

Maximal Hypoellipticity for Left-Invariant Differential Operators on Lie Groups

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Abstract. Given a maximal hypoelliptic differential operator of arbitrary order, we prove that its graph norm controls the Sobolev norm of the same order if the operator has left-invariant principal part and lower order terms with bounded coefficients. As an application, we obtain the essential self-adjointness on L^2 of Rumin's Laplacians on the contact complex of the Heisenberg groups.

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1. Introduction

The notion of maximal hypoellipticity for a differential operator of order m was first introduced by Helffer and Nourrigat [11] and it consists, for the smooth functions supported on a given bounded open set, on a control on the L^p -Sobolev norm of order m in terms of the graph norm in L^p of the operator, with a constant depending on the chosen set. It seems a natural question whether this local condition is equivalent to a global estimate when the operator and the norms involved enjoy some sort of translation invariance.

In this paper, we consider matrix-valued, differential operators defined in terms of left-invariant vector fields on Lie groups endowed with a left measure. We answer in the affirmative for operators with left-invariant principal part and with lower order terms whose coefficients are essentially bounded, and we characterize the domain of their closure. The matrix-valued case is aimed at considering differential operators acting on differential forms of arbitrary degree, see also [2, 3, 4]. Indeed, this result allows to obtain the essential self-adjointness on L^2 of Rumin's Laplacians on the contact complex of the Heisenberg groups.

Let G be a noncompact connected Lie group and identify its Lie algebra \mathfrak{g} with the algebra of left-invariant vector fields on G . For a positive integer k and multi-index $I = (I_1, \dots, I_k)$ of length $|I| = k$, we denote by X_I the differential operator $X_{I_1} \cdots X_{I_k}$, where $X_{I_j} \in \mathfrak{g}$. We shall consider operators $\mathcal{P}: C^\infty(G)^n \rightarrow C^\infty(G)^m$ for arbitrary $n, m \in \mathbb{N}$ of the form

$$\mathcal{P}(\alpha_1, \dots, \alpha_n) = \left(\sum_{i=1}^n \mathcal{P}_{i1} \alpha_i, \dots, \sum_{i=1}^n \mathcal{P}_{im} \alpha_i \right), \quad \alpha = (\alpha_1, \dots, \alpha_n) \in C^\infty(G)^n,$$

where \mathcal{P}_{ij} are (scalar-valued) differential operators on G given by

$$\mathcal{P}_{ij} = \sum_{0 \leq |I| \leq m} a_{ij}^I X_I, \tag{1}$$

for some measurable coefficients (a_{ij}^I) on G , vector fields X_{I_j} in \mathfrak{g} , and $m \in \mathbb{N}$. We shall identify \mathcal{P} with its components (\mathcal{P}_{ij}) . Following [4], we say that \mathcal{P} is left invariant if \mathcal{P}_{ij} is for every i, j , i.e. if \mathcal{P}_{ij} commutes with left translations. For $x \in G$, by left translation by x we mean the operator τ_x^* given by $(\tau_x^*u)(y) = u(\tau_x y)$ for every measurable function u and $y \in G$, where $\tau_x y = x^{-1}y$. We then extend τ_x^* on vector-valued functions by componentwise action. We say that \mathcal{P} has order $m \in \mathbb{N}$, $m \geq 1$, if all \mathcal{P}_{ij} have orders $\leq m$ and at least one has order equal to m . Given \mathcal{P}_{ij} as in (1), the operators

$$(\mathcal{P}_{ij})_m = \sum_{|I|=m} a_{ij}^I X_I, \quad (\mathcal{P}_{ij})_{<m} = \mathcal{P}_{ij} - (\mathcal{P}_{ij})_m,$$

will be called its *principal part* and its *lower order terms*, respectively. We then set

$$\mathcal{P}_m = ((\mathcal{P}_{ij})_m), \quad \mathcal{P}_{<m} = ((\mathcal{P}_{ij})_{<m}).$$

