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Low Dimensional Contact Lie Algebras

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Abstract. The aim of this work is to provide explicit calculations that describe any 7-dimensional contact nilpotent Lie algebra as a double extension of a 5-dimensional contact nilpotent Lie algebra. In particular, we describe an arbitrary $(2n+1)$-dimensional contact filiform Lie algebra as a double extension of a $(2n-1)$-dimensional contact nilpotent Lie algebra $\mathfrak{m}$ of nilindex n by a pair (D,θ).

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1. Introduction

A *contact* manifold is a $(2n+1)$-dimensional smooth manifold M equipped with a 1-form ν on M such that $\nu \wedge (d\nu)^n \neq 0$ pointwise over M. Such 1-form ν is called a *contact structure* on M. Gromov proved that there exists a contact structure on every odd dimensional connected non-compact Lie group (see [13]). However, such contact structures are non-necessary left invariant under left translations of the Lie group elements. In [7] Boothby and Wang proved that a semisimple (connected) Lie group admitting a left invariant contact structure is locally isomorphic with either the ortogonal group $O(3,\mathbb{R})$ or $SL(2,\mathbb{R})$. Then, an interesting and open problem is to determine which Lie groups admit a left invariant contact structure. An approach to solve this problem is to understand contact Lie groups through their Lie algebras. A *contact* Lie algebra $\mathfrak{g}$ is a $(2n+1)$-dimensional Lie algebra endowed with a 1-form $\alpha \in \mathfrak{g}^*$ such that $\alpha \wedge (d\alpha)^n \neq 0$. Hence a natural question is how to construct contact Lie algebras. In this work we focus on the construction of 7-dimensional indecomposable contact nilpotent Lie algebras, obtaining all of them as double extensions of 5-dimensional indecomposable contact nilpotent Lie algebras.

The first result concerning the classification of 3-dimensional contact Lie algebras appears in [16] and it was obtained by Kruglikov. In this work it is proved that almost every 3-dimensional Lie algebra admits a contact structure; moreover, it is also described a method to construct an $(\ell+1)$-dimensional non-degenerate Lie algebra from an ℓ-dimensional non-degenerate one. In this way, a $(2n+1)$-dimensional contact Lie algebra can be obtained from a symplectic Lie subalgebra of codimension 1 and in this case the process is called a *contactization.* Analogously, a $2n$-dimensional symplectic Lie algebra can be obtained from a contact Lie subalgebra of codimension 1 and in this case the process is called a *symplectization.*

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However, not every contact Lie algebra can be obtained in this way. For example, letting $\rho\colon \mathfrak{sl}_2 \to \mathbb{R}^2$ be the usual irreducible representation of $\mathfrak{sl}_2$ into $\mathbb{R}^2$, the Lie algebra $A_{5,40} = \mathfrak{sl}_2 \ltimes_\rho \mathbb{R}^2$ cannot be obtained from a symplectic Lie subalgebra of codimension 1.

In [10], Diatta provides a complete list of the 5-dimensional indecomposable non-isomorphic contact Lie algebras. He presents a contactization method to construct contact Lie groups via their Lie algebras, starting from exact symplectic ones. The method works fine for 3-dimensional contact Lie algebras, but it does not provide a constructive way to get a complete classification of the 5-dimensional indecomposable solvable contact Lie algebras as contactizations of exact symplectic Lie algebras. At first sight, from the exhaustive list of 5-dimensional indecomposable contact Lie algebras provided by the author, just the Lie algebras $A_{5,36}$ and $A_{5,37}$ can be obtained in this way. In order to get a complete classification, a characterization in terms of the derived ideal of a solvable contact Lie algebra $\mathfrak{g}$ has to be done.

In [17], Kutsak classifies the 7-dimensional nilmanifolds admitting a contact structure. This work is based in the classification of nilpotent Lie algebras of dimension 7 over algebraically closed fields and $\mathbb{R}$, provided by Gong (see [11]). Later, the list given by Kutsak was used by Capelletti–Montano et al in order to provide examples of K-contact manifolds with no Sasakian metric (see [8]).

On the other hand, in [12], Goze and Remm approached the classification problem of finite dimensional contact Lie algebras in terms of deformations. They prove that any $(2n+1)$-dimensional contact Lie algebra is a linear or quadratic deformation of the $(2n+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. They determine the closed 2-forms of the Chevalley-Eilenberg cohomology of the Heisenberg algebras which give linear or quadratic deformations, and they also find the classifications in dimension 3,5, or 7 of contact Lie algebras given in [10].

In [15], Khakimdjanov et al provide a geometric description of contact Lie groups, giving necessary and sufficient conditions for which a filiform Lie group admits a left invariant contact form. The authors point out that in [18] it is proved that every nilpotent symplectic Lie algebra is obtained from a sequence of symplectic double extensions. As a consequence, every contact nilpotent Lie algebra can be obtained by two operations, namely the symplectic double extension and the contactization.

The notion of *double extension* goes back to the work of Kac (see [14], §2, exercise 2.10) and it plays a significant role providing an inductive construction of finite dimensional Lie (super)algebras endowed with a geometric structure. For a deeper discussion of this subject, we refer the reader to [1], [3], [5] and [6]. However, it is not possible to provide an inductive construction that generates all the finite dimensional Lie algebras endowed with a contact structure: in [19], Rodríguez–Vallarte and Salgado exhibit a family of $(4n+1)$-dimensional indecomposable contact solvable Lie algebras that cannot be obtained neither as a suspension of a symplectic Lie algebra of codimension 1 or as a double extension of codimension 2; and they also show that a suitable double extension of a finite dimensional indecomposable contact Lie algebra is a contact Lie algebra again. On the other hand, in [2] Alvarez et al provide an inductive construction of the finite dimensional contact nilpotent Lie algebras. More precisely, they proved that for $n \geq 1$, every $(2n+3)$-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ can be obtained from the 3-dimensional Heisenberg

Lie algebra $\mathfrak{h}_3$, by applying a finite number of successive series of double extensions. Thus, the aim of this work is to apply the inductive construction presented in [2] to provide explicit calculations that describe a low-dimensional (i.e., a 5 and 7-dimensional) indecomposable contact nilpotent Lie algebra $\mathfrak{g}$ as a double extension of an indecomposable contact nilpotent Lie algebra $\mathfrak{m}$ of codimension 2. More specifically, we are able to express $\mathfrak{g}$ as a semidirect product $\mathfrak{m}(D,\theta) = \langle D\rangle \ltimes \mathfrak{m}_\theta(e)$, where $\mathfrak{m}_\theta(e)$ is a central extension of $\mathfrak{m}$ by a 2-closed form $\theta \in (\Lambda^2\mathfrak{m})^*$, and $D \in \mathrm{Der}(\mathfrak{m}_\theta(e))$ is a derivation. For dimension three it is well known that the Heisenberg Lie algebra $\mathfrak{h}_3$ is the only one indecomposable contact nilpotent Lie algebra, whereas for dimension five there are three non-isomorphic indecomposable contact nilpotent Lie algebras, namely $N_{5,1}, N_{5,2,2}$ and $N_{5,3,1}$ (which is the 5-dimensional Heisenberg Lie algebra $\mathfrak{h}_5$), and we describe each one of them as a double extension of $\mathfrak{h}_3$ by an appropiate pair (D,θ).

On the other hand, since for dimension seven there are 43 non-isomorphic indecomposable contact nilpotent Lie algebras (see [17] for more details), we express each one of them as a double extension of $\mathfrak{m}$ by a suitable pair (D,θ), where $\mathfrak{m}$ is one of the 5-dimensional contact nilpotent Lie algebras: $N_{5,1}, N_{5,2,2}$ and $N_{5,3,1}$. Now, to achieve our goal we shall proceed in the following way: given a contact nilpotent Lie algebra $\mathfrak{m}$ having dimension $n = 3,5$, first we shall determine the second scalar cohomology group $H^2(\mathfrak{m},\mathbb{R})$, and then we shall compute a central extension $\mathfrak{m}_\theta(e)$ of $\mathfrak{m}$ by a closed 2-form $\theta \in (\Lambda^2\mathfrak{m})^*$. Next, it follows from [19] (see Theorem 2.2) that if there exists a suitable derivation $D \in \mathrm{Der}(\mathfrak{m}_\theta(e))$ acting non-trivially on the center $\langle e\rangle$, then the double extension $\mathfrak{m}(D,\theta)$ is a $(n+2)$-dimensional nilpotent Lie algebra endowed with a contact form. Therefore, in order to find such D, it shall be necessary to compute the structure of the Lie algebra of derivations $\mathrm{Der}(\mathfrak{m}_\theta(e))$ of a central extension $\mathfrak{m}_\theta(e)$.

In order to obtain a double extension $\mathfrak{m}(D,\theta)$ endowed with a contact structure, according to Theorem 2.2 in [19], it is necessary to requiere the existence of a derivation $D \in \mathrm{Der}(\mathfrak{m}_\theta(e))$ acting non-trivially on the center $\langle e\rangle$ of $\mathfrak{m}_\theta(e)$. We must point out that for any 3-dimensional contact Lie algebra $\mathfrak{m}$ and for any 2-closed form $\theta \in (\Lambda^2\mathfrak{m})^*$ it is always possible to find a derivation D such that $\mathfrak{m}(D,\theta)$ is a 5-dimensional contact Lie algebra (see Table 4 from the Appendix). Indeed, in all the worked examples it is possible to find such derivation (see section §5, Tables 2 and 3). For this reason one may conjecture that this fact must be true in general. The precise statement can be found in Section §3, Conjecture 3.6.

For the sake of completeness the paper is organized as follows: in Section 2, we provide the basic definitions, we also explain how to obtain a central extension and the conditions satisfied by the derivations of these Lie algebras; in Section 3, we point out general facts about 3 and 5-dimensional contact Lie algebras; in Section 4 we describe inductively the contact nilpotent Lie algebras as double extensions, focusing our work on the description of a $(2n+1)$-dimensional contact filiform Lie algebra as double extension of a $(2n-1)$-dimensional contact nilpotent Lie algebra $\mathfrak{m}$ of nilindex n, Thus, as an application of the results given in the previous sections, in §5 we give a complete description of the 5 and 7-dimensional contact nilpotent Lie algebras as double extensions of contact nilpotent Lie algebras of codimension 2. Finally, in the Appendix we describe the 5 dimensional contact Lie algebras obtained from a 3-dimensional Lie algebra through double extension.

2. Preliminaries

Throughout the paper $\mathfrak{g}$ denotes a finite-dimensional real Lie algebra. For the usual definitions about Lie algebra cohomology, we follow [9].

Let $\mathfrak{g}$ be a Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. We denote by $C^q(\mathfrak{g}, V)$ $(q \geq 2)$, the set of alternating q-linear applications $\omega\colon \mathfrak{g} \times \cdots \times \mathfrak{g} \to V$. By definition $C^0(\mathfrak{g}, V) = V$ and $C^1(\mathfrak{g}, V) = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, V)$. We say that $\omega \in C^q(\mathfrak{g}, V)$ is a *q-form.*

Given a q-form $\omega \in C^q(\mathfrak{g}, V)$ $(q \geq 1)$, one can obtain a $(q+1)$-form via the formula

$$\begin{aligned} d\omega(x_1, \ldots, x_{q+1}) = {} & \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i+1} \rho(x_i) \omega(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1}) \\ & + \frac{1}{q+1} \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}), \end{aligned}$$

where $x_i \in \mathfrak{g}$ $(i = 1, \ldots, q+1)$ and the hat over an $x_i \in \mathfrak{g}$ means that this argument is omitted. If $q = 0$, then $\omega \in V$ and we set $d\omega(x) = \rho(x)\omega$ for all $x \in \mathfrak{g}$. The linear mapping $d\colon C^q(\mathfrak{g}, V) \to C^{q+1}(\mathfrak{g}, V)$ is called the *exterior differential.*

One also can define the following subspaces,

$$\begin{aligned} Z^q(\mathfrak{g}, V) &= \mathrm{Ker}(d\colon C^q(\mathfrak{g}, V) \to C^{q+1}(\mathfrak{g}, V)), \\ B^q(\mathfrak{g}, V) &= \mathrm{Im}(d\colon C^{q-1}(\mathfrak{g}, V) \to C^q(\mathfrak{g}, V)). \end{aligned}$$

We say that a q-form $\omega \in Z^q(\mathfrak{g}, V)$ is *closed*, and that a q-form $\eta \in B^q(\mathfrak{g}, V)$ is *exact.*

The *q-th group of cohomology with coefficients in the representation ρ* is defined by:

$$H^q(\mathfrak{g}, V) = \frac{Z^q(\mathfrak{g}, V)}{B^q(\mathfrak{g}, V)}, \quad q \geq 1, \quad \text{and} \quad H^0(\mathfrak{g}, V) = Z^0(\mathfrak{g}, V).$$

When $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is the trivial representation and $\dim_{\mathbb{R}} V = 1$, the q-th group of cohomology with coefficients in the representation ρ is called the q-th group of *scalar cohomology*, and in this case we denote $C^q(\mathfrak{g}, \mathbb{R})$ by $(\Lambda^q \mathfrak{g})^*$.

