Journal of Lie Theory
Volume **29** (2019) 839–879

An Algebraic Approach to Duflo's Polynomial Conjecture in the Nilpotent Case

Yoshinori Tanimura

Communicated by T. Kobayashi

Abstract. We introduce a new algebraic approach to Duflo's polynomial conjecture (Problem 1.1) in the nilpotent case. Duflo's polynomial conjecture is an algebraic abstraction of the problem about the center of the algebra of all invariant differential operators on a homogeneous linear bundle. In previous research on Duflo's polynomial conjecture in the nilpotent case, one already used an analytic approach to Corwin-Greenleaf's polynomial conjecture. Corwin-Greenleaf's polynomial conjecture is a restriction of Duflo's polynomial conjecture to the case where all differential operators are commutative each other. Especially, in the nilpotent case, there were no approach to Duflo's polynomial conjecture in the case where there exist two non-commutative invariant differential operators, in the knowledge of the author.

In this paper, we introduce a new approach to Duflo's polynomial conjecture in the case where invariant differential operators are not necessarily commutative. This approach is based on the split symmetrization map (Definition 1.3), which is a rustic and algebraic map introduced in this paper. Furthermore, by our new approach, we solve Duflo's polynomial conjecture completely in the 2-step nilpotent case and the special nilpotent Lie algebra case (Theorem 1.7).

Mathematics Subject Classification: 22E30, 22E25.

Key Words: Polynomial conjecture, Poisson algebra, F-method, harmonic analysis, Lie group, representation theory.

1. Introduction

The Laplacian is invariant under isometries. This property holds not only in the Euclidian space but also in general Riemannian manifolds. Furthermore, there may exist some other invariant differential operators on a homogeneous Riemannian manifold. Now we want to discuss the following question: on a general homogeneous line bundle, how many are there differential operators which are invariant under the group action, like the Laplacian.

Initially, this question is a matter of geometry and analysis. However, by using Lie algebras, this question is reduced to an algebraic problem. Michel Duflo formulated such a purely algebraic problem (Problem 1.1) in 1986, and we discuss this problem of Duflo in this paper. Previous principal research of Problem 1.1 and related works include:

(1) Duflo, Benoist, and Rouvière solved Problem 1.1 in the symmetric space case ([7, 17, 39]). These results resulted in Duflo's proposal.

ISSN 0949–5932 / $2.50

(2) Kobayashi and Oshima established the equivalence between representation theoretic property (uniform boundedness of multiplicities in the regular representations) and the ring structure (the commutativity of the ring of differential operators) for reductive Lie groups ([32]).

(3) Corwin and Greenleaf showed the same equivalence for nilpotent homogeneous line bundles in the specific case ([13]). Baklouti and Ludwig generalized Corwin-Greenleaf's result in the "common polarization" case ([6]). Finally, Fujiwara, Lion, Magneron, and Mehdi extended their results in the general nilpotent case ([23]).

(4) Corwin and Greenleaf studied Problem 1.1 in the case that the ring of all invarinat differential operators is commutative. They suggested Conjecture A.3 and proved their conjecture in the specific case ([12]). Baklouti and Ludwig generalized Corwin-Greenleaf's result in the "common polarization" case ([6]). Corwin-Greenleaf's method was extended to the general nilpotent case by Fujiwara in [21]. However, Conjecture A.3 is still unsolved.

Most of the previous research uses analytic consideration. Apart from Rouvière's one, these results concern the relationship between the commutativity of differential operators and multiplicity of induced representations. It may be noted that few papers treat the center of all differential operators on a nilpotent homogeneous line bundle in the case that the algebra is non-commutative, to the best of knowledge of the author.

In this paper, we consider the "split symmetrization map", which is rustic and algebraic, and suggest a new approach to Problem 1.1 in the nilpotent case. Moreover, we will completely solve Problem 1.1 in the case of that the Lie algebra is special nilpotent or 2-step nilpotent (Theorem 1.7). As we see in Section B.2, the split symmetrization map is motivated by the F-method which has recently been introduced by Kobayashi [27].

1.1. Problems, conjectures and the main theorem

Through this paper, let $\mathbf{K}$ be a field of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{K}$, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, and $\lambda\colon \mathfrak{h} \to \mathbf{K}$ a representation. We review that the symmetric algebra $S(\mathfrak{g})$ has a natural Poisson structure and we regard the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ as a Poisson algebra by the commutator. We define the following linear spaces.

$$\mathfrak{h}_\lambda := \{Y - \lambda(Y) \in \mathfrak{h} \oplus \mathbf{K} \mid Y \in \mathfrak{h}\},\ \tilde{S}_\lambda(\mathfrak{g}) := S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{h}_\lambda,\ \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) := \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{h}_\lambda.$$

Then $\tilde{S}_\lambda(\mathfrak{g})$ and $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})$ have natural representations of $\mathfrak{h}$. We denote by $\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ and $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h}$ the linear subspaces of all $\mathfrak{h}$-invariant elements of $\tilde{S}_\lambda(\mathfrak{g})$ and $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})$, respectively. These subspaces $\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ and $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h}$ have the Poisson structures induced by the ones of $S(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$, each other. Now $\tilde{S}_\lambda(\mathfrak{g})$ is isomorphic to the polynomial algebra on the affine space $\Gamma_\lambda := \{u \in \mathfrak{g}^* \mid u|_\mathfrak{h} = \lambda\}$, and $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h}$ is the algebraic abstraction of the algebra of differential operators on homogeneous line bundle (cf. $\mathcal{D}_\tau(G/H)$ in Conjeture A.3). We denote by $Z(A)$ the center of a Poisson algebra A. Duflo suggested the following problem in 1986.

Problem 1.1 (Duflo's polynomial conjecture, [18, Problem 3])**.**
Let $\rho := 1/2\ \mathrm{tr}_{\mathfrak{g}/\mathfrak{h}}\colon \mathfrak{h} \to \mathbf{K}$. Then is $Z(\tilde{S}_{\lambda+\rho}(\mathfrak{g})^\mathfrak{h})$ ring isomorphic to $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h})$?

We will summarize some previous research on this problem in Section A. In the following, we assume that $\mathfrak{g}$ is nilpotent. In this case, $\rho = 0$, and one also considers a Corwin-Greenleaf conjecture (Conjecture A.3), which is analogous of Duflo's conjecture. Although the main concern of the Corwin-Greenleaf conjecture is unitary representations, we consider the following algebraic generalization.

Conjecture 1.2 (The algebraic generalization of Conjecture A.3)**.** For a nilpotent Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a representation $\lambda \colon \mathfrak{h} \to \mathbf{K}$, if $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is abelian, then is the ring $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is isomorphic to $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$.

Although the assumption of Conjecture 1.2 is stronger than the one of Problem 1.1, Problem 1.1 does not implicate Conjecture 1.2. It is because the Poisson bracket of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is not necessarily obvious even if $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is abelian.
In this paper, we propose a new algebraic approach to the above problems. In Problem 1.1 and Conjecture 1.2, an isomorphism map is not explicitly given. So we will suggest such a candidate in this paper. First of all, we define the "split symmetrization" map.

Definition 1.3. For a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$, we define *split symmetrization map* by the following.

$$\sigma_{\mathfrak{q}}(PQ) = \sigma(P)\sigma(Q) \quad (P \in S(\mathfrak{q}), Q \in S(\mathfrak{h})).$$

Here, we denote by $\sigma \colon S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ the symmetrization map, which is defined as follows.

$$\sigma(X_0 \cdots X_{n-1}) = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} X_{s(0)} \cdots X_{s(n-1)} \quad (X_0, \ldots, X_{n-1} \in \mathfrak{g})$$

The split symmetrization map is a generalization of the symmetrization map. In fact, it is known from definition that $\sigma_{\{0\}} = \sigma_{\mathfrak{g}} = \sigma$. By Lemma 2.2, we get the following, immediately.

Proposition 1.4. *There exists the unique linear map* $\tilde{\sigma}_{\mathfrak{q}} \colon \tilde{S}_\lambda(\mathfrak{g}) \to \tilde{\mathcal{U}}_\lambda(\mathfrak{g})$ *induced by* $\sigma_{\mathfrak{q}} \colon S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$.

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\sigma_{\mathfrak{q}}} & \mathcal{U}(\mathfrak{g}) \\ \downarrow & \circlearrowleft & \downarrow \\ \tilde{S}_\lambda(\mathfrak{g}) & \xrightarrow[\tilde{\sigma}_{\mathfrak{q}}]{} & \tilde{\mathcal{U}}_\lambda(\mathfrak{g}). \end{array}$$

We expect that this map $\tilde{\sigma}_{\mathfrak{q}}$ gives the isomorphism in Problem 1.1 and Conjecture 1.2. In order to simplify the description in the following part,we fix some terminologies:

Definition 1.5. For a nilpotent Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a representation $\lambda \colon \mathfrak{h} \to \mathbf{K}$, and a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$, we say:

(1) the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies *the split Duflo property* if $\tilde{\sigma}_{\mathfrak{q}}(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) = \tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ and the restriction of $\tilde{\sigma}_{\mathfrak{q}}$ is a ring isomorphism from $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ to $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$;

(2) the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies *the split Corwin-Greenleaf property* if $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is non-commutative or the restriction of $\tilde{\sigma}_{\mathfrak{q}}$ is a ring isomorphism from $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ to $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$.

Here, the split Duflo property implies the split Corwin-Greenleaf property. The following conjecture is a sufficient condition of Problem 1.1 in the nilpotent case and Conjecture 1.2.

Conjecture 1.6. For a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a representation $\lambda\colon \mathfrak{h} \to \mathbf{K}$, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ such that the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisifies the split Duflo property.

Even though we assume "split Duflo property"in Conjecture 1.6, the statement is still a sufficient condition of Conjecture 1.2. Problem 1.1 and Conjecture 1.2 do not imply Conjecture 1.6.
Conjecture 1.6 is obvious if $\mathfrak{g}$ is abelian. Furthermore, if $\mathfrak{h} = 0$, Conjecture 1.6 follows from the classical result (Theorem 5.1). In Section 2.3, we will see the difficulties which occur in the general case but do not occur if $\mathfrak{h} = 0$.
In this paper, we will prove Conjecture 1.6 in some cases. The two conditions in the following theorem are fundamental to investigate topics of nilpotent Lie algebras.

Theorem 1.7. *Suppose that a nilpotent Lie algebra $\mathfrak{g}$ satisfies one of the following conditions.*

(i) $\mathfrak{g}$ *is* 2*-step nilpotent.*

(ii) *We can decompose* $\mathfrak{g} = \mathbf{K} \ltimes \mathbf{K}^n$*, i.e.,* $\mathfrak{g}$ *is special nilpotent Lie algebra.*

Then, for any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and any representation $\lambda\colon \mathfrak{h} \to \mathbf{K}$, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ such that the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.

We will prove more precise theorems. The condition (i) corresponds to Theorem 6.2 and the condition (ii) Theorem 6.1.
The condition (i) in this theorem is the case where even Corwin-Greenleaf's conjecture (Conjecture A.3) was not solved in previous research. Corwin and Greenleaf already solved their conjecture in the condition (ii). However, in previous research, Problem 1.1 was not solved even in the condition (ii).
Moreover, we will prove the symmetric case.

Theorem 1.8. *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric pair and $\mathfrak{q} := \mathfrak{g}^{-\sigma}$. i.e., $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$. Moreover, let $\lambda\colon \mathfrak{h} \to \mathbf{K}$ be a representation such that $\lambda([\mathfrak{q}, \mathfrak{q}]) = 0$. Then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Example 1.9. We consider group manifolds. That is, let $\mathfrak{g}'$ be a nilpotent Lie algebra, $\mathfrak{g} := \mathfrak{g}' \times \mathfrak{g}'$,

$$\mathfrak{h} := \{(X, X) \in \mathfrak{g} \mid X \in \mathfrak{g}'\}, \text{ and } \mathfrak{q} := \{(X, -X) \in \mathfrak{g} \mid X \in \mathfrak{g}'\}.$$

Then, for any representation $\lambda\colon \mathfrak{h} \to \mathbf{K}$, the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property since $[\mathfrak{q}, \mathfrak{q}] = [\mathfrak{h}, \mathfrak{h}] \subset \operatorname{Ker} \lambda$.

We will show Theorem 1.8 in Section 5.2. If $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and $\lambda = 0$, Rouvière generalized Theorem 1.8 in the non-nilpotent case.
We will prove Conjecture 1.6 in more conditions.

Theorem 1.10. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. Suppose that there exists an abelian ideal $\mathfrak{a} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{a}$. Then, for any representation $\lambda\colon \mathfrak{h} \to \mathbf{K}$, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ such that the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. For a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ included in $\mathfrak{a}$, the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property by Corollary 2.12. ■

In Section 7, we will consider the split Corwin-Greenleaf property and the split Duflo property for the Lie algebra

$$\mathfrak{n}_4 := \left\{ \begin{pmatrix} 0 & x_0 & -y_0 & z \\ 0 & 0 & t & y_1 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; t, x_0, x_1, y_0, y_1, z \in \mathbf{K} \right\}$$

in a practical manner. Finally, we will prove the follwing theorem.

Theorem 1.11. *For any subalgebra $\mathfrak{h} \subset \mathfrak{n}_4$ and any representation $\lambda\colon \mathfrak{h} \to \mathbf{K}$, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Corwin-Greenleaf property. Moreover, if $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \neq 1$, then we can choose $\mathfrak{q}$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

2. Basic considerations

In this section, we introduce the concepts mentioned in Section 1.1, more precisely. Furthermore, we clarify some bottlenecks to solve Conjecture 1.6.

2.1. The split symmetrization map

In this subsection, we consider some properties of the split symmetrization map, defined in Definition 1.3. In this paper, we use the following notations.

Remark 2.1. For a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a representation $\lambda\colon \mathfrak{h} \to \mathbf{K}$, we define the following notations.

(1) We denote by $\tilde{P}$ the element of $\tilde{S}_\lambda(\mathfrak{g})$ corresponding to $P \in S(\mathfrak{g})$.

(2) We also denote by $\tilde{A}$ the element of $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})$ corresponding to $A \in \mathcal{U}(\mathfrak{g})$.

(3) For a linear subspace $\mathfrak{q} \subset \mathfrak{g}$, we denote $\mathcal{U}(\mathfrak{q}) := \sigma(S(\mathfrak{q}))$. This $\mathcal{U}(\mathfrak{q})$ coinsides with the universal enveloping algebra of $\mathfrak{q}$ if $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$.

(4) For a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ and $X \in \mathfrak{g}$, we define $X_\mathfrak{q} \in \mathfrak{q}$, $X_\mathfrak{h} \in \mathfrak{h}$ by $X = X_\mathfrak{q} + X_\mathfrak{h}$.

Now we prove Proposition 1.4. This proposition follows from the following lemma.

Lemma 2.2. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda\colon \mathfrak{h} \to \mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h} \subset \mathfrak{g}$. Then the symmetrization map $\sigma_\mathfrak{q}$ satisfies $\sigma_\mathfrak{q}(S(\mathfrak{g})\mathfrak{h}_\lambda) \subset \mathcal{U}(\mathfrak{g})\mathfrak{h}_\lambda$.*

Proof. It is enough to show that, for $P \in S(\mathfrak{q})$, $Q \in S(\mathfrak{h})$, and $Y \in \mathfrak{h}$,

$$\sigma_{\mathfrak{q}}(PQ(Y - \lambda(Y))) = \sigma(P)\sigma(Q(Y - \lambda(Y))) \in \mathcal{U}(\mathfrak{g})\mathfrak{h}_\lambda.$$

So we must only to show $\sigma(S(\mathfrak{h})\mathfrak{h}_\lambda) \subset \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda$. Now we consider the following claim.

Claim 1. For $Y_0, \ldots, Y_{n-1}, Y \in \mathfrak{h}$, $(Y - \lambda(Y))Y_0 \cdots Y_{n-1} \in \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda$.

If this claim holds, $\mathfrak{h}_\lambda \cdot \mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda$ follows from this claim and

$$\sigma(S(\mathfrak{h})\mathfrak{h}_\lambda) = \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda\mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda.$$

In the remaining part of this proof, we show the above claim by induction on n. Since the $n = 0$ case is clear, it is enough to show the $n \geq 1$ case. Then,

$$(Y - \lambda(Y))Y_0 \cdots Y_{n-1} = Y_0(Y - \lambda(Y))Y_1 \cdots Y_{n-1} + [Y, Y_0]Y_1 \cdots Y_{n-1}.$$

By the induction hypothesis, $(Y - \lambda(Y))Y_1 \cdots Y_{n-1} \in \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda$. Moreover, since $[\mathfrak{h}, \mathfrak{h}] \subset \operatorname{Ker} \lambda \subset \mathfrak{h}_\lambda$, we obtain $[Y, Y_0]Y_1 \cdots Y_{n-1} \in \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda$ by the induction hypothesis. So

$$(Y - \lambda(Y))Y_0 \cdots Y_{n-1} = Y_0(Y - \lambda(Y))Y_1 \cdots Y_{n-1} + [Y, Y_0]Y_1 \cdots Y_{n-1} \in \mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda. \quad \blacksquare$$

Next, we investigate the induced map $\tilde{\sigma}_{\mathfrak{q}}$, defined in Proposition 1.4.

Proposition 2.3. *For a Lie algebra* $\mathfrak{g}$*, a subalgebra* $\mathfrak{h} \subset \mathfrak{g}$*, a representation* $\lambda\colon \mathfrak{h} \to \mathbf{K}$*, and a decomposition* $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$*,*

(i) $\sigma_{\mathfrak{q}}$ *and* $\tilde{\sigma}_{\mathfrak{q}}$ *are linear isomorphisms,*

(ii) $S(\mathfrak{g}) = S(\mathfrak{q}) \oplus S(\mathfrak{g})\mathfrak{h}_\lambda$, $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{q}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{h}_\lambda$*, and*

(iii) *all maps are linear isomorphisms in the commutative diagram*

$$\begin{array}{ccc} S(\mathfrak{q}) & \xrightarrow[\sigma = \sigma_{\mathfrak{q}}]{\cong} & \mathcal{U}(\mathfrak{q}) \\ \cong \downarrow & \circlearrowleft & \downarrow \cong \\ \tilde{S}_\lambda(\mathfrak{g}) & \xrightarrow[\tilde{\sigma}_{\mathfrak{q}}]{\cong} & \tilde{\mathcal{U}}_\lambda(\mathfrak{g}), \end{array}$$

introduced by Proposition 1.4.

Proof. Statement (iii) follows from (i) and (ii). $S(\mathfrak{g}) = S(\mathfrak{q}) \oplus S(\mathfrak{g})\mathfrak{h}_\lambda$ is obvious. Moreover, by the Poincarè-Birkhoff-Witt theorem, we immediately conclude that $\sigma_{\mathfrak{q}}$ is a linear isomorphism if we take the basis of each $\mathfrak{q}$ and $\mathfrak{h}$. So we get

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{q}) \cdot \mathcal{U}(\mathfrak{h}) = \mathcal{U}(\mathfrak{q}) \cdot ((\mathbf{K} \cdot 1) \oplus (\mathcal{U}(\mathfrak{h})\mathfrak{h}_\lambda)) = \mathcal{U}(\mathfrak{q}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{h}_\lambda,$$

and now we have proved (ii). By (ii) and Lemma 2.2, $\sigma_{\mathfrak{q}}(S(\mathfrak{g})\mathfrak{h}_\lambda) = \mathcal{U}(\mathfrak{g})\mathfrak{h}_\lambda$. Besides, $\sigma_{\mathfrak{q}}$ is a linear isomorphism. So $\tilde{\sigma}_{\mathfrak{q}}$ is also a linear isomorphism. $\blacksquare$

Remark 2.4. By Proposition 2.3, we can define the $\mathfrak{h}$-actions to $S(\mathfrak{q})$ and $\mathcal{U}(\mathfrak{q})$ by pulling-back from $\tilde{S}_\lambda(\mathfrak{g})$ and $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})$, each other. If $\mathfrak{q} \subset \mathfrak{g}$ is a subalgebra, we get Poisson isomorphisms $S(\mathfrak{q})^{\mathfrak{h}} \cong \tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ and $\mathcal{U}(\mathfrak{q})^{\mathfrak{h}} \cong \tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$. Furthermore, this $\mathfrak{h}$-action coincides with the natural one if $\mathfrak{q} \subset \mathfrak{g}$ is an ideal.

2.2. Easy examples

Before we confirm the basic strategy, we see some examples which are obvious or follow from the known fact immediately.

Proposition 2.5. *For a nilpotent Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a representation $\lambda\colon \mathfrak{h} \to \mathbf{K}$, and a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$, the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property if the quadruple satisfies one of the following conditions:* (i) *g is abelian;* (ii) $\mathfrak{h} = 0$.

Proof. If $\mathfrak{g}$ is abelian, we can identify $S(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})$, and this isomorphism coincides with $\sigma_{\mathfrak{q}}$. So it is clear that the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. If $\mathfrak{h} = 0$, the quadruple $(\mathfrak{g}, 0, 0, \mathfrak{g})$ satisfies the split Duflo property by the well-known classical fact (Theorem 5.1). ∎

2.3. The basic strategy

Our problem is about nilpotent Lie algebras. Since the nilpotency of Lie algebras is characterized by filtrations, we often use induction on dimensions when we solve problems of the nilpotent Lie algebras. Therefore, it is important how to reduce to a matter of subalgebras or quotient algebras. In this paper, if we simply say a *reduction*, we mean an operation to reduce the matter of subalgebras or quotient algebras. In this subsection, let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda\colon \mathfrak{h} \to \mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h} \subset \mathfrak{g}$.

Bottlenecks. As mentioned above, we already know that Conjecture 1.6 stands if $\mathfrak{h} = 0$, by the classical result. In this subsection, we clarify the bottlenecks to solve Conjecture 1.6 generally. There are three bottlenecks to judge the split Duflo property.