Let λ be a left Haar measure on G . For a measurable subset $\Omega \subseteq G$ and $p \in [1, \infty]$, we denote by $L^p(\Omega, \lambda)$ the classical Lebesgue spaces on Ω with respect to the measure λ , endowed with the usual norms which we denote by $\|\cdot\|_{L^p(\Omega, \lambda)}$. For $\mathbf{n} \in \mathbb{N}$, we denote by $L^{p, \mathbf{n}}(\Omega, \lambda) = (L^p(\Omega, \lambda))^{\mathbf{n}}$ the space of \mathbf{n} -vectors whose entries are in $L^p(\Omega, \lambda)$, endowed with the norm

$$\|\alpha\|_{L^{p, \mathbf{n}}(\Omega, \lambda)} = \sum_{i=1}^{\mathbf{n}} \|\alpha_i\|_{L^p(\Omega, \lambda)}.$$

If $\Omega = G$, we write $L^{p, \mathbf{n}}(\lambda)$. When there is no risk of confusion, we omit to specify λ and write simply L^p , $L^p(\Omega)$, $\|\cdot\|_p$, and so on. The space of smooth and compactly supported \mathbf{n} -vectors $(C_c^\infty(\Omega))^{\mathbf{n}}$ will be denoted $C_c^{\infty, \mathbf{n}}(\Omega)$.

We fix a family $\mathbf{X} = \{X_1, \dots, X_\nu\}$ of linearly independent left-invariant vector fields in \mathfrak{g} . We say that an operator \mathcal{P} as in (1) is a differential operator *along derivatives of \mathbf{X}* if $X_{I_j} \in \mathbf{X}$ for every I appearing in (1). If $u \in C_c^\infty(G)$, $p \in [1, \infty]$ and $\ell \geq 1$, we define

$$|u|_{\ell, p}^{\mathbf{X}} = \sum_{|I|=\ell} \|X_I u\|_p, \quad X_{I_j} \in \mathbf{X}, \quad j = 1, \dots, \ell$$

and for $\alpha \in C_c^{\infty, \mathbf{n}}(G)$, we set $|\alpha|_{\ell, p}^{\mathbf{X}} = \sum_{i=1}^{\mathbf{n}} |\alpha_i|_{\ell, p}^{\mathbf{X}}$.

Observe that we do not require a priori that \mathbf{X} satisfies Hörmander’s condition. If this is the case, however, and if $p \in (1, \infty)$, then $\sum_{\ell=0}^m |u|_{\ell, p}^{\mathbf{X}}$ is equivalent to the norm in the Sobolev space $L_m^p(\lambda)$ defined in [8] in terms of the vector fields in \mathbf{X} .

Definition 1.1. Let \mathcal{P} be a differential operator of degree m , $\Omega \subseteq G$ be open and $p \in [1, \infty]$. We say that \mathcal{P} is *\mathbf{X} -maximal hypoelliptic* in $L^p(\Omega)$ if there exists a constant $C(\Omega)$ such that for all α in $C_c^{\infty, \mathbf{n}}(\Omega)$

$$\sum_{\ell=0}^m |\alpha|_{\ell, p}^{\mathbf{X}} \leq C(\Omega) (\|\mathcal{P}\alpha\|_p + \|\alpha\|_p). \tag{2}$$

If G is a stratified group and the family \mathbf{X} is a basis of the first layer of its Lie algebra, for every bounded open $\Omega \subset G$ our definition of \mathbf{X} -maximal hypoellipticity in $L^2(\Omega)$ coincides with the maximal hypoellipticity on Ω considered in [4]. Outside stratified groups, examples of \mathbf{X} -maximal hypoelliptic operators in $L^2(\Omega)$ for some $\Omega \subset G$ are Rumin's Laplacians [15] on the group $\mathrm{SL}(2, \mathbb{R})$ and on the groups of [12, 9], when \mathbf{X} is a basis of the kernel of the contact 1-form of the group (see also [5]). The case $p \in (1, \infty)$ was also considered in [4]. Observe that if G is a symmetric, nonnegative, left-invariant and homogeneous differential operator of even order on a stratified group, and if \mathbf{X} is a basis of the first layer of \mathfrak{g} , then \mathbf{X} -maximal hypoellipticity in $L^2(\Omega)$ is equivalent to hypoellipticity in Ω (cf. [4, Theorem 1.1]).