A *contact structure* on a $(2n+1)$-dimensional Lie algebra $\mathfrak{g}$ is a 1-form $\alpha \in \mathfrak{g}^*$ such that $\alpha \wedge (d\alpha)^n \neq 0$. We say that a *contact* Lie algebra is a Lie algebra endowed with a contact structure. Observe that if $\mathfrak{g}$ is a contact Lie algebra, $\alpha \wedge (d\alpha)^n$ is a volume form for the corresponding Lie group.

A *symplectic structure* on a $2n$-dimensional Lie algebra $\mathfrak{g}$ is a closed 2-form $\omega \in (\Lambda^2 \mathfrak{g})^*$ such that ω has maximal rank, that is, $d\omega = 0$ and $\omega^n \neq 0$ is a volume form on the corresponding Lie group. We say that a Lie algebra endowed with a symplectic structure is a *symplectic* Lie algebra. Moreover, the symplectic structure is called *exact* if $\omega = d\varphi$ holds for some $\varphi \in \mathfrak{g}^*$.

We say that a finite dimensional Lie algebra $\mathfrak{g}$ is *decomposable* if it can be written as a direct sum of non-trivial ideals. Otherwise, we say that $\mathfrak{g}$ is *indecomposable.* It is easy to check that a decomposable Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$ is a contact Lie algebra if and only if $\mathfrak{m}$ is a contact Lie algebra and $\mathfrak{n}$ is an exact symplectic Lie algebra or vice versa.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and let $\theta \in (\Lambda^2\mathfrak{g})^*$ be a closed 2-form. Let $\mathrm{Span}_{\mathbb{F}}\{e\} := \langle e \rangle$ be a 1-dimensional real vector space spanned by an element e that does not belong to $\mathfrak{g}$. Then, letting

$$[x,y]_\theta = [x,y]_{\mathfrak{g}} + \theta(x,y)e \quad \text{for } x,y \in \mathfrak{g}, \text{ and } \quad [x,e]_\theta = 0 \quad \text{for } x \in \mathfrak{g}, \tag{1}$$

the real vector space $\mathfrak{g} \oplus \langle e \rangle$ is a Lie algebra. We denote this Lie algebra by $\mathfrak{g}_\theta(e)$ and we say that it is a *central extension* of $\mathfrak{g}$ by the closed 2-form θ. It is a well known fact that the elements of $H^2(\mathfrak{g}, \mathbb{R})$ are in a one to one correspondence with the isomorphism classes of central extensions of a given Lie algebra $\mathfrak{g}$.
On the other hand, given a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by the closed 2-form θ and a derivation $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$, the *double extension* of $\mathfrak{g}$ by the pair (D,θ) is the semidirect product $\mathfrak{g}(D,\theta) := \langle D \rangle \ltimes \mathfrak{g}_\theta(e)$ of the abelian Lie algebra $\langle D \rangle$ with $\mathfrak{g}_\theta(e)$ (see [19]). A direct calculation shows the following

Lemma 2.1. *Let $\mathfrak{g}$ be a n-dimensional real Lie algebra, and let $\mathfrak{g}_\theta(e) = \mathfrak{g} \oplus \langle e \rangle$ be a central extension of $\mathfrak{g}$ by a closed 2-form θ. Then the center of $\mathfrak{g}_\theta(e)$ is given by*

$$Z(\mathfrak{g}_\theta(e)) = (Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)) \oplus \langle e \rangle.$$

Consider a central extension of a n-dimensional real Lie algebra $\mathfrak{g}$ by a closed 2-form θ, $\mathfrak{g}_\theta(e)$. The following Lemma shows that it is a simple matter to characterize a derivation $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$:

Lemma 2.2. *A linear transformation $D \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ consists of a 4-tuple $(A,f,v,b) \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g}) \times \mathfrak{g}^* \times \mathfrak{g} \times \mathbb{R}$ such that*

$$D(x) = A(x) + f(x)e \quad \textit{for } x \in \mathfrak{g}, \quad \textit{and} \quad D(e) = v + be.$$

Moreover, such $D \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ is a derivation if and only if for all $x,y \in \mathfrak{g}$, the following conditions hold:

(1) $dA(x,y) = \theta(x,y)v$ *(in this case $\rho = \mathrm{ad}$),*

(2) $-df(x,y) = \theta(A(x),y) + \theta(x,A(y)) - b\,\theta(x,y)$ *(in this case $\rho = 0$),*

(3) $v \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$, *where* $\mathrm{Rad}(\theta) = \{z \in \mathfrak{g} \mid \theta(z,x) = 0, \ \forall x \in \mathfrak{g}\}$.

Corollary 2.3. *Let $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ be a derivation as in Lemma 2.2 above. It follows that,*

(1) *$D(e) \in Z(\mathfrak{g}_\theta(e))$.*

(2) *If $x \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$, then $A(x) \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$.*

(3) *If D is a nilpotent derivation such that $D(e) = be$ with $b \in \mathbb{R}$, then $A \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g})$ is a nilpotent derivation and clearly, $b = 0$.*

Proof. It suffices to prove (2). From condition (1) of Lemma 2.2 one have that for $\rho = \mathrm{ad}$ and for all $x,y \in \mathfrak{g}$,

$$dA(x,y) = [A(x),y]_{\mathfrak{g}} + [x,A(y)]_{\mathfrak{g}} - A([x,y]_{\mathfrak{g}}) = \theta(x,y)v.$$

Let $x \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$. It follows that $[A(x),y] = 0$ for all $y \in \mathfrak{g}$, that is, $A(x) \in Z(\mathfrak{g})$. Since $df(x,y) = -f([x,y])$ for all $x,y \in \mathfrak{g}$, condition (2) of Lemma 2.2 implies that $A(x) \in \mathrm{Rad}(\theta)$, which completes the proof. ∎

Now, given a derivation $D \in \mathrm{Der}_{\mathbb{R}}(\mathfrak{g}_\theta(e))$ a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by the closed 2-form θ, consider the double extension of $\mathfrak{g}$ by the pair (D,θ). We have the following result:

Lemma 2.4. *Let $\mathfrak{g}(D,\theta)$ be a double extension of $\mathfrak{g}$ by the pair (D,θ), and suppose $D(e)\neq 0$. Then $w\in Z(\mathfrak{g}(D,\theta))$ if and only if there exist $w_0\in\mathfrak{g}, r,s\in\mathbb{R}$ such that*

(1) $w = w_0 + re + sD$,

(2) $w_0 \in Z(\mathfrak{g})\cap \mathrm{Rad}(\theta)$,

(3) $s = 0$,

(4) $D(w_0+\alpha e) = 0$.

3. Contact Lie algebras of dimension lower or equal than 5

For the sake of completeness, the goal of this section is to provide a description as complete as possible of the 5-dimensional real non-isomorphic indecomposable contact Lie algebras that can be described as double extensions of 3-dimensional real Lie algebras. The results and techniques presented here will be helpful in order to understand the structure of low-dimensional real contact nilpotent Lie algebras. For a deeper discussion of the following results we refer the reader to [19].

The classification of 5-dimensional real contact Lie algebras is known (see [10] for more details): there are 3 non-solvable 5-dimensional contact Lie algebras, namely, $\mathfrak{af}_2\oplus\mathfrak{sl}_2, \mathfrak{af}_2\oplus\mathfrak{so}_3$ (both of them decomposable), and $\mathfrak{sl}_2\ltimes\mathbb{R}^2$ (indecomposable); whereas there are 24 indecomposable solvable 5-dimensional contact Lie algebras. Now, in order to describe a 5-dimensional indecomposable Lie algebra $\mathfrak{m}$ as a double extension of a 3-dimensional Lie algebra $\mathfrak{g}$, we shall include the following well known result:

Theorem 3.1. *Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra over the ground field $\mathbb{R}$, with basis $\{e_1,e_2,e_3\}$. Then, $\mathfrak{g}$ is isomorphic to:*

(1) $\mathfrak{a}_3 := \mathbb{R}^3$,

(2) $\mathfrak{h}_3$, $[e_1,e_2]=e_3$,

(3) $\mathfrak{q}_0$, $[e_1,e_2]=e_2$,

(4) $\mathfrak{q}^1_\lambda$, $[e_1,e_2]=\lambda e_2+e_3$, $[e_1,e_3]=-e_2+\lambda e_3$, $\lambda\in\mathbb{R}$,

(5) $\mathfrak{q}_\lambda$, $[e_1,e_2]=e_2$, $[e_1,e_3]=\lambda e_3$, $\lambda\neq 0$,

(6) $\mathfrak{p}$, $[e_1,e_2]=e_2$, $[e_1,e_3]=e_2+e_3$,

(7) $\mathfrak{sl}_2$, $[e_1,e_2]=2e_2$, $[e_1,e_3]=-2e_3$, $[e_2,e_3]=e_1$,
$\mathfrak{su}_2$ *or* $\mathfrak{so}_3$, $[e_1,e_2]=-e_3$, $[e_1,e_3]=e_2$, $[e_2,e_3]=-e_1$.

It is easy to check that:

Theorem 3.2. *With exception of $\mathfrak{q}_1$, every 3-dimensional non-abelian real Lie algebra is a contact Lie algebra.*

The next step is to compute the second scalar cohomology group $H^2(\mathfrak{g},\mathbb{R})$ for a 3-dimensional Lie algebra $\mathfrak{g}$ as those in Theorem 3.2. It is a simple matter to prove the following result:

Proposition 3.3. *Let $\mathfrak{g}$ be a Lie algebra of dimension lower or equal than 3. For $0 \leq q \leq 3$, the scalar cohomology groups $H^q(\mathfrak{g},\mathbb{F})$ are generated by the cohomology classes of the following representatives:*

$\mathfrak{g}$	$H^0(\mathfrak{g},\mathbb{F})$	$H^1(\mathfrak{g},\mathbb{F})$	$H^2(\mathfrak{g},\mathbb{F})$	$H^3(\mathfrak{g},\mathbb{F})$
$\mathfrak{a}_2$	$\mathbb{F}$	e^1, e^2	$e^1 \wedge e^2$	0
$\mathfrak{af}$	$\mathbb{F}$	e^1	0	0
$\mathfrak{a}_3$	$\mathbb{F}$	e^1, e^2, e^3	$e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3$	$e^1 \wedge e^2 \wedge e^3$
$\mathfrak{h}_3$	$\mathbb{F}$	e^1, e^2	$e^1 \wedge e^3, e^2 \wedge e^3$	$e^1 \wedge e^2 \wedge e^3$
$\mathfrak{q}_0$	$\mathbb{F}$	e^1, e^3	$e^1 \wedge e^3$	0
$\mathfrak{q}_0^1$	$\mathbb{R}$	e^1	$e^2 \wedge e^3$	$e^1 \wedge e^2 \wedge e^3$
$\mathfrak{q}_\lambda^1$, $\lambda \neq 0$	$\mathbb{R}$	e^1	0	0
$\mathfrak{q}_{-1}$	$\mathbb{F}$	e^1	$e^2 \wedge e^3$	0
$\mathfrak{q}_\lambda$, $\lambda \neq \{0,-1\}$	$\mathbb{F}$	e^1	0	0
$\mathfrak{p}$	$\mathbb{F}$	e^1	0	0
$\mathfrak{sl}_2$	$\mathbb{C}$	0	0	$e^1 \wedge e^2 \wedge e^3$
$\mathfrak{so}_3$	$\mathbb{R}$	0	0	$e^1 \wedge e^2 \wedge e^3$