1. Does $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ coincide with the image of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ under $\tilde{\sigma}_{\mathfrak{q}}$?
2. Does $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ coincide with the image of $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ under $\tilde{\sigma}_{\mathfrak{q}}$?
3. Does $\tilde{\sigma}_{\mathfrak{q}}$ preserve the multiplication on $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$?

In these checkpoints, the third appears even in the $\mathfrak{h} = 0$ case and is often removed by reductions. On the other hand, the other checkpoints do not appear in the $\mathfrak{h} = 0$ case, and disturb reductions which will be packaged after this section. In the following, we discuss why these checkpoints become more difficult in general case and present some sufficient conditions to remove these bottlenecks.

The first bottleneck. The first bottleneck is caused by that $\tilde{\sigma}_{\mathfrak{q}}$ is not necessarily $\mathfrak{h}$-equivariant. For example, the following proposition gives a sufficient condition to remove this bottleneck.

Proposition 2.6. *If $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q} \oplus C_{\mathfrak{h}}(\mathfrak{q})$, then $\tilde{\sigma}_{\mathfrak{q}}\colon \tilde{S}_\lambda(\mathfrak{g}) \to \tilde{\mathcal{U}}_\lambda(\mathfrak{g})$ is $\mathfrak{h}$-equivariant. Here, $C_{\mathfrak{h}}(\mathfrak{q}) := \{Y \in \mathfrak{h} \mid [Y, \mathfrak{q}] = 0\}$.*

Proof. This proposition follows from Lemma 2.7. ∎

Lemma 2.7. *If the linear decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ and $Y \in \mathfrak{g}$ satisfy the inclusion $[Y, \mathfrak{q}] \subset \mathfrak{q} \oplus C_{\mathfrak{h}}(\mathfrak{q})$, then $\sigma_{\mathfrak{q}} \circ \mathrm{ad}(Y) = \mathrm{ad}(Y) \circ \sigma_{\mathfrak{q}}$ on $S(\mathfrak{q})$.*

Proof. Let $X_0, ..., X_{n-1} \in \mathfrak{q}$. The $\mathrm{ad}(Y)$-equivariance follows from

$$\begin{aligned}
&\sigma_{\mathfrak{q}}(\mathrm{ad}(Y)(X_0 \cdots X_{n-1})) \\
=&\sum_{i=0}^{n-1} \sigma_{\mathfrak{q}}(X_0 \cdots \hat{X}_i \cdots X_{n-1}[Y, X_i]) \\
=&\sum_{i=0}^{n-1} \Big(\sigma(X_0 \cdots \hat{X}_i \cdots X_{n-1}[Y, X_i]_{\mathfrak{q}}) + \sigma(X_0 \cdots \hat{X}_i \cdots X_{n-1})[Y, X_i]_{\mathfrak{h}}\Big) \\
=&\sum_{i=0}^{n-1} \Big(\sigma(X_0 \cdots \hat{X}_i \cdots X_{n-1}[Y, X_i]_{\mathfrak{q}}) + \sigma(X_0 \cdots \hat{X}_i \cdots X_{n-1}[Y, X_i]_{\mathfrak{h}})\Big) \\
=&\sum_{i=0}^{n-1} \sigma(X_0 \cdots \hat{X}_i \cdots X_{n-1}[Y, X_i]) \\
=&\mathrm{ad}(Y)\sigma(X_0 \cdots X_{n-1}) = \mathrm{ad}(Y)\sigma_{\mathfrak{q}}(X_0 \cdots X_{n-1}).
\end{aligned}$$

Here, we use the following notation.

$$X_0 \cdots \hat{X}_i \cdots X_{n-1} = X_0 \cdots X_{i-1} X_{i+1} \cdots X_{n-1}. \qquad \blacksquare$$

The second bottleneck. In the present situation, the second bottleneck is essential. This bottleneck caused by that $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ and $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ are not necessarily generated by 1-degree elements. In fact, we can often avoid this bottleneck if $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is generated by 1-degree elements.

Proposition 2.8. *Suppose that $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ coincides with the image of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ under $\tilde{\sigma}_{\mathfrak{q}}$, and $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is generated by 1-degree elements, i.e., $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is generated by the projection of $U := \{X \in \mathfrak{q} \mid [\mathfrak{h}, X] \subset \mathfrak{h}_\lambda\}$ and $\mathbf{K} \cdot 1$ as ring. If $[U, \mathfrak{q}] \subset \mathfrak{q} \oplus C_{\mathfrak{h}}(\mathfrak{q})$, then $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ is the image of $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ under $\tilde{\sigma}_{\mathfrak{q}}$.*

Proof. Since $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is generated by U and $\mathbf{K} \cdot 1$,

$$Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) = \Big\{\tilde{P} \in \tilde{S}_\lambda(\mathfrak{g}) \mid P \in S(\mathfrak{g}), [X, P] \in S(\mathfrak{g}) \cdot \mathfrak{h}_\lambda \ (X \in U)\Big\}.$$

By the definition of $\sigma_{\mathfrak{q}}$, $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is also generated by U and $\mathbf{K} \cdot 1$. So,

$$Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}) = \Big\{\tilde{A} \in \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) \mid A \in \mathcal{U}(\mathfrak{g}), [X, A] \in \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{h}_\lambda \ (X \in U)\Big\}.$$

Furthermore, by Lemma 2.7 and $[U, \mathfrak{q}] \subset \mathfrak{q} \oplus C_{\mathfrak{h}}(\mathfrak{q})$, $\sigma_{\mathfrak{q}}$ and $\mathrm{ad}(X)$ is commutative on $S(\mathfrak{q})$ for all $X \in U$. Therefore,

$$\begin{aligned}
\tilde{\sigma}_{\mathfrak{q}}(Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})) &= \Big\{\tilde{\sigma}_{\mathfrak{q}}(\tilde{P}) \in \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) \mid P \in S(\mathfrak{g}),\ [X, P] \in S(\mathfrak{g}) \cdot \mathfrak{h}_\lambda \ (X \in U)\Big\} \\
&= \Big\{\tilde{\sigma}_{\mathfrak{q}}(\tilde{P}) \in \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) \mid P \in S(\mathfrak{g}),\ \sigma_{\mathfrak{q}}([X, P]) \in \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{h}_\lambda \ (X \in U)\Big\} \\
&= \Big\{\tilde{\sigma}_{\mathfrak{q}}(\tilde{P}) \in \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) \mid P \in S(\mathfrak{g}),\ [X, \sigma_{\mathfrak{q}}(P)] \in \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{h}_\lambda \ (X \in U)\Big\} \\
&= \Big\{\tilde{A} \in \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) \mid A \in \mathcal{U}(\mathfrak{g}),\ [X, A] \in \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{h}_\lambda \ (X \in U)\Big\} = Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}). \qquad \blacksquare
\end{aligned}$$

If $[\mathfrak{h}, \mathfrak{q}] = 0$, the assumption of Proposition 2.8 holds.

Proposition 2.9. *If $[\mathfrak{h}, \mathfrak{q}] = 0$, then $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h}$ is the image of $\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ and $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h})$ is the image of $Z(\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h})$ under $\tilde{\sigma}_\mathfrak{q}$.*

Proof. By Proposition 2.6 and $[\mathfrak{h}, \mathfrak{q}] = 0$, $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h}$ is the image of $\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ under $\tilde{\sigma}_\mathfrak{q}$. Moreover, since $C_\mathfrak{h}(\mathfrak{q}) = \mathfrak{h}$, $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{g} = \mathfrak{q} \oplus C_\mathfrak{h}(\mathfrak{q})$. Furthermore, $\tilde{S}_\lambda(\mathfrak{g}) = \tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ is generated by the projection of $\mathfrak{q}$ and $\mathbf{K} \cdot 1$ as ring. Hence, by Proposition 2.8, $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h})$ is the image of $Z(\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h})$ under $\tilde{\sigma}_\mathfrak{q}$. ∎

In this paper, we will prove Conjecture 1.6 in the 2-step nilpotent case by taking two steps. At first, we will prove that the assumption of Proposition 2.8 holds if the derived algebra is one dimensional (Theorem 5.5). Next, we will reduce the general 2-step nilpotent case to the one that the derived algebra is one dimensional.

Furthermore, if $[\mathfrak{q}, \mathfrak{q}] \subset \mathrm{Ker}\lambda$, the second bottleneck does not appear even if $\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ is not generated by 1-degree elements (cf. Theorem 1.8). Because the Poisson bracket of $\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ becomes trivial in this case. Moreover, if $\mathfrak{q}$ is abelian, the third bottleneck vanishes automatically. So we can conclude the split Duflo property in this case.

Proposition 2.10. *If $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^\mathfrak{h}$ is the image of $\tilde{S}_\lambda(\mathfrak{g})^\mathfrak{h}$ under $\tilde{\sigma}_\mathfrak{q}$ and $\mathfrak{q}$ is abelian, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. Since $\mathfrak{q}$ is abelian, the symmetrization map $\sigma\colon S(\mathfrak{q}) \to \mathcal{U}(\mathfrak{q})$ is an isomorphism between the Poisson algebras, whose Poisson brackets are trivial. So, by the commutative diagram in Proposition 2.3 and Remark 2.4, $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. ∎

Corollary 2.11. *If $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q} \oplus C_\mathfrak{h}(\mathfrak{q})$ and $[\mathfrak{q}, \mathfrak{q}] = 0$, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. This corollary follows from Proposition 2.6, 2.10. ∎

Corollary 2.12. *If there exists an abelian ideal of $\mathfrak{g}$ including $\mathfrak{q}$, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. Let $\mathfrak{a} \subset \mathfrak{g}$ be an abelian ideal including $\mathfrak{q}$. Then $\mathfrak{a} = \mathfrak{q} \oplus (\mathfrak{h} \cap \mathfrak{a})$ and $\mathfrak{a} \cap \mathfrak{h} \subset C_\mathfrak{h}(\mathfrak{q})$. So

$$[\mathfrak{h}, \mathfrak{q}] \subset [\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a} \subset \mathfrak{q} \oplus C_\mathfrak{h}(\mathfrak{q}).$$

Therefore, by Corollary 2.11, the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. ∎

The third bottleneck. The third bottleneck appears even if $\mathfrak{h} = 0$. We have few direct approaches to this bottleneck, and the most effective strategy is induction. If we already remove the first and second bottlenecks, since we can use all reductions packaged in the later sections, we can reduce the dimension like the $\mathfrak{h} = 0$ case. However, for the center $C(\mathfrak{g})$ of $\mathfrak{g}$, if $\dim \mathfrak{q} \cap C(\mathfrak{g}) = \dim \mathfrak{h} \cap C(\mathfrak{g}) = 1$, we cannot use the classical argument for the $\mathfrak{h} = 0$ case, and we have to consider some new approaches.

Other difficulties. The essential bottlenecks are the above three ones. However, there are other technical difficulties. For example, since the computation of two 1-degree elements by the Poisson brackets of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ or $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ may be a constant, we cannot use the arguments which is used in the $\mathfrak{h} = 0$ case. Moreover, since compatibility between a linear decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ and $C(\mathfrak{g})$ does not necessarily preserved if we take subalgebra or quotient algebra, the induction may not hold immediately.

3. Quotient reductions

In this section and the next one, we will package some "subreductions" and "quotient reductions." In the present situation, the most effective strategy for Conjecture 1.6 is that we use the induction on the dimension by the reductions packaged in this section and the next one. In this section, we introduce some quotient reductions. Through this section, let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda\colon \mathfrak{h} \to \mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h} \subset \mathfrak{g}$.

3.1. The $\mathfrak{h}$-quotient reduction

At first, we introduce the $\mathfrak{h}$-quotient reduction (Theorem 3.1). We can use this reduction even if we remove no bottlenecks in Section 2.3.

Theorem 3.1. *Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal included in* $\operatorname{Ker} \lambda$. *Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the quotient map, $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{a}$, $\bar{\mathfrak{h}} := \mathfrak{h}/\mathfrak{a}$, $\bar{\mathfrak{q}} := \pi(\mathfrak{q})$, and $\bar{\lambda}\colon \bar{\mathfrak{h}} \to \mathbf{K}$ the induced map of λ. Then, the following conditions are equivalent.*

(i) *The quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

(ii) *The quadruple $(\bar{\mathfrak{g}}, \bar{\mathfrak{h}}, \bar{\lambda}, \bar{\mathfrak{q}})$ satisfies the split Duflo property.*

Proof. This theorem follows from the following proposition. ∎

Proposition 3.2. *We assume the assumption of Theorem* 3.1.

(1) *For $\mathcal{M} = S, \mathcal{U}$, there exists the induced map $\tilde{\pi}\colon \tilde{\mathcal{M}}_\lambda(\mathfrak{g}) \to \tilde{\mathcal{M}}_\lambda(\bar{\mathfrak{g}})$ defined by the following commutative diagram.*

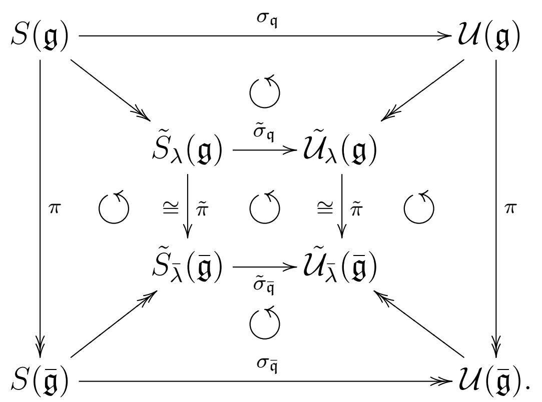
$$\begin{array}{ccc} \mathcal{M}(\mathfrak{g}) & \xrightarrow{\ \pi\ } & \mathcal{M}(\bar{\mathfrak{g}}) \\ \downarrow & \circlearrowleft & \downarrow \\ \tilde{\mathcal{M}}_\lambda(\mathfrak{g}) & \underset{\cong}{\xrightarrow{\ \tilde{\pi}\ }} & \tilde{\mathcal{M}}_{\bar{\lambda}}(\bar{\mathfrak{g}}). \end{array}$$

Moreover, $\tilde{\pi}$ is a homomorphism between $\mathfrak{h}$-modules, and $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is isomorphic to $\tilde{\mathcal{M}}_{\bar{\lambda}}(\bar{\mathfrak{g}})^{\bar{\mathfrak{h}}}$ as a Poisson algebra under $\tilde{\pi}$.

(2) *The following commutative diagram holds.*

$$\begin{array}{ccc} \tilde{S}_\lambda(\mathfrak{g}) & \xrightarrow{\ \tilde{\sigma}_{\mathfrak{q}}\ } & \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) \\ \cong \downarrow \tilde{\pi} & \circlearrowleft & \cong \downarrow \tilde{\pi} \\ \tilde{S}_{\bar{\lambda}}(\bar{\mathfrak{g}}) & \xrightarrow[\ \tilde{\sigma}_{\bar{\mathfrak{q}}}\ ]{} & \tilde{\mathcal{U}}_{\bar{\lambda}}(\bar{\mathfrak{g}}). \end{array}$$

Proof. When we prove (1), the diagram of statement (2) is the inner rectangle in the following commutative diagram.



In the following, we prove (1). At first, we assume the existence and the injectivity of $\tilde{\pi}$, and prove the others. Since the other maps in the commutative diagram in Statement 1 are surjective $\mathfrak{h}$-homomorphisms, $\tilde{\pi}$ is also a surjective $\mathfrak{h}$-homomorphism. Moreover, since π is a Poisson algebra homomorphism and the algebraic structure of $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ (resp. $\tilde{\mathcal{M}}_\lambda(\bar{\mathfrak{g}})^{\bar{\mathfrak{h}}}$) is compatible with the one of $\mathcal{M}_\lambda(\mathfrak{g})$ (resp. $\mathcal{M}_\lambda(\bar{\mathfrak{g}})$), $\tilde{\pi}$ is also a Poisson algebra homomorphism.

Next, we prove the existence and the injectivity of $\tilde{\pi}$. By

$$\pi(\mathcal{M}(\mathfrak{g})\mathfrak{h}_\lambda) \subset \mathcal{M}(\bar{\mathfrak{g}})\bar{\mathfrak{h}}_{\bar{\lambda}},\ \mathfrak{a} \subset \mathrm{Ker}\ \lambda \subset \mathfrak{h}_\lambda,$$

and Lemma 3.3,

$$\begin{aligned}\mathrm{Ker}\left(\mathcal{M}(\mathfrak{g}) \xrightarrow{\pi} \mathcal{M}(\bar{\mathfrak{g}}) \twoheadrightarrow \tilde{\mathcal{M}}_{\bar{\lambda}}(\bar{\mathfrak{g}})\right) &= \pi^{-1}(\mathcal{M}(\bar{\mathfrak{g}}) \cdot \bar{\mathfrak{h}}_{\bar{\lambda}}) = \mathcal{M}(\mathfrak{g}) \cdot \mathfrak{h}_\lambda + \mathcal{M}(\mathfrak{g}) \cdot \mathfrak{a} \\ &= \mathcal{M}(\mathfrak{g}) \cdot \mathfrak{h}_\lambda = \mathrm{Ker}(\mathcal{M}(\mathfrak{g}) \twoheadrightarrow \tilde{\mathcal{M}}_\lambda(\mathfrak{g})).\end{aligned}$$

Therefore $\tilde{\pi}$ exists and is injective. ■

Lemma 3.3. *For a Lie algebra* $\mathfrak{g}$ *and an ideal* $\mathfrak{a} \subset \mathfrak{g}$, $\mathrm{Ker}(\mathcal{M}(\mathfrak{g}) \twoheadrightarrow \mathcal{M}(\mathfrak{g}/\mathfrak{a})) = \mathcal{M}(\mathfrak{g}) \cdot \mathfrak{a}$ $(\mathcal{M} = S, \mathcal{U})$.

Proof. This lemma is clear. ■

3.2. The $\mathfrak{q}$-quotient reduction

Next, we introduce the $\mathfrak{q}$-quotient reduction (Theorem 3.4). Unlike the $\mathfrak{h}$-quotient reduction, we have to remove the second bottleneck in Section 2.3 to use the $\mathfrak{q}$-quotient reduction. This reduction may seem to be hard to use. However, this reduction is essential in the proof of the $\mathfrak{h} = 0$ case.

Theorem 3.4. *Suppose one of the following conditions.*

(i) $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *is the image of* $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *and* $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ *is the image of* $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ *under* $\tilde{\sigma}_{\mathfrak{q}}$.

(ii) $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *is the image of* $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *under* $\tilde{\sigma}_{\mathfrak{q}}$ *and the Poisson bracket of* $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *is trivial.*

Let $\mathfrak{a} \subset \mathfrak{q} \cap C(\mathfrak{g})$ be a subspace such that $\dim \mathfrak{a} \geq 2$. Moreover, for all $X \in \mathfrak{a} - \{0\}$ and the linear subspace $\langle X \rangle$ of $\mathfrak{g}$ generated by X, let $\pi_X : \mathfrak{g} \to \mathfrak{g}/\langle X \rangle$ be the quotient map, $\mathfrak{g}_X := \mathfrak{g}/\langle X \rangle$, $\mathfrak{h}_X := \pi_X(\mathfrak{h})$, $\lambda_X := \lambda \circ \pi_X|_{\mathfrak{h}}^{-1} : \mathfrak{h}_X \to \mathbf{K}$, and $\mathfrak{q}_X := \pi_X(\mathfrak{q})$. If the quadruple $(\mathfrak{g}_X, \mathfrak{h}_X, \lambda_X, \mathfrak{q}_X)$ satisfies the split Duflo property for all $X \in \mathfrak{a}$, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ also satisfies the split Duflo property.

Proof. If the Poisson bracket of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is trivial, then $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}} = Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$. So it is enough to show that $\tilde{\sigma}_{\mathfrak{q}}$ preserves the multiplication on $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$. Take $P, Q \in S(\mathfrak{g})$ such that $\tilde{P}, \tilde{Q} \in Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$. It is enough to show

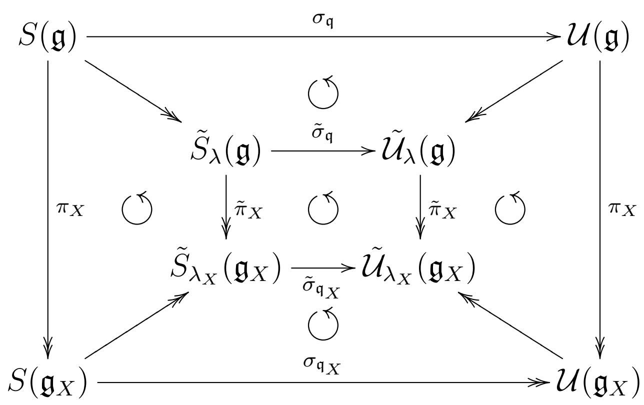
$$\tilde{R} := \tilde{\sigma}_{\mathfrak{q}}^{-1}(\tilde{\sigma}_{\mathfrak{q}}(P)\tilde{\sigma}_{\mathfrak{q}}(Q)) - \tilde{P}\tilde{Q} = 0 \in \tilde{S}_\lambda(\mathfrak{g}).$$

Since $X \in C(\mathfrak{g})$, $\sigma_{\mathfrak{q}}(S(\mathfrak{g}) \cdot X) = \mathcal{U}(\mathfrak{g}) \cdot X$. Hence, by Lemma 3.5, $\tilde{R} \in \tilde{S}_\lambda(\mathfrak{g}) \cdot \tilde{X}$. So $\tilde{R}$ is divisible by $\tilde{X}$. On the other hand, since we took an arbitrary $X \in \mathfrak{a}$, $\dim \mathfrak{a} \geq 2$, and $\tilde{S}_\lambda(\mathfrak{g})$ is ring isomorphic to $S(\mathfrak{q})$, we conclude that $\tilde{R} = 0$. ■

Lemma 3.5. *Take a non-zero element $X \in \mathfrak{q} \cap C(\mathfrak{g})$ such that the quadruple $(\mathfrak{g}_X, \mathfrak{h}_X, \lambda_X, \mathfrak{q}_X)$ defined in Theorem 3.4 satisfies the split Duflo property. Then, for $P, Q \in S(\mathfrak{g})$ such that $\tilde{P}, \tilde{Q} \in Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$,*

$$\tilde{\sigma}_{\mathfrak{q}}^{-1}(\tilde{\sigma}_{\mathfrak{q}}(\tilde{P})\tilde{\sigma}_{\mathfrak{q}}(\tilde{Q})) - \tilde{P}\tilde{Q} \in \tilde{S}_\lambda(\mathfrak{g}) \cdot \tilde{X}.$$

Proof. There exists the induced map $\tilde{\pi}_X$ which is characterized by the following commutative diagram.