If \mathcal{P} is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$, then it is also in $L^p(\Omega_0)$ for any open $\Omega_0 \subseteq \Omega$. Moreover, if \mathcal{P} is left invariant and \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$ for some $\Omega \subseteq G$, then for any $x \in G$ it is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega_x)$, where $\Omega_x = \tau_x \Omega$ is a neighbourhood of x ; this is a consequence of the left invariance of \mathcal{P} and of the norm. It seems a natural question whether a left-invariant differential operator \mathcal{P} which is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$ for some open $\Omega \subset G$ is also \mathbf{X} -maximal hypoelliptic in $L^p(G)$. In other words, whether the local estimate (2) implies the global estimate

$$\sum_{\ell=0}^m |\alpha|_{\ell,p}^{\mathbf{X}} \leq C (\|\mathcal{P}\alpha\|_p + \|\alpha\|_p), \quad \alpha \in C_c^{\infty, \mathbf{n}}(G),$$

for some constant C independent of α . In this paper we prove the following.

Theorem 1.2. *Let $p \in [1, \infty)$ and \mathcal{P} be a matrix-valued differential operator of degree m along derivatives of \mathbf{X} . Assume that*

- (1) \mathcal{P}_m is left invariant;
- (2) $\mathcal{P}_{< m}$ has L^∞ -coefficients.

If \mathcal{P} is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$ for some $\Omega \subseteq G$, then it is \mathbf{X} -maximal hypoelliptic in $L^p(G)$.

We shall prove Theorem 1.2 by proving that the statement holds when \mathcal{P} itself is left invariant, and that maximal hypoellipticity is preserved under L^∞ -perturbations of the lower order terms. This is an analogous result to [7, Theorem 7] for local estimates. Theorem 1.2 will allow us to characterize the domain of the closure of Rumin's Laplacians on test horizontal forms on the contact complex of the Heisenberg groups, and to obtain their essential self-adjointness on L^2 .

We observe that the study of maximal hypoellipticity of matrix-valued operators cannot be reduced to that of scalar-valued operators. Indeed, if all components \mathcal{P}_{ij} of \mathcal{P} are \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$ for some Ω , then \mathcal{P} is also, but the converse is not true. To see this, consider the operator

$$\mathcal{P} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} : C^{\infty, 2}(\mathbb{R}^n) \rightarrow C^{\infty, 2}(\mathbb{R}^n),$$

where Δ is the Laplacian on \mathbb{R}^n , and let $\mathbf{X} = \{\partial_1, \dots, \partial_n\}$ be the canonical basis of vector fields on \mathbb{R}^n . Then, \mathcal{P} is \mathbf{X} -maximal hypoelliptic in $L^2(\mathbb{R}^n)$ since Δ is, but its off-diagonal components are not.

2. From local to global estimates

We begin by proving a Nash-type inequality.

Lemma 2.1. *Given $m \geq 2$ and $p \in [1, \infty]$, there exists a constant $C(\nu, m)$ such that for every $\ell \leq m - 1$ and $\epsilon \in (0, 1]$*

$$|\alpha|_{\ell, p}^{\mathbf{X}} \leq \epsilon^{m-\ell} |\alpha|_{m, p}^{\mathbf{X}} + C(\nu, m) \epsilon^{-\ell} \|\alpha\|_p \quad \forall \alpha \in C_c^{\infty, \mathbf{n}}(G).$$