From Proposition 3.3 above it follows:

Theorem 3.4. *Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra with basis $\{e_1, e_2, e_3\}$, as in Theorem 3.1, and let $\{e^1, e^2, e^3\}$ be its dual basis. Let $H^2(\mathfrak{g},\mathbb{R})$ be its second scalar cohomology group. Then*

(1) $\dim H^2(\mathfrak{g},\mathbb{R}) = 3$ *if and only if* $\mathfrak{g} = \mathfrak{a}_3$. *In this case,*

$$H^2(\mathfrak{a}_3,\mathbb{R}) = \langle [e^1 \wedge e^2],\ [e^1 \wedge e^3],\ [e^2 \wedge e^3] \rangle.$$

(2) $\dim H^2(\mathfrak{g},\mathbb{R}) = 2$ *if and only if* $\mathfrak{g} = \mathfrak{h}_3$. *In this case,*

$$H^2(\mathfrak{h}_3,\mathbb{R}) = \langle [e^1 \wedge e^3],\ [e^2 \wedge e^3] \rangle.$$

(3) $\dim H^2(\mathfrak{g},\mathbb{R}) = 1$ *if and only if $\mathfrak{g}$ is one of the following Lie algebras:* $\mathfrak{q}_{-1}, \mathfrak{q}_0, \mathfrak{q}_0^1$. *In this case,*

$$H^2(\mathfrak{q}_{-1},\mathbb{R}) = \langle [e^2 \wedge e^3]] \rangle, \quad H^2(\mathfrak{q}_0,\mathbb{R}) = \langle [e^1 \wedge e^3] \rangle, \quad H^2(\mathfrak{q}_0^1,\mathbb{R}) = \langle [e^2 \wedge e^3] \rangle.$$

(4) $\dim H^2(\mathfrak{g},\mathbb{R}) = 0$ *if $\mathfrak{g}$ is different from* $\mathfrak{a}_3, \mathfrak{h}_3, \mathfrak{q}_{-1}, \mathfrak{q}_0, \mathfrak{q}_0^1$.

Suppose now that $\mathfrak{g}$ is a 3-dimensional Lie algebra with basis $\{e_1, e_2, e_3\}$ as in Theorem 3.1, and suppose also that such $\mathfrak{g}$ is endowed with a contact form. From Theorem 3.4 we can compute a central extension $\mathfrak{g}_\theta(e_4)$ of $\mathfrak{g}$ by a closed 2-form θ. Now, for such central extensions, it follows from [19], Theorem 2.2, that if there exists a suitable derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e_4))$ acting non-trivially on $\langle e_4 \rangle$, then the double extension $\mathfrak{g}(D,\theta)$ is also endowed with a contact form. Thus, in order to prove that a 5-dimensional indecomposable contact Lie algebra $\mathfrak{m}$ can be obtained as a double extension of a 3-dimensional Lie algebra $\mathfrak{g}$ as those given in Theorem 3.1, it is necessary to compute the structure of the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g}_\theta(e_4))$ for all possible central extensions $\mathfrak{g}_\theta(e_4)$. Since the computation of $\mathrm{Der}(\mathfrak{g}_\theta(e_4))$ has been previously done in [19] (see Appendix A), we can state the following result:

Theorem 3.5. *Let $\mathfrak{g}$ be a 3-dimensional non-abelian real Lie algebra. Then for any 2-closed form θ, there exists a suitable derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ such that $\mathfrak{g}(D,\theta) = \langle D\rangle \ltimes \mathfrak{g}_\theta(e)$ is a contact Lie algebra. If $\mathfrak{g} = \mathfrak{a}_3$, then for any non-trivial 2-closed form θ, there exists a suitable derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ such that $\mathfrak{g}(D,\theta) = \langle D\rangle \ltimes \mathfrak{g}_\theta(e)$ is a contact Lie algebra.*

Proof. We shall proceed by cases, considering each one the 3-dimensional contact Lie algebras $\mathfrak{g}$ given in Theorem 3.1. First, for such $\mathfrak{g}$, one can choose a representative θ of the second scalar cohomology group, $H^2(\mathfrak{g},\mathbb{R})$, in order to construct a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by θ. Since the structure of the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g}_\theta(e))$ is known (see for instance [19], Appendix A), a straightforward calculation allows us to choose a derivation acting non-trivially in the central element of $\mathfrak{g}_\theta(e)$, and hence, through the double extension process, one can construct a 5-dimensional contact Lie algebra $\mathfrak{g}(D,\theta)$. More specifically, in Table 4 of the Appendix we describe the 5-dimensional contact Lie algebras obtained in this way, exhibiting in each case a suitable derivation. Since Theorem 3.1 does not hold for non-contact Lie algebras, it suffices to focus on the remaining cases: $\mathfrak{g} = \mathfrak{a}_3$ and $\mathfrak{g} = \mathfrak{q}_1$.
If either $\mathfrak{g} = \mathfrak{a}_3$ or $\mathfrak{g} = \mathfrak{q}_1$, it remains to prove the assertion. In both cases, from Theorem 3.4 we can compute a central extension $\mathfrak{g}_\theta(e_4)$ of $\mathfrak{g}$ for any 2-closed form θ and the corresponding Lie algebra of derivations $\mathrm{Der}(\mathfrak{g}_\theta(e_4))$ (see [19] for more details). Then one can choose a suitable derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e_4))$ and by a direct calculation, one can show that $\mathfrak{g}(D,\theta)$ is a contact Lie algebra with contact form $\eta \in \mathfrak{g}^*$. More specifically: for $\mathfrak{g} = \mathfrak{a}_3$ consider $\theta = e^1 \wedge e^2$, hence choosing $D \in \mathrm{Der}((\mathfrak{a}_3)_\theta(e_4))$ as

$$[D] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

it follows that $\eta = e^3 + e^4 \in \mathfrak{a}_3(D,\theta)^*$ is a contact form, whereas for $\mathfrak{g} = \mathfrak{q}_1$ choosing $D \in \mathrm{Der}((\mathfrak{q}_1)_\theta(e_4))$ as

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

it follows that $\eta = e^2 + e^4 \in \mathfrak{q}_1(D,\theta)^*$ is a contact form. ■

Observe that given a contact Lie algebra $\mathfrak{g}$ one may pose the problem of determining when the double extension $\mathfrak{g}(D,\theta)$ is a contact Lie algebra again. For a 3-dimensional contact Lie algebra one can deduce the conditions for which the double extension is a contact Lie algebra again (see [19] for more details). Thus, letting $e^* \in (\mathfrak{g}_\theta(e))^*$ be such that $\langle e^*, e\rangle = 1$ and $\langle e^*, x\rangle = 0$ for every $x \in \mathfrak{g}$, one may conjecture that the following statement is true:

Conjecture 3.6. Let $\mathfrak{g}$ be a finite dimensional contact Lie algebra with a contact structure $\alpha \in \mathfrak{g}^*$, let θ be a closed 2-form and consider a central extension $\mathfrak{g}_\theta(e)$ of $\mathfrak{g}$ by θ. Then there exists a derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$, $D(e) = v + be$ for some $b \in \mathbb{R}$, $v \in \mathfrak{g}$, such that the double extension $\mathfrak{g}(D, \theta)$ of $\mathfrak{g}$ by the pair (D, θ) is a contact Lie algebra with contact form $\beta := \alpha + \lambda e^*$ for some $\lambda \in \mathbb{R}$.

Remark 3.7. Some evidence for this claim comes from Theorem 2.2 in [19], since this result states that if there exists a derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ such that $\beta(D(e)) \neq 0$, then the double extension $\mathfrak{g}(D, \theta)$ of $\mathfrak{g}$ by the pair (D, θ) is a contact Lie algebra with contact form $\beta := \alpha + \lambda e^*$ for some $\lambda \in \mathbb{R}$. Next, we present some cases in which this situation can be easily corroborated.

Since $\mathfrak{g}$ is a finite dimensional contact Lie algebra with a contact structure $\alpha \in \mathfrak{g}^*$, there exists a contact basis $\mathcal{B} = \{x_1, \ldots, x_{2n}, x_{2n+1}\}$ of $\mathfrak{g}$ such that $\alpha = x^*_{2n+1}$ and $d\alpha = -\sum_{i=1}^n x^*_{2i-1} \wedge x^*_{2i}$ (see [12]). Letting $\widehat{\mathcal{B}} = \{x_1, \ldots, x_{2n}, x_{2n+1}, e\}$ be a basis of $\mathfrak{g}_\theta(e)$, Lemma 2.2 allows us to characterize $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ as a 4-tuple $(A, f, a, b) \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g}) \times \mathfrak{g}^* \times \mathbb{R} \times \mathbb{R}$ such that

$$D(x) = A(x) + f(x)e \ \text{ for } x \in \mathfrak{g}, \quad D(e) = ax_{2n+1} + be \ \text{ for } a \in \mathbb{R},$$

and satisfying conditions (1), (2) and (3) stated in Lemma 2.2.
With no loss of generality, we shall denote in the same way a linear transformation and its matrix representation in the chosen basis. Hence, $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$ is given by

$$D = \left(\begin{array}{c|c} \begin{matrix} \\ A \\ \\ \end{matrix} & \begin{matrix} 0 \\ \\ a \end{matrix} \\ \hline f & b \end{array}\right),$$

and conditions (1), (2) and (3) of Lemma 2.2 are satisfied.

Now, we can consider the following cases:

Case 1: If $\theta = 0$ and $a \neq 0$, it is clear that we may choose the derivation

$$[D] = \left(\begin{array}{c|c} \begin{matrix} \\ A \\ \\ \end{matrix} & \begin{matrix} 0 \\ \\ a \end{matrix} \\ \hline 0 & b \end{array}\right)$$

where Lemma 2.2 implies that $A \in \mathrm{Der}(\mathfrak{g})$, $\theta(A(x), y) + \theta(x, A(y)) - b\theta(x, y) = 0$ and $ax_{2n+1} \in Z(\mathfrak{g}) \cap \mathrm{Rad}(\theta)$. Letting $\beta = \alpha$ we have

$$d\beta = -\sum_{i=1}^{n} x^*_{2i-1} \wedge x^*_{2i} - \sum_{i=1}^{2n+1} t_{i,2n+1} D^* \wedge x^*_i - aD^* \wedge e^*$$

for some $t_{i,2n+1} \in \mathbb{R}$ $(i = 1, \ldots, 2n+1)$. Therefore

$$\beta \wedge (d\beta)^{n+1} = (-1)^{n+1}(n+1)!\, a \left(x^*_1 \wedge x^*_2 \wedge \cdots \wedge x^*_{2n+1} \wedge D^* \wedge e^*\right) \neq 0.$$

Case 2: If $\theta = 0$ and $a = 0$, we can choose a derivation of the form

$$[D] = \left(\begin{array}{ccc|c} 0 & & 0 & \\ & \ddots & & 0 \\ 0 & & 0 & \\ \hline & 0 & & b \end{array}\right),$$

with $b \neq 0$. Hence, letting $\beta = \alpha + \lambda e^*$ we have that $\beta(D(e)) = \lambda b$. Therefore, by Theorem 2.2 in [19] we can choose $\lambda \neq 0$ and the claim follows.

Case 3: If $\theta \neq 0$ and if there exists a semisimple derivation

$$[D] = \left(\begin{array}{ccc|c} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_{2n+1} & \\ \hline & 0 & & b \end{array}\right),$$

where $a_1, \ldots, a_{2n+1} \in \mathbb{R}$, the system of equations to solve in order to get a contact double extension is given by $a_i + a_j = b$ if $\theta(x_i, x_j) \neq 0$, and $a_k = a_i + a_j$ if $c_{i,j}^k \neq 0$ for $1 \leq i, j, k \leq 2n+1$. Hence, letting again that $\beta = \alpha + \lambda e^*$, the claim follows from Theorem 2.2 in [19].