Now, for $\mathcal{M} = S, \mathcal{U}$, $\tilde{\pi}_X : \tilde{\mathcal{M}}_\lambda(\mathfrak{g}) \twoheadrightarrow \tilde{\mathcal{M}}_{\lambda_X}(\mathfrak{g}_X)$ is $\mathfrak{h}$-equivariant under the identification $\mathfrak{h} \cong \mathfrak{h}_X$, and the restriction $\tilde{\sigma}_{\mathfrak{q}} : \tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}} \to \tilde{\mathcal{M}}_{\lambda_X}(\mathfrak{g}_X)^{\mathfrak{h}_X}$ is a ring isomorphism. Therefore, $\tilde{\pi}_X(\tilde{P}), \tilde{\pi}_X(\tilde{Q}) \in Z(\tilde{S}_{\lambda_X}(\mathfrak{g}_X)^{\mathfrak{h}_X})$. Furthermore, since $(\mathfrak{g}_X, \mathfrak{h}_X, \lambda_X, \mathfrak{q}_X)$ satisfies the split Duflo property,

$$\tilde{\pi}_X(\tilde{\sigma}_{\mathfrak{q}}(\tilde{P})\tilde{\sigma}_{\mathfrak{q}}(\tilde{Q})) = \tilde{\sigma}_{\mathfrak{q}_X}(\tilde{\pi}_X(\tilde{P}))\tilde{\sigma}_{\mathfrak{q}_X}(\tilde{\pi}_X(\tilde{Q})) = \tilde{\sigma}_{\mathfrak{q}_X}(\tilde{\pi}_X(\tilde{P})\tilde{\pi}_X(\tilde{Q})) = \tilde{\pi}_X\tilde{\sigma}_{\mathfrak{q}}(\tilde{P}\tilde{Q}).$$

Hence,

$$\tilde{\pi}_X(\tilde{\sigma}_{\mathfrak{q}}^{-1}(\tilde{\sigma}_{\mathfrak{q}}(\tilde{P}) \cdot \tilde{\sigma}_{\mathfrak{q}}(\tilde{Q})) - \tilde{P}\tilde{Q}) = \tilde{\sigma}_{\mathfrak{q}}^{-1} \circ \pi_X(\tilde{\sigma}_{\mathfrak{q}}(\tilde{P}) \cdot \tilde{\sigma}_{\mathfrak{q}}(\tilde{Q}) - \tilde{\sigma}_{\mathfrak{q}}(\tilde{P}\tilde{Q})) = 0 \in \tilde{S}_{\lambda_X}(\mathfrak{g}_X).$$

So we conclude $\tilde{\sigma}_{\mathfrak{q}}^{-1}(\tilde{\sigma}_{\mathfrak{q}}(\tilde{P})\tilde{\sigma}_{\mathfrak{q}}(\tilde{Q})) - \tilde{P}\tilde{Q} \in \tilde{S}_\lambda(\mathfrak{g}) \cdot \tilde{X}$ by the natural identification $S(\mathfrak{q}) \cong \tilde{S}_\lambda(\mathfrak{g})$ and $\operatorname{Ker} \pi_X = S(\mathfrak{g}) \cdot X$. ■

4. Subreductions

4.1. The compatibility between a linear subspace and a linear decomposition

In this section, we introduce some subreductions. In this first subsection, we prepare a concept of linear algebra before introduce the subreductions.

Definition 4.1. For a linear space V, we say that a subspace $U \subset V$ and a linear decomposition $V = W_0 \oplus W_1$ are *compatible* if $U = (U \cap W_0) \oplus (U \cap W_1)$.

Remark 4.2. Let V be a finite dimensional linear space.

(1) For a subspace $U \subset V$ and a decomposition $V = W_0 \oplus W_1$, if $U \subset W_0$ or $W_0 \subset U$, then U and $V = W_0 \oplus W_1$ are compatible.

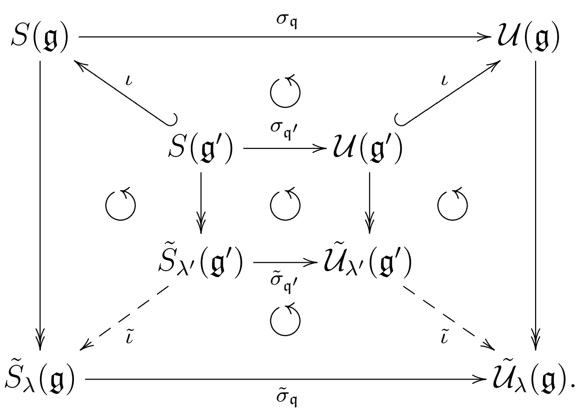
(2) For a sequence $U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset V$ of subspaces and a subspace $W \subset V$, there exists a linear complement W' of $W \subset V$ such that the decomposition $V = W \oplus W'$ is compatible with all $U_0, U_1, \ldots, U_{n-1}$.

(3) For a decomposition $V = U_0 \oplus U_1$ and a subspace $W \subset V$, there exists a linear complement W' of $W \subset V$ such that the decomposition $V = W \oplus W'$ is compatible with U_0 and U_1.

4.2. The main concept of subreduction

In this subsection, we see the main concept of the $(\mathfrak{h}, \mathfrak{q})$-subreduction and the $\mathfrak{h}$-subreduction. In the following part of Section 4, we consider the following situation.

Setting 4.3. Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda \colon \mathfrak{h} \to \mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h} \subset \mathfrak{g}$. We take a subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ compatible with the decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ and let $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{g}'$, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$, $\lambda' := \lambda|_{\mathfrak{h}'}$. We denote by $\iota \colon \mathfrak{g}' \hookrightarrow \mathfrak{g}$ the inclusion, and by $\iota \colon S(\mathfrak{g}') \hookrightarrow S(\mathfrak{g})$ and $\iota \colon \mathcal{U}(\mathfrak{g}') \hookrightarrow \mathcal{U}(\mathfrak{g})$ the induced inclusions. There exists the induced map $\tilde{\iota}$ defined by the following commutative diagram.



The following proposition is the essence of subreductions.

Proposition 4.4. *Suppose Setting* 4.3 *and that* $\tilde{\mathcal{M}}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'}$ *is linear isomorphic to* $\tilde{\mathcal{M}}_{\lambda}(\mathfrak{g})^{\mathfrak{h}}$ *under* $\tilde{\iota}$ *for* $\mathcal{M} = S, \mathcal{U}$. *Then the following are equivalent.*

(i) *The quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *satisfies the split Duflo property.*

(ii) *The quadruple* $(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q}')$ *satisfies the split Duflo property.*

Proof. In the Setting 4.3, the inclusion $\iota\colon \mathcal{M}(\mathfrak{g}') \hookrightarrow \mathcal{M}(\mathfrak{g})$ is a Poisson algebra isomorphism. So, by the compatibility of the algebraic structures of $\mathcal{M}(\mathfrak{g})$ and $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$, $\tilde{\mathcal{M}}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'}$ is isomorphic to $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ as a Poisson algebra under $\tilde{\iota}$. ■

4.3. The $(\mathfrak{h}, \mathfrak{q})$-subreduction

In this subsection, we introduce the $(\mathfrak{h}, \mathfrak{q})$-subreduction (Theorem 4.5). We connot use this subreduction unless we solve the first bottleneck in Section 2.3 at least partially. However, we can use this subreduction even if we do not remove the second bottleneck.

Theorem 4.5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda\colon \mathfrak{h} \to \mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h} \subset \mathfrak{g}$. Take $Y \in \mathfrak{h}$ such that*

$$\tilde{\sigma}_{\mathfrak{q}}(\tilde{S}_\lambda(\mathfrak{g})^{\mathrm{ad}(Y)}) = \tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathrm{ad}(Y)}, \quad \dim[Y, \mathfrak{g}] - \dim[Y, \mathfrak{g}] \cap \mathrm{Ker}\ \lambda = 1.$$

Moreover, we suppose that $\mathfrak{g}' := \mathrm{ad}(Y)^{-1}(\mathrm{Ker}\ \lambda)$ is a subalgebra and let $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{g}'$. Then, the following conditionsare equivalent.

(i) *The quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

(ii) *The quadruple $(\mathfrak{g}', \mathfrak{h}, \lambda, \mathfrak{q}')$ satisfies the split Duflo property.*

Proof. This theorem follows from Proposition 4.4, 4.6, and Remark 4.8. ■

Proposition 4.6. *Under the assumption of Theorem 4.5, $\tilde{\mathcal{M}}_\lambda(\mathfrak{g}')^{\mathfrak{h}} = \tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ for $\mathcal{M} = S, \mathcal{U}$.*

Proof. It is enough to show $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathrm{ad}(Y)} \subset \tilde{\mathcal{M}}_\lambda(\mathfrak{g}')$ for $\mathcal{M} = S, \mathcal{U}$. Since $\tilde{\sigma}_{\mathfrak{q}}(\tilde{S}_\lambda(\mathfrak{g})^{\mathrm{ad}(Y)}) = \tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathrm{ad}(Y)}$ and $Y \in \mathfrak{h}$, we can assume that $\mathcal{M} = S$. Since the restriction $S(\mathfrak{q}) \overset{\cong}{\to} \tilde{S}_\lambda(\mathfrak{g})$ of the quotient map is a linear isomorphism, it is enough to show $P \in S(\mathfrak{g}')$ for $P \in S(\mathfrak{q})$ such that $\tilde{P} \in \tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$.

By Lemma 4.7 and $\dim[Y, \mathfrak{g}] - \dim[Y, \mathfrak{g}] \cap \mathrm{Ker}\ \lambda = 1$, $\dim \mathfrak{g} - \dim \mathfrak{g}' = 1$ and $\dim \mathfrak{q} - \dim \mathfrak{q}' = 1$. Take $X_0 \in \mathfrak{q} - \mathfrak{q}'$ and a basis $X_1, \ldots, X_{m-1}$ of $\mathfrak{q}'$. Then, by $P \in S(\mathfrak{q})$ and $\tilde{P} \in \tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$,

$$S(\mathfrak{g})\mathfrak{h}_\lambda \ni [Y, P] = \sum_{i=0}^{m-1} \frac{\partial P}{\partial X_i}[Y, X_i].$$

Here, since $S(\mathfrak{g})\mathfrak{h}_\lambda \subset S(\mathfrak{g})$ is a prime ideal, $[Y, X_0] \notin \mathfrak{h}_\lambda$, and $[Y, X_i] \in \mathfrak{h}_\lambda$ $(i \geq 1)$, we obtain $\partial P/\partial X_0 \in S(\mathfrak{g})\mathfrak{h}_\lambda$. So,

$$\frac{\partial P}{\partial X_0} \in S(\mathfrak{q}) \cap (S(\mathfrak{g})\mathfrak{h}_\lambda) = 0.$$

Therefore, $P \in S(\mathfrak{g}')$. ■

Lemma 4.7. *For a finite dimensional linear space V, a linear subspace $W \subset V$, and a linear map $f\colon V \to V$,*

$$\dim f(V) - \dim f(V) \cap W = \dim V - \dim f^{-1}(W).$$

Proof. By the dimension theorem,

$$\dim V = \dim f(V) + \dim \mathrm{Ker}\ f, \ \dim f(V) \cap W = \dim f^{-1}(W) + \dim \mathrm{Ker}\ f|_{f^{-1}(W)}.$$

Since $\operatorname{Ker} f = f^{-1}(0) \subset f^{-1}(W)$,

$$\operatorname{Ker} f|_{f^{-1}(W)} = f^{-1}(W) \cap \operatorname{Ker} f = \operatorname{Ker} f.$$

So, $$\dim V - \dim f(V) = \dim \operatorname{Ker} f = \dim f(V) \cap W - \dim f^{-1}(W).$$ ■

Remark 4.8. Suppose Setting 4.3 and $\mathfrak{h} \subset \mathfrak{g}'$. Then we can regard $\tilde{\iota}$ as the inclusion and $\sigma_{\mathfrak{q}'} = \sigma_{\mathfrak{q}}|_{S(\mathfrak{g}')}$, $\tilde{\sigma}_{\mathfrak{q}'} = \tilde{\sigma}_{\mathfrak{q}}|_{\tilde{S}_\lambda(\mathfrak{g}')}$.

4.4. The $\mathfrak{h}$-subreduction. Next, we introduce the $\mathfrak{h}$-subreduction. We can use this reduction even if we do not remove the first bottleneck in Section 2.3.

Theorem 4.9. *Suppose Setting* 4.3 *and* $\mathfrak{q} \subset \mathfrak{g}'$. *If there exists a linear complement* $\mathfrak{l}$ *of* $\mathfrak{h}' \subset \mathfrak{h}$ *such that* $[\mathfrak{l}, \mathfrak{q}] = 0$, *then the following are equivalent.*

(i) *The quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *satisfies the split Duflo property.*

(ii) *The quadruple* $(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q})$ *satisfies the split Duflo property.*

Proof. By Proposition 4.4, it is enough to show that $\tilde{\mathcal{M}}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'}$ is linear isomorphic to $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ under $\tilde{\iota}$ for $\mathcal{M} = S, \mathcal{U}$. We review that we defined $\mathcal{U}(\mathfrak{q}) := \sigma(S(\mathfrak{q}))$ in Notation 2.1. Since both $\tilde{\mathcal{M}}_{\lambda'}(\mathfrak{g}')$ and $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})$ are isomorphic to $\mathcal{M}(\mathfrak{q})$ under the restrictions of the quotient maps, $\tilde{\iota}\colon \tilde{\mathcal{M}}_{\lambda'}(\mathfrak{g}') \to \tilde{\mathcal{M}}_\lambda(\mathfrak{g})$ is surjective. Moreover, by Proposition 2.3,

$$\mathcal{M}(\mathfrak{g}') \cap (\mathcal{M}(\mathfrak{g})\mathfrak{h}_\lambda) = (\mathcal{M}(\mathfrak{q}) \oplus \mathcal{M}(\mathfrak{g}')\mathfrak{h}'_{\lambda'}) \cap (\mathcal{M}(\mathfrak{g})\mathfrak{h}_\lambda) = \mathcal{M}(\mathfrak{g}')\mathfrak{h}'_{\lambda'}.$$

So $\tilde{\iota}\colon \tilde{\mathcal{M}}_{\lambda'}(\mathfrak{g}') \to \tilde{\mathcal{M}}_\lambda(\mathfrak{g})$ is injective. Furthermore, by $[\mathfrak{l}, \mathfrak{q}] = 0$, for $Y \in \mathfrak{h}'$, $Z \in \mathfrak{l}$, and $X_0, \ldots, X_{n-1} \in \mathfrak{q}$,

$$[Y + Z, X_0 \cdots X_{n-1}] = [Y, X_0 \ldots X_{n-1}] \text{ in } \mathcal{M}(\mathfrak{g}).$$

Therefore, $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}} = \tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}'}$. Hence, $\tilde{\mathcal{M}}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'}$ is linear isomorphic to $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ under $\tilde{\iota}$. ■

Corollary 4.10. *Let* $\mathfrak{q}, \mathfrak{h}$ *be Lie algebras and* $\mathfrak{g} = \mathfrak{q} \times \mathfrak{h}$ *the product of Lie algebras. Then, for all representation* $\lambda\colon \mathfrak{h} \to \mathbf{K}$, $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *satisfies the split Duflo property.*

Proof. By Theorem 4.9, it is enough to show that $(\mathfrak{q}, 0, 0, \mathfrak{q})$ satisfies the split Duflo property. It is clear by Proposition 2.5. ■

4.5. The $(\mathfrak{q}, \mathfrak{q})$-subreduction. In this subsection, we introduce the $(\mathfrak{q}, \mathfrak{q})$-subreduction (Theorem 4.12). This subreduction is out of the concept of Proposition 4.4. It is because this subreduction needs more precise evaluation. When we use this subreduction, we must solve the first and the second bottlenecks of Section 2.3. However, we need this subsection when we prove the $\mathfrak{h} = 0$ case. The essence of the $(\mathfrak{q}, \mathfrak{q})$-subreduction is the following proposition.

Proposition 4.11. *Suppose Setting* 4.3 *and the following three conditions.*

(1) $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *is the image of* $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *under* $\tilde{\sigma}_{\mathfrak{q}}$.

(2) $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ *is the image of* $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ *under* $\tilde{\sigma}_{\mathfrak{q}}$.

(3) *There exists a section* $s\colon Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \to \tilde{S}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'}$ *of* $\tilde{\iota}$.

If the quadruple $(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q}')$ *satisfies the split Duflo property, then the quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *also satisfies the split Duflo property.*

Proof. It is enough to show that $\tilde{\sigma}_{\mathfrak{q}}$ preserves the multiplication on $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$. Since there exists the section s, for any $\tilde{A} \in Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$, there exists the corresponding element $A \in S(\mathfrak{g}')$ to $\tilde{A}$. Then, since

$$[B, A] \in S(\mathfrak{g}') \cap (S(\mathfrak{g})\mathfrak{h}_\lambda) = S(\mathfrak{g}')\mathfrak{h}'_{\lambda'} \quad (B \in S(\mathfrak{g}')),$$

we obtain $s(\tilde{A}) \in Z(\tilde{S}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'})$.

Take $A, B \in S(\mathfrak{g}')$ such that $\tilde{A}, \tilde{B} \in Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$. The argument above implies that $s(\tilde{A}), s(\tilde{B}) \in Z(\tilde{S}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'})$. Therefore, since the quadruple $(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q}')$ satisfies the split Duflo property,

$$\sigma_{\mathfrak{q}'}(A)\sigma_{\mathfrak{q}'}(B) - \sigma_{\mathfrak{q}'}(AB) \in \mathcal{U}(\mathfrak{g}')\mathfrak{h}'_{\lambda'} \subset \mathcal{U}(\mathfrak{g})\mathfrak{h}_\lambda.$$

Hence, $\tilde{\sigma}_{\mathfrak{q}}(\tilde{A})\tilde{\sigma}_{\mathfrak{q}}(\tilde{B}) - \tilde{\sigma}_{\mathfrak{q}}(\widetilde{AB}) = 0$. ■

Now we introduce the $(\mathfrak{q}, \mathfrak{q})$-subreduction.

Theorem 4.12. *Suppose Setting* 4.3, $\mathfrak{h} \subset \mathfrak{g}'$, $\dim \mathfrak{g}/\mathfrak{g}' = 1$, *and* $[\mathfrak{h}, \mathfrak{q}] \subset \operatorname{Ker} \lambda$. *Moreover, we assume that there exists* $Y \in \mathfrak{q}$ *such that*

$$[Y, \mathfrak{g}] \not\subset \operatorname{Ker} \lambda, \ [Y, \mathfrak{g}'] \subset \operatorname{Ker} \lambda.$$

If the quadruple $(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q}')$ *satisfies the split Duflo property, then the quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *also satisfies the split Duflo property.*

Proof. Since $\operatorname{Ker} \lambda \subset \mathfrak{g}$ is an ideal, we can assume $\operatorname{Ker} \lambda = 0$ by the $\mathfrak{h}$-quotient reduction (Theorem 3.1). Then, by Proposition 2.9, $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is the image of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ and $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ is the image of $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ under $\tilde{\sigma}_{\mathfrak{q}}$. So, if we prove Proposition 4.13, we can conclude this theorem by Proposition 4.11. ■

Proposition 4.13. *Under the assumption of Theorem 4.12,* $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \subset \tilde{S}_{\lambda'}(\mathfrak{g}')$.

Proof. Take $P \in S(\mathfrak{q})$ such that $\tilde{P} \in Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$. It is enough to show that $P \in S(\mathfrak{g}')$. Take $X_0 \in \mathfrak{q} - \mathfrak{q}'$ and a basis $X_1, \ldots, X_{m-1}$ of $\mathfrak{q}'$. Since $[\mathfrak{h}, \mathfrak{q}] \subset \operatorname{Ker} \lambda$, we obtain $Y \in \tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$. So, by $\tilde{P} \in Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$,

$$S(\mathfrak{g})\mathfrak{h}_\lambda \ni [Y, P] = \sum_{i=0}^{m-1} \frac{\partial P}{\partial X_i}[Y, X_i].$$

Since $S(\mathfrak{g})\mathfrak{h}_\lambda \subset S(\mathfrak{g})$ is a prime ideal, $[Y, X_0] \notin \mathfrak{h}_\lambda$, and $[Y, X_i] \in \mathfrak{h}_\lambda$ $(i \geq 1)$, we obtain $\partial P/\partial X_0 \in S(\mathfrak{g})\mathfrak{h}_\lambda$. Therefore,

$$\frac{\partial P}{\partial X_0} \in S(\mathfrak{q}) \cap (S(\mathfrak{g})\mathfrak{h}_\lambda) = 0.$$

Hence $P \in S(\mathfrak{g}')$. ■

4.6. The special $\mathfrak{h}$-subreduction. In this subsection, we introduce the special $\mathfrak{h}$-subreduction. If the Poisson bracket of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is trivial, the second bottleneck is removed automatically if we remove the first bottleneck. This fact is a fundamental idea in Theorem 1.8, 1.10. The special $\mathfrak{h}$-subreduction is also based on this idea. The essence of the special $\mathfrak{h}$-subreduction is the following proposition.