Proof. Observe first that it is enough to prove the statement when $\mathbf{n} = 1$, i.e. for $\alpha = u \in C_c^\infty(G)$. Moreover, since the family of vector fields \mathbf{X} plays no role, we write $|u|_{\ell, p}$ instead of $|u|_{\ell, p}^{\mathbf{X}}$. Let $\epsilon \in (0, 1]$. The starting point is the inequality

$$\|X_j u\|_p \leq \epsilon \|X_j^2 u\|_p + 2\epsilon^{-1} \|u\|_p, \quad (3)$$

which can be found in the proof of [14, Lemma III.3.3], by considering the left-translation as a representation of G onto $L^p(\lambda)$. Taking the sum over $j = 1, \dots, \nu$ we get

$$|u|_{1, p} \leq \epsilon |u|_{2, p} + 2\nu\epsilon^{-1} \|u\|_p. \quad (4)$$

By induction on ℓ , we now prove that for $\epsilon > 0$

$$|u|_{\ell, p} \leq \epsilon |u|_{\ell+1, p} + C(\nu, \ell) \epsilon^{-\ell} \|u\|_p. \quad (5)$$

If $\ell = 1$, this is (4). Hence, assume (5) holds for $\ell - 1 \leq m - 2$. By (4)

$$\begin{aligned} |u|_{\ell, p} &= \sum_{|I|=\ell-1} |X_I u|_{1, p} \leq \sum_{|I|=\ell-1} (\epsilon |X_I u|_{2, p} + 2\nu\epsilon^{-1} \|X_I u\|_p) \\ &= \epsilon |u|_{\ell+1, p} + 2\nu\epsilon^{-1} |u|_{\ell-1, p}. \end{aligned} \quad (6)$$

Since for any $\delta > 0$

$$|u|_{\ell-1, p} \leq \delta |u|_{\ell, p} + C(\nu, \ell - 1) \delta^{-(\ell-1)} \|u\|_p$$

by the inductive assumption, if $\delta = \epsilon/(4\nu)$ then by (6)

$$|u|_{\ell, p} \leq \epsilon |u|_{\ell+1, p} + \frac{1}{2} |u|_{\ell, p} + C(\nu, \ell - 1) 2^{2\ell-1} \nu^\ell \epsilon^{-\ell} \|u\|_p,$$

and thus $|u|_{\ell, p} \leq 2\epsilon |u|_{\ell+1, p} + C(\nu, \ell - 1) 2^{2\ell} \nu^\ell \epsilon^{-\ell} \|u\|_p$

for any $\epsilon > 0$. This is (5) with 2ϵ instead of ϵ . The statement is then proved applying (5) backwards from $\ell = m - 1$. \blacksquare

By means of Lemma 2.1, we can prove that maximal hypoellipticity is preserved under L^∞ -perturbations of lower order terms.

Proposition 2.2. *Let \mathcal{P} be a differential operator of degree m along derivatives of \mathbf{X} such that $\mathcal{P}_{< m}$ has $L^\infty(\Omega)$ coefficients for some open $\Omega \subseteq G$. Let $p \in [1, \infty]$. Then \mathcal{P} is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$ if and only if \mathcal{P}_m is.*

Proof. Let A be the maximum of the $L^\infty(\Omega)$ norms of $\mathcal{P}_{<m}$'s coefficients, and let $\alpha \in C_c^{\infty, n}(\Omega)$. Then, by Lemma 2.1, for $\epsilon \in (0, 1]$

$$\begin{aligned} \|\mathcal{P}_{<m}\alpha\|_p &\leq C(A, \mathbf{m}) \left(\sum_{\ell=1}^{m-1} |\alpha|_{\ell, p}^{\mathbf{X}} + \|\alpha\|_p \right) \\ &\leq C'(A, \mathbf{m}) \left(\sum_{\ell=1}^{m-1} \left(\epsilon^{m-\ell} |\alpha|_{m, p}^{\mathbf{X}} + C(\nu, m) \epsilon^{-\ell} \|\alpha\|_p \right) + \|\alpha\|_p \right) \\ &\leq C(A, \mathbf{m}, \nu, m) \left(\epsilon |\alpha|_{m, p}^{\mathbf{X}} + \epsilon^{-m+1} \|\alpha\|_p \right). \end{aligned}$$