Now, following the notation and classification of the 5-dimensional indecomposable solvable Lie algebras $\mathfrak{m}$ presented in [4], where $\{f_1, \ldots, f_5\}$ is a basis for $\mathfrak{m}$ and $\{f^1, \ldots, f^5\}$ is the dual basis, for each $\mathfrak{m}$, a suitable change of basis can be provided in order to recognize each $\mathfrak{m}$ as a double extension $\mathfrak{g}(D, \theta)$ of a 3-dimensional contact Lie algebra $\mathfrak{g}$ by a pair (D, θ), where θ is a closed 2-form and $D \in \mathrm{Der}(\mathfrak{g}_\theta(e_4))$ acts non-trivially on $\langle e_4 \rangle$, as it is stated in Theorem 2.2 of [19].

Theorem 3.8. *Let $\mathfrak{m}$ be a 5-dimensional indecomposable contact Lie algebra with basis $\{f_1, \ldots, f_5\}$. Then, with exception of $A_{5,35}^{ab}$, $A_{5,39}$ and $A_{5,40}$, $\mathfrak{m}$ can be obtained as a double extension of a 3-dimensional contact Lie algebra $\mathfrak{g}$.*

Remark 3.9. Theorem 3.8 above says that $A_{5,35}^{ab}$, $A_{5,39}$ and $A_{5,40}$ are the unique 5-dimensional indecomposable contact Lie algebras that cannot be obtained as a double extension of a 3-dimensional contact Lie algebra. More specifically,

(a) Let $A_{5,35}^{ab}$ be the Lie algebra with brackets given by $[e_1, e_4] = be_1$, $[e_1, e_5] = ae_1$, $[e_2, e_4] = e_2$, $[e_3, e_5] = e_2$, $[e_3, e_4] = e_3$, and $[e_2, e_5] = -e_2$. It is clear that $A_{5,35}^{ab}$ is the semidirect product of the 3-dimensional real abelian Lie algebra $\mathfrak{a}_3 = \langle e_1, e_2, e_3 \rangle$ by $\langle e_4, e_5 \rangle$, where $\langle e_4, e_5 \rangle$ acts on $\mathfrak{a}_3$ via $\mathrm{Der}(\mathfrak{a}_3)$. Observe that $A_{5,35}^{ab}$ is a family of indecomposable solvable Lie algebras having trivial center. Letting $\{e^1, \ldots, e^5\}$ be a basis of $(A_{5,35}^{ab})^*$, it is easy to verify that both $\alpha = e^1 + e^2 \in (A_{5,35}^{ab})^*$ and $\alpha = e^1 + e^3 \in (A_{5,35}^{ab})^*$ define contact structures on $A_{5,35}^{ab}$. Clearly, $A_{5,35}^{ab}$ cannot be constructed as a double extension of a 3-dimensional contact Lie algebra $\mathfrak{g}$ by a pair (D, θ) since it is not splittable by a 1-dimensional abelian subalgebra. In [19] this example is generalized to a family of $(4n+1)$-dimensional contact solvable Lie algebras.

(b) Let $A_{5,39}$ be the Lie algebra with the brackets $[e_1, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_2, e_4] = e_2$, $[e_1, e_5] = -e_2$ and $[e_4, e_5] = -e_3$. An easy calculation shows that $A_{5,39}$

can be obtained as a double extension of the 3-dimensional Lie algebra $\mathfrak{q}_1$, however $\mathfrak{q}_1$ is not a contact Lie algebra.

(c) Let $A_{5,40}$ be the Lie algebra with the brackets $[e_1, e_2] = 2e_1$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = 2e_3$, $[e_2, e_4] = e_4$, $[e_3, e_5] = e_4$, $[e_1, e_4] = e_5$, and $[e_2, e_5] = -e_5$. It is easy to check that $A_{5,40} = \mathfrak{sl}_2 \ltimes_\rho \mathbb{R}^2$ is a contact Lie algebra. Suppose now that $A_{5,40}$ can be written as a double extension of a 3-dimensional Lie algebra, say, $A_{5,40} = \langle D \rangle \ltimes \mathfrak{sl}_2(e)$ for some $D \in \mathrm{Der}(\mathfrak{sl}_2(e))$. Since $\langle e \rangle$ is a 1-dimensional trivial $\mathfrak{sl}_2$-module, it follows that the irreducible $\mathfrak{sl}_2$-module $\mathbb{R}^2$ can be splitted as $\langle D \rangle \oplus \langle e \rangle$ and clearly, it is not possible.

Then, from Theorem 3.8 and Remark 3.9 it follows that not every $(2n+3)$-dimensional indecomposable contact Lie algebra $\mathfrak{m}$ can be obtained as a double extension of a $(2n+1)$-dimensional contact Lie algebra $\mathfrak{g}$.

4. Contact nilpotent Lie algebras as double extensions

In this section we use the results obtained in Section 2 for providing a description of an arbitrary contact nilpotent Lie algebra as a double extension of a contact nilpotent Lie algebra of codimension 2. For a deeper discussion of the results presented here, we refer the reader to [2]. Finally, in subsection 4.1 below, we provide explicit calculations in order to describe a $(2p+1)$-dimensional contact filiform Lie algebra as a double extension of a $(2p-1)$-dimensional contact nilpotent Lie algebra.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of nilindex $k+1$. Then, it is clear that $0 \neq \mathfrak{g}^k \subseteq Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$. Moreover, if such $\mathfrak{g}$ is also a contact Lie algebra it is well known that $\dim Z(\mathfrak{g}) \leq 1$, and hence

$$\dim \mathfrak{g}^k = \dim Z(\mathfrak{g}) = 1.$$

Now, from [12] we recall the following result:

Lemma 4.1. *Let $\mathfrak{g}$ be a contact Lie algebra with contact form $\alpha \in \mathfrak{g}^*$. Then there exists a basis $\{\alpha_1, \dots, \alpha_{2n+1}\}$ of $\mathfrak{g}^*$ with the properties $\alpha_{2n+1} = \alpha$ and $d\alpha_{2n+1} = -\alpha_1 \wedge \alpha_2 - \cdots - \alpha_{2n-1} \wedge \alpha_{2n}$. Moreover, if $\{x_1, \dots, x_{2n+1}\}$ is the dual basis of $\mathfrak{g}^*$, it follows that*

$$[x_{2k-1}, x_{2k}] = x_{2n+1} + \sum_{s=1}^{2n} C^s_{2k-1,2k} x_s, \quad k = 1, \dots, n,$$

$$[x_i, x_j] = \sum_{s=1}^{2n} C^s_{i,j} x_s, \quad 1 \leq i < j \leq 2n+1, \ (i,j) \neq (2k-1, 2k),$$

where $C^s_{i,j} \in \mathbb{R}$ for all $i, j, s = 1, \dots, 2n$.

Suppose now that $\mathfrak{g}$ is a $(2n+1)$-dimensional contact Lie algebra and let us fix a basis as in Lemma 4.1 above. Supposing that $C^s_{2k-1,2k} = C^s_{i,j} = 0$ for all $s = 1, \dots, 2n, k = 1, \dots, n$, and $1 \leq i < j \leq 2n+1$ with $(i,j) \neq (2k-1, 2k)$, it is clear that $\mathfrak{g}$ is the $(2n+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. On the other hand, supposing that either $C^s_{2k-1,2k} \neq 0$ or $C^s_{i,j} \neq 0$ for some $s = 1, \dots, 2n, k = 1, \dots, n$ and $1 \leq i < j \leq 2n+1$ with $(i,j) \neq (2k-1, 2k)$, it follows that:

Lemma 4.2. *Let $\mathfrak{g}$ be a $(2n+1)$-dimensional contact Lie algebra with a contact form $\alpha \in \mathfrak{g}^*$. Consider the basis $\{\alpha_1, \ldots, \alpha_{2n+1}\}$ of $\mathfrak{g}^*$ such that $\alpha_{2n+1} = \alpha$ and $d\alpha_{2n+1} = -\alpha_1 \wedge \alpha_2 - \cdots - \alpha_{2n-1} \wedge \alpha_{2n}$. If $\mathfrak{g}$ is nilpotent, then $Z(\mathfrak{g}) = \langle x_{2n+1} \rangle$.*

In [12] (see Theorem 20), it is proved that a contact Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}$ is a central extension of a symplectic nilpotent Lie algebra. Lemma 4.2 above can be used to provide another proof for this result:

Theorem 4.3. *Let $\mathfrak{g}$ be a $(2n+1)$-dimensional nilpotent Lie algebra. Then $\mathfrak{g}$ is a contact Lie algebra if and only if it is a central extension of a nilpotent symplectic Lie algebra.*

Now, one can state the following:

Theorem 4.4. *Let $\mathfrak{g}$ be a $(2n+1)$-dimensional contact nilpotent Lie algebra $(n \geq 2)$. Then $\mathfrak{g}$ is a double extension of the Heisenberg Lie algebra $\mathfrak{h}_{2n-1}$ by a pair $(D, 0)$ where $D \in \operatorname{Der}((\mathfrak{h}_{2n-1})_0(e))$, if and only if, $\mathfrak{g}$ is a double extension of the abelian Lie algebra $\mathbb{R}^{2n-1}$ by a pair (T, θ) where $T \in \operatorname{Der}((\mathbb{R}^{2n-1})_\theta(z))$ and $\theta \in \Lambda^2(\mathbb{R}^{2n-1})^*$ is a closed 2-form of maximal rank.*

The main result of this section is the following:

Theorem 4.5. *Let $\mathfrak{g}$ be a $(2n+1)$-dimensional contact nilpotent Lie algebra $(n \geq 2)$ with a contact form $\beta \in \mathfrak{g}^*$. Then, there exist*

(1) *a $(2n-1)$-dimensional contact nilpotent Lie algebra $\mathfrak{h}$ with a contact form $\alpha \in \mathfrak{h}^*$,*

(2) *a 2-closed form $\theta \in (\Lambda^2\mathfrak{h})^*$, and*

(3) *a nilpotent derivation $D \in \operatorname{Der}(\mathfrak{h}_\theta(e))$ of the central extension $\mathfrak{h}_\theta(e)$ of $\mathfrak{h}$ by θ,*

such that $\mathfrak{g}$ is isomorphic to the double extension $\mathfrak{h}(D, \theta)$ of $\mathfrak{h}$ by the pair (D, θ). Moreover, there exists $\lambda \in \mathbb{R}$ such that $\beta = \alpha + \lambda e^*$.

Corollary 4.6. *For $n \geq 2$, every $(2n+1)$-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ can be obtained from the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$, by applying a finite number of successive series of double extensions by pairs (D, θ), where D is a nilpotent derivation.*

Contact Filiform Lie algebras. Now, we focus on providing explicit calculations that allow us to express a $(2n+1)$-dimensional contact filiform Lie algebra as a double extension of a $(2n-1)$-dimensional contact nilpotent Lie algebra $\mathfrak{m}$ of nilindex n. So, we shall introduce the notion of a formal deformation of a Lie algebra, following Goze [12].

A *formal deformation* $\mathfrak{g}_t$ of a Lie algebra $\mathfrak{g}_0$ is a Lie algebra whose Lie bracket μ_t satisfies

$$\mu_t(x, y) = \mu_0(x, y) + t\phi_1(x, y) + t^2\phi_2(x, y) + \cdots + t^n\phi_n(x, y) + \cdots$$

where $\mu_0(x, y)$ is the Lie bracket of $\mathfrak{g}_0$ and the maps ϕ_t are bilinear on $\mathfrak{g}_0$ with values on $\mathfrak{g}_0$ (the Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_t$ have the same underlying space). The formal deformation is called *quadratic* if $\phi_i = 0$ for any $i \geq 3$. It is called *linear* if $\phi_i = 0$ for any $i \geq 2$.