Proposition 4.14. *Suppose Setting* 4.3 *and the following three conditions.*

(1) $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *is the image of* $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *under* $\tilde{\sigma}_{\mathfrak{q}}$.

(2) *The Poisson bracket of* $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ *is trivial.*

(3) *There exists a section* $s\colon \tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}} \to \tilde{S}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'}$ *of* $\tilde{\iota}$.

If the quadruple $(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q}')$ *satisfies the split Duflo property, then the quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *also satisfies the split Duflo property.*

Proof. If we replace $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ with $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ in the proof of Proposition 4.11, it becomes the proof of this theorem. ∎

Now we introduce the special $\mathfrak{h}$-subreduction.

Theorem 4.15. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra,* $\mathfrak{h} \subset \mathfrak{g}$ *a subalgebra,* $\lambda\colon \mathfrak{h} \to \mathbf{K}$ *a representation, and* $\mathfrak{q}$ *a linear complement of* $\mathfrak{h} \subset \mathfrak{g}$. *Suppose that*

$$[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q} \oplus C_{\mathfrak{h}}(\mathfrak{q}), \ [\mathfrak{q}, \mathfrak{q}] \subset \operatorname{Ker} \lambda.$$

Let $\mathfrak{g}' := \mathfrak{q} + [\mathfrak{h}, \mathfrak{q}] + [\mathfrak{q}, \mathfrak{q}]$, $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{g}'$, *and* $\lambda' := \lambda|_{\mathfrak{h}'}$. *If the quadruple* $(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q})$ *satisfies the split Duflo property, then the quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *also satisfies the split Duflo property.*

Proof. It is enough to show the assumption of Proposition 4.14. By Proposition 2.6 and $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q} \oplus C_{\mathfrak{h}}(\mathfrak{q})$, $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is the image of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ under $\tilde{\sigma}_{\mathfrak{q}}$. Moreover, since $[\mathfrak{q}, \mathfrak{q}] \subset \operatorname{Ker} \lambda$, $[P, Q] \in S(\mathfrak{q})\operatorname{Ker} \lambda$ for all $P, Q \in S(\mathfrak{q})$. So the Poisson bracket of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is trivial. Furthermore, since $\tilde{\iota}$ coincides with the $\mathfrak{h}'$-homomorphism defined by the natural linear isomorphism $\tilde{S}_{\lambda'}(\mathfrak{g}') \cong S(\mathfrak{q}) \cong \tilde{S}_\lambda(\mathfrak{g})$, there exists the section $s\colon \tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}} \to \tilde{S}_{\lambda'}(\mathfrak{g}')^{\mathfrak{h}'}$ of $\tilde{\iota}$. ∎

5. Application of our packaged reductions

5.1. The classical case: $\mathfrak{h} = 0$. We prove the $\mathfrak{h} = 0$ case by the reductions, which are packaged in the previous sections.

Theorem 5.1 ([15, Théorème 1]). *For a nilpotent Lie algebra* $\mathfrak{g}$, $Z(S(\mathfrak{g}))$ *is ring isomorphic to* $Z(\mathcal{U}(\mathfrak{g}))$ *under the symmetrization map* $\sigma\colon S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$.

Proof. It is enough to show that the quadruple $(\mathfrak{g}, 0, 0, \mathfrak{g})$ satisfies the split Duflo property by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 0, 1, 2$, it is clear that the quadruple $(\mathfrak{g}, 0, 0, \mathfrak{g})$ satisfies the split Duflo property. So it is enough to show the $\dim \mathfrak{g} \geq 3$ case.

At first, we assume that $\dim C(\mathfrak{g}) \geq 2$. We remark that $[0, \mathfrak{g}] = 0$ and $\dim C(\mathfrak{g}) \geq 2$. Moreover, by the induction hypothesis, for any $X \in C(\mathfrak{g})$ and $\mathfrak{g}_X := \mathfrak{g}/\langle X \rangle$, the

quadruple $(\mathfrak{g}_X, 0, 0, \mathfrak{g}_X)$ satisfies the split Duflo property. Therefore, by the $\mathfrak{q}$-quotient reduction (Theorem 3.4), the quadruple $(\mathfrak{g}, 0, 0, \mathfrak{g})$ also satisfies the split Duflo property.
Next, we assume that $\dim C(\mathfrak{g}) = 1$. By Lemma 5.2, there exists $Y \in \mathfrak{g}$ such that $\Im \mathrm{ad}(Y) = C(\mathfrak{g})$. Let $\mathfrak{g}' := \mathrm{Ker}\, \mathrm{ad}(Y)$. Now we remark that

$$\dim \mathfrak{g}/\mathfrak{g}' = \dim C(\mathfrak{g}) = 1, \quad [Y, \mathfrak{g}] \neq 0, \quad [Y, \mathfrak{g}'] = 0.$$

Moreover, by the induction hypothesis, the quadruple $(\mathfrak{g}', 0, 0, \mathfrak{g}')$ satisfies the split Duflo property. So, the $(\mathfrak{q}, \mathfrak{q})$-subreduction (Theorem 4.12) implies that the quadruple $(\mathfrak{g}, 0, 0, \mathfrak{g})$ also satisfies the split Duflo property. ■

Lemma 5.2 (Kirillov's Lemma)**.** *For a non-abelian nilpotent Lie algebra $\mathfrak{g}$ such that $\dim C(\mathfrak{g}) = 1$, there exists $X \in \mathfrak{g}$ such that $\Im \mathrm{ad}(X) = C(\mathfrak{g})$.*

Proof. We denote by $\mathfrak{g} = D_0\mathfrak{g} \supsetneq D_1\mathfrak{g} \supsetneq \cdots \supsetneq D_n\mathfrak{g} = 0$ the lower central series *i.e.*

$$D_k\mathfrak{g} = [\mathfrak{g}, D_{k-1}\mathfrak{g}] \quad (k = 1, \ldots, n).$$

Since $0 \neq D_{n-1}\mathfrak{g} \subset C(\mathfrak{g})$ and $\dim C(\mathfrak{g}) = 1$, $D_{n-1}\mathfrak{g} = C(\mathfrak{g})$.
Take $X \in D_{n-2}\mathfrak{g} - D_{n-1}\mathfrak{g}$. Since $X \notin D_{n-1}\mathfrak{g} = C(\mathfrak{g})$,

$$0 \neq \Im \mathrm{ad}(X) \subset [D_{n-2}\mathfrak{g}, \mathfrak{g}] = D_{n-1}\mathfrak{g} = C(\mathfrak{g}).$$

So, since $\dim C(\mathfrak{g}) = 1$, we obtain $\Im \mathrm{ad}(X) = C(\mathfrak{g})$. ■

Remark 5.3. In the solvable case and the semisimple case, $Z(S(\mathfrak{g}))$ is not necessarily ring isomorphic to $Z(\mathcal{U}(\mathfrak{g}))$ under σ. It means that we cannot simply generalize Conjecture 1.6 to the non-nilpotent case. Duflo showed that $Z(S(\mathfrak{g}))$ is ring isomorphic to $Z(\mathcal{U}(\mathfrak{g}))$ under $\sigma \circ D$ for a certain endomorphism $D\colon S(\mathfrak{g}) \to S(\mathfrak{g})$ in the solvable case ([16, Théorème IV.1]) and the semisimple case ([16, Théorème V.2]). We may generalize Conjecture 1.6 to the general case by referring this Duflo's result.

5.2. The symmetric case. In this subsection, we prove Theorem 1.8.

Proof of Theorem 1.8. We use induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1, 2$, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property since $\mathfrak{g}$ is abelian (Proposition 2.5). So it is enough to prove the $\dim \mathfrak{g} \geq 3$ case.
At first, we assume that $\mathfrak{a} := C(\mathfrak{g}) \cap \mathrm{Ker}\, \lambda \neq 0$. By the $\mathfrak{h}$-quotient reduction (Theorem 3.1), our problem is reduced to the matter of $\mathfrak{g}/\mathfrak{a}$ and $\mathfrak{h}/\mathfrak{a}$. So, by the induction hypothesis, the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.

Secondly, we assume that $\mathfrak{a} = 0$ and $C(\mathfrak{g}) \cap \mathfrak{h} \neq 0$. Let $\mathfrak{h}'$ be a linear complement of $C(\mathfrak{g}) \cap \mathfrak{h} \subset \mathfrak{h}$ such that $\mathrm{Ker}\, \lambda \subset \mathfrak{h}'$. Then, since $[\mathfrak{q}, \mathfrak{q}], [\mathfrak{h}, \mathfrak{h}] \subset \mathrm{Ker}\, \lambda$, $\mathfrak{g}' := \mathfrak{q} \oplus \mathfrak{h}'$ is a subalgebra of $\mathfrak{g}$. Therefore, by the $\mathfrak{h}$-subreduction (Theorem 4.9), if the quadruple $(\mathfrak{g}', \mathfrak{h}', \lambda|_{\mathfrak{h}'}, \mathfrak{q})$ satisfies the split Duflo property, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ also satisfies the split Duflo property. By the induction hypothesis, the quadruple $(\mathfrak{g}', \mathfrak{h}', \lambda|_{\mathfrak{h}'}, \mathfrak{q})$ satisfies the split Duflo property.

Thirdly, we assume that $C(\mathfrak{g})\cap\mathfrak{h}=0$ and $\dim C(\mathfrak{g})\geq 2$. By Lemma 5.4, $C(\mathfrak{g})\subset\mathfrak{q}$. By the induction hypothesis, for any $X\in C(\mathfrak{g})$, the quadruple $(\mathfrak{g}_X,\mathfrak{h}_X,\lambda_X,\mathfrak{q}_X)$ in Theorem 3.4 satisfies the split Duflo property. Moreover, by Proposition 2.6 and $[\mathfrak{h},\mathfrak{q}]\subset\mathfrak{q}$, $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is the image of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ under $\tilde{\sigma}_{\mathfrak{q}}$. Furthermore, since $[\mathfrak{q},\mathfrak{q}]\subset\operatorname{Ker}\lambda$, the Poisson bracket of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is trivial. So, by the $\mathfrak{q}$-quotient reduction (Theorem 3.4), the quadruple $(\mathfrak{g},\mathfrak{h},\lambda,\mathfrak{q})$ also satisfies the split Duflo property.

Finally, we assume that $C(\mathfrak{g})\cap\mathfrak{h}=0$ and $\dim C(\mathfrak{g})=1$. By Lemma 5.4, $C(\mathfrak{g})\subset\mathfrak{q}$. By the special $\mathfrak{h}$-subreduction (Theorem 4.15), we can assume that $\mathfrak{h}=[\mathfrak{q},\mathfrak{q}]$. By Kirillov's Lemma (Lemma 5.2), there exists $Y\in\mathfrak{g}$ such that $\Im\operatorname{ad}(Y)=C(\mathfrak{g})$. Then, since

$$[Y_{\mathfrak{q}},\mathfrak{q}]\subset\mathfrak{h}\cap\mathfrak{q}=0,\ [Y_{\mathfrak{q}},\mathfrak{h}]=[Y_{\mathfrak{q}},[\mathfrak{q},\mathfrak{q}]]=[\mathfrak{q},[Y_{\mathfrak{q}},\mathfrak{q}]]+[\mathfrak{q},[\mathfrak{q},Y_{\mathfrak{q}}]]=0,$$

we can assume that $Y\in\mathfrak{h}$. Since

$$(\Im\operatorname{ad}(Y))\cap(\operatorname{Ker}\lambda)\subset C(\mathfrak{g})\cap\mathfrak{h}=0,$$

we obtain $\mathfrak{g}':=\operatorname{ad}(Y)^{-1}(\operatorname{Ker}\lambda)=\operatorname{Ker}\operatorname{ad}(Y)$. Now $\mathfrak{g}'\subset\mathfrak{g}$ is a subalgebra, and, by the induction hypothesis, $(\mathfrak{g}',\mathfrak{h},\lambda,\mathfrak{q}\cap\mathfrak{g}')$ satisfies the split Duflo property. Moreover, by Propostion 2.6 and $[\mathfrak{h},\mathfrak{q}]\subset\mathfrak{q}$, $\tilde{\sigma}_{\mathfrak{q}}$ is $\mathfrak{h}$-equivariant. Furthermore,

$$\dim[Y,\mathfrak{g}]-\dim[Y,\mathfrak{g}]\cap\operatorname{Ker}\lambda=\dim C(\mathfrak{g})-\dim 0=1.$$

So, by the $(\mathfrak{h},\mathfrak{q})$-subreduction (Theorem 4.5), the quadruple $(\mathfrak{g},\mathfrak{h},\lambda,\mathfrak{q})$ satisfies the split Duflo property.

Lemma 5.4. *Let $(\mathfrak{g},\mathfrak{h},\sigma)$ be a symmetric pair and $\mathfrak{q}:=\mathfrak{g}^{-\sigma}$, i.e., $[\mathfrak{h},\mathfrak{q}]\subset\mathfrak{q}$ and $[\mathfrak{q},\mathfrak{q}]\subset\mathfrak{h}$. Then the center $C(\mathfrak{g})$ is compatible with the linear decomposition $\mathfrak{g}=\mathfrak{q}\oplus\mathfrak{h}$.*

Proof. Take an arbitrary $Z\in C(\mathfrak{g})$. It is enough to show that $Z_{\mathfrak{q}},Z_{\mathfrak{h}}\in C(\mathfrak{g})$. Take an arbitrary $X\in\mathfrak{q}$. Then

$$0=[X,Z]=[X,Z_{\mathfrak{q}}]+[X,Z_{\mathfrak{h}}].$$

Therefore,

$$[X,Z_{\mathfrak{q}}]=-[X,Z_{\mathfrak{h}}]\in\mathfrak{h}\cap\mathfrak{q}=0.$$

Similarly, for an arbitrary $Y\in\mathfrak{h}$,

$$[Y,Z_{\mathfrak{q}}]=-[Y,Z_{\mathfrak{h}}]\in\mathfrak{q}\cap\mathfrak{h}=0.$$

So $Z_{\mathfrak{q}},Z_{\mathfrak{h}}\in C(\mathfrak{g})$. ■

5.3. In the case where the derived algebra is 1-dimensional

In this subsection, we consider the case where the derived algebra is 1-dimensional, like the Heisenberg algebra. Although this case is included in Theorem 6.2, we prove this case here. In fact, when we prove Theorem 6.2, we will reduce the general case to this case. Moreover, we can prove this case by our packaged reductions.

Theorem 5.5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra such that $\dim[\mathfrak{g},\mathfrak{g}]\leq 1$, $\mathfrak{h}\subset\mathfrak{g}$ a subalgebra, $\lambda\colon\mathfrak{h}\to\mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h}\subset\mathfrak{g}$ such that the decomposition $\mathfrak{g}=\mathfrak{q}\oplus\mathfrak{h}$ and the linear subspace $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}$ are compatible. Then:*

(i) *$\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is generated by $\mathbf{K}\cdot 1$ and the projection of $U := \{X \in \mathfrak{q} \mid [\mathfrak{h}, X] \in \operatorname{Ker}\lambda\}$ as ring;*

(ii) *The quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. We proceed by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1, 2$, then the conclusion of this theorem holds by Remark 5.6. So it is enough to show the $\dim \mathfrak{g} \geq 3$ case.
At first, we assume that $[\mathfrak{g}, \mathfrak{g}] \subset \operatorname{Ker}\lambda$. Then, by the $\mathfrak{h}$-quotient reduction (Theorem 3.1) and Proposition 3.2, we can reduce our problem to the matter of the abelian algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Therefore, by Remark 5.6, we obtain the conclusion of this theorem.
Next, we assume that $[\mathfrak{g}, \mathfrak{g}] \not\subset \operatorname{Ker}\lambda$ and $[\mathfrak{h}, \mathfrak{q}] \neq 0$. Then, there exists $Y \in \mathfrak{h}$ such that $[Y, \mathfrak{q}] \neq 0$. Now $\mathfrak{g}' := \operatorname{ad}(Y)^{-1}(\operatorname{Ker}\lambda)$ is a subalgebra of $\mathfrak{g}$ by $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}'$ and the quadruple $(\mathfrak{g}', \mathfrak{h}, \lambda, \mathfrak{q} \cap \mathfrak{g}')$ satisfies the conclusion of this theorem by the induction hypothesis. Moreover, by Lemma 5.7, $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is the image of $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ under $\tilde{\sigma}_{\mathfrak{q}}$. Furthermore,

$$\dim[Y, \mathfrak{g}] - \dim[Y, \mathfrak{g}] \cap \operatorname{Ker}\lambda = \dim[\mathfrak{g}, \mathfrak{g}] - \dim[\mathfrak{g}, \mathfrak{g}] \cap \operatorname{Ker}\lambda = 1 - 0 = 1.$$

So, by the $(\mathfrak{h}, \mathfrak{q})$-subreduction (Theorem 4.5) and Proposition 4.6, the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the conclusion of this theorem.
Finally, we assume that $[\mathfrak{h}, \mathfrak{q}] = 0$. Then, since $\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}} = \tilde{S}_\lambda(\mathfrak{g})$ is generated by $U = \mathfrak{q}$, it is enough to show that the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. If $[\mathfrak{q}, \mathfrak{q}] = 0$, it is clear by Corollary 2.11. So we can assume that $[\mathfrak{q}, \mathfrak{q}] \neq 0$. By $[\mathfrak{q}, \mathfrak{q}] \neq 0$, there exists $Y \in \mathfrak{q}$ such that $[Y, \mathfrak{g}] \neq 0$. Let $\mathfrak{g}' := \operatorname{Ker}\operatorname{ad}(Y)$. Then $\mathfrak{g}' \subset \mathfrak{g}$ is a 1-codimensional ideal. Moreover, since $[\mathfrak{h}, \mathfrak{q}] = 0$, we obtain $\mathfrak{h} \subset \mathfrak{g}'$. Furthermore,

$$[Y, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] \not\subset \operatorname{Ker}\lambda, \quad [Y, \mathfrak{g}'] = 0.$$

So, by the $(\mathfrak{q}, \mathfrak{q})$-subreduction, if the quadruple $(\mathfrak{g}', \mathfrak{h}, \lambda, \mathfrak{q} \cap \mathfrak{g}')$ satisfies the split Duflo property, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ also satisfies the split Duflo property. Now, by the induction hypothesis, the quadruple $(\mathfrak{g}', \mathfrak{h}, \lambda, \mathfrak{q} \cap \mathfrak{g}')$ satisifies the split Duflo property. ■

Remark 5.6. If $\mathfrak{g}$ is abelian, then any linear subspaces $\mathfrak{q}, \mathfrak{h} \subset \mathfrak{g}$ are Lie subalgebras and the conclusion of Theorem 5.5 holds for any representation $\lambda \colon \mathfrak{h} \to \mathbf{K}$.

Lemma 5.7. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda \colon \mathfrak{h} \to \mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h} \subset \mathfrak{g}$. If the linear subspace $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ and the decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ are compatible, then $\tilde{\sigma}_{\mathfrak{q}} \colon \tilde{S}_\lambda(\mathfrak{g}) \to \tilde{\mathcal{U}}_\lambda(\mathfrak{g})$ is $\mathfrak{h}$-equivariant.*

Proof. This lemma follows from Proposition 2.6. ■

6. Proof of Theorem 1.7

6.1. Special Lie algebras

In this subsection, we prove the split Duflo property in the special Lie algebra case.

Theorem 6.1. *Let* $\mathfrak{g} := \mathbf{K} \ltimes \mathbf{K}^n$ *be nilpotent,* $\mathfrak{h} \subset \mathfrak{g}$ *a subalgebra,* $\lambda \colon \mathfrak{h} \to \mathbf{K}$ *a representation, and* $\mathfrak{q}$ *a linear complement of* $\mathfrak{h} \subset \mathfrak{g}$*. Suppose that the decomposition* $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ *is compatible with* $\mathfrak{g}' := 0 \ltimes \mathbf{K}^n \subset \mathfrak{g}$*. Then the quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *satisfies the split Duflo property.*

Proof. At first, we assume that $\mathfrak{h} \not\subset \mathfrak{g}'$. Then we obtain $\mathfrak{q} \subset \mathfrak{g}'$ by the compatibility of $\mathfrak{g}' \subset \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$. So,

$$[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{g}' = (\mathfrak{q} \cap \mathfrak{g}') \oplus (\mathfrak{h} \cap \mathfrak{g}') \subset \mathfrak{q} \oplus C_{\mathfrak{h}}(\mathfrak{q}), \quad [\mathfrak{q}, \mathfrak{q}] = 0.$$

Therefore, by Corollary 2.11, the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.

Next, we assume that $\mathfrak{h} \subset \mathfrak{g}'$ and $[\mathfrak{h}, \mathfrak{g}] \subset \operatorname{Ker} \lambda$. Since $\operatorname{Ker} \lambda \subset \mathfrak{g}$ is an ideal, we can reduce our problem to the matter of $\mathfrak{g}/\operatorname{Ker} \lambda$ and $\mathfrak{h}/\operatorname{Ker} \lambda$ by the $\mathfrak{h}$-quotient reduction (Theorem 3.1). So we can assume that $[\mathfrak{h}, \mathfrak{g}] = \operatorname{Ker} \lambda = 0$. By Proposition 2.5, if $\mathfrak{g}$ is abelian, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. So we can assume that $\mathfrak{g}$ is non-abelian. Then, by $[\mathfrak{h}, \mathfrak{g}] = 0$, there exist $X \in \mathfrak{q} - \mathfrak{g}'$ and $Y \in \mathfrak{q} \cap \mathfrak{g}'$ such that $[Y, X] \neq 0$. Then $[Y, \mathfrak{g}] \neq 0 = \operatorname{Ker} \lambda$ and $[Y, \mathfrak{g}'] = 0$. Therefore, by the $(\mathfrak{q}, \mathfrak{q})$-subreduction (Theorem 4.12), if the quadruple $(\mathfrak{g}', \mathfrak{h}, \lambda, \mathfrak{q} \cap \mathfrak{g}')$ satisfies the split Duflo property, then the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ also satisfies the split Duflo property. By Proposition 2.5, the quadruple $(\mathfrak{g}', \mathfrak{h}, \lambda, \mathfrak{q} \cap \mathfrak{g}')$ satisfies the split Duflo property since $\mathfrak{g}'$ is abelian.