Therefore, if we choose $\epsilon = (2C(A, \mathbf{m}, \nu, m) C(\Omega))^{-1}$ we get

$$\|\mathcal{P}_{<m}\alpha\|_p \leq \frac{1}{2C(\Omega)} |\alpha|_{m, p}^{\mathbf{X}} + C(\nu, m, A, \Omega) \|\alpha\|_p.$$

Suppose now that \mathcal{P} is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$. Then

$$\begin{aligned} \|\mathcal{P}\alpha\|_p &\leq \|\mathcal{P}_m\alpha\|_p + \|\mathcal{P}_{<m}\alpha\|_p \\ &\leq \|\mathcal{P}_m\alpha\|_p + \frac{1}{2C(\Omega)} |\alpha|_{m, p}^{\mathbf{X}} + C(\nu, m, A, \Omega) \|\alpha\|_p \\ &\leq \|\mathcal{P}_m\alpha\|_p + \frac{1}{2} (\|\mathcal{P}\alpha\|_p + \|\alpha\|_p) + C(\nu, m, A, \Omega) \|\alpha\|_p, \end{aligned}$$

from which we obtain the inequality $\|\mathcal{P}\alpha\|_p \leq C(\|\mathcal{P}_m\alpha\|_p + \|\alpha\|_p)$. Thus, also \mathcal{P}_m is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$.

Analogously, one can prove that if \mathcal{P}_m is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$, then $\|\mathcal{P}_m\alpha\|_p \leq C'(\|\mathcal{P}\alpha\|_p + \|\alpha\|_p)$ for every $\alpha \in C_c^{\infty, n}(\Omega)$, so that \mathcal{P} is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$ as well. ■

The last ingredient in the proof of Theorem 1.2 is a covering lemma. Its proof is an easy generalization of the proof of [1, Lemma 1], and is omitted.

Lemma 2.3. *Let G be a locally compact second-countable group, and let U, V be two open relatively compact subsets of G such that $U \subset V$. Then, there exists a countable family $\mathfrak{U} = \{x_k : k \in \mathbb{N}\} \subset G$ and $n \in \mathbb{N}$ such that*

- (1) $G = \bigcup_k x_k U$;
- (2) for all $x_0 \in \mathfrak{U}$, $x_0 V \cap x V$ is non-empty for at most n elements $x \in \mathfrak{U}$.

Property (2) of Lemma 2.3 will be referred to as *bounded overlap property*.

Proof of Theorem 1.2. By Proposition 2.2, it is enough to prove the statement when \mathcal{P} is left invariant. For notational convenience, for every open $\Omega \subseteq G$ and $0 \leq \ell \leq m$ we define

$$\{u\}_{\ell, p}^{\mathbf{X}} = \sum_{|I|=\ell} \|X_I u\|_p^p, \quad u \in C_c^\infty(\Omega), \quad \{\alpha\}_{\ell, p}^{\mathbf{X}} = \sum_{i=1}^n \{\alpha_i\}_{\ell, p}^{\mathbf{X}}, \quad \alpha \in C_c^{\infty, n}(\Omega).$$

All the vector fields are meant to belong to \mathbf{X} . Under this new notation, the \mathbf{X} -maximal hypoellipticity of \mathcal{P} in $L^p(\Omega)$ is equivalent to saying that

$$\sum_{\ell=0}^m \{\alpha\}_{\ell,p}^{\mathbf{X}} \leq C(\Omega) (\|\mathcal{P}\alpha\|_p^p + \|\alpha\|_p^p)$$

for all $\alpha \in C_c^{\infty,n}(\Omega)$. By Lemma 2.1, there exists a constant $\kappa = \kappa(p, \nu, m)$ such that for $\ell \leq m - 1$ and $\epsilon > 0$

$$\{\alpha\}_{\ell,p}^{\mathbf{X}} \leq \kappa(\epsilon^{p(m-\ell)}\{\alpha\}_{m,p}^{\mathbf{X}} + \epsilon^{-p\ell}\|\alpha\|_p^p), \quad \alpha \in C_c^{\infty,n}(G). \tag{7}$$

Since \mathcal{P} is left invariant, we may assume that Ω is an open neighbourhood of the identity $e \in G$.