Let $L_n = \langle e_0, \ldots, e_{2n} \rangle$ be the $(2n+1)$-dimensional filiform Lie algebra with brackets $\mu_1(e_0, e_i) = e_{i+1}$ for $i = 1, \ldots, 2n-1$. Clearly, $Z(L_n) = \langle e_{2n} \rangle$. In [15] Khakimdjanov et al give necessary and sufficient conditions for which a $(2n+1)$-dimensional filliform Lie algebra is a contact one. In [12] Goze and Remm summarize this result as follows:

Theorem 4.7. *Any $(2n+1)$-dimensional filiform contact Lie algebra $\mathfrak{g}_{2n+1}$ is a linear deformation of the filiform Lie algebra L_n whose bracket is isomorphic to*

$$\mu_{1,2n+1} = \mu_1 + \sum_{i=1}^{n-1} a_{i,2i+2} \Psi_{i,2i+2}, \quad \textit{where}$$

$$\Psi_{k,s}(e_i, e_j) = (-1)^k \binom{j-k-1}{k-i} (\operatorname{ad} e_0)^{j-2k-1+i} (e_s) \qquad \begin{array}{c} 0 \leq i < j-1 \leq 2n-1, \\ j-k-1 \geq k-i \geq 1, \end{array}$$

$$\Psi_{k,s}(e_k, e_j) = e_{s+j-k-1},$$

and the constants $a_{j,2j+2} \in \mathbb{R}$ satisfy for $i = 1, \ldots, n-1$

$$A_i := \sum_{k=0}^{i-1} (-1)^k a_{n-i+k,2(n-i+k)+2} \binom{k}{2i-k-2} \neq 0. \tag{2}$$

Remark 4.8. For $\mathfrak{g}_{2n+1}$ as in Theorem 4.7 above, let $C^{2n}_{\ell,j}$ denote the structure constants accompanying the central element e_{2n}. For future convenience, we shall show that the constants $C^{2n}_{\ell,j}$ are not zero for any pair of integers (ℓ, j) such that $1 \leq \ell < j \leq 2n$. From Equation (2) follows that $k \geq 2i - k - 2$, thus $k \geq i - 1$. Then, for $k = i - 1$ it follows that $n - i + k = n - 1$. Therefore, for the pair of integers $(n-1, 2(n-1)+2)$ we have that

$$\begin{aligned}\Psi_{n-1,2(n-1)+2}(e_l, e_j) &= (-1)^{n-1} \binom{j-(n-1)-1}{(n-1)-l} e_{2(n-1)+2+j-2(n-1)-1+l} \\ &= (-1)^{n-1} \binom{j-n}{n-1-l} e_{l+j+1}\end{aligned}$$

$$\Psi_{n-1,2(n-1)+2}(e_{n-1}, e_j) = e_{2(n-1)+2+j-(n-1)-1} = e_{n+j}.$$

Since $j - n \geq n - 1 - l$, it follows that $l + j + 1 \geq 2n$. Hence, for any $j > n-1$ we have that $n + j \geq 2n$, which proves our claim.

Observe that we can now rewrite the brackets for the filiform Lie algebra $\mathfrak{g}_{2n+1}$ as follows:

$$[e_0, e_j] = e_{j+1} \text{ for } 1 \leq j \leq 2n-1, \text{ and } [e_i, e_j] = c_{i,j}\ e_{i+j+1} \text{ for } 1 \leq i < j \leq 2n,$$

where

$$c_{i,j} = \left(a_{i,2i+2} + \sum_{k=i+1}^{\lfloor \frac{i+j-1}{2} \rfloor} a_{k,2k+2} (-1)^k \binom{j-k-1}{k-i} \right) \qquad \begin{array}{c} j-k-1 \geq k-i \geq 1, \\ i+j+1 \leq 2n. \end{array}$$

Now, applying the following change of basis

$$x_{2k-1} = \frac{1}{c_{k-1,2n-k}} e_{k-1} \text{ for } 1 \leq k \leq n, \quad x_{2k} = e_{2n-k} \text{ for } 1 \leq k \leq n, \text{ and } x_{2n+1} = e_{2n},$$

and appropriately labeling the indexes of the constants $c_{k-1,2n-k}$, we have

Corollary 4.9. *Let $\mathfrak{h}$ be the $(2n+1)$-dimensional filiform Lie algebra defined on the basis $\{x_1,\ldots,x_{2n+1}\}$ by the following brackets:*

$$
\begin{aligned}
[x_{2k-1},x_{2k}] &= x_{2n+1} && 1\le k\le n,\\
[x_1,x_{2l-1}] &= \begin{cases} \dfrac{1}{c_{l-1,2n-l}}x_{2l+1} & \text{if } 2<l\le n-1,\\ \dfrac{1}{c_{n-1,n}}x_{2n} & \text{if } l=n,\end{cases}\\
[x_1,x_{2l}] &= x_{2(l-1)} && 1<l\le n,\\
[x_2,x_l] &= 0 && 1<l\le 2n+1,\\
[x_{2k-1},x_{2(k+1)}] &= \frac{c_{k-1,2p-k-1}}{c_{k-1,2p-k}}x_2 && 1\le k\le n-1.
\end{aligned}
$$

Then $\mathfrak{h}$ is isomorphic to the $(2n+1)$-dimensional filiform Lie algebra $\mathfrak{g}_{2n+1}$.

From Corollary 4.9 above, it is clear that we have a closed 2-form θ given by

$$\theta=\sum_{k=2}^{n-1}\frac{c_{k-1,2n-k-1}}{c_{k-1,2n-k}}\left(x^{2k-1}\wedge x^{2k+2}\right).$$

Hence, since $D=\operatorname{ad}(x_1)$ is a derivation, we can state the following:

Theorem 4.10. *Let $\mathfrak{h}$ be the $(2n+1)$-dimensional contact filiform Lie algebra defined on the basis $\{x_1,\ldots,x_{2n+1}\}$ by the brackets given in Corollary 4.9 above. Then $\mathfrak{h}$ can be obtained as a contact double extension $\mathfrak{m}(D,\theta)$ of a $(2n-1)$-dimensional contact nilpotent Lie algebra $\mathfrak{m}$ by a pair (D,θ), where $D\in\operatorname{Der}((\mathfrak{m})_\theta(x))$ is a derivation and $\theta\in(\Lambda^2\mathfrak{m})^*$ is a closed 2-form. Moreover, the nilindex of $\mathfrak{m}$ is n.*

Proof. Let $\mathfrak{m}=\langle x_3,\ldots,x_{2n-1}\rangle$ be the $(2n-1)$-dimensional Lie algebra defined by the following brackets:

$$
\begin{aligned}
[x_{2k-1},x_{2k}] &= x_{2n+1} && 2\le k\le n,\\
[x_{2k-1},x_{2l-1}] &= \begin{cases} \dfrac{c_{k-1,l-1}}{c_{k-1,2n-k}c_{l-1,2n-l}}x_{2(k+l-1)+1} & \text{if } \begin{array}{c}4\le k+l-1\le n-1,\\ 2\le k<l\le n,\end{array}\\ \dfrac{c_{k-1,l-1}}{c_{k-1,2n-k}c_{l-1,2n-l}}x_{2[2n-(k+l-1)]} & \text{if } \begin{array}{c}n\le k+l-1\le 2n-2,\\ 2\le k<l\le n,\end{array}\end{cases}\\
[x_{2k-1},x_{2l}] &= \frac{c_{k-1,2n-l}}{c_{k-1,2n-k}}x_{2(l-k)} && 2\le k<l-1\le n-1,\\
[x_{2k},x_l] &= 0 && 4\le 2k<l\le n.
\end{aligned}
$$

From the brackets $[x_{2k-1},x_{2k}]= x_{2n+1}$ for any $2\le k\le n$, it is clear that $\mathfrak{m}$ is a contact Lie algebra with contact form given by $\alpha=x^*_{2n+1}\in\mathfrak{m}^*$. Thus, what it is left is to show that $\mathfrak{m}$ has nilindex n, but this is done by a simple computation:

$$
\begin{aligned}
\mathfrak{g}^k &= \{x_{2i-1}\}_{i=2k+3}^{n}\cup\{x_{2i}\}_{i=2}^{n}\cup\{x_{2n+1}\} && \text{for } 1\le k\le \left\lfloor\frac{n+1}{2}\right\rfloor-2,\\
\mathfrak{g}^{\lfloor\frac{n+1}{2}\rfloor-2+k} &= \{x_{2i}\}_{i=2}^{n-2(k-1)}\cup\{x_{2n+1}\} && \text{for } 1\le k\le\left\lfloor\frac{n}{2}\right\rfloor,\\
\mathfrak{g}^{n-1} &= \{x_{2n+1}\}\quad\text{and}\quad \mathfrak{g}^n=\{0\}.
\end{aligned}
$$

∎

Remark 4.11. In [15] Khakimdjanov et al characterized through deformations the $(2n+1)$-dimensional contact filiform Lie algebras. In contrast, Theorem 4.10 above provides explicit calculations that allow to identify an arbitrary $(2n+1)$-dimensional contact filiform Lie algebra as the double extension of a $(2n-1)$-dimensional contact nilpotent Lie algebra $\mathfrak{m}$ of nilindex n by a pair (D, θ).

5. Low dimensional contact nilpotent Lie algebras

For $n = 5$ and 7, the aim of this section is to present an inductive construction of the n-dimensional indecomposable contact nilpotent Lie algebras via a double extension. The underlying result behind this is Theorem 4.5 introduced in [2]: for $k \geq 1$, every finite $(2k+1)$-dimensional contact nilpotent Lie algebra can be obtained as a double extension of a contact nilpotent Lie algebra $\mathfrak{m}$ of codimension 2 by a pair (D, θ). Thus, following the notation and classification of the low-dimensional nilpotent Lie algebras given by Gong (see [11]), for each indecomposable contact nilpotent Lie algebra $\mathfrak{g}$ having dimension 5 and 7, our goal is to explicitly identify it as a double extension of an indecomposable contact nilpotent Lie algebra $\mathfrak{m}$ of codimension 2 by a pair (D, θ), where $\theta \in (\Lambda^2\mathfrak{m})^*$ is a 2-closed form on $\mathfrak{m}$ and $D \in \mathrm{Der}((\mathfrak{m})_\theta(e))$ is a derivation of a central extension $(\mathfrak{m})_\theta(e)$ of $\mathfrak{m}$.

First, we shall study the 3 and 5-dimensional indecomposable contact nilpotent Lie algebras. The Heisenberg Lie algebra $\mathfrak{h}_3$ is the only one 3-dimensional Lie algebra of this type, and in this case, the second scalar cohomology group $H^2(\mathfrak{h}_3, \mathbb{R})$ and the Lie algebra of derivations $\mathrm{Der}((\mathfrak{h}_3)_\theta(e))$ are known. On the other hand, from the work of Kutsak (see [17]), it follows that there are three 5-dimensional indecomposable contact nilpotent Lie algebras: $N_{5,1}, N_{5,2,2}$ and $N_{5,3,1}$ (where $N_{5,3,1}$ is the 5-dimensional Heisenberg Lie algebra). In this case we describe each one of them as a double extension of $\mathfrak{h}_3$ by a pair (D, θ) with $\theta \in (\Lambda^2\mathfrak{h}_3)^*$ and $D \in \mathrm{Der}((\mathfrak{h}_3)_\theta(e))$.

Finally, in order to describe any 7-dimensional indecomposable contact nilpotent Lie algebra as a double extension of a 5-dimensional one, for $\mathfrak{g} = N_{5,1}, N_{5,2,2}$ and $N_{5,3,1}$ we shall to compute the second scalar cohomology group $H^2(\mathfrak{g}, \mathbb{R})$ and the Lie algebra of derivations $\mathrm{Der}((\mathfrak{g})_\theta(e))$. In each one of these cases, we obtain $H^2(\mathfrak{g}, \mathbb{R})$ by straightforward calculations, whereas we use the *LieAlgebras* package from Maple(c) to compute the structure of $\mathrm{Der}(\mathfrak{g}_\theta(e))$.

3 and 5-dimensional contact nilpotent Lie algebras. First, from Theorem 3.2 clearly follows that the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ is the unique 3-dimensional contact nilpotent Lie algebra.

Table 1: 3-dimensional contact nilpotent Lie algebras

Lie Algebra	Lie Product
$\mathfrak{h}_3$	$[e_1, e_2] = e_3$

Now, let $\mathfrak{g}$ be a 5-dimensional contact nilpotent Lie algebra spanned by $\langle e_1, \dots e_5 \rangle$. As a consequence of Theorem 4.5, every 5-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ is a contact double extension $\mathfrak{h}_3(D, \theta)$ of the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3 = \langle e_1, e_2, e_3 \rangle$ by a pair (D, θ), where $\theta \in H^2(\mathfrak{h}_3, \mathbb{R})$ is a closed 2-form and $D \in \mathrm{Der}((\mathfrak{h}_3)_\theta(e_4))$ is a derivation of the central extension $(\mathfrak{h}_3)_\theta(e_4)$.

