(-\frac{1}{2} p_{jj}(p_{ki} + p_{ik}) + \frac{1}{4}(p_{ij} + p_{ji})(p_{kj} + p_{jk}) \right) E_k. \tag{6}$$

$$R(E_i, E_{n+1})E_{n+1} = \sum_{r=1}^{n} \left(\sum_{j=1}^{n} \frac{1}{4}(p_{ji} + p_{ij})(p_{rj} - p_{jr}) - \frac{1}{2} p_{ji}(p_{rj} + p_{jr}) \right) E_r. \tag{7}$$

By definition $$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv)|_{r=s=0}.$$

So by a direct computation we get

$$\begin{aligned} g_y(u, v) &= \tilde{a}(u, v) + \tilde{a}(X, u)\tilde{a}(X, v) - \frac{\tilde{a}(X, y)\tilde{a}(y, u)\tilde{a}(y, v)}{\tilde{a}(y, y)^{\frac{3}{2}}} \\ &\quad + \frac{1}{\sqrt{\tilde{a}(y, y)}} \{(\tilde{a}(X, u)\tilde{a}(y, v) + \tilde{a}(X, y)\tilde{a}(u, v) + \tilde{a}(X, v)\tilde{a}(y, u)\}. \end{aligned} \tag{8}$$

According to the formula (8) we have

$$g_{E_j}(E_j, E_j) = (1 + \tilde{a}(X, E_j))^2 = (1 + x_j)^2 \tag{9}$$

$$\begin{aligned} g_{E_j}(E_i, E_i) &= 1 + \tilde{a}(X, E_i)^2 + \tilde{a}(X, E_j) \\ &= 1 + x_i^2 + x_j \end{aligned} \tag{10}$$

$$\begin{aligned} g_{E_j}(E_j, E_i) &= \tilde{a}(X, E_i)(1 + \tilde{a}(X, E_j)) \\ &= x_i(1 + x_j) \end{aligned} \tag{11}$$

$$g_{E_j}(R(E_i, E_j)E_j, E_i) = \left(\frac{1}{4}(p_{ij} + p_{ji})^2 - p_{ii}p_{jj} \right)(1 + x_j) + x_i\tilde{a}(X, A) \tag{12}$$

where
$$\begin{aligned} A &= R(E_i, E_j)E_j, E_i \\ &= \sum_{k=1}^{n} \left(-\frac{1}{2} p_{jj}(p_{ki} + p_{ik}) + \frac{1}{4}(p_{ij} + p_{ji})(p_{kj} + p_{jk}) \right) E_k. \end{aligned}$$

Also for $j = n + 1$ we have

$$g_{E_{n+1}}(E_{n+1}, E_{n+1}) = (1 + x_{n+1})^2, \tag{13}$$

$$g_{E_{n+1}}(E_i, E_i) = 1 + x_i^2 + x_{n+1}, \tag{14}$$

$$g_{E_{n+1}}(E_{n+1}, E_i) = x_i(1 + x_{n+1}), \tag{15}$$

$$\begin{aligned} & g_{E_{n+1}}(R(E_i, E_{n+1})E_{n+1}, E_i) = \\ & (1 + x_{n+1}) \sum_{j=1}^{n} \left(\frac{1}{4}(p_{ji} + p_{ij})(p_{ij} - p_{ji}) - \frac{1}{2} p_{ji}(p_{ij} + p_{ji}) \right) \qquad (16) \\ & + x_i \left(\tilde{a}(X, B) + \sum_{j=1}^{n} \frac{1}{4}(p_{ji} + p_{ij})(p_{(n+1)j} - p_{j(n+1)}) - \frac{1}{2} p_{ji}(p_{(n+1)j} + p_{j(n+1)}) \right), \end{aligned}$$

where
$$\begin{aligned} B &= R(E_i, E_{n+1})E_{n+1} \\ &= \sum_{r=1}^{n}\left(\sum_{j=1}^{n}\frac{1}{4}(p_{ji}+p_{ij})(p_{rj}-p_{jr})-\frac{1}{2}p_{ji}(p_{rj}+p_{jr})\right)E_r. \end{aligned}$$

Substituting (9), (10), (11), (12), (13), (14), (15), (16) in equation (1) completes the proof. ■