Let U, V be open sets such that $U \subset V \subset \Omega$. By Lemma 2.3, there exists a countable family (x_k) of elements of G such that $G = \bigcup_k x_k U$ and each $x_k V$ intersects at most n other $x_k V$. We set $U_k := x_k U$ and $V_k := x_k V$.

Let now $\psi \in C_c^\infty(V)$ be such that $\psi \equiv 1$ on U , $\psi \equiv 0$ on V^c and ψ takes values in $[0, 1]$; set $\psi_k = \tau_{x_k}^* \psi$. Let $\alpha \in C_c^{\infty,n}(G)$. Then

$$\sum_{\ell=0}^m \{\alpha\}_{\ell,p}^{\mathbf{X}} \leq \sum_k \sum_{\ell=0}^m \{(\alpha\psi_k)\}_{\ell,p}^{\mathbf{X}} = \sum_k \sum_{\ell=0}^m \{\tau_{x_k}^*(\alpha\psi_k)\}_{\ell,p}^{\mathbf{X}}$$

since $\psi_k \equiv 1$ on U_k , (U_k) is a covering of G , and the L^p norm and \mathbf{X} are left invariant.

Observe that $\tau_{x_k}^*(\alpha\psi_k)$ is supported in V . By the \mathbf{X} -maximal hypoellipticity of \mathcal{P} in $L^p(V)$,

$$\begin{aligned} \sum_{\ell=0}^m \{\alpha\}_{\ell,p}^{\mathbf{X}} &\leq C(V) \sum_k \left[\|\tau_{x_k}^*(\alpha\psi_k)\|_p^p + \|\mathcal{P}(\tau_{x_k}^*(\alpha\psi_k))\|_p^p \right] \\ &\leq C(V, n) \left[\|\alpha\|_p^p + \sum_k \|\mathcal{P}(\alpha\psi_k)\|_p^p \right], \end{aligned} \tag{8}$$

where we used the bounded overlap property of the family (V_k) , and the left invariance of \mathcal{P} and of the L^p -norm. By the Leibniz formula,

$$\mathcal{P}(\alpha\psi_k) = \psi_k \mathcal{P}\alpha + \mathcal{R}(\psi_k; \alpha)$$

where $\mathcal{R}(\psi_k; \alpha)$ is an \mathbf{m} -vector supported in V_k whose entries are sum of terms having at least one derivative of ψ_k , at most $m - 1$ derivatives of the components of α , and all the derivatives are along the vector fields in \mathbf{X} . By the bounded overlap property and (8)

$$\sum_{\ell=0}^m \{\alpha\}_{\ell,p}^{\mathbf{X}} \leq C(V, n) \left[\|\mathcal{P}\alpha\|_p^p + \|\alpha\|_p^p + \sum_k \|\mathcal{R}(\psi_k; \alpha)\|_p^p \right].$$

If E denotes $\max_{|I| \leq m} \|X_I \psi\|_\infty$, then by (7) we obtain that for $\epsilon \in (0, 1)$

$$\begin{aligned} \sum_k \|\mathcal{R}(\psi_k; \alpha)\|_p^p &\leq C(\mathbf{m}, m, E) n \sum_{\ell=0}^{m-1} \{\alpha\}_{\ell,p}^{\mathbf{X}} \\ &\leq C(\mathbf{m}, m, E, n, \kappa) \left(\epsilon^p \{\alpha\}_{m,p}^{\mathbf{X}} + C \epsilon^{-p(m-1)} \|\alpha\|_p^p \right). \end{aligned}$$