For the sake of completeness let us observe that according to Theorem 3.4, every closed 2-form θ can be written as $\theta = a\theta_{12} + b\theta_{13} + c\theta_{23}$ for some scalars $a, b, c \in \mathbb{R}$, where θ_{ij} stands for a non-trivial generator $[e^i \wedge e^j]$ $(i, j = 1, 2, 3)$ of $H^2(\mathfrak{h}_3, \mathbb{R})$. From [19] we know the structure of the Lie algebra of derivations $\mathrm{Der}((\mathfrak{h}_3)_\theta(e_4))$. More precisely,

Proposition 5.1. *For the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$, assume $\theta = a\theta_{13} + b\theta_{23}$, $[\theta] \in H^2(\mathfrak{h}_3, \mathbb{F})$ for some $a, b \in \mathbb{R}$, and consider a central extension $(\mathfrak{h}_3)_\theta(e_4) = \langle e_1, e_2, e_3, e_4 \rangle$ with brackets $[e_1, e_2] = e_3$, $[e_1, e_3] = ae_4$, $[e_2, e_3] = be_4$. Then $D \in \mathrm{Der}((\mathfrak{h}_3)_\theta(e_4))$ if and only if*

$$[D] = \begin{pmatrix} D_{11} & D_{12} & 0 & 0 \\ D_{21} & D_{22} & 0 & 0 \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix},$$

where $D_{ij} \in \mathbb{R}$ $(i, j = 1, \ldots, 4)$ satisfy the following relations:

(1) $D_{33} = D_{11} + D_{22}$,

(2) $aD_{34} = bD_{34} = 0$,

(3) $D_{43} = aD_{32} - bD_{31}$,

(4) $aD_{44} = a(D_{11} + D_{33}) + bD_{21}$,

(5) $bD_{44} = aD_{12} + b(D_{22} + D_{33})$.

Table 2: 5-dimensional contact nilpotent Lie algebras described as a double extension of a 3-dimensional contact nilpotent Lie algebra.

Lie Algebra	Lie Product	Double extension data
$N_{5,1}$	$[e_1, e_2] = e_5$ $[e_3, e_4] = e_5$ $[e_1, e_3] = e_2$ $[e_1, e_4] = -e_3$	$\left(\mathfrak{h}_3, 0, [D] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right)$
$N_{5,2,2}$	$[e_1, e_2] = e_5$ $[e_3, e_4] = e_5$ $[e_1, e_4] = -e_2$	$\left(\mathfrak{h}_3, 0, [D] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right)$
$\mathfrak{h}_5 = N_{5,3,1}$	$[e_1, e_2] = e_5$ $[e_3, e_4] = e_5$	$\left(\mathfrak{h}_3, 0, [D] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)$

In Table 2 above, we describe each 5-dimensional contact nilpotent Lie algebra as a double extension $\mathfrak{h}_3(D, \theta)$ of the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ by a pair (D, θ). The double extension data $\mathfrak{h}_3(D, \theta)$ is given by the triple $(\mathfrak{h}_3, \theta, D)$, where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg Lie algebra spanned by $\langle e_3, e_4, e_5 \rangle$ with $[e_3, e_4] = e_5$, e_2 is the central element of the central extension of $\mathfrak{h}_3$ by θ, and $D = \mathrm{ad}(e_1)$.

7-dimensional contact nilpotent Lie algebras. The classification of all 7-dimensional indecomposable nilpotent Lie algebras over the real numbers has been done in [11]. On the other hand, the existence of contact structures on these indecomposable nilpotent Lie algebras has been discussed in [17]. Here we follow the notation and classification introduced in [11], and the exhaustive list of the non-isomorphic 7-dimensional indecomposable contact nilpotent Lie algebras given in [17]; that is, each 7-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ is introduced in accordance with the sequence of dimensions of the upper central series (*UCSD*, for short). More specifically, the Lie algebras having a UCSD 1,4,7 are listed as follows: (147A), (147B), (147C), etc. Thus, in the first column of Table 5, we present the USCD of a 7-dimensional contact nilpotent Lie algebra $\mathfrak{g}$ with brackets defined on a basis $\{e_1, \ldots, e_7\}$. Again, as a consequence of Theorem 4.5, each one of these Lie algebras is a contact double extension $\mathfrak{m}(D, \theta)$ of a 5-dimensional contact nilpotent Lie algebra $\mathfrak{m}$ by a pair (D, θ), where $\theta \in H^2(\mathfrak{m}, \mathbb{R})$ is a closed 2-form and $D \in \mathrm{Der}(\mathfrak{m}_\theta(e))$ is a derivation of the central extension $\mathfrak{m}_\theta(e)$. The double extension data $\mathfrak{m}(D, \theta)$ is given by the triple $(\mathfrak{m}, \theta, D)$.
Observe that from Table 2 follows that there are three non-isomorphic 5-dimensional contact nilpotent Lie algebras, namely $N_{5,1}, N_{5,2,2}$ and $N_{5,3,1}$ (which is the 5-dimensional Heisenberg Lie algebra $\mathfrak{h}_5$). Hence, for the sake of completeness we shall compute the second scalar cohomology groups of these Lie algebras. A straightforward calculation proves that:

Theorem 5.2. *Let $\mathfrak{g}$ be a 5-dimensional real contact nilpotent Lie algebra as those in Table 2, with basis $\{e_1, \ldots, e_5\}$ and $\{e^1, \ldots, e^5\}$ as its dual basis. Let $H^2(\mathfrak{g}, \mathbb{R})$ be its second scalar cohomology group. Then,*

$$H^2(N_{5,1}, \mathbb{R}) = \{[e^1 \wedge e^5 + e^2 \wedge e^4], [e^3 \wedge e^4 - e^1 \wedge e^2], [e^4 \wedge e^5 - e^2 \wedge e^3]\},$$
$$H^2(N_{5,2,2}, \mathbb{R}) = \{[e^1 \wedge e^5 + e^2 \wedge e^3], [e^2 \wedge e^4], [e^1 \wedge e^3], [e^1 \wedge e^2 - e^3 \wedge e^4]\},$$
$$H^2(N_{5,3,1}, \mathbb{R}) = \{[e^1 \wedge e^2 - e^3 \wedge e^4], [e^1 \wedge e^3], [e^1 \wedge e^4], [e^2 \wedge e^3], [e^2 \wedge e^4]\}.$$

Let $\mathfrak{g}$ be either $N_{5,1}$ or $N_{5,2,2}$. Given an arbitrary closed 2-form $\theta \in H^2(\mathfrak{g}, \mathbb{R})$, one can compute the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g}_\theta(e_6))$ of a central extension $\mathfrak{g}_\theta(e_6)$. On the other hand, if $\mathfrak{g} = N_{5,3,1}$, the computation of $\mathrm{Der}(\mathfrak{g}_\theta(e_6))$ is not a simple task since it requires to solve a system of 12 linear equations involving 21 parameters: 5 of them correspond to an arbitrary closed 2-form $\theta \in H^2(\mathfrak{g}, \mathbb{R})$, and the remaining correspond to an arbitrary derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e_6))$. On the other hand, since the derivations $D \in \mathrm{Der}(\mathfrak{g}_\theta(e_6))$ must be nilpotent, for $\mathfrak{g} = N_{5,1}$, $N_{5,2,2}$ it will be enough to exhibit the structure of the nilpotent derivations $D \in \mathrm{Der}(\mathfrak{g}_\theta(e_6))$, whereas for $\mathfrak{g} = N_{5,3,1}$ it will be enough to consider a couple of specific 2-closed forms to get a suitable central extension $(\mathfrak{h}_5)_\theta(e)$ and hence, a suitable nilpotent derivation $D \in \mathrm{Der}((\mathfrak{h}_5)_\theta(e))$ (see for instance Lemma 5.3 below and Table 3). Thus, denoting again $e^i \wedge e^j$ $(1 \leq i < j \leq 5)$ by θ_{ij}, we have the following result:

Lemma 5.3. *Let $\mathfrak{g}$ be a 5-dimensional real contact nilpotent Lie algebra as those in Table 2, with basis $\{e_1, \ldots, e_5\}$. Hence,*

(1) *For $\mathfrak{g} = N_{5,1}$ let $\theta \in H^2(N_{5,1}, \mathbb{R})$ be a closed 2-form given by*

$$\theta = \alpha_1(\theta_{15} + \theta_{24}) + \alpha_2 (\theta_{34} - \theta_{12}) + \alpha_3 (\theta_{45} - \theta_{23})$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$*. Then a nilpotent derivation* $D \in \mathrm{Der}((N_{5,1})_\theta(e_6))$ *has the following matrix representation in the chosen basis:*

$$[D] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ D_{21} & 0 & D_{23} & 0 & 0 & 0 \\ D_{31} & 0 & 0 & -D_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ D_{51} & D_{23} & -D_{31} & D_{54} & 0 & 0 \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & 0 \end{pmatrix}$$

where $D_{ij} \in \mathbb{R}$ $(i, j = 1, \ldots, 6)$ *satisfy the following relations:*

(a) $D_{62} = -\alpha_1 D_{31} - \alpha_2 D_{23} - \alpha_3 D_{21}$,

(b) $D_{63} = -\alpha_1 D_{54} + \alpha_3 D_{51} - \alpha_2 D_{31} - \alpha_1 D_{21}$, *and*

(c) $D_{65} = \alpha_3 D_{31} + \alpha_1 D_{23}$.

(2) *For* $\mathfrak{g} = N_{5,2,2}$ *let* $\theta \in H^2(N_{5,2,2}, \mathbb{R})$ *be a closed 2-form given by*

$$\theta = \alpha_1(\theta_{15} + \theta_{23}) + \alpha_2\theta_{24} + \alpha_3\theta_{13} + \alpha_4(\theta_{12} - \theta_{34})$$

where $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$*. Then a nilpotent derivation* $D \in \mathrm{Der}((N_{5,2,2})_\theta(e_6))$ *has the following matrix representation in the chosen basis:*

$$[D] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ D_{21} & 0 & D_{23} & D_{24} & 0 & 0 \\ D_{31} & 0 & 0 & 0 & 0 & 0 \\ D_{23} & 0 & 0 & 0 & 0 & 0 \\ D_{51} & D_{52} & D_{53} & D_{54} & 0 & 0 \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & 0 \end{pmatrix}$$

where $D_{ij} \in \mathbb{R}$ $(i, j = 1, \ldots, 6)$ *satisfy the following relations:*

(a) $D_{31} = -\frac{\alpha_2}{\alpha_1} D_{23}, \alpha_1 \neq 0$,

(b) $D_{52} = -D_{24} - \frac{\alpha_2}{\alpha_1} D_{23}$,

(c) $D_{53} = -\frac{2\alpha_4}{\alpha_1} D_{23} - D_{21}$,

(d) $D_{62} = -\alpha_1 D_{54} - \alpha_4 D_{24} - \frac{\alpha_2\alpha_4}{\alpha_1} D_{23} - \alpha_2 D_{21}$, *and*

(e) $D_{65} = \alpha_1 D_{24} + \alpha_2 D_{23}$.

(3) *For* $\mathfrak{g} = N_{5,3,1}$*, Table 5 shows that it is sufficient to consider the 2-closed forms* $\theta_1 = \alpha_1\theta_{13} + \alpha_2\theta_{14} + \alpha_3\theta_{24}$ *and* $\theta_2 = \theta_{23}$*, where* $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$*. In the chosen basis, a derivation* $D \in \mathrm{Der}((N_{5,3,1})_{\theta_1}(e_6))$ *has the following matrix representation:*

$$[D] = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ D_{21} & D_{22} & D_{23} & D_{24} & 0 & 0 \\ D_{31} & D_{32} & D_{33} & D_{34} & 0 & 0 \\ D_{41} & -D_{13} & D_{43} & D_{44} & 0 & 0 \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{11} + D_{22} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{11} + D_{22} \end{pmatrix}$$

where $D_{ij} \in \mathbb{R}$ $(i, j = 1, \ldots, 6)$ *satisfy the following relations:*

(a) $D_{56} = \frac{\alpha_3 D_{23} - \alpha_1 D_{14} + \alpha_2 D_{13}}{\alpha_1 \alpha_3}$,

(b) $D_{65} = \alpha_1 \alpha_3 D_{56}$,

(c) $D_{31} = -D_{24} + D_{56}$,

(d) $D_{32} = D_{14} + \alpha_3 D_{56}$,

(e) $D_{33} = D_{22} + \frac{\alpha_2}{\alpha_3} D_{12}$,

(f) $D_{34} = \frac{\alpha_2}{\alpha_1}(D_{33} - D_{11}) - \frac{\alpha_3}{\alpha_1} D_{21}$,

(g) $D_{41} = D_{23} - \alpha_1 D_{56}$,

(h) $D_{43} = -\frac{\alpha_1}{\alpha_3} D_{12}$,

(i) $D_{44} = D_{11} - \frac{\alpha_2}{\alpha_3} D_{12}$.