Finally, we assume that $\mathfrak{h} \subset \mathfrak{g}'$ and $[\mathfrak{h}, \mathfrak{g}] \not\subset \operatorname{Ker} \lambda$. Let $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{g}'$. By Proposition 2.5, the quadruple $(\mathfrak{g}', \mathfrak{h}, \lambda, \mathfrak{q}')$ satisfies the split Duflo property. So, by Proposition 4.4 and Remark 4.8, it is enough to show that $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}} \subset \tilde{\mathcal{M}}_\lambda(\mathfrak{g}')$ for $\mathcal{M} = S, \mathcal{U}$.

We prove $\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}} \subset \tilde{\mathcal{M}}_\lambda(\mathfrak{g}')$ by contradiction. We assume that there exists $A \in \mathcal{M}(\mathfrak{g})$ such that $\tilde{A} \in \tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}} - \tilde{\mathcal{M}}_\lambda(\mathfrak{g}')$. Since $[\mathfrak{h}, \mathfrak{q}] \not\subset \operatorname{Ker} \lambda$ and $[\mathfrak{h}, \mathfrak{q}'] = 0$, there exists $X \in \mathfrak{q} - \mathfrak{q}'$ and $Y \in \mathfrak{h}$ such that $[Y, X] \notin \operatorname{Ker} \lambda$. We denote $\mathcal{M}(X) := \mathcal{M}(\langle X \rangle)$. Then

$$\begin{aligned}\mathcal{M}(\mathfrak{g}) &= \mathcal{M}(X)\mathcal{M}(\mathfrak{g}') = \mathcal{M}(X) \cdot (\mathcal{M}(\mathfrak{q}') \oplus (\mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda)) \\ &= (\mathcal{M}(X)\mathcal{M}(\mathfrak{q}')) \oplus (\mathcal{M}(X)\mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda) = (\mathcal{M}(X)\mathcal{M}(\mathfrak{q}')) \oplus \mathcal{M}(\mathfrak{g})\mathfrak{h}_\lambda.\end{aligned}$$

So $\mathcal{M}(X) \otimes \mathcal{M}(\mathfrak{q}') \overset{\cong}{\to} \tilde{\mathcal{M}}_\lambda(\mathfrak{g})$, $X^k \otimes B \mapsto \widetilde{X^k B}$ is a linear isomorphism. Therefore, since $\tilde{A} \notin \tilde{\mathcal{M}}_\lambda(\mathfrak{g}')$, there exist $k \geq 1$ and $A_k, \ldots, A_0 \in \mathcal{M}(\mathfrak{q}')$ such that $A_k \neq 0$ and

$$A := X^k A_k + X^{k-1} A_{k-1} + \cdots + X A_1 + A_0$$

is corresponding to $\tilde{A} \in \tilde{\mathcal{M}}_\lambda(\mathfrak{g})$. Moreover, since $\mathfrak{g}'$ is abelian, $[Y, A_k] = 0$. Hence, there exist $B_{k-2}, \ldots, B_0 \in \mathcal{M}(\mathfrak{g}')$ such that

$$[Y, A] = k X^{k-1} [Y, X] A_k + X^{k-2} B_{k-2} + \cdots + X B_1 + B_0. \tag{$*$}$$

Now, since $\tilde{A} \in \tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$, $[Y, A] \in \mathcal{M}(\mathfrak{g})\mathfrak{h}_\lambda$. Moreover, since $\mathcal{M}(\mathfrak{g})\mathfrak{h}_\lambda = \mathcal{M}(X)\mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda$ and $\mathcal{M}(X) \otimes \mathcal{M}(\mathfrak{g}') \to \mathcal{M}(\mathfrak{g})$ is injective, $\mathcal{M}(X) \otimes \mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda \to \mathcal{M}(\mathfrak{g})\mathfrak{h}_\lambda$ is bijective. Hence, by Equation $(*)$, $[Y, X] A_k \in \mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda$. Since $\mathfrak{g}'$ is abelian, we can identify $\mathcal{U}(\mathfrak{g}') \cong S(\mathfrak{g}')$. So $\mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda \subset \mathcal{M}(\mathfrak{g}')$ is a prime ideal. Therefore, since

$$[Y, X] \notin \mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda \text{ and } [Y, X] A_k \in \mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda,$$

$A_k \in \mathcal{M}(\mathfrak{g}')\mathfrak{h}_\lambda$. However, it contradicts $A_k \notin \mathcal{M}(\mathfrak{q}') - \{0\}$. ■

6.2. 2-step nilpotent Lie algebras

In this subsection, we prove the split Duflo property in the 2-step nilpotent case.

Theorem 6.2. *Let* $\mathfrak{g}$ *be a* 2*-step nilpotent Lie algebra,* $\mathfrak{h} \subset \mathfrak{g}$ *a subalgebra,* $\lambda\colon \mathfrak{h} \to \mathbf{K}$ *a representation, and* $\mathfrak{q}$ *a linear complement of* $\mathfrak{h} \subset \mathfrak{g}$*. Suppose that the decomposition* $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ *is compatible with the subspace* $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$*. Then the quadruple* $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ *satisfies the split Duflo property.*

To prove this theorem, we need some complicated preparation and discussion. So we will give a proof of this theorem later. Furthermore, since our notation will be too complicated, we will cut out some discussion as lemmas with simple notation at the end of this subsection.

In the following, let $\mathfrak{g}$ be a 2-step non-abelian nilpotent Lie algebra over $\mathbf{K}$, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda\colon \mathfrak{h} \to \mathbf{K}$ a representation, and $\mathfrak{q}$ a linear complement of $\mathfrak{h} \subset \mathfrak{g}$. We suppose that the decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ is compatible with $[\mathfrak{g}, \mathfrak{g}]$. Take a linear complement $\mathfrak{g}_0$ of $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, and we put

$$\mathfrak{q}_0 := \mathfrak{q} \cap \mathfrak{g}_0,\ \mathfrak{q}_1 := \mathfrak{q} \cap [\mathfrak{g}, \mathfrak{g}],\ \mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0,\ \mathfrak{h}_1 := \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}].$$

Let $R := S(\mathfrak{q}_1)$ and $Q = Q(R)$ be the field of quotients of R. If we regard $P \in S(\mathfrak{q}_1)$ as an element of Q, we denote P by P'. For $X \in [\mathfrak{g}, \mathfrak{g}]$, we denote $X'_\mathfrak{q} := (X_\mathfrak{q})'$.

For a linear space V over $\mathbf{K}$, we denote $V_Q := V \otimes Q$. For a linear map $f\colon V \to W$ over $\mathbf{K}$, we denote $f_Q := f \otimes \mathrm{id}_Q\colon V_Q \to W_Q$. For a linear space V over Q, we denote the symmetric algebra of V by $S^Q(V)$. For a Lie algebra $\mathfrak{l}$ over Q, we denote the universal enveloping algebra of $\mathfrak{l}$ by $\mathcal{U}^Q(\mathfrak{l})$.

By using a symbol W, we define a 2-step nilpotent Lie algebra $\mathfrak{g}^Q := (\mathfrak{g}_0)_Q \oplus QW$ over Q with the bracket determined by

$$[X, Y] = ([X, Y]'_\mathfrak{q} + \lambda([X, Y]_\mathfrak{h}))W \text{ for } X, Y \in \mathfrak{g}_0.$$

Moreover, we define a subspace $\mathfrak{q}^Q := (\mathfrak{q}_0)_Q \subset \mathfrak{g}^Q$, a subalgebra $\mathfrak{h}^Q := (\mathfrak{h}_0)_Q \oplus QW$, and a representation $\lambda^Q\colon \mathfrak{h}^Q \to Q$ by

$$\lambda^Q(W) = 1 \text{ and } \lambda^Q(X) = \lambda(X)\ \ (X \in \mathfrak{h}_0).$$

We define a surjective linear map $\alpha\colon \mathfrak{g}_Q \to \mathfrak{g}^Q$ by

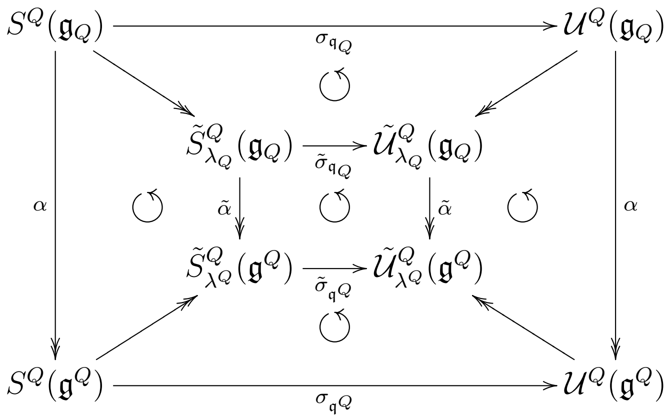
$$\alpha(X + Z) = X + (Z'_\mathfrak{q} + \lambda(Z_\mathfrak{h}))W \quad (X \in \mathfrak{g}_0, Z \in [\mathfrak{g}, \mathfrak{g}]).$$

Since, for any $X_0, X_1 \in \mathfrak{g}_0$ and $Z_0, Z_1 \in [\mathfrak{g}, \mathfrak{g}]$,

$$\alpha([X_0 + Z_0, X_1 + Z_1]) = ([X_0, X_1]'_\mathfrak{q} + \lambda([X_0, X_1]_\mathfrak{h}))W = [\alpha(X_0 + Z_0), \alpha(X_1 + Z_1)],$$

the map α is a Lie algebra homomorphism. We also denote the induced Poisson algebra homomorphisms by $\alpha\colon S^Q(\mathfrak{g}_Q) \to S^Q(\mathfrak{g}^Q)$ and $\alpha\colon \mathcal{U}^Q(\mathfrak{g}_Q) \to \mathcal{U}^Q(\mathfrak{g}^Q)$.

Proposition 6.3. *There exists the induced homomorphisms $\tilde{\alpha}$ such that all rectangles of the following diagram are commutative.*



Proof. We remark that $\alpha(\mathfrak{h}_Q) \subset \mathfrak{h}^Q$, $\lambda_Q = \lambda^Q \circ \alpha$, $\alpha^{-1}(\mathfrak{q}^Q) \subset \mathfrak{q}_Q$, and the subspace $[\mathfrak{g}_Q, \mathfrak{g}_Q] \subset \mathfrak{g}_Q$ is compatible with the decomposition $\mathfrak{g}_Q = \mathfrak{q}_Q \oplus \mathfrak{h}_Q$. So, by Lemma 6.8, there exists $\tilde{\alpha}$ such that all rectangles of the above diagram are commutative. Here, the correspondence between our situation and Lemma 6.8 is written in the following table.

Our situation	Lemma 6.8
$(\mathfrak{g}_Q, \mathfrak{h}_Q, \lambda_Q, \mathfrak{q}_Q)$	$(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$
$(\mathfrak{g}^Q, \mathfrak{h}^Q, \lambda^Q, \mathfrak{q}^Q)$	$(\mathfrak{g}', \mathfrak{h}', \lambda', \mathfrak{q}')$
α	α
$[\mathfrak{g}, \mathfrak{g}]$	$\mathfrak{a}$

■

Proposition 6.4. *The map $\tilde{\alpha}$ is injective on $\tilde{\mathcal{M}}_\lambda(\mathfrak{g}) \subset \tilde{\mathcal{M}}^Q_{\lambda_Q}(\mathfrak{g}_Q)$ for $\mathcal{M} = S, \mathcal{U}$.*

Proof. By the commutative diagram in Proposition 6.3, it is enough to prove the $\mathcal{M} = S$ case. We remark that

$$\widetilde{\alpha(X)} = \tilde{X} \in \mathfrak{q}^Q,\ \widetilde{\alpha(Z)} = Z' \in Q \text{ for } X \in \mathfrak{q}_0, Z \in \mathfrak{q}_1.$$

Therefore, the composition of $\alpha\colon S^Q(\mathfrak{g}_Q) \to S^Q(\mathfrak{g}^Q)$ and the projection $S^Q(\mathfrak{g}^Q) \to \tilde{S}^Q_{\lambda^Q}(\mathfrak{g}^Q)$ is injective on $S(\mathfrak{q})$. So, by the natural identification $S(\mathfrak{q}) \cong \tilde{S}_\lambda(\mathfrak{g})$, $\tilde{\alpha}$ is injective on $\tilde{S}_\lambda(\mathfrak{g})$. ■

Proposition 6.5. *$\tilde{\mathcal{M}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}$ is the image of $\tilde{\mathcal{M}}^Q_{\lambda_Q}(\mathfrak{g}_Q)^{\mathfrak{h}_Q}$ under $\tilde{\alpha}$ for $\mathcal{M} = S, \mathcal{U}$.*

Proof. By Lemma 5.7, $\tilde{\sigma}_{\mathfrak{q}_Q}$ and $\tilde{\sigma}_{\mathfrak{q}^Q}$ are $\mathfrak{h}$-equivariant. Therefore, by the commutative diagram in Proposition 6.3, it is enough to prove the $\mathcal{M} = S$ case. Since $\alpha\colon S^Q(\mathfrak{g}_Q) \to S^Q(\mathfrak{g}^Q)$ is a surjective Poisson algebra homomorphism we have $\tilde{\alpha}\left(\tilde{S}^Q_{\lambda_Q}(\mathfrak{g}_Q)^{\mathfrak{h}_Q}\right) \subset \tilde{S}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}$. So, it is enough to show $\tilde{S}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q} \subset \tilde{\alpha}\left(\tilde{S}^Q_{\lambda_Q}(\mathfrak{g}_Q)^{\mathfrak{h}_Q}\right)$. By Theorem 5.5 and $\dim_Q[\mathfrak{g}^Q, \mathfrak{g}^Q] = 1$, the ring $\tilde{S}_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}$ is generated by $Q \cdot 1$ and the projection of $U := \{X \in \mathfrak{q}^Q \mid [\mathfrak{h}^Q, X] = 0\}$. So, it is enough to show that the projection of U is contained in $\tilde{\alpha}\left(\tilde{S}^Q_{\lambda_Q}(\mathfrak{g}_Q)^{\mathfrak{h}_Q}\right)$.

Take any $X^Q \in U$. Then there exist $X_i \in \mathfrak{q}_0$, $P_i \in R$, $Q_i \in R-\{0\}$ $(i = 0, \dots, n-1)$ such that

$$X^Q = \sum_{i=0}^{n-1} \frac{P_i'}{Q_i'} X_i.$$

Now we put

$$Q := Q_0 ... s Q_{n-1}, \quad T_i := Q_0 ... Q_{i-1} Q_{i+1} ... Q_{n-1} \ (i = 0, ..., n-1), \quad P := \sum_{i=0}^{n-1} P_i T_i X_i.$$

Then, by Lemma 6.6, $\tilde{\alpha}\left(\frac{P}{Q'}\right) = \sum_{i=0}^{n-1} \frac{P_i' T_i'}{Q'} \tilde{X}_i = \sum_{i=0}^{n-1} \frac{P_i'}{Q_i'} \tilde{X}_i = X^Q.$

Hence, it is enough to show that $\widetilde{[Y, P]} = 0$ for any $Y \in \mathfrak{h}_0$. We remark that

$$S^Q(\mathfrak{g}^Q) \cdot (\mathfrak{h}^Q)_{\lambda^Q} \ni [Y, X^Q] = \sum_{i=0}^{n-1} \frac{P_i'}{Q_i'} \left([Y, X_i]_{\mathfrak{q}}' + \lambda([Y, X_i]_{\mathfrak{h}})\right) W.$$

So, by $\lambda^Q(W) = 1$, $\quad \sum_{i=0}^{n-1} \frac{P_i'}{Q_i'} \left([Y, X_i]_{\mathfrak{q}}' + \lambda([Y, X_i]_{\mathfrak{h}})\right) = 0.$

Therefore, $\quad \sum_{i=0}^{n-1} P_i' T_i' \left([Y, X_i]_{\mathfrak{q}}' + \lambda([Y, X_i]_{\mathfrak{h}})\right) = 0.$

Thus, $\quad \sum_{i=0}^{n-1} P_i T_i \left([Y, X_i]_{\mathfrak{q}} + \lambda([Y, X_i]_{\mathfrak{h}})\right) = 0.$

Since $P_i, T_i \in S(\mathfrak{q}_1)$, $\quad \widetilde{[Y, P]} = \sum_{i=0}^{n-1} \tilde{P}_i \tilde{T}_i \left(\widetilde{[Y, X_i]}_{\mathfrak{q}} + \lambda([Y, X_i]_{\mathfrak{h}})\right) = 0.$ ■

Lemma 6.6. *For any $P \in S(\mathfrak{q}_1)$, $\tilde{\alpha}(\tilde{P}) = P'$.*

Proof. For $X_0, \dots, X_{n-1} \in \mathfrak{q}_1$,

$$\alpha(X_0 \cdots X_{n-1}) = (X_0' W) \cdots (X_{n-1}' W) = X_0' \cdots X_{n-1}' W^n.$$

Therefore, since $\lambda^Q(W) = 1$, $\tilde{\alpha}(\tilde{X}_0 \cdots \tilde{X}_{n-1}) = X_0' \cdots X_{n-1}'$. ■

Proposition 6.7. $\tilde{\alpha}(Z(\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}})) = \tilde{\alpha}(\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \cap Z\left(\tilde{\mathcal{M}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right)$ *for $\mathcal{M} = S, \mathcal{U}$.*

Proof. By Proposition 6.4 and Lemma 6.9.1,

$$\tilde{\alpha}\left(\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}\right) \cap Z\left(\tilde{\mathcal{M}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right) \subset \tilde{\alpha}\left(Z(\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}})\right).$$

Moreover, by Proposition 6.5 and Lemma 6.9.2,

$$\tilde{\alpha}\left(Z(\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}})\right) \subset \tilde{\alpha}\left(Z\left(\tilde{\mathcal{M}}^Q_{\lambda_Q}(\mathfrak{g}_Q)^{\mathfrak{h}_Q}\right)\right) \subset Z\left(\tilde{\mathcal{M}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right).$$

So, $\tilde{\alpha}(Z(\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}})) = \tilde{\alpha}(\tilde{\mathcal{M}}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \cap Z\left(\tilde{\mathcal{M}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right)$. ■

Proof of Theorem 6.2. By Lemma 5.7, it is enough to show that $Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ is ring isomorphic to $Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ under $\tilde{\sigma}_{\mathfrak{q}}$.