Theorem 2.5. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{a}\oplus_p \mathfrak{r}$ where the matrix $p=(p_{ij})$ is skew-symmetric. Let $\{E_1,...,E_{n+1}\}$ be an orthonormal basis for $\mathfrak{g}$ and equip the left invariant metric $\tilde{a}$ and the associated Lie group G with Lie algebra $\mathfrak{g}$. Let F be a Randers metric defined by the Riemannian metric $\tilde{a}$ and the vector field $X=cE_{n+1}$. Then the flag curvature of F is zero.*

Proof. By using formula (3), (5) we have

$$\nabla X = -\frac{1}{2}c\sum_{j=1}^{n}\left(\sum_{i=1}^{n}(p_{ij}+p_{ji})E_i\right)\otimes\theta_j.$$

Therefore the Chern connection of F coincides with the Levi-Civita connection of $\tilde{a}$. By using the formulas (2), (3), (4) and (5) we get

$$\nabla_{E_i}E_j = 0, \quad \nabla_{E_i}E_{n+1}=0,$$
$$\nabla_{E_{n+1}}E_i=\sum_{k=1}^{n}p_{ki}E_k, \quad \nabla_{E_{n+1}}E_{n+1}=0.$$

A simple computation for the curvature tensor shows that

$$R(E_i,E_j)E_j=0, \quad \text{and} \quad R(E_i,E_{n+1})E_{n+1}=0.$$

So $K(P,E_j)=K(P,E_{n+1})=0$. ■

Corollary 2.6. *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{a}\oplus_p \mathfrak{r}$ and let $\{E_1,...,E_{n+1}\}$ be an orthonormal basis for $\mathfrak{g}$ and equip the left invariant Riemannian metric $\tilde{a}$ on the associated Lie group G with Lie algebra $\mathfrak{g}$. Then the sectional curvature of $\tilde{a}$ is given by*

$$\begin{aligned} K(E_i,E_j) &= \frac{1}{4}(p_{ij}+p_{ji})^2-p_{ii}p_{jj} \\ K(E_i,E_{n+1}) &= -\frac{1}{4}\sum_{k=1}^{n}(p_{ki}+p_{ik})(3p_{ki}-p_{ik}) \end{aligned}$$

3. Homogeneous geodesics

Let $(M=G/H,F)$ be a homogeneous Finsler manifold with a fixed origin o, $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and H, respectively, and

$$\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$$

a reductive decomposition. A homogeneous geodesic through the origin $o \in M = G/H$ is a geodesic $\gamma(t)$ which is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(o), \qquad t \in R$$

where Z is a nonzero vector of $\mathfrak{g}$. A nonzero vector $Z \in \mathfrak{g} - \{0\}$ for which $\gamma(t) = \exp(tZ)(o)$, $t \in R$, is a geodesic is called a geodesic vector. Thus geodesic vectors are in one-to-one correspondence with homogeneous geodesics through the origin o. There are many interesting results about homogeneous geodesic [1, 6, 7, 8, 17].

The following is a simple criterion for a vector to be a geodesic vector in the Finslerian case [10].

Lemma 3.1. *A vector $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if*

$$g_{X_{\mathfrak{m}}}(X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}}) = 0, \qquad \forall Z \in \mathfrak{m}. \tag{17}$$

We are going to calculate the above criterion for the left invariant Randers metric on semi-direct product $\mathfrak{a} \oplus_p \mathfrak{r}$. Let F be a left invariant Randers metric on Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus_p \mathfrak{r}$ defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{n+1} x_i E_i$, i.e,

$$F(x, y) = \sqrt{\tilde{a}(y, y)} + \tilde{a}(X(x), y).$$

According to formula (8) we have

$$g_y(y, [y, z]) = \tilde{a}\left(X + \frac{y}{\sqrt{\tilde{a}(y, y)}}, [y, z]\right) F(y). \tag{18}$$