On the other hand, a derivation $D \in \mathrm{Der}((N_{5,3,1})_{\theta_2}(e_6))$ *has the following matrix representation:*

$$[D] = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} & 0 & 0 \\ 0 & D_{22} & D_{23} & 0 & 0 & 0 \\ 0 & D_{14} & D_{33} & 0 & 0 & 0 \\ D_{23} & D_{42} & D_{43} & D_{11} + D_{22} - D_{33} & 0 & 0 \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{11} + D_{22} & -D_{13} - D_{42} \\ D_{61} & D_{62} & D_{63} & D_{64} & 0 & D_{22} + D_{33} \end{pmatrix}$$

where $D_{ij} \in \mathbb{R}$ $(i, j = 1, \dots, 6)$.

It is important to mention that our list of 7-dimensional contact Lie algebras is slightly different to the one given by Kutsak in [17]. The main remarks are the following:

(1) In the lists provided by Kutsak and Gong, there are a few equivalent Lie algebras, namely

Kutsak (see [17])	Gong (see [11])
$1357S(0)$	$1357P$
$1357S(\lambda),\ \lambda \neq 0, 1, -1$	$1357S\left(\left(\frac{\lambda+1}{\lambda-1}\right)^2\right)$
$1357S_2$	$1357S(0)$

(2) The Lie algebras $1357QRS_1(-1)$ and $12457N_2(1)$ are in fact contact Lie algebras, as can be seen in Table 5.

(3) The Lie algebras $1357N(0)$, $12457L$ and $12457N(1)$ are not contact Lie algebras. Let $\mathfrak{g}$ denote any of these Lie algebras, a straightforward computation shows that for an arbitrary 1-form $\omega = \sum_{i=1}^{7} \alpha_i e^i \in \mathfrak{g}^*$, necessarily $(d\omega)^3 = 0$.

Finally, as a consequence of Theorem 4.5 and Lemma 5.3, we can describe in Table 5 below each 7-dimensional contact nilpotent Lie algebra as a contact double extension of a 5-dimensional contact nilpotent Lie algebra of codimension 2. In what follows $\mathfrak{h}_5$ represents the 5-dimensional Heisenberg Lie algebra spanned by $\langle e_3, e_4, e_5, e_6, e_7 \rangle$ with brackets $[e_3, e_4] = [e_5, e_6] = e_7$, e_2 is the central element of a central extension of $\mathfrak{h}_5$ by a closed 2-form θ and the derivation $D \in \mathrm{Der}((\mathfrak{h}_5)_\theta(e_2))$ is given by $\mathrm{ad}(e_1)$.

Table 3: 7-dimensional contact nilpotent Lie algebras as double extensions

UCSD	Base Change	Lie product	Double extension data $(\mathfrak{m}, \theta, [D])$
$\mathfrak{h}_7 = (17)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_3, e_4] = e_7$ $[e_5, e_6] = e_7$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}\right)$
(157)	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = e_2$ $[e_3, e_4] = e_7$ $[e_5, e_6] = e_7$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix}\right)$
$(147A)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = e_5$ $[e_3, e_4] = e_7$ $[e_1, e_6] = -e_4$ $[e_5, e_6] = e_7$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&-1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}\right)$
$(147A_1)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = e_4$ $[e_3, e_4] = e_7$ $[e_1, e_6] = -e_5$ $[e_5, e_6] = e_7$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}\right)$
$(147B)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = e_4$ $[e_3, e_4] = e_7$ $[e_1, e_5] = e_2$ $[e_5, e_6] = e_7$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}\right)$
$(147D)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&1&0\\ 0&0&-2&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = \frac{1}{2}e_6 + \frac{1}{2}e_2$ $[e_3, e_4] = e_7$ $[e_1, e_5] = e_4$ $[e_5, e_6] = e_7$ $[e_3, e_5] = \frac{1}{2}e_2$	$\left(\mathfrak{h}_5, \frac{1}{2}\ \theta_{35}, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ \frac{1}{2}&0&0&0&0&0\\ 0&0&0&0&0&1\\ \frac{1}{2}&0&0&0&0&0 \end{pmatrix}\right)$
$(147E)$ $\lambda \neq 0, 1$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0\\ 0&0&1-\lambda&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&\lambda&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = \frac{1}{\lambda-1}e_6$ $[e_3, e_4] = e_7$ $[e_1, e_5] = \frac{1}{\lambda}e_4$ $[e_5, e_6] = e_7$ $[e_3, e_5] = \frac{1}{\lambda(1-\lambda)}e_2$	$\left(\mathfrak{h}_5, \frac{1}{\lambda(1-\lambda)}\ \theta_{35}, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&\frac{1}{\lambda}&0&0&0\\ 0&0&0&0&0&0\\ \frac{1}{\lambda-1}&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}\right)$
$(147E_1)$ $\lambda > 1$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&-\lambda&0\\ 0&0&-2&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&\frac{2}{\lambda}&0\\ 0&-\lambda&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = -\frac{\lambda}{2}e_2 + \frac{1}{\lambda}e_5$ $[e_3, e_4] = e_7$ $[e_1, e_6] = -\frac{1}{\lambda}e_4$ $[e_5, e_6] = e_7$ $[e_3, e_6] = -\frac{1}{2\lambda}e_5$	$\left(N_{5,2,2}, 0, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&-\frac{1}{\lambda}&0&0\\ \frac{1}{\lambda}&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ -\frac{\lambda}{2}&0&0&0&0&0 \end{pmatrix}\right)$

Table 3: (continued)

$(1457B)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = e_2$ $[e_3, e_4] = e_7$ $[e_1, e_4] = -e_3$ $[e_5, e_6] = e_7$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(137B)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = e_2$ $[e_3, e_4] = e_7$ $[e_4, e_5] = -e_6$ $[e_5, e_6] = e_7$	$\left(N_{5,2,2}, 0, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(137B_1)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_4] = e_6$ $[e_3, e_4] = e_7$ $[e_3, e_5] = e_6$ $[e_5, e_6] = e_7$ $[e_4, e_5] = -e_2$ $[e_1, e_3] = e_2$	$\left(N_{5,2,2}, -\theta_{45}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(137D)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = e_5$ $[e_3, e_4] = e_7$ $[e_1, e_4] = e_2$ $[e_5, e_6] = e_7$ $[e_3, e_6] = e_2$	$\left(\mathfrak{h}_5, \theta_{36}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(1357A)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_4] = e_2$ $[e_3, e_4] = e_7$ $[e_1, e_5] = e_4$ $[e_5, e_6] = e_7$ $[e_3, e_5] = e_2$	$\left(\mathfrak{h}_5, \theta_{35}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(1357C)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_4] = e_2$ $[e_3, e_4] = e_7$ $[e_1, e_5] = e_4$ $[e_5, e_6] = e_7$ $[e_3, e_5] = e_2$ $[e_1, e_3] = e_4$	$\left(\mathfrak{h}_5, \theta_{35}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(1357D)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_6] = -e_3$ $[e_3, e_4] = e_7$ $[e_3, e_6] = e_5$ $[e_5, e_6] = e_7$ $[e_4, e_6] = e_2$	$\left(N_{5,2,2}, \theta_{46}, \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(1357F)$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_4] = e_3$ $[e_3, e_4] = e_7$ $[e_1, e_5] = -e_6$ $[e_5, e_6] = e_7$ $[e_1, e_6] = e_2$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\right)$

Table 3: (continued)

$(1357F_1)$	$\begin{pmatrix} 0&1&0&1&0&0&0\\ 1&0&0&0&0&-1&0\\ 1&0&0&0&0&0&0\\ 0&0&-1&1&0&0&0\\ 0&0&0&0&1&-1&0\\ 0&1&0&0&0&0&0\\ 0&0&-1&0&0&0&-1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_6]=e_2$ $[e_3,e_4]=e_7$ $[e_3,e_6]=-e_4$ $[e_5,e_6]=e_7$ $[e_4,e_6]=e_5$	$\left(N_{5,1},0,\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0 \end{pmatrix}\right)$
$(1357H)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&1&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_2+e_5$ $[e_3,e_4]=e_7$ $[e_1,e_6]=e_3$ $[e_5,e_6]=e_7$ $[e_3,e_6]=-e_5$	$\left(N_{5,2,2},0,\begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0 \end{pmatrix}\right)$
$(1357J)$	$\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=-e_6$ $[e_3,e_4]=e_7$ $[e_1,e_6]=e_2$ $[e_5,e_6]=e_7$ $[e_4,e_5]=e_3$	$\left(N_{5,2,2},0,\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0 \end{pmatrix}\right)$
$(1357L)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&\frac{1}{2}&0&0&0&0\\ 0&-2&0&1&0&0&0\\ 0&\frac{1}{2}&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_6$ $[e_3,e_4]=e_7$ $[e_1,e_5]=e_3+4e_6$ $[e_5,e_6]=e_7$ $[e_4,e_5]=-2e_2$ $[e_1,e_3]=2e_2$	$\left(\mathfrak{h}_5,-2\theta_{45},\begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&4&0&0&0\\ 0&0&0&0&0&1\\ 2&0&0&0&0&0 \end{pmatrix}\right)$
$(1357M)$ $\lambda\neq 0,1$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1-\lambda&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&\lambda&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_6$ $[e_3,e_4]=e_7$ $[e_1,e_5]=\frac{1-\lambda}{\lambda}e_3$ $[e_5,e_6]=e_7$ $[e_4,e_5]=-\frac{1}{\lambda}e_2$ $[e_1,e_3]=\frac{1}{1-\lambda}e_2$	$\left(\mathfrak{h}_5,-\frac{1}{\lambda}\ \theta_{45},\begin{pmatrix} 0&0&\frac{1-\lambda}{\lambda}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ \frac{1}{1-\lambda}&0&0&0&0&0 \end{pmatrix}\right)$
$(1357N)$ $\lambda\neq 0$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&-\lambda&0&1&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=-\frac{1}{\lambda}e_3$ $[e_3,e_4]=e_7$ $[e_1,e_5]=e_6+\frac{1}{\lambda}e_3$ $[e_5,e_6]=e_7$ $[e_4,e_5]=-\frac{1}{\lambda}e_2$ $[e_1,e_3]=e_2$	$\left(\mathfrak{h}_5,-\frac{1}{\lambda}\ \theta_{45},\begin{pmatrix} 0&-\frac{1}{\lambda}&\frac{1}{\lambda}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix}\right)$
$(1357P)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=e_3$ $[e_3,e_4]=e_7$ $[e_3,e_5]=-e_6$ $[e_5,e_6]=e_7$ $[e_4,e_5]=-e_2$ $[e_1,e_3]=e_2$	$\left(N_{5,2,2},-\theta_{45},\begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix}\right)$

Table 3: (continued)