By $\dim_Q[\mathfrak{g}^Q, \mathfrak{g}^Q] = 1$ and Theorem 5.5,

$$\tilde{\sigma}_{\mathfrak{q}^Q} \colon Z\left(\tilde{S}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right) \to Z\left(\tilde{\mathcal{U}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right)$$

is a linear isomorphism. So, by Proposition 6.7,

$$\tilde{\sigma}_{\mathfrak{q}^Q} \circ \tilde{\alpha}(Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}})) = \tilde{\sigma}_{\mathfrak{q}^Q}\left(\tilde{\alpha}(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \cap Z\left(\tilde{S}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right)\right)$$
$$= (\tilde{\alpha} \circ \tilde{\sigma}_{\mathfrak{q}})(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \cap Z\left(\tilde{\mathcal{U}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right) = \tilde{\alpha}(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \cap Z\left(\tilde{\mathcal{U}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right) = \tilde{\alpha}(Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})).$$

Therefore, we obtain the following commutative diagram.

$$\begin{array}{ccc} Z\left(\tilde{S}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right) & \xrightarrow[\tilde{\sigma}_{\mathfrak{q}^Q}]{\cong} & Z\left(\tilde{\mathcal{U}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right) \\ \uparrow \tilde{\alpha} & \circlearrowleft & \uparrow \tilde{\alpha} \\ Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) & \xrightarrow[\tilde{\sigma}_{\mathfrak{q}}]{\cong} & Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}). \end{array}$$

By Theorem 5.5, $\tilde{\sigma}_{\mathfrak{q}^Q} \colon Z\left(\tilde{S}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right) \to Z\left(\tilde{\mathcal{U}}^Q_{\lambda^Q}(\mathfrak{g}^Q)^{\mathfrak{h}^Q}\right)$ is a ring isomorphism. Hence, $\tilde{\sigma}_{\mathfrak{q}} \colon Z(\tilde{S}_\lambda(\mathfrak{g})^{\mathfrak{h}}) \to Z(\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}})$ is also a ring isomorphism. ■

6.2.1. Key lemmas

Lemma 6.8. *Let $\mathfrak{g}$, $\mathfrak{g}'$ be Lie algebras, $\mathfrak{h} \subset \mathfrak{g}$, $\mathfrak{h}' \subset \mathfrak{g}'$ subalgebras, $\lambda \colon \mathfrak{h} \to \mathbf{K}$, $\lambda' \colon \mathfrak{h}' \to \mathbf{K}$ representations, $\mathfrak{q}$, $\mathfrak{q}'$ linear complements of $\mathfrak{h} \subset \mathfrak{g}$, $\mathfrak{h}' \subset \mathfrak{g}'$ respectively, and $\alpha \colon \mathfrak{g} \to \mathfrak{g}'$ a Lie algebra homomorphism. We also denote by $\alpha \colon S(\mathfrak{g}) \to S(\mathfrak{g}')$ and $\alpha \colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}')$ the induced homomorphisms by α.*

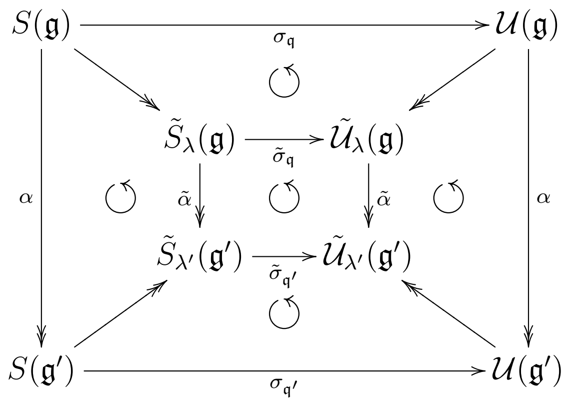
(1) *If $\alpha(\mathfrak{h}) \subset \mathfrak{h}'$ and $\lambda = \lambda' \circ \alpha$, then there exist the unique homomorphisms $\tilde{\alpha}$ such that the following diagrams are commutative.*

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\alpha} & S(\mathfrak{g}') \\ \downarrow & \circlearrowleft & \downarrow \\ \tilde{S}_\lambda(\mathfrak{g}) & \xrightarrow[\tilde{\alpha}]{} & \tilde{S}_{\lambda'}(\mathfrak{g}') \end{array} \qquad \begin{array}{ccc} \mathcal{U}(\mathfrak{g}) & \xrightarrow{\alpha} & \mathcal{U}(\mathfrak{g}') \\ \downarrow & \circlearrowleft & \downarrow \\ \tilde{\mathcal{U}}_\lambda(\mathfrak{g}) & \xrightarrow[\tilde{\alpha}]{} & \tilde{\mathcal{U}}_{\lambda'}(\mathfrak{g}') \end{array}$$

(2) *Suppose that there exists a subalgebra $\mathfrak{a} \subset C(\mathfrak{g})$ such that $\mathfrak{a}$ is compatible with the decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$, and $\alpha^{-1}(\mathfrak{h}') = \mathfrak{a} + \mathfrak{h}$. Moreover, suppose that $\alpha^{-1}(\mathfrak{q}') \subset \mathfrak{q}$. Then the following diagram is commutative.*

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\sigma_{\mathfrak{q}}} & \mathcal{U}(\mathfrak{g}) \\ \alpha \downarrow & \circlearrowleft & \downarrow \alpha \\ S(\mathfrak{g}') & \xrightarrow[\sigma_{\mathfrak{q}'}]{} & \mathcal{U}(\mathfrak{g}') \end{array}$$

(3) *Suppose that* $\alpha(\mathfrak{h}) \subset \mathfrak{h}'$, $\lambda = \lambda' \circ \alpha$, *there exists the above subalgebra* $\mathfrak{a} \subset C(\mathfrak{g})$, *and* $\alpha^{-1}(\mathfrak{q}') \subset \mathfrak{q}$. *Then the following diagram is commutative.*



Proof. (3) follows immediately from (1) and (2). If $\alpha(\mathfrak{h}) \subset \mathfrak{h}'$ and $\lambda = \lambda' \circ \alpha$, then $\alpha(\mathfrak{h}_\lambda) \subset \mathfrak{h}_{\lambda'}$, and the induced maps $\tilde{\alpha}$ in (1) are well-defined.

Now we prove (2). Let $\mathfrak{q}_0 := \alpha^{-1}(\mathfrak{q}')$ and $\mathfrak{q}_1 := \mathfrak{q} \cap \mathfrak{a}$. Then, since $\mathfrak{a}$ is compatible with the decomposition $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$,

$$\mathfrak{g} = \alpha^{-1}(\mathfrak{q}') \oplus \alpha^{-1}(\mathfrak{h}') = \mathfrak{q}_0 \oplus (\mathfrak{a} + \mathfrak{h}) = \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \mathfrak{h}.$$

So $S(\mathfrak{g})$ is generated by $S(\mathfrak{q}_0) \cdot S(\mathfrak{q}_1) \cdot S(\mathfrak{h})$. Therefore, it is enough to show that

$$\sigma_{\mathfrak{q}'} \circ \alpha(P_0P_1Q) = \alpha \circ \sigma_{\mathfrak{q}}(P_0P_1Q)$$

for any $P_0 \in S(\mathfrak{q}_0)$, $P_1 \in S(\mathfrak{q}_1)$, $Q \in S(\mathfrak{h})$. Since $\mathfrak{q}_1 \subset \mathfrak{a} \subset C(\mathfrak{g})$,

$$\sigma_{\mathfrak{q}}(P_0P_1Q) = \sigma(P_0P_1)\sigma(Q) = \sigma(P_0)\sigma(P_1)\sigma(Q) = \sigma(P_0)\sigma(P_1Q).$$

Since α is a homomorphism, $\alpha \circ \sigma = \sigma \circ \alpha$. Moreover, since $\mathfrak{q}_0 = \alpha^{-1}(\mathfrak{q}')$ and α is a homomorphism, we obtain $\alpha(P_0) \in S(\mathfrak{q}')$. Furthermore, since $\alpha^{-1}(\mathfrak{h}) = \mathfrak{q}_1 \oplus \mathfrak{h}$, we obtain $\alpha(P_1Q) \in S(\mathfrak{h}')$. Therefore,

$$\begin{aligned}\alpha \circ \sigma_{\mathfrak{q}}(P_0P_1Q) &= \alpha(\sigma(P_0)\sigma(P_1Q)) = \alpha(\sigma(P_0))\alpha(\sigma(P_1Q)) \\ &= \sigma(\alpha(P_0))\sigma(\alpha(P_1Q)) = \sigma_{\mathfrak{q}'}(\alpha(P_0)\alpha(P_1Q)) = \sigma_{\mathfrak{q}'} \circ \alpha(P_0P_1Q).\end{aligned}$$ ■

Lemma 6.9. *For Poisson algebras* A, B *and a Poisson algebra homomorphism* $f\colon A \to B$ *we have:*

(1) *If* f *is injective, then* $f(A) \cap Z(B) \subset f(Z(A))$*;*

(2) *If* f *is surjective, then* $f(Z(A)) \subset Z(B)$*.*

Proof. This lemma is proved by fundamental argument of Poisson algebras. ■

7. An example: 4 × 4 upper triangular matrices

In this section, we investigate the Lie algebra $\mathfrak{n}_4$ of all 4×4 upper triangular matrices whose diagonal components are 0, in a practical manner. Finally, we will prove Theorem 1.11.

In the following, let $\mathfrak{h} \subset \mathfrak{n}_4$ be a subalgebra and $\lambda\colon \mathfrak{h} \to \mathbf{K}$ a representation. We take a basis T, X_0, X_1, Y_0, Y_1, Z of $\mathfrak{n}_4$ as follows.

$$T := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; X_0 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; X_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_0 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; Y_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; Z := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, the Lie algebraic structure of $\mathfrak{n}_4$ is characterized by the following equations.

$$[T, X_0] = Y_0, \; [T, X_1] = Y_1, \; [X_0, Y_1] = [X_1, Y_0] = Z.$$

Especially, $[\mathfrak{n}_4, \mathfrak{n}_4] = \langle Y_0, Y_1, Z \rangle$, $C(\mathfrak{n}_4) = \langle Z \rangle$. We put $\mathfrak{m} := \langle X_0, X_1, Y_0, Y_1, Z \rangle$.
At first, we remark that $\mathfrak{n}_4$ is a 3-step nilpotent Lie algebra. So, we cannot solve even the first bottleneck in Section 2.3 by general consideration. The $\mathfrak{h}$-quotient reduction can be used even such a situation.

Lemma 7.1. *If $Z \in \operatorname{Ker} \lambda$, then there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. Let $\bar{\mathfrak{n}}_4 := \mathfrak{n}_4 / C(\mathfrak{n}_4)$, $\bar{\mathfrak{h}} := \mathfrak{h} / C(\mathfrak{n}_4)$, $\pi\colon \mathfrak{n}_4 \to \bar{\mathfrak{n}}_4$ be the quotient map, $\bar{\lambda}\colon \bar{\mathfrak{h}} \to \mathbf{K}$ the induced representation by λ. Then, $\bar{\mathfrak{n}}_4$ is 2-step nilpotent. So, by Theorem 1.7, there exists a linear complement $\bar{\mathfrak{q}}$ of $\bar{\mathfrak{h}} \subset \bar{\mathfrak{n}}_4$ such that the quadruple $(\bar{\mathfrak{n}}_4, \bar{\mathfrak{h}}, \bar{\lambda}, \bar{\mathfrak{q}})$ satisfies the split Duflo property. Let $\mathfrak{q}$ be a linear complement of $\mathfrak{h} \subset \mathfrak{n}_4$ such that $\pi(\mathfrak{q}) = \bar{\mathfrak{q}}$. Then, by the $\mathfrak{h}$-quotient reduction (Theorem 3.1), the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. ■

If $\mathfrak{h}$ is contained in the center of $\mathfrak{n}_4$, we can remove the first and the second bottlenecks by Proposition 2.9 and use all packaged reductions.

Lemma 7.2. *If $\mathfrak{h} \subset C(\mathfrak{n}_4)$, then there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. Since $\dim C(\mathfrak{n}_4) = 1$, $\mathfrak{h} = 0$ or $C(\mathfrak{n}_4)$. If $\mathfrak{h} = 0$, the conclusion follows from Proposition 2.5. So we can assume that $\mathfrak{h} = C(\mathfrak{n}_4)$. Moreover, if $\lambda = 0$, the conclusion follows from Lemma 7.1. So we can assume that $\lambda \neq 0$.
Let $\mathfrak{q} := \langle T, X_0, X_1, Y_0, Y_1 \rangle$, $\mathfrak{n}_4' := \langle T, X_0, Y_0, Y_1, Z \rangle$, $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{n}_4'$ and $Y := Y_0$. Then, since $\mathfrak{n}_4' = \langle X_0 \rangle \ltimes \langle T, Y_0, Y_1, Z \rangle$ and the decomposition $\mathfrak{n}_4' = \mathfrak{q}' \oplus \mathfrak{h}$ is compatible with the abelian ideal $\langle T, Y_0, Y_1, Z \rangle$, the quadruple $(\mathfrak{n}_4', \mathfrak{h}, \lambda, \mathfrak{q}')$ satisfies the split Duflo property by Theorem 6.1. Moreover,

$$[\mathfrak{h}, \mathfrak{q}] = 0, \; [Y, \mathfrak{n}_4] = C(\mathfrak{n}_4), \; [Y, \mathfrak{n}_4'] = 0, \; \operatorname{Ker} \lambda = 0.$$

So, by the $(\mathfrak{q}, \mathfrak{q})$-subreduction (Theorem 4.12), the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ also satisfies the split Duflo property. ■

The fundamental feature of $\mathfrak{n}_4$ is the 1-dimensional center. In general, if a n-step Lie algebra $\mathfrak{g}$ has 1-dimensional center, the $n-1$th lower central algebra $D_{n-1}\mathfrak{g}$ is the center and all $X \in D_{n-2}\mathfrak{g} - D_{n-1}\mathfrak{g}$ satisfies $[X, \mathfrak{g}] = D_{n-1}\mathfrak{g}$. So, we can often use the $(\mathfrak{h}, \mathfrak{q})$-subreduction if $\mathfrak{h} \cap (D_{n-2}\mathfrak{g} - D_{n-1}\mathfrak{g}) \neq 0$.

Lemma 7.3. *If $\mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \not\subset C(\mathfrak{n}_4)$, then there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. By Lemma 7.1, we can assume that $Z \notin \mathrm{Ker}\, \lambda$.
Take $Y \in \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] - C(\mathfrak{n}_4)$. Then, there exists the unique $a, b, c \in \mathbf{K}$ such that

$$Y = aY_0 + bY_1 + cZ, \quad (a, b) \neq (0, 0).$$

Now, $\dim[Y, \mathfrak{n}_4] - \dim[Y, \mathfrak{n}_4] \cap \mathrm{Ker}\, \lambda = \dim C(\mathfrak{n}_4) - \dim 0 = 1 - 0 = 1$.
Moreover, take a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the decomposition $\mathfrak{n}_4 = \mathfrak{q} \oplus \mathfrak{h}$ is compatible with both $C(\mathfrak{n}_4)$ and $\langle T, Y_0, Y_1, Z \rangle$. Then, by the commutative diagram in Proposition 2.3, Lemma 2.7, and $[Y, \mathfrak{n}_4] \subset C(\mathfrak{n}_4)$, $\tilde{\sigma}_{\mathfrak{q}}$ is $\mathrm{ad}(Y)$-equivariant. So, by the $(\mathfrak{h}, \mathfrak{q})$-subreduction (Theorem 4.5), the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property if and only if the quadruple $(\mathfrak{n}_4', \mathfrak{h}, \lambda, \mathfrak{q}')$ satisfies the split Duflo property. Here, we define the subalgebra $\mathfrak{n}_4'$ and its subspace $\mathfrak{q}'$ by

$$\mathfrak{n}_4' := \mathrm{ad}(Y)^{-1}(\mathrm{Ker}\, \lambda) = \langle T, aX_0 - bX_1, Y_0, Y_1, Z \rangle, \quad \mathfrak{q}' := \mathfrak{q} \cap \mathfrak{n}_4'.$$

Since the decomposition $\mathfrak{n}_4 = \mathfrak{q} \oplus \mathfrak{h}$ is compatible with $\langle T, Y_0, Y_1, Z \rangle$, the decomposition $\mathfrak{n}_4' = \mathfrak{q}' \oplus \mathfrak{h}$ is also compatible. Therefore, by Theorem 6.1 and $\mathfrak{n}_4' = \langle aX_0 - bX_1 \rangle \ltimes \langle T, Y_0, Y_1, Z \rangle$, the quadruple $(\mathfrak{n}_4', \mathfrak{h}, \lambda, \mathfrak{q}')$ satisfies the split Duflo property. It suffices to apply Theorem 4.5. ■

By the above three lemmas, we already prove the $\mathfrak{h} \subset [\mathfrak{n}_4, \mathfrak{n}_4]$ case. In this situation, there is no general method to deal with the $\mathfrak{h} \not\subset [\mathfrak{n}_4, \mathfrak{n}_4]$ case. So, in the following, we use proper features of $\mathfrak{n}_4$ explicitly. By the proper feature of $\mathfrak{n}_4$, we can use Theorem 1.10 to prove the following lemma.

Lemma 7.4. *If $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \geq 2$, then there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.*

Proof. At first, we suppose that $\mathfrak{h} \subset \mathfrak{m}$. Then, there exists $X_0', X_1' \in \mathfrak{h}$ such that

$$X_0' - X_0, X_1' - X_1 \in [\mathfrak{n}_4, \mathfrak{n}_4].$$

So, $\mathfrak{a} := \langle T, Y_0, Y_1, Z \rangle \subset \mathfrak{n}_4$ is an abelian ideal such that $\mathfrak{n}_4 = \mathfrak{h} + \mathfrak{a}$. Therefore, by Theorem 1.10, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property.
Next, we suppose that $\mathfrak{h} \not\subset \mathfrak{m}$. Then, there exists $S_0, S_1 \in \mathfrak{h}$ and $(p, q) \in \mathbf{K}^2 - \{(0, 0)\}$ such that $S_0 - T \in \mathfrak{m}$ and $S_1 - (pX_0 + qX_1) \in [\mathfrak{n}_4, \mathfrak{n}_4]$. Then,

$$[S_0, S_1] - (pY_0 + qY_1) \in C(\mathfrak{n}_4), \quad [S_1, [S_0, S_1]] = 2pqZ.$$

If $pq \neq 0$, since $Z \in [\mathfrak{h}, \mathfrak{h}] \subset \mathrm{Ker}\, \lambda$, the conclusion follows from Lemma 7.1. So we can assume that $pq = 0$. By the symmetry of the condition, we can assume that $q = 0, p \neq 0$. Then,

$$S_1 - pX_0 \in [\mathfrak{n}_4, \mathfrak{n}_4], \quad [S_0, S_1] - pY_0 \in C(\mathfrak{n}_4).$$

Therefore, $\mathfrak{a} := \langle X_1, Y_1, Z \rangle$ is an abelian ideal of $\mathfrak{n}_4$ such that $\mathfrak{n}_4 = \mathfrak{h} + \mathfrak{a}$. Hence, by Theorem 1.10, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{n}_4$ such that the quadruple $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. ■

By now we finished the argumentation in the case of $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \neq 1$. In the following, we consider the case $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] = 1$. In this case, it is difficult to check the split Duflo property. So we only check the split Corwin-Greenleaf property.

Lemma 7.5. *If* $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] = 1$*, then there exists a linear complement* $\mathfrak{q}$ *of* $\mathfrak{h} \subset \mathfrak{n}_4$ *such that the quadruple* $(\mathfrak{n}_4, \mathfrak{h}, \lambda, \mathfrak{q})$ *satisfies the split Corwin-Greenleaf property.*

Proof. By Lemmas 7.1, 7.3, we can assume that $Z \notin \mathrm{Ker}\ \lambda$ and $\mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \subset C(\mathfrak{n}_4)$. It is enough to show that $\tilde{\mathcal{U}}_\lambda(\mathfrak{n}_4)^{\mathfrak{h}}$ is non-abelian. We discuss the $\mathfrak{h} \subset \mathfrak{m}$ case and the $\mathfrak{h} \not\subset \mathfrak{m}$ case, separately.

The case $\mathfrak{h} \subset \mathfrak{m}$. By $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] = 1$ and $\mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \subset C(\mathfrak{n}_4)$, there exists $S \in \mathfrak{h} - [\mathfrak{n}_4, \mathfrak{n}_4]$, and $\mathfrak{h} = \langle S, Z \rangle$ or $\langle S \rangle$. Since $\mathfrak{h} \subset \mathfrak{m}$, there exists the unique $a, b, c, d, e \in \mathbf{K}$ such that

$$S = aX_0 + bX_1 + cY_0 + dY_1 + eZ, \quad (a, b) \neq (0, 0).$$

Now we put $A := bX_0 - aX_1 + dY_0 - cY_1$, and $B := 4abTZ + (aY_0 + bY_1)^2 \in \mathcal{U}(\mathfrak{n}_4)$.

Then, $$[S, A] = [S, B] = 0, \quad [A, B] = -2(a^2 + b^2)(aY_0 - bY_1)Z.$$

Since $[S, A] = [S, B] = 0$, we obtain $\tilde{A}, \tilde{B} \in \tilde{\mathcal{U}}_\lambda(\mathfrak{n}_4)^{\mathfrak{h}}$. On the other hand, by $aY_0 - bY_1 \notin \mathfrak{h}$, $Z \notin \mathrm{Ker}\ \lambda$, and $(a, b) \neq (0, 0)$, we obtain $[\tilde{A}, \tilde{B}] \neq 0$. So, $\tilde{\mathcal{U}}_\lambda(\mathfrak{n}_4)^{\mathfrak{h}}$ is non-abelian.

The case $\mathfrak{h} \not\subset \mathfrak{m}$. By $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] = 1$ and $\mathfrak{h} \not\subset \mathfrak{m}$, there exists $S \in \mathfrak{h}$ and $a, b, c, d, e \in \mathbf{K}$ such that

$$S = T + aX_0 + bX_1 + cY_0 + dY_1 + eZ.$$

Moreover, since $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] = 1$ and $\mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \subset C(\mathfrak{n}_4)$, $\mathfrak{h} = \langle S \rangle$ or $\langle S, Z \rangle$. At first, we assume that $(a, b) \neq (0, 0)$. We put

$$A := 2bX_0Z - Y_0^2 + 2dY_0Z,\ B := 2aX_1Z - Y_1^2 + 2cY_1Z \in \mathcal{U}(\mathfrak{n}_4).$$

Then $$[S, A] = [S, B] = 0, \quad [A, B] = 4\left(aY_0 - bY_1 - (ad - bc)Z\right) Z^2.$$

Since $[S, A] = [S, B] = 0$, we obtain $\tilde{A}, \tilde{B} \in \tilde{\mathcal{U}}_\lambda(\mathfrak{n}_4)^{\mathfrak{h}}$. On the other hand, by $(a, b) \neq (0, 0)$, $aY_0 - bY_1 \notin \mathfrak{h}$, and $Z \notin \mathrm{Ker}\ \lambda$, we obtain $[\tilde{A}, \tilde{B}] \neq 0$. So, $\tilde{\mathcal{U}}_\lambda(\mathfrak{n}_4)^{\mathfrak{h}}$ is non-abelian.

Next, we assume that $(a, b) = (0, 0)$. We put

$$C := X_0Y_1 - X_1Y_0 - (cX_0 - dX_1)Z \in \mathcal{U}(\mathfrak{n}_4).$$

Then $$[S, C] = [S, Y_0] = 0,\ [C, Y_0] = -Y_0Z + dZ^2.$$

Since $[S, C] = [S, Y_0] = 0$, we obtain $\tilde{Y}_0, \tilde{C} \in \tilde{\mathcal{U}}_\lambda(\mathfrak{n}_4)^{\mathfrak{h}}$. On the other hand, by $Y_0 \notin \mathfrak{h}$ and $Z \notin \mathrm{Ker}\ \lambda$, we obtain $[\tilde{C}, \tilde{Y}_0] \neq 0$. So, $\tilde{\mathcal{U}}_\lambda(\mathfrak{n}_4)^{\mathfrak{h}}$ is non-abelian. ■

Now we summarize the above lemmas and prove Theorem 1.11.