We used again the bounded overlap property of the sets (V_k) and left invariance of the vector fields in \mathbf{X} . By choosing $\epsilon^p = (2C(\mathbf{m}, m, E, n, \kappa)C(V, n))^{-1}$, we obtain the global estimate as desired. \blacksquare

3. Rumin's Laplacians on the Heisenberg groups

For $p \in (1, \infty)$ and $m \in \mathbb{N}$, consider the Sobolev space $L_m^{p,n}(\mathbf{X})$ of \mathbf{n} -vectors α whose norm

$$\|\alpha\|_{L_m^{p,n}(\mathbf{X})} = \sum_{\ell=0}^m |\alpha|_{\ell,p}^{\mathbf{X}}$$

is finite, see also [8]. Let \mathcal{P} be a matrix-valued differential operator of degree m along derivatives of \mathbf{X} , with left-invariant principal part and lower order terms with L^∞ -coefficients. If \mathcal{P} is \mathbf{X} -maximal hypoelliptic in $L^p(\Omega)$ for some $\Omega \subseteq G$, then by Theorem 1.2 there exists $C > 0$ such that

$$C^{-1} \|\alpha\|_{L_m^{p,n}(\mathbf{X})} \leq \|\mathcal{P}\alpha\|_p + \|\alpha\|_p \leq C \|\alpha\|_{L_m^{p,n}(\mathbf{X})}$$

for all α in $C_c^{\infty,n}(G)$. In other words, the norm of $L_m^{p,n}(\mathbf{X})$ and the graph norm of \mathcal{P} in $L^{p,n}$ are equivalent on $C_c^{\infty,n}(G)$. Thus, if \mathcal{P} is also closable in $L^{p,n}$ and $\overline{\mathcal{P}}$ stands for its closure,

$$\text{Dom}(\overline{\mathcal{P}}) = \overline{C_c^{\infty,n}(G)}^{L_m^{p,n}(\mathbf{X})}. \quad (9)$$

As we explain now, this fact allows to prove the essential self-adjointness on L^2 of Rumin's Laplacians on the contact complex of the Heisenberg groups. First, we briefly recall their construction, referring the reader to [16, 2, 3] for a thorough treatment and details.

Let $G = \mathbb{H}^n$ be the Heisenberg group identified with \mathbb{R}^{2n+1} through exponential coordinates. Let $\mathcal{B} = \{X_1, \dots, X_{2n}, T\}$ be the standard basis of its Lie algebra, see e.g. [6], where $\mathbf{X} = \{X_1, \dots, X_{2n}\}$ is a basis of the first layer. We endow \mathfrak{g} with an inner product for which \mathcal{B} is orthonormal. Denote by $\{\theta_1, \dots, \theta_{2n}, \theta\}$ the dual basis of \mathcal{B} . We define $\Lambda^0(\mathfrak{g}) = \mathbb{R}$, and

$$\Lambda^k(\mathfrak{g}) = \text{span} \{\theta_{i_1} \wedge \dots \wedge \theta_{i_k}, 1 \leq i_1 < \dots < i_k \leq 2n+1\}, \quad k = 1, \dots, 2n+1,$$

where $\theta_{2n+1} = \theta$. Denote by \mathcal{I}^* the ideal generated by θ and by \mathcal{J}^* the annihilator of \mathcal{I}^* . Both \mathcal{I}^* and \mathcal{J}^* are graded, i.e.