$(1357P_1)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_5] = -e_3$ $[e_3, e_4] = e_7$ $[e_3, e_5] = e_6$ $[e_5, e_6] = e_7$ $[e_4, e_5] = e_2$ $[e_1, e_3] = e_2$	$\left(N_{5,2,2}, \theta_{45}, \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(1357QRS_1)$ $\lambda \neq 0, 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_4] = e_6$ $[e_3, e_4] = e_7$ $[e_1, e_5] = \frac{1-\lambda}{\lambda} e_3$ $[e_5, e_6] = e_7$ $[e_3, e_5] = \frac{1}{\lambda(1-\lambda)} e_6$ $[e_1, e_3] = \frac{1}{1-\lambda} e_2$ $[e_4, e_5] = -\frac{1}{\lambda} e_2$	$\left(\begin{matrix} N_{5,2,2}, \\ -\frac{1}{\lambda}\theta_{45}, \end{matrix} \begin{pmatrix} 0 & 0 & \frac{1-\lambda}{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(1357R)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_6] = -e_3$ $[e_3, e_4] = e_7$ $[e_3, e_6] = e_2$ $[e_5, e_6] = e_7$ $[e_4, e_6] = e_2$ $[e_1, e_3] = e_5$	$\left(\begin{matrix} \mathfrak{h}_5, \\ \theta_{36} + \theta_{46}, \end{matrix} \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(1357S)$ $\lambda \neq 0, 1$	$\begin{pmatrix} 1 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = (1-\lambda) e_5$ $[e_3, e_4] = e_7$ $[e_1, e_5] = \frac{1}{1-\lambda} e_7$ $[e_5, e_6] = e_7$ $[e_1, e_6] = -e_3$ $[e_4, e_6] = e_2$ $[e_3, e_6] = e_2 - \lambda(1-\lambda) e_5$	$\left(\begin{matrix} N_{5,2,2}, \\ \theta_{36}+\theta_{46}, \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\lambda} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(13457C)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_3] = -e_4$ $[e_3, e_4] = e_7$ $[e_1, e_4] = e_5$ $[e_5, e_6] = e_7$ $[e_1, e_6] = e_3$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(13457E)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_4] = e_5$ $[e_3, e_4] = e_7$ $[e_1, e_6] = e_3$ $[e_5, e_6] = e_7$ $[e_3, e_6] = -e_5$ $[e_1, e_3] = -e_4$	$\left(N_{5,2,2}, 0, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$
$(13457G)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1, e_2] = e_7$ $[e_1, e_4] = e_5$ $[e_3, e_4] = e_7$ $[e_1, e_6] = e_3$ $[e_5, e_6] = e_7$ $[e_3, e_6] = -e_2$ $[e_1, e_3] = -e_4 - e_5$	$\left(\mathfrak{h}_5, -\theta_{36}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$

Table 3: (continued)

$(13457I)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_2+e_6$ $[e_3,e_4]=e_7$ $[e_1,e_5]=-e_3$ $[e_5,e_6]=e_7$ $[e_3,e_5]=e_6$ $[e_1,e_3]=-e_4$	$\left(N_{5,2,2}, 0, \begin{pmatrix} 0&0&-1&0&0&0\\ -1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0 \end{pmatrix}\right)$
$(12457D)$	$\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_6]=e_2$ $[e_3,e_4]=e_7$ $[e_3,e_5]=e_4$ $[e_5,e_6]=e_7$ $[e_4,e_5]=-e_2$ $[e_1,e_5]=e_3$	$\left(N_{5,2,2}, -\theta_{45}, \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0 \end{pmatrix}\right)$
$(12457E)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_2$ $[e_3,e_4]=e_7$ $[e_1,e_6]=-e_4$ $[e_5,e_6]=e_7$ $[e_3,e_5]=e_2$ $[e_1,e_3]=-e_6$ $[e_3,e_6]=-e_2$	$\left(\mathfrak{h}_5, \theta_{35}-\theta_{36}, \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&0&0\\ -1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0 \end{pmatrix}\right)$
$(12457G)$	$\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=e_3$ $[e_3,e_4]=e_7$ $[e_1,e_6]=e_2$ $[e_5,e_6]=e_7$ $[e_3,e_5]=e_4$ $[e_1,e_3]=-e_2$ $[e_4,e_5]=-e_2$	$\left(N_{5,2,2}, -\theta_{45}, \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ -1&0&0&1&0&0 \end{pmatrix}\right)$
$(12457I)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=e_2$ $[e_3,e_4]=e_7$ $[e_1,e_6]=-e_3$ $[e_5,e_6]=e_7$ $[e_3,e_6]=e_5$ $[e_1,e_3]=e_4$ $[e_4,e_6]=e_2$	$\left(N_{5,2,2}, \theta_{46}, \begin{pmatrix} 0&0&0&-1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}\right)$
$(12457J)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=e_2$ $[e_3,e_4]=e_7$ $[e_1,e_6]=-e_3$ $[e_5,e_6]=e_7$ $[e_3,e_6]=e_5$ $[e_1,e_3]=e_2+e_4$ $[e_4,e_6]=e_2$	$\left(N_{5,2,2}, \theta_{46}, \begin{pmatrix} 0&0&0&-1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 1&0&1&0&0&0 \end{pmatrix}\right)$

Table 3: (continued)

$(12457J_1)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=e_2$ $[e_3,e_4]=e_7$ $[e_1,e_6]=e_3$ $[e_5,e_6]=e_7$ $[e_3,e_6]=-e_5$ $[e_1,e_3]=e_4$ $[e_4,e_6]=-e_2$ $[e_1,e_4]=e_7$	$\left(\begin{matrix} N_{5,2,2}, \\ -\theta_{46}, \end{matrix} \begin{pmatrix} 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}\right)$
$(12457N)$ $\lambda \neq 1$	$\begin{pmatrix} 1&1&0&0&0&0&0\\ 0&0&0&0&1&1&0\\ 0&0&1&0&0&0&0\\ 0&0&1&1&-1&0&0\\ 0&0&0&0&1&0&0\\ 0&1-\lambda&0&0&0&0&0\\ 0&0&0&1&1&1&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=e_2+e_7$ $[e_3,e_4]=e_7$ $[e_1,e_6]=\frac{1}{1-\lambda}(e_3+e_4)$ $[e_5,e_6]=e_7$ $[e_3,e_6]=\frac{1}{\lambda-1}(e_5-2e_4)$ $[e_1,e_3]=e_4$ $[e_4,e_6]=\frac{1}{\lambda-1}e_2$ $[e_1,e_4]=e_7$	$\left(\begin{matrix} N_{5,2,2}, \\ \frac{1}{1-\lambda}\,\theta_{46}, \end{matrix} \begin{pmatrix} 0&0&0&\frac{1}{1-\lambda}&0&0\\ 1&0&0&\frac{1}{1-\lambda}&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&1&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}\right)$
$(12457N_2)$ $\lambda \geq 0$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&\lambda&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=\lambda e_2+e_7$ $[e_3,e_4]=e_7$ $[e_1,e_6]=e_4$ $[e_5,e_6]=e_7$ $[e_3,e_5]=-e_2$ $[e_1,e_3]=e_6$ $[e_3,e_6]=\lambda e_4+e_5$ $[e_1,e_4]=-e_2$	$\left(\begin{matrix} N_{5,2,2}, \\ -\theta_{35}, \end{matrix} \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&1\\ 0&-1&\lambda&0&0&0 \end{pmatrix}\right)$
$(12357C)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_5$ $[e_3,e_4]=e_7$ $[e_1,e_5]=e_2$ $[e_5,e_6]=e_7$ $[e_3,e_6]=e_5$ $[e_1,e_3]=e_4$ $[e_4,e_6]=e_2$	$\left(\begin{matrix} N_{5,2,2}, \\ \theta_{46}, \end{matrix} \begin{pmatrix} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}\right)$
$(123457C)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_5$ $[e_3,e_4]=e_7$ $[e_1,e_5]=e_2$ $[e_5,e_6]=e_7$ $[e_1,e_6]=e_3$ $[e_1,e_3]=-e_4$	$\left(\mathfrak{h}_5, 0, \begin{pmatrix} 0&0&0&1&0&0\\ -1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix}\right)$
$(123457F)$	$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&1&1&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_4]=e_5-e_2$ $[e_3,e_4]=e_7$ $[e_1,e_5]=e_2$ $[e_5,e_6]=e_7$ $[e_1,e_6]=e_3$ $[e_1,e_3]=-e_4-e_5$ $[e_3,e_6]=-e_2$	$\left(\begin{matrix} \mathfrak{h}_5, \\ -\theta_{36}, \end{matrix} \begin{pmatrix} 0&0&0&1&0&0\\ -1&0&0&0&0&0\\ -1&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&-1&1&0&0&0 \end{pmatrix}\right)$

Table 3: (continued)

(123457I) $\lambda \neq 0,1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$[e_1,e_2]=e_7$ $[e_1,e_5]=e_2$ $[e_3,e_4]=e_7$ $[e_1,e_6]=\frac{\lambda-1}{\lambda}e_3$ $[e_5,e_6]=e_7$ $[e_3,e_6]=\frac{1}{\lambda(1-\lambda)}e_5$ $[e_1,e_3]=\frac{1}{1-\lambda}e_4$ $[e_4,e_6]=\frac{1}{\lambda}e_2$ $[e_1,e_4]=e_5$	$\left(N_{5,2,2}, \frac{1}{\lambda}\ \theta_{46}, \begin{pmatrix} 0 & 0 & 0 & \frac{\lambda-1}{\lambda} & 0 & 0 \\ \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}\right)$

6. Appendix

Given a 3-dimensional Lie algebra $\mathfrak{g}$ as those of Theorem 3.1, one can choose a representative of its second scalar cohomology group $H^2(\mathfrak{g},\mathbb{R})$ in order to get a central extension $\mathfrak{g}_\theta(e)$. Since the structure of the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g}_\theta(e))$ is known (see [19]), choosing a suitable derivation $D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$, the double extension process allow us to construct a 5-dimensional contact Lie algebra $\mathfrak{g}(D,\theta)$. Hence, for the sake of completeness, in Table 6 below we present a list of 5-dimensional contact Lie algebras obtained as a double extension from a 3-dimensional Lie algebra $\mathfrak{g}$.

Table 4: 5-dimensional contact Lie algebras obtained from a 3-dimensional Lie algebra through double extension

$\mathfrak{g}(D,\theta)$	Representative of $[\theta] \in H^2(\mathfrak{g},\mathbb{F})$	$D \in \mathrm{Der}(\mathfrak{g}_\theta(e))$	Contact form η
$\mathfrak{h}_3(D,\theta)$	$\theta=\theta_{13}$	$[D]=\begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&1&0&1 \end{pmatrix}$	$\eta=e^2+e^4$
$\mathfrak{h}_3(D,\theta)$	$\theta=\theta_{23}$	$[D]=\begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}$	$\eta=e^3$
$\mathfrak{h}_3(D,\theta)$	$\theta=\theta_{13}+\theta_{23}$	$[D]=\begin{pmatrix} 1&1&0&0 \\ -1&-1&0&0 \\ 0&0&0&0 \\ 2&1&0&0 \end{pmatrix}$	$\eta=e^4$
$\mathfrak{h}_3(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&0&1 \\ 0&0&0&1 \end{pmatrix}$	$\eta=e^3+e^4$
$\mathfrak{q}_0(D,\theta)$	$\theta=\theta_{13}$	$[D]=\begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&1&1 \end{pmatrix}$	$\eta=e^2+e^4$
$\mathfrak{q}_0(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&1 \\ 0&0&0&0 \end{pmatrix}$	$\eta=e^2+e^3$
$\mathfrak{q}_0^1(D,\theta)$	$\theta=\theta_{23}$	$[D]=\begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&2 \end{pmatrix}$	$\eta=e^4$

Table 4 (continued)

$\mathfrak{q}_0^1(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}$	$\eta=e^2+e^3+e^4$
$\mathfrak{q}_\lambda^1(D,\theta)$, $\lambda\neq 0$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&1&1&0\\ 0&-1&1&0\\ 0&0&0&1 \end{pmatrix}$	$\eta=e^3+e^4$
$\mathfrak{q}_{-1}(D,\theta)$	$\theta=\theta_{23}$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&-1&0\\ 1&0&0&-1 \end{pmatrix}$	$\eta=e^1+e^4$
$\mathfrak{q}_{-1}(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}$	$\eta=e^2+e^3+e^4$
$\mathfrak{q}_1(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&1 \end{pmatrix}$	$\eta=e^2+e^4$
$\mathfrak{q}_\lambda(D,\theta)$, $\lambda\neq\{0,1,-1\}$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$	$\eta=e^2+e^3+e^4$
$\mathfrak{p}(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$	$\eta=e^2+e^4$
$\mathfrak{sl}_2(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 0&-1&0&0\\ 0&0&0&0\\ 2&0&0&0\\ 0&0&0&1 \end{pmatrix}$	$\eta=e^1+e^4$
$\mathfrak{so}_3(D,\theta)$	$\theta=0$	$[D]=\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix}$	$\eta=e^1+e^4$

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