Proof of Theorem 1.11. By Lemma 7.2-7.4, if $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] \neq 1$, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ such that the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Duflo property. Moreover, by Lemma 7.5, even if $\dim \mathfrak{h} - \dim \mathfrak{h} \cap [\mathfrak{n}_4, \mathfrak{n}_4] = 1$, there exists a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ such that the quadruple $(\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{q})$ satisfies the split Corwin-Greenleaf property. ■

A. Historical and geometrical background of the polynomial conjecture

In this section, we review the historical and geometrical background of Duflo's polynomial conjecture (Conjecture 1.1),

A.1. Duflo's commutative conjecture and polynomial one

In August 1986, Toshio Oshima organized a conference "Analysis on homogeneous spaces" in Katata, Shiga Prefecture. In this conference, Michel Duflo suggested nine problems. The polynomial conjecture (Problem 1.1) was one of the nine problems. Although Duflo's description is purely algebraic, his problem has the following geometrical meaning. That is, if $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, then, for the Lie groups G and H corresponding to the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and the representation $\chi\colon H \to \mathbf{K}^\times$ such that $\mathrm{d}\chi = -\lambda$, $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is the algebra of the all G-invariant differential operators on the homogeneous line bundle $G \times_{(H,\chi)} \mathbf{K} \to G/H$. So, as mentioned at the beginning of this paper, Duflo's polynomial conjecture is an attempt to describe the central element of the differential operator algebra by polynomials.

The following problem is another problem of Duflo's nine problem which has been studied extensively.

Problem A.1 (Duflo's Commutative Conjecture, [18, Problem 6])**.** Let G be a connected affine algebraic group on an algebraically closed field $\mathbf{K}$ of characteristic 0 and $H \subset G$ a connected affine algebraic subgroup. We define an affine subspace $\Gamma := \{u \in \mathfrak{g}^* \mid u|_{\mathfrak{h}} = \rho - \lambda\}$. Take an orbit Ω of the coadjoint representation $G \curvearrowright \mathfrak{g}^*$ such that $\Gamma \cap \Omega \neq \emptyset$ (Ω has a natural symplectic structure.) and an irreducible component ω of $\Omega \cap \Gamma$. Then are the following conditions equivalent?

(i) The algebra $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ is commutative.

(ii) $\omega \subset \Omega$ is Lagrangian. *i.e.*, The generic dimension of $H\backslash\omega$ is 0.

Problem 1.1 is a generalization of the positive results [17, 7, 39] when $\mathfrak{g}/\mathfrak{h}$ is a symmetric pair and Problem A.1 is a generalization of the positive results [24] when $\mathfrak{g}$ and $\mathfrak{h}$ are the complexification of compact real Lie algebras. It is natural that an orbit of the coadjoint representation appears in the setting of Problem A.1 when we consider the coadjoint orbit method. We call the unitary dual $\hat{G}$ of a Lie group G the set of all isomorphic classes of the irreducible unitary representations of G.

Fact A.2 (The coadjoint orbit method, [25, 8])**.** Let G be a 1-connected exponential solvable Lie group.

(1) For any linear form $u \in \mathfrak{g}^*$, there exists a connected closed subgroup $H_u \subset G$ such that:

(a) $u([\mathfrak{h}_u, \mathfrak{h}_u]) = 0$. *i.e.*, $u|_{\mathfrak{h}_u}$ is a representation of $\mathfrak{h}_u$.

(b) We define the unitary representation $\chi_u \colon H_u \to \mathbf{C}^\times$ by $\mathrm{d}\chi_u = iu|_{\mathfrak{h}_u}$. Then the induced representation $\rho(u) := L^2\text{-}\mathrm{Ind}_{H_u}^G \chi_u$ is an irreducible unitary representation.

(2) The isomorphic class of $\rho(u)$ in 1 is independent of the choice of H_u.

(3) The map $\rho\colon \mathfrak{g}^* \to \hat{G}$ defined by 1,2 is invariant by the coadjoint representation $G \curvearrowright \mathfrak{g}^*$, and the induced map $\hat{\rho}\colon G\backslash\mathfrak{g}^* \to \hat{G}$ is bijective.

A.2. The background of the commutative conjecture

Following the conjecture of Duflo in the previous subsection and the results of Benoist, Corwin and Greenleaf proposed the following conjecture.

Conjecture A.3 ([12, Subsection 8.5, Conjecture 1.2]). Let G be a 1-connected nilpotent Lie group, $H \subset G$ a closed subgroup and $\chi\colon H \to \mathbf{C}^\times$ a unitary character. We denote by $\tau := \mathrm{Ind}_H^G \chi$ the representation of G induced from χ and by $\mathcal{D}_\tau(G/H)$ the $\mathbf{C}$-algebra of all G-invariant differential operators of the associated line bundle $G \times_{(H,\chi)} \mathbf{C} \to G/H$. Since χ is unitary, $f := -i \cdot \mathrm{d}\chi \in \mathfrak{g}^*$, and we define an affine subspace $\Gamma_\tau := \{u \in \mathfrak{g}^* \mid u|_{\mathfrak{h}} = f\}$. Furthermore, $\mathbf{C}[\Gamma_\tau]^H$ denotes the set of all fixed points of the representation of H on $\mathbf{C}[\Gamma_\tau] := \{p|_{\Gamma_\tau} \mid p \in \mathbf{C}[\mathfrak{g}^*]\}$, which is defined by the composition of the coadjoint representation. Then the following are equivalent:

(i) τ has finite multiplicities.

(ii) The algebra $\mathcal{D}_\tau(G/H)$ is commutative.

(iii) The algebra $\mathcal{D}_\tau(G/H)$ is isomorphic to the algebra $\mathbf{C}[\Gamma_\tau]^H$.

The equivalence of (i) and (ii) was already proved. In fact, (i) $\Rightarrow$ (ii) was proved in [12, Theorem 1.1], which suggests this conjecture, and (ii) $\Rightarrow$ (i) was proved in [23, Corollary 5.3]. The equivalence (ii) and (iii) is a particular case of Duflo's polynomial conjecture (Problem 1.1). Since (iii) $\Rightarrow$ (ii) is clear, the problem is (ii) $\Rightarrow$ (iii).

While the condition (ii) in Duflo's commutativity conjecture (Problem A.1) is about coadjoint orbits, the condition (i) in Conjecture A.3 is about multiplicities. The relationship between coadjoint orbits, multiplicities, and commutativity is suggested before Conjecture A.3.

For the reductive case, Kobayashi raised a general problem in the late 1980s on what is the most general framework for which one could expect a reasonable and detailed analysis of function spaces on G/H. As a solution to this problem, Kobayashi and Oshima established a finiteness criterion for multiplicities of the regular representation in the space of sections for homogeneous vector bundles, as well as a uniformly boundedness criterion, by purely geometric languages. Their results and a sketch of were announced in "the international workshop in Denmark in 1991" organized by Pedersen and in the proceedings paper of the conference. Finally, the seminal results were published in [32] as below.

Setting A.4 (The setting in [32]). Let G be a connected semisimple real Lie group with finite center, $H \subset G$ a closed subgroup with finite connected components. We denote by $G_{\mathbf{C}}$ and $H_{\mathbf{C}}$ the comlexified of G and H, respectively. Take a maximal compact Lie group $K \subset G$, a minimal parabolic subgroup $P \subset G$ and a Borel subgroup $B \subset G_{\mathbf{C}}$. We will consider the following conditions.

(HP) The action $H \curvearrowright G/P$ has an open orbit.

(HB) The action $H_{\mathbf{C}} \curvearrowright G_{\mathbf{C}}/B$ has an open orbit.

We denote by $\hat{G}_{\mathrm{ad}}$ the set of all isomorphic classes of irreducible admissible representations of G and by $\hat{H}_f$ the set of all isomorphic classes of irreducible finite dimensional representations of H. For $\pi \in \hat{G}_{\mathrm{ad}}$ and $\tau \in \hat{H}_f$, $c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau)$ denotes the multiplicity of the underlying $(\mathfrak{g}, K)$-representation π_K of π occurring in $\mathrm{Ind}_H^G \tau$.

Fact A.5 (Finite Multiplicity Theorem, [32, Theorem A])**.** Assume Setting A.4.

(1) If (HP) holds, then $c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau) < \infty$ for all $\pi \in \hat{G}_{\mathrm{ad}}$ and $\tau \in \hat{H}_f$.

(2) Suppose that G and H are defined algebraically over $\mathbf{R}$. If (HP) does not hold, then for any algebraic $\tau \in \hat{H}_f$, there exists $\pi \in \hat{G}_{\mathrm{ad}}$ such that $c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau) = \infty$.

Fact A.6 (Uniformly Bounded Multiplicity Theorem, [32, Theorem B])**.** Assume Setting A.4.

(1) If (HB) holds, $c_{\mathfrak{g},K}$ is uniformly bounded; i.e.

$$\sup_{\tau \in \hat{H}_f} \sup_{\pi \in \hat{G}_{\mathrm{ad}}} \frac{1}{\dim \tau} c_{\mathfrak{g},K}(\pi, \mathrm{Ind}_H^G \tau) < \infty$$

(2) If G and H are defined algebraically over $\mathbf{R}$, (HB) and the uniform boundedness of $c_{\mathfrak{g},K}$ are equivalent.

Fact A.7 ([32, Remark 1.3])**.** (HB) and the commutativity of the algebra of all G-invariant differential operators on G/H are equivalent.

In the nilpotent and solvable cases, the irreducible decomposition by the coadjoint orbit method was already known.

Fact A.8 (Irreducible Decomposition, [14, 20])**.** Let G be a 1-connected exponential Lie group, $H \subset G$ a connected closed subgroup and $\chi\colon H \to \mathbf{C}^\times$ a unitary character. We denote $\tau := L^2\text{-}\mathrm{Ind}_H^G \chi$, $\lambda := -i \cdot \mathrm{d}\chi \in \mathfrak{h}^*$ and $\Gamma_\tau := \{u \in \mathfrak{g}^* \mid u|_{\mathfrak{h}} = \lambda\}$. We define $m\colon \hat{G} \to \mathbf{N}$ as the following, namely for any coadjoint orbit $\Omega \in G\backslash\mathfrak{g}^*$, $m(\hat{\rho}(\Omega))$ is the number of the H-orbits contained in $\Omega \cap \Gamma_\tau$ where $\hat{\rho}$ is the bijection defined in Fact A.2. Then there exists a "good" measure on $\hat{G}$ and τ is decomposed into the following direct sum:

$$\tau \cong \int_{\hat{G}}^{\oplus} m(\pi)\pi \, \mathrm{d}\mu(\pi)$$

Here, $k\pi$ is the k times direct sum of the G-representation π.

Furthermore, Corwin and Greenleaf proposed the following theorem 4 years before Conjecture A.3.

Fact A.9 ([10, Theorem 1.2])**.** Assume the setting in Conjecture A.3. Then the following are equivalent.

(i) τ has finite multiplicities.

(ii) For almost all $u \in \Gamma_\tau$, $2\dim(\mathfrak{g}(u) + \mathfrak{h}) = \dim \mathfrak{g} + \dim \mathfrak{g}(u)$, where $\mathfrak{g}(u) := \{X \in \mathfrak{g} \mid u([X, Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}$.

Condition (ii) of this fact is corresponding to the condition (ii) of Duflo's commutative conjecture (Problem A.1). In fact, this linearization plays an essential role in the proofs of both (i) $\Rightarrow$ (ii) and (i) $\Leftarrow$ (ii) in Conjecture A.3.

A.3. Background and previous research on the polynomial conjecture

As described above, the polynomial conjecture is more detailed than the commutative conjecture and still unsolved. However, there is a candidate for the isomorphism from $\mathcal{D}_\tau(G/H)$ to $\mathbf{C}[\Gamma_\tau]^H$. In fact, the prototype of this candidate was already given in Benoist's paper [7]. After this paper, Hidenori Fujiwara constructed the map from $\mathcal{D}_\tau(G/H)$ to the field of meromorphic functions by reference to the Abstract Plancherel Formula [38] proposed by Richard Penney. The problem is whether the image of this candidate map is polynomials or not. In this paragraph, we review this candidate map.

In the following part of this paragraph, we assume the setting in Conjecture A.3. For $u \in \mathfrak{g}^*$, $\pi_u := \rho(u)$ where ρ is in the coadjoint orbit method (Fact A.2). The basic idea of Benoist is to make the following mapping p_A by diagonalizing $\mathrm{d}\pi(A)$ for $A \in \mathcal{U}(\mathfrak{g}_\mathbf{C})$ realizing the element of $\mathcal{D}_\tau(G/H)$.

$$\text{“}p_A\colon \Gamma_\tau \to \mathbf{C},\ u \mapsto \text{(the eigenvalue of } \mathrm{d}\pi_u(A))\text{”}$$

Now we review more precisely. We denote by V_π^∞ the set of all C^∞-vectors of the representation space V_π of a unitary representation π, by $V_\pi^{-\infty}$ the set of all anti-linear forms of V_π^∞ and π^* the dual representation of π. For a unitary character $\kappa\colon H \to \mathbf{C}^\times$, $(V_\pi^{-\infty})^{H,\kappa} := \{a \in V_\pi^{-\infty} \mid \pi^*(h)a = \kappa(h)a \text{ for all } h \in H.\}$ Under this notation, the following fact holds for the induced representation $\tau := L^2\text{-}\mathrm{Ind}_H^G\chi$.

Fact A.10 ([38, 19])**.** Under the irreducible decomposition of τ in Fact A.8 we have

$$m(\pi) = \dim(V_\pi^{-\infty})^{H,\chi}, \ \text{ for } \mu\text{-a.e. } \pi \in \hat{G}.$$

In fact, the inequality $m(\pi) \le \dim(V_\pi^{-\infty})^{H,\chi}$ is shown in [38] and the following theorem is shown in [19].

Fact A.11 ([19])**.** Suppose that τ has finite multiplicities. Take a linear form $u \in \mathfrak{g}^*$ and a connected closed subgroup $H_u \subset G$ in Fact A.2.1. We regard $\pi := \rho(u)$ as the induced representation from H_u. For the coadjoint orbit Ω of u, we denote by $C_1, \ldots, C_{m(\pi)}$ the H-orbits contained in $\Omega \cap \Gamma_\tau$ and take $g_1, \ldots, g_k \in G$ such that $g_k u \in C_k$ $(k = 1, \ldots, m(\pi))$. Then $a_\pi^1, \ldots, a_\pi^{m(\pi)} \in (V_\pi^{-\infty})^{H,\chi}$ defined as the following are linearly independent.

$$\langle a_\pi^k, \varphi\rangle = \int_{H/H\cap g_k H_u g_k^{-1}} \overline{\varphi(hg_k)\chi(h)}\,\mathrm{d}[h] \quad (\varphi \in V_\pi^\infty,\ k = 1, \ldots, m(\pi))$$

Here, the measure of $H/H \cap g_k H_u g_k^{-1}$ is the H-invariant measure. Moreover, these $a_\pi^1, \ldots, a_\pi^{m(\pi)}$ depend on the irreducible representation π only up to a scalar multiple.

Furthermore, the following is known.

Fact A.12 ([21, Théorème 1])**.** Suppose that τ has finite multiplicities and we use the notations in Fact A.11. Then, for any $A \in \mathcal{U}(\mathfrak{g}_\mathbf{C})$ realizing an element of $\mathcal{D}_\tau(G/H)$, there exist constants $\lambda_\pi^1(A), \ldots, \lambda_\pi^{m(\pi)}(A) \in \mathbf{C}$ such that

$$\mathrm{d}\pi^*(A)a_\pi^k = \lambda_\pi^k(A) \cdot a_\pi^k \quad (k = 1, \ldots, m(\pi)).$$

By this fact, we can define the above map p_A by $p_A(u) := \lambda^k_{\rho(u)}(A)$ for $u \in C_k$. By the definition of a^k_π, we can immediately show that p_A is H-invariant. Moreover, we get the following fact from observations in [12]. Here, $\mathcal{U}(\mathfrak{g}_{\mathbf{C}}, \tau)$ denotes the set of all elements of $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ realizing the elements of $\mathcal{D}_\tau(G/H)$.

Fact A.13 ([21, 12])**.** (1) For $A \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}}, \tau)$, p_A is a meromorphic function on Γ_τ.
(2) The set $\{p_A \mid A \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}}, \tau)\}$ generates the field $\mathbf{C}(\Gamma_\tau)^H$ of all H-invariant meromorphic functions on Γ_τ.

So the problem is whether p_A is a polynomial or not. In [7], Benoist showed the polynomial conjecture in the symmetric case. In the paper, he used the inequality $m(\pi) \leq 1$, which holds in the symmetric case. Corwin and Greenleaf showed the polynomial conjecture with the assumption that we can take one polarization of $u \in \Gamma_\tau$ independent of the choice of u.

A.4. Supplements

Generalization of Conjecture A.3. Conjecture A.3 is about the nilpotent case. In fact, there is a counterexample of the polynomial conjecture in the completely solvable case.

Fact A.14 ([5])**.** Let $\mathfrak{g} := \langle T, X, Y, Z\rangle_{\mathbf{R}}$ be a Lie algebra with the bracket $[T, X] = X$, $[T, Y] = Y$, $[T, Z] = 2Z$, $[X, Y] = Z$ ($\mathfrak{g}$ is completely solvable) and $\mathfrak{h} := \mathbf{R}T$. G and H denote the 1-connected Lie group corresponding to $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let χ be the trivial representation of H and $\tau := \mathrm{Ind}_H^G \chi$ ($\Gamma_\tau = \mathfrak{h}^\perp$). Then the following conditions hold:
(1) τ has infinite multiplicities of continuous type.
(2) $\mathcal{D}_\tau(G/H) = \mathbf{C} \cdot 1$
(3) $\mathbf{C}[\Gamma_\tau]^H = \mathbf{C}[\mathfrak{h}^\perp] = \mathbf{C}$

v́spm1 Moreover, a^k_π in Fact A.11 is defined in the nilpotent case and its generalization to the solvable case is still not justified.

The dual conjecture for restriction cases. There are many dualities between induced representations and restrictions of representations (for example, the Frobenius reciprocity). For Duflo's conjecture about induced representations, formalization and proof of dual problems about restrictions have also been carried out. We will leave the main line so we will only review a brief summary here.
In the paper [32] and [29], which studies the semisimple case, Kobayashi and Oshima formulated and proved the dual theorems about restrictions corresponding to their results about induced representations.
On the other hand,in the nilpotent case, Corwin and Greenleaf proposed the following problem in the paper, in which they suggest Conjecture A.3.

Problem A.15 ([12, Subsection 8.7])**.** Formulate the dual proposition about restrictions corresponding to Conjecture A.3.

This problem is studied by Ali Baklouti, Hidenori Fujiwara, Jean Ludwig and so on. In fact, the commutative conjecture in the restriction case is already formulated and proved in [2]. Furthermore, the polynomial conjecture in the restriction case is already formulated and proved in some individual cases, including the cases that we restrict to normal subgroups or $\mathfrak{g}$ is 3-step nilpotent or lower, see [4].

B. Derivation of split symmetrization map by the F-method

In this section, we see the derivation of the split symmetrization map by the F-method.

B.1. Overview of the F-method in the original case. Before we derive the split symmetrization map, we review the F-method in the original case. The F-method was invented by Toshiyuki Kobayashi in Spring 2010. This novel method was used to determine symmetry breaking operators, which are differential operators equivariant with respect to group actions. The prototype of the F-method is the fundamental differential operators on the isotropic cone in connection with the minimal representations, see [31]. In the semisimple case, the branching law [26] of generalized Verma modules is the algebraic counterparts of the F-method. The original F-method was formulated for a complex reductive Lie group and its parabolic subgroup and this formulation is written in [34]. In this framework, there are applications to the conformal geometry [33, 30], and the complex geometry [35]. Furthermore, T.Kobayashi applied the F-method to find more generalized operators such as non-local operators in [28] (integral operators, for example). In the following part of this subsection, we review the formulation of the F-method in [34].

Symmetry breaking operator. The F-method is to decide symmetry breaking operator.

Definition B.1 (Differential operator and symmetry breaking operator, [34, 33])**.**

(1) Let $\pi\colon \mathcal{V} \to X$ and $\varpi\colon \mathcal{W} \to Y$ be C^∞-vector bundles of manifolds and $p\colon Y \to X$ a smooth map. $\Gamma(\mathcal{V})$ and $\Gamma(\mathcal{W})$ denote the set of all smooth sections of π and ϖ, respectively. Then a linear map $T\colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ is called a *differential operator* if T satisfies the following:

$$p(\operatorname{supp} Ts) \subset \operatorname{supp} s \quad (s \in \Gamma(\mathcal{V}))$$

Here, $\operatorname{supp} s$ denotes the support of $s \in \Gamma(\mathcal{V})$.

(2) Suppose that two Lie groups $G' \subset G$ act on vector bundles ϖ and π equivariantly, respectively, and that p is G'-equivariant by these actions. Then a differential operator $T\colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ is called a *symmetry breaking operator* if T is G'-equivariant.

$$\begin{array}{ccc} \mathcal{W} & & \mathcal{V} \\ {\scriptstyle G'\curvearrowright\varpi}\big\downarrow & & {\scriptstyle G\curvearrowright\pi}\big\downarrow \\ Y & \xrightarrow{\ p\ } & X \end{array}$$

Remark B.2 ([37])**.** The definition of differential operators in Definition B.1 is equivalent to the standard definition of differential operators if $\pi = \varpi$.

The original F-method. The "F" of "F-method" is the initial letter of "Fourier transform." The basic idea of the F-method is that we describe the differential operators by polynomials using the Fourier transform. In this paragraph, we consider in the following setting.