By using Lemma 3.1 and eq.(18) a vector $y = \sum_{i=1}^{n+1} y_i E_i$ of $\mathfrak{a} \oplus_p \mathfrak{r}$ is a geodesic vector if and only if

$$\tilde{a}\left(\sum_{k=1}^{n+1} x_k E_k + \frac{\sum_{k=1}^{n+1} y_k E_k}{\sqrt{\sum_{k=1}^{n+1} y_k^2}}, [\sum_{i=1}^{n+1} y_i E_i, E_j]\right) = 0, \tag{19}$$

for each $j = 1, ..., n+1$. For each $1 \le j \le n$ we have

$$\begin{aligned} &\tilde{a}\Big(\sum_{k=1}^{n+1} x_k E_k, y_{n+1}(\sum_{i=1}^{n} p_{ij} E_i)\Big) + \tilde{a}\Big(\frac{\sum_{k=1}^{n+1} y_k E_k}{\sqrt{\sum_{k=1}^{n+1} y_k^2}}, y_{n+1}(\sum_{i=1}^{n} p_{ij} E_i)\Big) \\ &= y_{n+1}\Big(\sum_{i=1}^{n} p_{ij} x_i + \frac{1}{\sqrt{\sum_{k=1}^{n+1} y_k^2}} \sum_{i=1}^{n} p_{ij} y_i\Big) = 0. \end{aligned} \tag{20}$$

And for $j = n+1$, we have

$$\begin{aligned} &\tilde{a}\Big(\sum_{k=1}^{n+1} x_k E_k, y_i(-\sum_{l=1}^{n} p_{li} E_l)\Big) + \tilde{a}\Big(\frac{\sum_{k=1}^{n+1} y_k E_k}{\sqrt{\sum_{k=1}^{n+1} y_k^2}}, y_i(-\sum_{l=1}^{n} p_{li} E_l)\Big) \\ &= \sum_{i,k=1}^{n} p_{ki} y_i x_k + \frac{1}{\sqrt{\sum_{k=1}^{n+1} y_k^2}} \sum_{i,k=1}^{n} p_{ki} y_i y_k = 0. \end{aligned} \tag{21}$$

Thus we have
$$y_{n+1}\Big(\sum_{i=1}^{n} p_{ij}x_i + \frac{1}{\sqrt{\sum_{k=1}^{n+1} y_k^2}}\sum_{i=1}^{n} p_{ij}y_i\Big) = 0, \tag{22}$$
$$\sum_{i,k=1}^{n} p_{ki}y_ix_k + \frac{1}{\sqrt{\sum_{k=1}^{n+1} y_k^2}}\sum_{i,k=1}^{n} p_{ki}y_iy_k = 0. \tag{23}$$

Assume that $y_{n+1} \neq 0$. Then we have
$$\sum_{i=1}^{n} p_{ij}x_i + \frac{1}{\sqrt{\sum_{k=1}^{n+1} y_k^2}}\sum_{i=1}^{n} p_{ij}y_i = 0,$$
for $j = 1, ..., n$ or equivalently,
$$\sum_{i=1}^{n} p_{ij}(Cx_i + y_i) = 0, \tag{24}$$
for $j = 1, ..., n$, where $C = \sqrt{\sum_{k=1}^{n+1} y_k^2}$. In other words, the $CX + Y$ lies in the null-space of P^t. Thus if $\det(p_{ij}) \neq 0$, we have $y_i = -Cx_i$. Thus $(-Cx_1, ..., -Cx_n, y_{n+1})$ is the only geodesic vector. Next assume that $y_{n+1} = 0$. Then we have
$$\sum_{i,k=1}^{n} p_{ki}y_iy_k + C\sum_{i,k=1}^{n} p_{ki}y_ix_k = 0. \tag{25}$$

If (p_{ij}) is skew-symmetric plus a positive diagonal matrix, i.e. $p_{ij} = -p_{ji}$, $i \neq j$ and $p_{ii} > 0$, $i = 1, ..., n$, and $x_i = 0$, i=1,...,n (i.e. $X = X_{n+1}E_{n+1}$), then we have y_i's are all zero. Thus we have the following

Theorem 3.2. *If the matrix (p_{ij}) is skew-symmetric matrix plus a positive diagonal matrix and non-singular, then the Lie group with Lie algebra $\mathfrak{a} \oplus_p \mathfrak{r}$ equipped with the Randers metric F defined by Riemannian metric $\tilde{a}$ and $X = x_{n+1}E_{n+1}$ has only one geodesic vector.*

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