$$\mathcal{I}^* = \bigoplus_{k=0}^{2n+1} \mathcal{I}^k, \quad \mathcal{J}^* = \bigoplus_{k=0}^{2n+1} \mathcal{J}^k,$$

where (if one sets $\Lambda^{-2}(\mathfrak{g}) = \Lambda^{-1}(\mathfrak{g}) = \{0\}$)

$$\begin{aligned} \mathcal{I}^k &= \{\theta \wedge \beta + d\theta \wedge \gamma: \beta \in \Lambda^{k-1}(\mathfrak{g}), \gamma \in \Lambda^{k-2}(\mathfrak{g})\}, \\ \mathcal{J}^k &= \{\alpha \in \Lambda^k(\mathfrak{g}): \theta \wedge \alpha = d\theta \wedge \alpha = 0\}. \end{aligned}$$

For $k = 0, \dots, n$, one has $\mathcal{J}^k = \{0\}$ and $\mathcal{I}^{2n+1-k} = \Lambda^{2n+1-k}(\mathfrak{g})$. Then, let

$$E_0^k = \begin{cases} \Lambda^k(\mathfrak{g})/\mathcal{I}^k & \text{if } k = 0, \dots, n, \\ \mathcal{I}^k & \text{if } k = n+1, \dots, 2n+1. \end{cases}$$

If $k = 0, \dots, n-1$, we denote by $d_c: C^\infty E_0^k \rightarrow C^\infty E_0^{k+1}$ the map induced by the exterior differential on the quotient spaces, while if $k = n+1, \dots, 2n+1$, we let d_c be the usual exterior differential. Then, see [15], there exists a left-invariant homogeneous differential operator of order two $D: C^\infty E_0^n \rightarrow C^\infty E_0^{n+1}$ such that the contact complex induced by d_c and D has the same cohomology as that of De Rham. If δ_c is the formal adjoint of d_c in L^2 , and D^* that of D , we define Rumin's Laplacian on $C^\infty E_0^k$ as

$$\Delta_{\mathbb{H},k} = \begin{cases} (n-k)d_c\delta_c + (n-k+1)\delta_c d_c & \text{if } k = 0, \dots, n-1, \\ (d_c\delta_c)^2 + D^*D & \text{if } k = n, \\ DD^* + (\delta_c d_c)^2 & \text{if } k = n+1, \\ (n-k+1)d_c\delta_c + (n-k)\delta_c d_c & \text{if } k = n+2, \dots, 2n+1. \end{cases}$$

We recall that $\Delta_{\mathbb{H},k}$ is symmetric, left invariant and homogeneous, of order four if $k = n$ or $k = n+1$, of order two otherwise. We also recall that it is \mathbf{X} -maximal hypoelliptic in $L^2(\Omega)$ for a suitable neighbourhood Ω of the identity, see [15] and also [2, 4].

Corollary 3.1. *For every $k = 0, \dots, 2n+1$, $\Delta_{\mathbb{H},k}$ with domain $C_c^\infty E_0^k$ is essentially self-adjoint on $L^2 E_0^k$.*

Proof. Let $(\Delta_{\mathbb{H},k})_0$ be the restriction of $\Delta_{\mathbb{H},k}$ to $C_c^\infty E_0^k$. Since $(\Delta_{\mathbb{H},k})_0$ is symmetric and densely defined in $L^2 E_0^k$, it is closable. Thus, by (9) and the density of test functions in the Sobolev spaces on the Heisenberg groups (cf. [10]),

$$\begin{aligned} \text{Dom}(\overline{(\Delta_{\mathbb{H},k})_0}) &= L_2^2 E_0^k & \text{if } k \neq n, n+1, \\ \text{Dom}(\overline{(\Delta_{\mathbb{H},k})_0}) &= L_4^2 E_0^k & \text{if } k = n, n+1. \end{aligned} \tag{10}$$

By [13, Theorem I.3.29], the operator $\Delta_{\mathbb{H},k}$ defined in the distributional sense on the spaces (10) is self-adjoint. Thus, $\Delta_{\mathbb{H},k}$ and $\overline{(\Delta_{\mathbb{H},k})_0}$ coincide, hence the closure of $(\Delta_{\mathbb{H},k})_0$ is self-adjoint. This implies that $(\Delta_{\mathbb{H},k})_0$ is essentially self-adjoint. ■

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