Setting B.3. Let G be a Lie group, $P \subset G$ a closed subgroup, $G' \subset G$ a Lie subgroup and $H' \subset H$ a closed subgroup such that $H' \subset G'$. We define $X := G/H$

and $Y := G'/H'$. Let V and W be finite dimensional H- and H'-modules, respectively. We define $\mathcal{V} := G \times_H V$ and $\mathcal{W} := G' \times_{H'} W$ and denote by $\mathrm{Diff}_{G'}(\mathcal{V}, \mathcal{W})$ the set of all symmetry breaking operators from $\Gamma(\mathcal{V})$ to $\Gamma(\mathcal{W})$. We define a G-module $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V^* := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V^*$.

The original F-method was formulated in the case of a complex reductive Lie group and its parabolic subgroup.

Fact B.4 ([34, Theorem 4.1])**.** Assume Setting B.3. Suppose that G is a connected complex reductive Lie group and $H = P \subset G$ is a parabolic subgroup. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}_+$ be the Gelfand-Naimark decomposition with respect to $\mathfrak{p}$. Suppose that $H' = P'$ satisfies $P' = L' \exp(\mathfrak{n}'_+)$ where $\mathfrak{l}' := \mathfrak{l} \cap \mathfrak{p}'$ and $\mathfrak{n}'_+ := \mathfrak{n}_+ \cap \mathfrak{p}'$. Let μ be the inner tensor product of the dual representation of V and $\det(\mathrm{Ad}_P \colon \mathfrak{n}_+ \to \mathfrak{n}_+)$, and π_μ the holomorphic induced representation $P \uparrow G$ of μ. Then we have:

(1) There are the following natural linear isomorphisms:

$$\mathrm{Diff}_{G'}(\mathcal{V}, \mathcal{W}) \cong (\mathbf{C}[\mathfrak{n}_+] \otimes \mathrm{Lin}(V, W))^{L', \widehat{d\pi_\mu}(\mathfrak{n}'_+)} \cong \mathrm{Hom}_{L'}(V \otimes \mathbf{C}[\mathfrak{n}_+], W)^{\widehat{d\pi_\mu}(\mathfrak{n}'_+)}$$

where Lin denotes the set of all linear maps and $\mathrm{Hom}_{L'}$ the set of all L'-homomorphisms.

(2) If $\mathfrak{n}_+$ is commutative, then the following diagram is commutative.

$$\begin{array}{ccccc}
\mathrm{Lin}(W^*, \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^*)) & \xrightarrow{F_C \otimes \mathrm{id}} & \mathbf{C}[\mathfrak{n}_+] \otimes \mathrm{Lin}(V, W) & \xleftarrow{\mathrm{Symb} \otimes \mathrm{id}} & \mathrm{Diff}_0(\mathfrak{n}_-) \otimes \mathrm{Lin}(V, W) \\
\text{inclusion} \uparrow & & \circlearrowleft & & \uparrow \text{inclusion} \\
\mathrm{Hom}_{P'}(W^*, \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^*)) & & \xrightarrow{\quad D_{X \to Y} \quad} & & \mathrm{Diff}_{G'}(\mathcal{V}, \mathcal{W})
\end{array}$$

where $\mathrm{Diff}_0(\mathfrak{n}_+)$ is the set of all differential operators on $\mathfrak{n}_+$ with constant coefficients.

The Fourier transform is the map $\mathrm{Symb} \colon \mathrm{Diff}_0(\mathfrak{n}_-) \to \mathbf{C}[\mathfrak{n}_+]$. This map is defined by the canonical pairing of $\mathfrak{n}_+$ and $\mathfrak{n}_-$. In the following part of this subsection, we review the symbols $D_{X \to Y}$, Symb, $\widehat{d\pi_\mu}$ and F_C in Fact B.4.

Fact B.5 (Duality Theorem, [34, Theorem 2.9])**.** Under Setting B.3, there exists the unique linear isomorphism

$$D_{X \to Y} \colon \mathrm{Hom}_{H'}(W^*, \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^*)) \xrightarrow{\cong} \mathrm{Diff}_{G'}(\mathcal{V}, \mathcal{W})$$

such that for any $\varphi \in \mathrm{Hom}_{H'}(W^*, \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V^*)$, $F \in C^\infty(G, V)^H$ and $w^* \in W^*$,

$$\langle D_{X \to Y}(\varphi) F, w^* \rangle = \sum_j \langle A_j F, v_j^* \rangle,$$

where $\varphi(w^*) = \sum_j A_j \otimes v_j^* \in \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V^*$ $(A_j \in \mathcal{U}(\mathfrak{g}), v_j^* \in V^*)$.

Definition B.6 (The Weyl algebra)**.** $\mathcal{D}(E)$ denotes the algebra of all the differential operators with polynomial coefficients on a n-dimensional linear space E. This algebra $\mathcal{D}(E)$ is called *the Weyl algebra* on E.

Definition B.7 (The algebraic Fourier transform Symb, [34])**.** *The algebraic Fourier transform* Symb: $\mathcal{D}(E) \to \mathcal{D}(E^*)$ is defined as the unique ring isomorphism such that the following equation holds for a coordinate $(z_1, \dots, z_n)$ on E and its dual coordinate $(\zeta_1, \dots, \zeta_n)$:

$$\mathrm{Symb}\,(z_k) = \frac{\partial}{\partial \zeta_k}, \quad \mathrm{Symb}\left(\frac{\partial}{\partial z_k}\right) = -\zeta_k.$$

Notation B.8. In Fact B.4, the representation space of π_μ is the sections $\Gamma(\mathcal{V}^* \otimes \Omega_X)$ where $\mathcal{V}^* := G \times_P V^*$ and Ω_X is the canonical bundle on $X = G/P$. So we can regard the differential representation $\mathrm{d}\pi_\mu$ of π_μ is the map $\mathrm{d}\pi_\mu \colon \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V^*)$ by identifying $\mathfrak{n}_-$ as the tangent space at $P \in G/P$. By using the canonical pairing of $\mathfrak{n}_-$ and $\mathfrak{n}_+$, we can define the algebraic Fourier transform Symb: $\mathcal{D}(\mathfrak{n}_-) \to \mathcal{D}(\mathfrak{n}_+)$. Now we define the following representation by using this map Symb.

$$\widehat{\mathrm{d}\pi_\mu} := (\mathrm{Symb} \otimes \mathrm{id}) \circ \mathrm{d}\pi_\mu \colon \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V^*)$$

Definition B.9 ([34, Corollary 3.12])**.** The $(\mathfrak{g}, P)$-isomorphism $F_C \colon \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^*) \xrightarrow{\cong} \mathbf{C}[\mathfrak{n}_+] \otimes V^*, (u \otimes v^*) \mapsto \widehat{\mathrm{d}\pi_\mu}(u)(1 \otimes v^*)$ is called the algebraic Fourier transform on the generalized Verma module $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V^*$.

B.2. The F-method on a single homogeneous bundle In this subsection, we apply the F-method to a single homogeneous bundle and derive the symmetrization map by the F-method. Let G be a real Lie group, $H \subset G$ a connected closed subgroup and $\chi \colon H \to \mathbf{R}$ a representation. If we want to consider the case that χ is a unitary representation, then we can use the same discussion by complexification. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebra of G and H, respectively. Moreover, we put $\lambda := -\mathrm{d}\chi \colon \mathfrak{h} \to \mathbf{R}$. From a geometrical viewpoint, $\mathcal{U}(\mathfrak{g})$ and $S(\mathfrak{g})$ are interpreted as follows:

The $\mathbf{R}$-algebra $\mathcal{U}(\mathfrak{g})$

- The algebra of all left G-invariant differential operators on G. Its multiplication is composition.
- The algebra of all differential operators at the unit of G. Its multiplication is convolution.

The $\mathbf{R}$-algebra $S(\mathfrak{g})$

- The algebra of all differential operators at the origin of $\mathfrak{g}$. Its multiplication is convolution.
- The polynomial algebra $\mathbf{R}[\mathfrak{g}^*]$ on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

The "F" of the F-method means the Fourier transform. In fact, the equivalence of the two interpretations of $S(\mathfrak{g})$ is a conclusion of the Fourier transform. The essence of the F-method is technical exchange of the above interpretations.

If we regard $S(\mathfrak{g})$ as the polynomial algebra $\mathbf{R}[\mathfrak{g}^*]$, then $\tilde{S}_\lambda(\mathfrak{g})$ is a polynomial algebra $\mathbf{R}[\Gamma]$ on the affine subspace $\Gamma := \{u \in \mathfrak{g}^* \mid u|_{\mathfrak{h}} = \lambda\}$.

If we regard $\mathcal{U}(\mathfrak{g})$ as the algebra of all left G-invariant differential operators on G, then we can regard $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}}$ as the left G-invariant differential operators on the homogeneous line bundle $G \times_{(H,\chi)} \mathbf{R} \to G/H$. Now we see this interpretation in the strict sense. Let $\mathcal{E}(G)$ be the space of all C^∞-functions on G. Then, the space of all sections of the homogeneous line bundle $G \times_{(H,\chi)} \mathbf{R} \to G/H$ can be identified as

$$\mathcal{E}_\chi(G/H) := \{\varphi \in \mathcal{E}(G) \mid \varphi(gh) = \chi(h)^{-1} \cdot \varphi(g) \quad (g \in G, h \in H)\}.$$

If we regard $\mathcal{U}(\mathfrak{g})$ as the algebra of all left G-invariant differential operators on G,

$$\tilde{\mathcal{U}}_\lambda(\mathfrak{g}) = \{A|_{\mathcal{E}_\chi(G/H)} \mid A \in \mathcal{U}(\mathfrak{g})\},$$
$$\tilde{\mathcal{U}}_\lambda(\mathfrak{g})^{\mathfrak{h}} = \{A|_{\mathcal{E}_\chi(G/H)} \mid A \in \mathcal{U}(\mathfrak{g}),\ A(\mathcal{E}_\chi(G/H)) \subset \mathcal{E}_\chi(G/H)\}.$$

Moreover, if we regard $\mathcal{U}(\mathfrak{g})$ as the algebra of all differential operators at the unit of G and $S(\mathfrak{g})$ as the algebra of all differential operators at the origin of $\mathfrak{g}$, we obtain a linear isomorphism from $S(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g})$ by the local diffeomorphism from the neighborhood of the origin of $\mathfrak{g}$ to the one of the unit of G. For example, the exponential map $\exp\colon \mathfrak{g} \to G$ is such a local diffeomorphism. However, the differentiation $\mathrm{d}\exp$ does not induce the map from $\tilde{S}_\lambda(\mathfrak{g})$ to $\tilde{\mathcal{U}}_\lambda(\mathfrak{g})$. Now, we take a linear complement $\mathfrak{q}$ of $\mathfrak{h} \subset \mathfrak{g}$ and define the local diffeomorphism $\alpha_\mathfrak{q}\colon \mathfrak{g} \to G$ by

$$\alpha_\mathfrak{q}(X+Y) = \exp(X) \cdot \exp(Y)\ (X \in \mathfrak{q}, Y \in \mathfrak{h}).$$

In fact, we can derivate the symmetrization map $\sigma_\mathfrak{q}$ by the map $\alpha_\mathfrak{q}$.

Proposition B.10. *The covariant differentiation* $\mathrm{d}\alpha_\mathfrak{q}\colon S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ *of* $\alpha_\mathfrak{q}$ *is the split symmetrization map* $\sigma_\mathfrak{q}$.

Proof. If we regard $S(\mathfrak{g})$ as the algebra of all differentiations at the origin of $\mathfrak{g}$, for $X_0, \ldots, X_{n-1} \in \mathfrak{g}$,

$$(X_0 \cdots X_{n-1})(\varphi) := \left.\frac{\partial}{\partial t_0}\right|_{t_0=0} \cdots \left.\frac{\partial}{\partial t_{n-1}}\right|_{t_{n-1}=0} \varphi(t_0X_0 + \cdots + t_{n-1}X_{n-1}) \quad (\varphi \in \mathcal{E}(\mathfrak{g})).$$

Similarly, if we regard $\mathcal{U}(\mathfrak{g})$ as the algebra of all differentiation at the unit of $\mathfrak{g}$, for $X_0, \ldots, X_{n-1} \in \mathfrak{g}$ and $\varphi \in \mathcal{E}(G)$,

$$(X_0 \cdots X_{n-1})(\varphi) := \left.\frac{\partial}{\partial t_0}\right|_{t_0=0} \cdots \left.\frac{\partial}{\partial t_{n-1}}\right|_{t_{n-1}=0} \varphi(\exp(t_0X_0) \cdots \exp(t_{n-1}X_{n-1})).$$

So, for $X_0, \ldots, X_{m-1} \in \mathfrak{q}$, $Y_0, \ldots, Y_{n-1} \in \mathfrak{h}$, and $\varphi \in \mathcal{E}(G)$,

$$\begin{aligned} &\mathrm{d}\alpha_\mathfrak{q}(X_0 \cdots X_{m-1} Y_0 \cdots Y_{n-1})(\varphi) = (X_0 \cdots X_{m-1} Y_0 \cdots Y_{n-1})(\varphi \circ \alpha_\mathfrak{q}) \\ &= \prod_{i=0}^{m-1} \left.\frac{\partial}{\partial s_i}\right|_{s_i=0} \prod_{j=0}^{n-1} \left.\frac{\partial}{\partial t_j}\right|_{t_j=0} \varphi \circ \alpha_\mathfrak{q} \left(\sum_{i=0}^{m-1} s_i X_i + \sum_{j=0}^{n-1} t_j Y_j \right) \\ &= \prod_{i=0}^{m-1} \left.\frac{\partial}{\partial s_i}\right|_{s_i=0} \prod_{j=0}^{n-1} \left.\frac{\partial}{\partial t_j}\right|_{t_j=0} \varphi \left(\exp\left(\sum_{i=0}^{m-1} s_i X_i \right) \exp\left(\sum_{j=0}^{n-1} t_j Y_j \right) \right). \end{aligned} \qquad (*)$$

By Fact B.11,

$$(*) = \sigma(X_0 \cdots X_{m-1})\sigma(Y_0 \cdots Y_{n-1})\varphi = \sigma_\mathfrak{q}(X_0 \cdots X_{m-1} Y_0 \cdots Y_{n-1})\varphi.$$

So, we obtain $\mathrm{d}\alpha_\mathfrak{q} = \sigma_\mathfrak{q}\colon S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$. ■

Fact B.11 ([11, Lemma 3.3.1])**.** The covariant differentiation $\mathrm{d}\exp\colon S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ of $\exp$ coincides with the symmetrization map σ.

Acknowledgement. The author appreciates the referees' precise, attentive and rapid reviews.

Funding. This work is supported by the Program for Leading Graduate Schools, MEXT, Japan.

References

[1] L. Auslander, B. Kostant: *Polarization and unitary representations of solvable Lie groups*, Invent. Math. 14 (1971) 255–354.

[2] A. Baklouti, H. Fujiwara: *Commutativité des opérateurs différentiels sur l'espace des représentations restreintes d'un groupe de Lie nilpotent*, J. Math. Pures Appl. (9) 83(1) (2004) 137–161.

[3] A. Baklouti, H. Fujiwara, J. Ludwig: *Analysis of restrictions of unitary representations of a nilpotent Lie group*, Bull. Sci. Math. 129(3) (2005) 187–209.

[4] A. Baklouti, H. Fujiwara, J. Ludwig: *The polynomial conjecture for restrictions of some nilpotent Lie groups representations*, J. Lie Theory 29(2) (2019) 311–341.

[5] A. Baklouti, H. Fujiwara, J. Ludwig: *Monomial representations of discrete type of an exponential solvable Lie group*, Proceedings in Mathematics and Statistics, Springer, to appear.

[6] A. Baklouti, J. Ludwig: *Invariant differential operators on certain nilpotent homogeneous spaces*, Monatshefte Math 134(1) (2001) 19–37.

[7] Y. Benoist: *Analyse harmonique sur les espaces symétriques nilpotents*, J. Funct. Anal. 59(2) (1984) 211–253.

[8] P. Bernat, N. Conze, M. Duflo, M. Lévy-Nahas, M. Raïs, P. Renouard, M. Vergne: *Représentations des Groupes de Lie Résolubles*, Monographies de la Société Mathématique de France 4, Dunod, Paris (1972).

[9] P. Bonnet: *Transformation de Fourier des distributions de type positif sur un groupe de Lie unimodulaire*, J. Funct. Anal. 55(2) (1984) 220–246.

[10] L. Corwin, F. P. Greenleaf: *A canonical approach to multiplicity formulas for induced and restricted representations of nilpotent Lie groups*, Comm. Pure Appl. Math. 41(8) (1988) 1051–1088.

[11] L. J. Corwin, F. P. Greenleaf: *Representations of Nilpotent Lie Groups and their Applications. I: Basic Theory and Examples*, Cambridge Studies in Advanced Mathematics 18, Cambridge University Press, Cambridge (1990).

[12] L. Corwin, F. P. Greenleaf: *Commutativity of invariant differential operators on nilpotent homogeneous spaces with finite multiplicity*, Comm. Pure Appl. Math. 45(6) (1992) 681–748.

[13] L. Corwin, F. P. Greenleaf: *Spectral decomposition of invariant differential operators on certain nilpotent homogeneous spaces*, J. Funct. Anal. 108(2) (1992) 374–426.

[14] L. Corwin, F. P. Greenleaf, G. Grélaud: *Direct integral decompositions and multiplicities for induced representations of nilpotent Lie groups*, Trans. Amer. Math. Soc. 304(2) (1987) 549–583.

[15] J. Dixmier: *Sur l'algèbre enveloppante d'une algèbre de Lie nilpotente*, Arch. Math. 10 (1959) 321–326.

[16] M. Duflo: *Caractères des groupes et des algèbres de Lie résolubles*, Ann. Sci. École Norm. Sup. (4) 3 (1970) 23–74.

[17] M. Duflo: *Opérateurs différentiels bi-invariants sur un groupe de Lie*, Ann. Sci. École Norm. Sup. (4) 10(2) (1977) 265–288.

[18] M. Duflo: *Open problems in representation theory of Lie groups*, in: *Conference on Analysis on Homogeneous Spaces*, T. Oshima (ed.), August 25-30, Kataka, Japan (1986) 1–5.

[19] H. Fujiwara: *Représentations monomiales des groupes de Lie nilpotents*, Pacific J. Math. 127(2) (1987) 329–352.

[20] H. Fujiwara: *Représentations monomiales des groupes de Lie résolubles exponentiels*, in: *The Orbit Method in Representation Theory*, Copenhagen 1988, Progress in Mathematics 82, Birkhäuser, Boston (1990) 61–84.

[21] H. Fujiwara: *Analyse harmonique pour certaines représentations induites d'un groupe de Lie nilpotent*, J. Math. Soc. Japan 50(3) (1998) 753–766.

[22] H. Fujiwara, G. Lion, B. Magneron: *Algebras of functions associated with monomial representations of nilpotent Lie groups*, Prépub. Math. Univ. Paris 13 (2002) 2002-2.

[23] H. Fujiwara, G. Lion, B. Magneron, S. Mehdi: *A commutativity criterion for certain algebras of invariant differential operators on nilpotent homogeneous spaces*, Math. Ann. 327(3) (2003) 513–544.

[24] V. Guillemin, S. Sternberg: *Multiplicity-free spaces*, J. Differential Geom. 19(1) (1984) 31–56.

[25] A. A. Kirillov: *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk 17 (4/106) (1962) 57–110.

[26] T. Kobayashi: *Restrictions of generalized Verma modules to symmetric pairs*, Transform. Groups 17(2) (2012) 523–546.

[27] T. Kobayashi: *F-method for constructing equivariant differential operators*, in: *Geometric Analysis and Integral Geometry*, Contemp. Math. 598, American Mathematical Society, Providence (2013) 139–146.

[28] T. Kobayashi: *F-method for symmetry breaking operators*, Differential Geom. Appl. 33(suppl.) (2014) 272–289.

[29] T. Kobayashi: *Shintani functions, real spherical manifolds, and symmetry breaking operators*, in: *Developments and Retrospectives in Lie Theory*, Developments in Mathematics 37, Springer, Cham (2014) 127–159.

[30] T. Kobayashi, T. Kubo, M. Pevzner: *Conformal Symmetry Breaking Operators for Differential Forms on Spheres*, Lecture Notes in Mathematics 2170, Springer, Singapore (2016).

[31] T. Kobayashi, G. Mano: *The Schrödinger Model for the Minimal Representation of the Indefinite Orthogonal Group* $\mathrm{O}(p,q)$, Memoirs of the American Mathematical Society 213, Number 1000 (2011) 132 pp.

[32] T. Kobayashi, T. Oshima: *Finite multiplicity theorems for induction and restriction*, Adv. Math. 248 (2013) 921–944.

[33] T. Kobayashi, B. Ørsted, P. Somberg, V. Souček: *Branching laws for Verma modules and applications in parabolic geometry I*, Adv. Math. 285 (2015) 1796–1852.

[34] T. Kobayashi, M. Pevzner: *Differential symmetry breaking operators. I: General theory and F-method*, Selecta Math. (N.S.) 22(2) (2016) 801–845.

[35] T. Kobayashi, M. Pevzner: *Differential symmetry breaking operators. II: Rankin-Cohen operators for symmetric pairs*, Selecta Math. (N.S.) 22(2) (2016) 847–911.

[36] G. W. Mackey: *The Theory of Unitary Group Representations*, Lectures in Mathematics, University of Chicago Press, Chicago (1976).

[37] J. Peetre: *Une caractérisation abstraite des opérateurs différentiels*, Math. Scand. 7 (1959) 211–218.

[38] R. Penney: *Abstract Plancherel theorems and a Frobenius reciprocity theorem*, J. Funct. Anal. 18 (1975) 177–190.

[39] F. Rouvière: *Espaces symétriques et méthode de Kashiwara-Vergne*, Ann. Sci. École Norm. Sup. (4) 19(4) (1986) 553–581.

Yoshinori Tanimura
Grad. School of Math. Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
yotabaito@gmail.com

Received January 9, 2019
and in final form May 18, 2019